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FACTORIZATION AND THE GENERALIZED VENEZIANO MODEL WITH SATELLITES

David J. Gross ^{*)}
CERN - Geneva

A B S T R A C T

We construct the most general form of the Veneziano N-point functions with satellites that factorize along the leading trajectory. A large class of these generalizations factorize along the non-leading trajectories. The degeneracy of the daughter levels is greatly increased.

^{*)} Address after Sept. 1: Palmer Physical Laboratory, Princeton University, Princeton.

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1. INTRODUCTION

Soon after Veneziano ¹⁾ proposed a model for the four-point scattering amplitude that incorporates duality ²⁾ a prescription was given ^{3),4)} that generalizes this model to any number of external spinless particles. One of the most remarkable features of the multi-particle Veneziano formula is its factorization properties. These are particularly simple for the particles which lie on the leading trajectory. At the poles which occur when this trajectory passes through an integer the residue of the N-point function is simply a product of two such functions ^{3),4)}. Recently it has been shown that the residue of the poles that lie on the lower trajectories can be factorized into a finite sum of terms ^{5),6)}. In fact this factorization determines the level structure of the particle states contained in the N-point functions and their degeneracy (which increases rapidly with growing mass, M , as e^M).

The above mentioned N-point amplitudes were constructed by generalizing the simple form for the four-point amplitude

$$\begin{aligned}
 A(s, t) &= \text{const.} \int_0^1 dx \, x^{-\alpha(s)-1} (1-x)^{-\alpha(t)-1} + (\text{cyclic terms}) = \\
 &= \text{const.} \frac{\Gamma(-\alpha(s)) \Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))} + (\text{cyclic terms}).
 \end{aligned}
 \tag{1.1}$$

However, it is clear ^{1),7)} that a combination of such terms, such as

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{j=0}^m C_{n,m}^j \left[\frac{\Gamma(n-\alpha(s)) \Gamma(m-\alpha(t)) + \Gamma(m-\alpha(s)) \Gamma(n-\alpha(t))}{\Gamma(n+j-\alpha(s)-\alpha(t))} \right],
 \tag{1.2}$$

(where the $C_{n,m}^j$ are such that the series converges), has essentially the same properties as (1.1). Similar generalizations can easily be written for the N-point functions.

This type of generalization makes the Veneziano amplitude extremely non-unique. In most applications it has been assumed, without much justification, that these additional terms are not present. The problem studied in this paper is whether the requirement of factorization removes some, or all, of this ambiguity.

In Section 1 we discuss the N-point function with satellite terms [satellites give only additional non-leading trajectories, e.g., in (1.2) $C_{n,m}^j = \delta_{jm} C_{n,m}$). We find that it is indeed possible to factorize these amplitudes along the parent trajectory (without introducing degeneracies). We determine the most general form of the N-point function, with satellites, that factorizes. This general solution has many parameters, in fact in each successive N-point function, an almost arbitrary function of $N-3$ variables may appear.

In Section 2, we discuss the factorization along the lower trajectories. We find that it is possible, for a wide class of the above models, to factor the residues into a finite number of terms independent of the number of external particles considered. Thus, our conclusion is that one can generalize the multi-particle Veneziano amplitude, and introduce as many parameters as desired by adding satellites and preserve factorization.

There is however a price that must be paid for the additional freedom of these generalizations. The degeneracy of the daughter levels is now greatly increased. In this we agree with the conjecture of Bardakçi and Mandelstam ⁶⁾ that the no-satellite formula is unique if one demands the minimum number of factorizable daughter trajectories.

2. SATELLITES AND FACTORIZATION OF THE LEADING TRAJECTORY

The N-point amplitude for the scattering of identical scalar bosons is simply expressed in terms of the "dual" diagram, Fig. 1. This is the closed polygon, whose sides are the momenta of the external particles. To each internal diagonal line, connecting

vertices i and j , is associated an integration variable

$$u_{ij} \begin{cases} i=1 & j=2,3,\dots,N-2 \\ i=2,3,\dots,N & j=i+1,\dots,N-1 \end{cases} \quad (2.1)$$

and a trajectory

$$\alpha_{ij} = a + b [p_i + p_{i+1} + \dots + p_j]^2. \quad (2.2)$$

The N -point function is then given by an integral over the u_{ij} 's, where the integrand is just

$$\prod u_{ij}^{-\alpha_{ij}-1},$$

subject to the duality conditions. These conditions, which enforce the absence of forbidden coincident poles, state that a variable u must equal one whenever any of the variables which are dual to it (i.e., correspond to lines that intersect the line u) vanish.

The number of independent variables is reduced to $N-3$, which can be chosen to the $N-3$ non-intersected lines u_{1j} , $j=2,\dots,N-1$ corresponding to the particular tree diagram of Fig. 2. Finally we have ⁴⁾

$$B_N = \prod_{(i,j)} \int_0^1 du_{ij} u_{ij}^{-\alpha_{ij}-1} \prod_{(2 \leq k < j \leq N-1)} \delta \left[u_{kj} + \prod_{1 \leq p < k < q \leq j} u_{pq} \prod_{k < r \leq j < s \leq N-1} u_{rs} - 1 \right], \quad (2.3)$$

and the amplitude itself is a sum of $\frac{(N-1)!}{2}$ such terms, corresponding to cyclic permutations of the external lines.

If one performs the integrations over all the variables, except the u_{1j} , this form reduces to that given in Refs. 4).

The explicit solution for u_{kj} in terms of u_{1j} will be needed in the following, it is:

$$u_{k,j} = \frac{(1 - u_{1,k-1} u_{1,k} \dots u_{1,j})(1 - u_{1,k} u_{1,k+1} \dots u_{1,j-1})}{(1 - u_{1,k-1} u_{1,k} \dots u_{1,j-1})(1 - u_{1,k} u_{1,k+1} \dots u_{1,j})} \quad (2.4)$$

We now wish to modify (2.3) by adding satellites. To do this we multiply the integrand of (2.3) by

$$S_N = e^{F_N(u_{ij})}, \quad (2.5)$$

obtaining a new function B'_N .

In order to preserve duality, crossing, and the analytic properties of B'_N , we must require $S_N(u_{ij})$ to be an analytic function of the u_{ij} 's within the range of integration, and to be symmetric under a cyclic change of variables. For convenience, we exclude the case where S_N vanishes within the range of integration (it cannot vanish at ends without altering the pole structure), so that F_N is analytic. Any cyclic symmetric function F_N can be written in the manifestly symmetric form

$$\begin{aligned} F_N(u_{ij}) = & F_N(u_{12}, u_{13}, \dots, u_{1N-2}) + F_N(u_{23}, u_{24}, \dots, u_{2N-1}) + \\ & + F_N(u_{34}, u_{35}, \dots, u_{3N-1}, u_{12}) + \dots + F_N(u_{k,k+1}, \dots, u_{k,N-1}, u_{1,k-1}, \dots, u_{k-2,k-1}) \\ & + \dots + F_N(u_{2N-1}, u_{3N-1}, \dots, u_{N-2,N-1}). \end{aligned} \quad (2.6)$$

The normalization of the four-point function is, of course, arbitrary, and therefore we choose

$$F_4(u_{12}=0, u_{23}=1) = F_4(u_{12}=1, u_{23}=0) = 0 \quad (2.7)$$

The normalization of F_N ($N > 4$) will be determined through factorization.

We can therefore write

$$F_4(u_{12}, u_{23} = 1 - u_{12}) = 2u_{12}(1 - u_{12})[\tilde{F}_4(u_{12}) + \tilde{F}_4(u_{23})] \quad (2.8)$$

With this form the four-point function is

$$B_4'(s, t) = \text{const.} \int_0^1 dx x^{-\alpha_{12}-1} (1-x)^{-\alpha_{23}-1} \exp\{2x(1-x)[\tilde{F}_4(x) + \tilde{F}_4(1-x)]\}. \quad (2.9)$$

When the f_4 is expanded in a power series in x , one recovers, upon integrating, the series given by (1.2) with $C_{nm}^j = \delta_{jm} C_{nm}$.

The residue of B_N at a pole corresponding to the first particle on the leading trajectory (which is identical with the scalar external particles), is simply a product of two B_N functions. For B_N' to factorize at this pole it is necessary and sufficient that when the variable u_{1j} vanishes, F_N can be written as a sum of the F_{j+1} and F_{N-j+1} corresponding to the residues of the pole in α_{1j} (see Fig. 3). When $u_{1j} = 0$, all variables dual to it equal one, namely $u_{p,q} = 1$ for $2 \leq p < q \leq N-1$ when $u_{1j} = 0$.

Therefore we have the following condition:

$$\begin{aligned} F_N(u_{ij})|_{u_{1j}=0} &= F_N(u_{12}, \dots, u_{1j-1}, 0, u_{1j+1}, \dots, u_{1N-2}) + \\ &+ F_N(u_{23}, \dots, u_{2j}, 1, \dots, 1) + \dots + F_N(u_{k-1k+1}, \dots, u_{kj}, 1, \dots, 1, u_{1k-2}, \dots, u_{k-2k-1}) + \\ &+ \dots + F_N(u_{j+1j+2}, \dots, u_{j+1N-1}, 0, u_{2j}, \dots, u_{j-1j}) + \dots + F_N(u_{q-1q+1}, \dots, u_{q-1q-1}, 1, \dots, 1, u_{j+1q-1}, \dots, u_{q-2q-1}) + \\ &+ \dots + F_N(1, \dots, 1, u_{j+1N-1}, \dots, u_{N-2N-1}). = \end{aligned}$$

$$\begin{aligned}
 &= f_{j+1}(u_{12}, \dots, u_{2j-1}) + f_{j+1}(u_{23}, \dots, u_{2j}) + \dots + \\
 &+ f_{j+1}(u_{k+1}, \dots, u_{kj}, u_{1k-1}, \dots, u_{k-2k-1}) + \dots + f_{j+1}(u_{2j}, \dots, u_{j-1,j}) + \\
 &+ f_{N-j+1}(u_{j+1,j+2}, \dots, u_{j+1,N-1}) + f_{N-j+1}(u_{j+2,j+3}, \dots, u_{j+2,N-1}, u_{1j+1}) + \\
 &+ \dots + f_{N-j+1}(u_{q+1}, \dots, u_{qN-1}, u_{1q-1}, \dots, u_{q-2q-1}) + \dots + \\
 &+ f_{N-j+1}(u_{1j+1}, \dots, u_{2N-1}).
 \end{aligned} \tag{2.10}$$

It is clear that this will be satisfied if f has the following properties:

$$f_N(x_1, \dots, x_{k-1}, 0, x_{k+1}, \dots, x_{N-3}) = f_{k+2}(x_1, \dots, x_{k-1}) + f_{N-k}(x_k, \dots, x_{N-3}) \tag{2.11}$$

and

$$f_N(1, x_2, \dots, x_{N-3}) = f(x_2, 1, \dots, x_{N-3}) = \dots = f_N(x_2, x_3, \dots, x_{N-3}, 1) = f_{N-1}(x_2, \dots, x_{N-3}), \tag{2.12}$$

where we define $f_3 = 0$.

Using the fact that f_N itself can be chosen, without loss of generality, to be cyclic symmetric $[f_N = (1/N)f_{N-}]$, one can show that this is the most general solution of (2.10).

Due to these conditions, $f_N(x_1, \dots, x_{N-3})$ is completely determined whenever any $x_i = 0$ or 1, by the values of f_i , $i = 4, \dots, N-1$. It is simple to construct the general solution of (2.11) and (2.12). It is

$$f_N(x_1, \dots, x_{N-3}) = \sum_{k=1}^{N-3} \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq N-3} x_{i_1} x_{i_1+1} \dots x_{i_2} x_{i_2+1} \dots x_{i_k} (1-x_{i_1})(1-x_{i_2}) \dots (1-x_{i_k}) \tilde{f}_{k+3}(x_{i_1}, \dots, x_{i_k}) \tag{2.13}$$

The functions $\tilde{f}_N$, $N=4, \dots, \infty$, are completely arbitrary.

Once factorization has been achieved for the lowest member of the leading trajectory it is then automatic for all the states on the leading trajectory. This is because $F_N(u_{ij})$ does not depend on any of the external momenta, and therefore the residue of the spin J particle, when $\alpha_{1j} = J$, will be proportional to

$$\exp\{F_N(u_{ij})\} |_{u_{ij}=0}$$

which already factorizes.

In summary, we can modify the N -point function B_N by introducing a weight function $\exp[F_N(u_{ij})]$, where F_N is given by Eqs. (2.6) and (2.13), without affecting the factorization along the leading trajectory. At this stage we have at our disposal, for each successive N -point function, a completely arbitrary function $\tilde{f}_N$ of $N-3$ variables.

3. FACTORIZATION OF THE LOWER TRAJECTORIES

Factorization for the daughter trajectories does not follow automatically, as it does for the leading trajectory, even for the unmodified N -point function. However, it has been shown, Refs. 5), 6), that the residue of B_N , when $\alpha_{ij} = J$, can be written as a sum of factorizable terms, i.e., the daughter levels are degenerate and the number of these terms does not depend on the number of external lines. The last point is the crucial one for a consistent physical interpretation of the daughter poles as narrow resonances. It is far from obvious that our modified N -point function can be analogously factorized. We shall show that for a wide class of modifications this is possible. The degeneracy of the daughter levels will, however, be greatly increased.

To simplify things we shall adopt the notation of Fubini and Veneziano (FV) ⁵⁾. We label the particles $p_0, \dots, p_{s+1}$,

$q_0, \dots, q_{r+1}$, according to the tree diagram of Fig. 4, and look at the $(s+1)$ body [or $(r+1)$ body] pole at $\alpha(s) = N$. We will distinguish four types of variables. First we have z , which is associated with the pole under discussion. Then we have variables corresponding to lines that are completely to the left of z in the dual diagram. These are (see Fig. 5) $x_1, x_2, \dots, x_r$ and the variables

$$l_{pq} = \frac{(1 - x_p x_{p+1} \dots x_{q-1})(1 - x_{p-1} x_p \dots x_q)}{(1 - x_p x_{p+1} \dots x_q)(1 - x_{p-1} x_p \dots x_{q-1})}, \quad 1 \leq p < q \leq r+1 \quad (3.1)$$

where we have used (2.4), and we define $x_0 = z$, $x_{r+1} = 0$. Similarly we have "right" variables, whose lines are completely to the right of z , namely $y_1, y_2, \dots, y_s$ and

$$r_{pq} = \frac{(1 - y_p y_{p+1} \dots y_{q-1})(1 - y_{p-1} y_p \dots y_q)}{(1 - y_p y_{p+1} \dots y_q)(1 - y_{p-1} y_p \dots y_{q-1})} \quad 1 \leq p < q \leq s+1 \quad (3.2)$$

where $y_0 = z$, $y_{s+1} = 0$.

Finally we have dual variables, whose lines intersect z (see Fig. 5)

$$d_{pq} = \frac{(1 - s_p z \sigma_{q-1})(1 - s_{p-1} z \sigma_q)}{(1 - s_p z \sigma_q)(1 - s_{p-1} z \sigma_{q-1})}, \quad \begin{matrix} p=1 \dots r+1 \\ q=1 \dots s+1 \end{matrix} \quad (3.3)$$

where, following FV, we define

$$s_i = x_1 x_2 \dots x_i, \quad \sigma_i = y_1 y_2 \dots y_i, \quad s_0 = \sigma_0 = 1. \quad (3.4)$$

Fubini and Veneziano have shown that the B_{r+s+4} function can be written as

$$B_{r+s+4} = \int dx_1 \dots dx_r \varphi(x, p) \int dy_1 \dots dy_s \varphi(y, q) \int_0^1 dz z^{-N(s)-1} \phi(z; x, p; y, q), \quad (3.5)$$

where

$$\phi(z; x, p; y, q) = \exp \left\{ \sum_{n=1}^{\infty} \frac{z^n}{n} (1 - \alpha - P_{\mu}^{(n)} Q_{\mu}^{(n)}) \right\}, \quad (3.6)$$

$$P_{\mu}^{(n)} = \sqrt{26} \sum_{i=0}^r (p_i)^n p_{\mu}^i, \quad Q_{\mu}^{(n)} = \sqrt{26} \sum_{i=0}^s (q_i)^n q_{\mu}^i, \quad (3.7)$$

and $\varphi(x, p) [\varphi(y, q)]$ is the usual integrand of the $r+3$ ($s+3$) point function. The coefficient of z^N in the expansion of ϕ , in powers of z , is then just the residue of the pole at $\alpha(s) = N$, and can be written as a sum of terms that are products of a function of the left variables and vectors (x and p) times a function of the right variables and vectors (y and q).

To prove that one can factorize our modified function, we must show that $F_{r+s+4}[\bar{x} \ell; z; y, r; d]$ can be expanded in a power series in z , whose coefficients are sums of factorizable terms, the number of which is independent of r and s . To this end we expand the variables, ℓ_{1q} , r_{1q} and d_{pq} (the other variables are independent of z) around $z=0$,

$$\ell_{1q} = \frac{1 - p_{q-1}}{1 - p_{q-1} x_q} \left[1 + (1 - x_q) \sum_{n=1}^{\infty} (p_{q-1})^n z^n \right], \quad (3.8)$$

$$r_{1q} = \frac{1 - q_{q-1}}{1 - q_{q-1} y_q} \left[1 + (1 - y_q) \sum_{n=1}^{\infty} (q_{q-1})^n z^n \right], \quad (3.9)$$

$$d_{pq} = 1 + (1 - x_p)(1 - y_q) \sum_{n=1}^{\infty} (p_{q-1})^n (q_{q-1})^n \frac{1 - (x_p y_q)^n}{1 - x_p y_q} z^n. \quad (3.10)$$

$p=1 \dots r+1$
 $q=1 \dots s+1$

Let us consider the simplest satellite term, i.e., we choose $\tilde{f}_4$ to be a constant λ , and $\tilde{f}_N$ ($N \geq 5$) to vanish. In this case it is a simple and tedious exercise to insert the expansion (3.8)-(3.10) in Eqs. (2.6) and (2.13). We find

$$F_{r+s+4}(x, z, y) = F_{r+3}(x) + F_{s+3}(y) + \lambda z(1-z) +$$

$$+ \lambda \sum_{n=1}^{\infty} z^n \left\{ L_n(x) + R_n(y) + \sum_{m=0}^{n-1} \left[D_{nm}^1(x) D_{nm}^1(y) + \frac{1}{2}(n+m)(n-m-1) D_{nm}^2(x) D_{nm}^2(y) \right] \right\}$$

(3.11)

where

$$F_{r+3}(x) = \lambda \left[\sum_{i=1}^r x_i(1-x_i) + \sum_{2 \leq p < q \leq r+1} l_{pq}(1-l_{pq}) \right] \Big|_{z=0},$$

$$L_n(x) = \sum_{q=2}^{r+1} \frac{1-s_{q-1}}{1-s_{q-1}x_q} (1-x_q) s_{q-1}^n \left[1 - \frac{1-s_{q-1}}{1-s_{q-1}x_q} (n+1-x_q(n-1)) \right],$$

and

$$D_{nm}^i(x) = \sum_{q=1}^{r+1} (1-x_p)^i (s_{p-1})^n x_p^m,$$

with similar expression for $F_{s+3}(y)$, $R_n(y)$ and $D_{nm}^i(y)$. Thus

$$\left(\frac{d}{dz} \right)^n F_{r+s+4} \Big|_{z=0}$$

is a sum of $2n+1$ factorizable terms. The residue of the pole at $\alpha(s) = n$, will be proportional to

$$\exp\{F_{r+3}(x) + F_{s+3}(y)\} \times \sum_{p(n)} \prod_{j=1}^n \frac{(A_j)^{i_j}}{(i_j)!}$$

$$A_j = \frac{1}{j} (1-a - P_{\mu}^{(j)} \cdot Q_{\mu}^{(j)}) + \left(\frac{d}{dz} \right)^j F_{r+s+4}(x, z, y) \Big|_{z=0},$$

(3.12)

where $P(n)$ is the set of partitions of n into integers i ; such that $\sum j \cdot i_j = n$. The number of levels is greatly increased (determination of the exact number is a very difficult problem), but this number is certainly finite and independent of r and s .

Furthermore if $\tilde{f}_4(x)$ is taken to be a finite polynomial in x , or even an arbitrary analytic function of x , one can easily show, by expanding $\tilde{f}_4$ in a power series in z collecting and

summing terms, that $(d/dz)^j F_{r+s+4}(x,z,y)_{z=0}$ can be written as a sum of a number of factorizable terms (independent of r and s). Similarly, if $\tilde{f}_5, \tilde{f}_6, \dots, \tilde{f}_N$ are arbitrary, the same factorization takes place. For each new parameter that is introduced in this fashion, the number of degenerate daughter levels increases accordingly.

The only part of our general solution of Section 2 that does not factorize is when an infinite number of $\tilde{f}$'s are non-vanishing. This is because as we increase r or s , more and more $\tilde{f}$'s appear and the number of factorizable terms in $(d/dz)^j F_{r+s+4}(x,z,y)_{z=0}$ will increase without limit.

As in the factorization of the unmodified N-point function ^{5),6)} many of the daughters have imaginary coupling constants (ghosts), and some of the daughters are spurious due to linear dependence among the vertices.

4. CONCLUSIONS

We have found the general form of the generalized Veneziano model with satellites that factorizes along the leading trajectory. This generalization is determined by an arbitrary set of functions $\tilde{f}_N$, $N=4,5,\dots$, of $N-3$ variables. If all but a finite number of these functions vanish, then one has complete factorization at all levels. The number of degenerate daughters is greatly increased. This makes it unlikely that these generalizations can be used to cure the divergences found when loop diagrams are constructed from the factorized N-point functions ⁸⁾, since these divergences are due to the large number of particle states.

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FIG.1 'DUAL' DIAGRAM FOR THE N-POINT FUNCTION

FIG.2 TREE DIAGRAM CORRESPONDING TO THE VARIABLES U_{1j}

FIG.3 DIAGRAM FOR THE POLE AT $\alpha_{1j}=0$

FIG.4 TREE DIAGRAM CORRESPONDING TO THE FV VARIABLES

FIG.5 'DUAL' DIAGRAM CORRESPONDING TO THE FV VARIABLES