Pair creation in inhomogeneous fields from worldline instantons

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Abstract. We show how to do semiclassical nonperturbative computations within the worldline approach to quantum field theory using “worldline instantons”. These worldline instantons are classical solutions to the Euclidean worldline loop equations of motion, and are closed spacetime loops parametrized by the proper-time. Specifically, we compute the imaginary part of the one loop effective action in scalar and spinor QED using worldline instantons, for a wide class of inhomogeneous electric field backgrounds.

Keywords: Pair production; semiclassical approximation.

PACS: 11.27.+d, 12.20.Ds

INTRODUCTION

As is well-known, quantum field theory allows the spontaneous creation of electron-positron pairs from vacuum in external electric fields. This effect has been considered already in the early days of quantum electrodynamics [1, 2], and Schwinger used effective action methods to obtain a simple closed-form expression for the production rate in the constant field case [3]. Although spontaneous pair creation is of potential interest for many branches of physics the chances for its direct experimental verification hitherto seemed very remote. This is due to the exponential smallness of the production rate for field strengths below the critical value

\[ E_c = \frac{m^2 c^3}{\hbar \alpha} \approx (1.3 \times 10^{18}) \text{V/m}, \]

which is far above the electric fields which can be produced in the laboratory macroscopically. However, given the rapid progress of laser technology it seems not any more impossible that in the near future pair production might be observable in laser fields. Both the optical laser system POLARIS [4] under construction at the Jena high-intensity laser facility and the X-ray free electron lasers to be constructed at SLAC [5] and DESY [6] are expected to reach laser field strengths missing \( E_c \) only by a few orders of magnitude. Moreover, it has been argued that for focused laser pulses substantial pair creation should set in already somewhat below critical field strength [7].

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1 Talk given by C. S. at X Mexican Workshop on Particles and Fields, Morelia, Mexico, Nov. 6 – 12, 2005 (to appear in the conference proceedings).
Laser fields cannot usually be treated in the constant field approximation, and this is particularly true for the pair creation process due to its nonperturbative nature. Much effort has gone into developing methods for the calculation of pair creation rates in inhomogeneous fields, mostly based on WKB \[8, 9, 10, 11, 12, 13\]. In this talk I will present a substantially different approach \[14\] based on Feynman’s worldline path integral formalism \[15, 16\] and work done by Affleck et al. in 1982 for the constant field case in scalar QED \[17\]. The path integral representing the imaginary part of the one loop effective action is calculated in a semiclassical approximation around a closed classical “instanton” trajectory. The worldline action evaluated on this solution directly gives the Schwinger exponent of the imaginary part of the effective action in the weak field approximation.

**PAIR CREATION IN ELECTRIC FIELDS**

Let us start with Schwinger’s well-known formula for the imaginary parts of the scalar and spinor QED effective Lagrangians in a constant electric field (at one loop) \[3\]:

\[
\text{Im}\mathcal{L}^{(1)}_{\text{scalar}}(E) = \frac{m^4}{16\pi^3} \beta^2 \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\exp\left[ -\frac{\pi n}{\beta} \right]}{n^2}
\]

\[
\text{Im}\mathcal{L}^{(1)}_{\text{spinor}}(E) = \frac{m^4}{8\pi^3} \beta^2 \sum_{n=1}^{\infty} \frac{1}{n^2} \exp\left[ -\frac{\pi n}{\beta} \right]
\]

where \( \beta = \frac{eE}{m^2} \). The first term in each series gives (up to a factor of 2) directly the total pair-production rates per volume per time \[18\]. The higher order \( n \geq 2 \) terms are statistics dependent and contain the information on the coherent production of \( n \) pairs by the field. All terms are exponentially suppressed for \( E < E_{\text{cr}} \). Higher loop corrections have also been considered \[19, 20\]. They can be neglected for subcritical fields as far as the total pair production rate is concerned \[21\].

The formulas \[1\] are commonly derived from the standard propertime representation for the one-loop effective Lagrangian. For spinor QED, this is the Euler-Heisenberg Lagrangian \[2\]:

\[
\mathcal{L}^{(1)}_{\text{spinor}}(E) = -\frac{1}{8\pi^2} \int_0^{\infty} dT \frac{e^{2m^2T}}{T^3} \left[ \frac{eET}{\tan(eET)} + \frac{1}{3}(eET)^2 - 1 \right]
\]

The \( n \)th term in the series for \( \text{Im}\mathcal{L}_{\text{spinor}} \) in \[1\] is generated by the pole at \( T = \frac{n\pi}{eE} \) of the integrand in \[2\].
SCHWINGER’S FORMULA FROM WORLDLINE INSTANTONS

Feynman in 1950 [16] presented, “as an alternative to the formulation of second quantization”, a formula representing the scalar QED effective action \( \Gamma_{\text{scalar}}[A] \) in terms of first-quantized worldline path integrals. In the quenched approximation (i.e. with only one scalar loop but any number of photons) it reads

\[
\Gamma^{(\text{quenched})}_{\text{scalar}}[A] = \int d^4x \mathcal{L}^{(\text{quenched})}_{\text{scalar}}[A] = \int_0^\infty \frac{dT}{T} e^{-m^2 T} \int \mathcal{D}x(\tau) e^{-S[x(\tau)]} \tag{3}
\]

Here the path integral is over all closed loops in spacetime with a given period \( T \) in the proper-time of the loop scalar. The worldline action \( S = S_0 + S_e + S_i \) has three parts,

\[
S_0 = \int_0^T d\tau \frac{x^2}{4}, \\
S_e = ie \int_0^T \dot{x}^\mu A_\mu(x(\tau)), \\
S_i = -\frac{e^2}{8\pi^2} \int_0^T d\tau_1 \int_0^T d\tau_2 \frac{\dot{x}(\tau_1) \cdot \dot{x}(\tau_2)}{(x(\tau_1) - x(\tau_2))^2} \tag{4}
\]

Of these \( S_e \) incorporates the interaction with the external field, while \( S_i \) takes all internal photon exchanges in the loop into account.

Feynman generalized this “worldline representation” to spinor QED in [16]. Although it has always been considered an interesting alternative to standard quantum field theory, only in recent years it has gained some popularity as a calculational tool. Presently there exist, among others, the following approaches to the calculation of this type of path integral:

- The “string-inspired” approach [22, 23, 24, 20, 25] (see [26] for a review) which aims at an analytical calculation using appropriate worldline Green’s functions.
- Semiclassical calculation using a stationary phase approximation [17].
- Variational methods [27].
- Numerical calculation using Monte Carlo methods [28, 29]. This approach in principle applies to effective actions in arbitrary backgrounds, and has been applied to the pair creation process in [30].

We will expand here on the first approach, which is inspired by instanton methods in field theory. The idea is to calculate \( \text{Im}\mathcal{L}_{\text{scalar}} \) for weak fields using an extremal trajectory of the worldline path integral for a stationary phase approximation. In [17] it was shown that for the case of a constant electric field in the \( z \) direction this extremal action trajectory (“worldline instanton”) is given by a circle in the (euclidean) \( t - z \) plane:

\[
x_{\text{extremal}}(\tau) = \frac{m}{eE} (0, 0, \cos(2\pi \tau), \sin(2\pi \tau)) \quad (T = 1) \tag{5}
\]
At leading order in the stationary phase approximation, the exponent of the imaginary part of the effective Lagrangian is given by the worldline action of this trajectory,

$$\text{Im} \mathcal{L}_{\text{scalar}}^{(\text{quenched})}(E) \sim e^{-S[x_{\text{extremal}}]}$$

This is easily evaluated to be

$$(S_0 + S_e)[x_{\text{extremal}}] = \frac{m^2}{eE}, \quad S_i[x_{\text{extremal}}] = -\alpha \pi. \quad (7)$$

The contribution of $S_0 + S_e$ just reproduces the first of the exponentials in Schwinger’s one-loop formula (1), while the higher order ones are generated by the “multi-instantons” where the same circle is traversed $n$ times. More surprising is the simplicity of the $S_i$ term, which represents the contribution of all the higher loop corrections involving arbitrary photon exchanges in the loop. According to [17], this is the exact all-orders result in the weak field limit, including renormalization effects:

$$\text{Im} \mathcal{L}_{\text{scalar}}^{(\text{quenched})}(E) \xrightarrow{\beta \to 0} \frac{m^4}{8\pi^3} \beta^2 \exp\left(-\frac{\pi}{\beta} + \alpha \pi\right) \quad (8)$$

At the two-loop level this remarkable formula has been independently confirmed [31].

**INHOMOGENEOUS BACKGROUND FIELDS**

Despite of the simplicity and elegance of this worldline instanton approach it appears that the work of [17] has never been extended either to spinor QED or to more general backgrounds. As we will now show, at least at the one-loop level the method generalizes to a large class of inhomogeneous backgrounds straightforwardly. Let us return to Feynman’s formula (3), omitting the photon exchange term $S_i$:

$$\Gamma^{(1-\text{loop})}[A] = \int_0^\infty \frac{dT}{T} e^{-m^2 T} \int \mathcal{D}x \ e^{-\int_0^T d\tau \left(\frac{\dot{x}^2}{2} + ie A \cdot \dot{x}\right)} \quad (9)$$

Rescaling $\tau = Tu$, this becomes

$$\Gamma^{(1-\text{loop})}[A] = \int_0^\infty \frac{dT}{T} e^{-m^2 T} \int \mathcal{D}x \ e^{-\left(\frac{T}{\beta} \int_0^1 du \dot{x}^2 + ie \int_0^1 du A \cdot \dot{x}\right)} \quad (10)$$

The $T$ integral has a stationary point at

$$T_0^2 = \frac{\int du \dot{x}^2}{m^2} \quad (11)$$
leading to

\[
\text{Im} \Gamma_{\text{scalar}}^{(1-\text{loop})} = \frac{1}{m} \sqrt{\frac{2\pi}{t_0}} \text{Im} \int \mathcal{D}x \ e^{-\left( \frac{m}{\sqrt{\int du \dot{x}^2 + ie \int_0^1 du A \cdot \dot{x}}} \right)}
\]  

(12)

The new worldline action,

\[
S = m \sqrt{\int du \dot{x}^2 + ie \int_0^1 du A \cdot \dot{x}}
\]  

(13)

is stationary if

\[
m \frac{\ddot{x}_\mu}{\sqrt{\int du \dot{x}^2}} = ieF_{\mu\nu}\dot{x}_\nu
\]  

(14)

Contracting with \( \dot{x}^\mu \) yields \( x^2 = \text{constant} \equiv a^2 \) so that

\[
m \ddot{x}_\mu = ieF_{\mu\nu}\dot{x}_\nu
\]  

(15)

Now specialize to a \textit{time-dependent electric field directed in the} \( x_3 \) \textit{direction}. Choose a gauge where

\[
A_3 = A_3(x_4); \quad A_\mu = 0 \text{ for } \mu \neq 3.
\]  

(16)

Since \( F_{\mu1} = F_{\mu2} = 0 \), the stationarity conditions together with the periodicity imply that \( x_1 \) and \( x_2 \) must be constant. Hence one is down to an equation for \( x_3 \) and \( x_4 \):

\[
\ddot{x}_3 = \frac{iea}{m} F_{34} \dot{x}_4, \quad \ddot{x}_4 = -\frac{iea}{m} F_{34} \dot{x}_3.
\]  

(17)

In terms of \( A_3 \) this can be further reduced to

\[
\ddot{x}_3 = -\frac{iea}{m} A_3(x_4), \quad |\dot{x}_4| = a \sqrt{1 + \left( \frac{eA_3(x_4)}{m} \right)^2}.
\]  

(18)

As an example, let us consider the following single-pulse electric background [12, 11],

\[
E(t) = E \text{ sech}^2(\omega t)
\]  

(19)
For this background the solution of (17) turns out to be very simple:

\[
\begin{align*}
    x_3(u) &= -\frac{1}{\omega} \frac{1}{\sqrt{1+\gamma^2}} \text{arcsinh}[\gamma \cos(2n\pi u)] \\
    x_4(u) &= \frac{1}{\omega} \arcsin \left( \frac{\gamma}{\sqrt{1+\gamma^2}} \sin(2n\pi u) \right)
\end{align*}
\]

(20)

Here \(\gamma \equiv \frac{m_0\omega}{eE}\) denotes the “adiabaticity parameter” \cite{8} and the integer \(n \in \mathbb{Z}^+\) counts the number of times the closed path is traversed. The worldline action (13) evaluated on this instanton is

\[
S_0 = n \frac{m^2 \pi}{eE} \left( \frac{2}{1 + \sqrt{1 + \gamma^2}} \right)
\]

(21)

In fig. 1 we plot the the instanton trajectories for various values of the parameter \(\gamma\). In the static limit \(\gamma \to 0\) we recover the circular paths \cite{5} of the constant field case. In the short-pulse limit \(\gamma \to \infty\) the instantons shrink in size and become elongated. Thus the instanton action decreases with increasing \(\gamma\), leading to a local enhancement of the pair creation rate as compared to the case of a constant field with magnitude \(E\).

**FIGURE 1.** Plot of the worldline instanton paths \cite{20} in the \((x_3,x_4)\) plane for the case of a time-dependent electric field \(E(t) = E \text{sech}^2(\omega t)\). The paths are shown for various values of the adiabaticity parameter \(\gamma\). \(x_3, x_4\) have been expressed in units of \(\frac{m}{eE}\).
The case of a spatially inhomogeneous electric field in the \( z \) direction can be treated completely analogously. The (euclidean) gauge potential can be chosen as

\[
A_4 = A_4(x_3); \quad A_\mu = 0 \quad \text{for } \mu \neq 4.
\]  

This leads to instanton equations differing from eqs. (18) simply by the interchange \( 3 \leftrightarrow 4 \).

As an example, let us consider the spatial analogue of (19), i.e. a single-bump electric field depending only on \( x_3 \):

\[
E(x_3) = E \text{sech}^2(kx_3)
\]  

The instanton solutions are obtained from the single-pulse ones (20) \textit{mutatis mutandis}:

\[
x_3(u) = \frac{m}{eE} \frac{1}{\gamma} \arcsinh \left( \frac{\gamma}{\sqrt{1 - \gamma^2}} \sin(2\pi n u) \right)
\]

\[
x_4(u) = \frac{m}{eE} \frac{1}{\gamma \sqrt{1 - \gamma^2}} \arcsin \left( \gamma \cos(2\pi n u) \right)
\]  

with \( \gamma = \frac{mk}{eE} \). The stationary action is

\[
S_0 = \frac{m^2 \pi}{eE} \left( \frac{2}{1 + \sqrt{1 - \gamma^2}} \right)
\]  

These solutions are plotted in fig. 2 for various values of \( \gamma \). Note that again they reduce to the constant field circles for \( \gamma \to 0 \), but they grow in size with increasing \( \gamma \) and become infinitely large when \( \gamma \to 1 \). Physically, this is where the width of the electric field becomes too small for a virtual pair to extract from it the energy necessary to turn real.

The instanton action increases with increasing \( \gamma \), leading to a lower local pair production rate as compared to the constant field case. Thus we see a rule emerging here which we believe holds quite generally:

- \textit{Temporal inhomogeneity increases the pair production rate}
- \textit{Spatial inhomogeneity decreases the pair production rate}
FIGURE 2. Plot of the instanton paths for the case of a space-dependent field $E(x) = Esech^2(kx)$ for various values of the adiabaticity parameter $\gamma$.

THE SPINOR LOOP CASE

A path integral representation for the one loop effective action in spinor QED [16] can be obtained from the scalar QED one [7] by multiplication by a global factor of $-\frac{1}{2}$ and the insertion of the following “spin factor” $S[x,A]$,

$$S[x,A] = \text{tr}_T e^{\frac{i}{2} \sigma^\mu \nu \int_0^T d\tau F_{\mu \nu}(x(\tau))}$$

For the two-dimensional background fields considered above the path ordering has no effect, since the $F_{\mu \nu}(x(\tau))$’s at different proper-times commute. The spin factor then reduces to

$$S[x,A] = 4 \cos \left[ eT \int_0^1 du E(x(u)) \right]$$

Since the exponent of the spin factor (26) is purely imaginary it affects neither the determination of the stationary point $T_0$ nor the instanton equations (14). Therefore it remains only to evaluate the spin factor on the instanton solutions for the scalar loop. For the class of backgrounds of the form $A_3(x_4)$ or $A_4(x_3)$ considered above the result turns out to be [14]

$$S[x,A] = 4(-1)^n$$

Thus at least for this class of inhomogeneous backgrounds we find that $\text{Im} \mathcal{L}_{\text{scalar}}^{(1)}$ and $\text{Im} \mathcal{L}_{\text{spinor}}^{(1)}$ differ by the the same simple sign changes as in the constant field case, eq. (1).
SUMMARY, WORK IN PROGRESS

We have shown that the worldline instanton approach holds considerable promise as a tool for calculating pair creation rates in inhomogeneous backgrounds in scalar and spinor QED. Although the class of backgrounds which we have considered here is also amenable to a treatment by WKB methods [10, 11, 12, 13] the worldline approach offers a number of distinct advantages: (i) it bypasses the momentum space integrals which usually would have to be done in this type of calculation (ii) it uses proper-time instead of time, which seems more natural in the treatment of an intrinsically relativistic effect such as pair creation (iii) it provides a framework for the inclusion of radiative corrections.

Clearly we have not fully explored here the potential of this method. The instanton equations in the form (14) generalize immediately to the case of an arbitrary electromagnetic background field. While closed-form solutions can be expected only in special cases, the numerical integration of these equations poses no problems in principle. However, the instanton provides only the exponents of the Schwinger exponentials. We have not discussed here the prefactors of the Schwinger exponentials (1), which in the present approach involve the determinant of fluctuations around the worldline instanton. In the constant field case the determinant computation is straightforward, since the fluctuation problem is Gaussian [17]. As will be shown in a forthcoming paper [32] for inhomogeneous background fields the determinant can be computed using the Gelfand-Yaglom technique, since the fluctuation operator is an ordinary differential operator, depending only on the proper-time.

ACKNOWLEDGMENTS

We are very grateful to Don Page for helpful correspondence. We acknowledge the support of the NSF US-Mexico Collaborative Research Grant 0122615.
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