Unveiling Hidden Patterns in CMB Anisotropy Maps

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ABSTRACT

Bianchi VII$_h$ models have been recently proposed to explain potential anomalies in the CMB anisotropy as observed by WMAP. We investigate the statistical isotropy of embedded Bianchi VII$_h$ templates in the CMB anisotropy maps to determine whether the existence of a hidden Bianchi template in the WMAP data is consistent with the previous null detection of the bipolar power spectrum in the WMAP first year maps. We compute the bipolar power spectrum for low density Bianchi VII$_h$ models embedded in the background CMB anisotropy maps with the power spectrum that best fits the first year WMAP data. By examining statistical isotropy of these maps, we put a limit of $(\sigma_H H_0) \geq 2.55 \times 10^{-10}(99.9\% CL)$ on the shear parameter in Bianchi VII$_h$ models which is (marginally) consistent with Jaffe et al. (2005) but not with the enhanced Bianchi templates of Cayon et al. (2006). In this paper we also present a detailed review of Bianchi models and their properties.

1. Introduction

Friedmann-Robertson-Walker (FRW) models are simplest models of the expanding Universe which are spatially homogeneous and isotropic. When proposed, the principal justification for studying these models was their mathematical tractability rather than observational evidence. However, we now have observational evidence from the isotropy of CMB and large scale structure that the structure of the universe at large scales must be very close to that of a FRW model. Still though, there is a freedom to choose homogeneous models which are initially anisotropic and, as the time goes on, become more isotropic and asymptotically tend to a FRW model. Bianchi models provide a generic description
of homogeneous anisotropic cosmologies and they are classified into 10 equivalence classes (Ellis & MacCallum 1969). Among these models it is reasonable to consider only those types that encompass FRW models. These are types I and VII$_0$ in the case of $k = 0$, V and VII$_h$ in the case of $k = -1$, and IX in the case of $k = +1$. Bianchi models which do not admit FRW solutions become highly anisotropic at large times. Type IX models recollapse after a finite time and hence do not approach arbitrarily near to isotropy. Also models of type VII$_h$ will not in general approach isotropy (Collins & Hawking 1973b). The most general Bianchi types that admit FRW models are Bianchi types VII$_h$ and IX. These two types contain types I, V, VII$_0$ as special sub-cases. An interesting feature of these models is that they resemble a universe with a vorticity. It is interesting to determine bounds on universal rotation from cosmological observations because the absence of such a rotation is a prediction of many models of the early universe.

CMB anisotropy is a powerful tool to study the evolution of vorticity in the universe because models with vorticity have clear signatures in the CMB. The vortex patterns which are imprinted by the unperturbed anisotropic expansion are roughly constructed out of two parts: production of pure quadrupole variations (or focusing of the quadrupole pattern into a hot spot in open models) and a spiral pattern which is the signature of VII$_0$ and VII$_h$ models (Collins & Hawking 1973b; Doroshkevich et al. 1975; Barrow et al. 1985). This temperature anisotropy pattern for Bianchi VII$_h$ universes is of the form

$$\Delta T^B_T(\theta, \phi) = f_1(\theta) \sin \phi + f_2(\theta) \cos \phi.$$  

(1)

The functions $f_1(\theta)$ and $f_2(\theta)$ depend on details of the particular Bianchi model and must be computed numerically (Barrow et al. 1985). There are distinct features in each Bianchi model. In a pioneering work, analytical arguments were used to find upper bounds on the amount of shear and vorticity in the universe today, from the absence of any detected CMB anisotropy (Collins & Hawking 1973b). A detailed numerical analysis of such models used
experimental limits on the dipole and quadrupole to refine limits on universal rotation (Barrow et al. 1985). After the first detection of CMB anisotropy by COBE-DMR, Bianchi models were again studied by fitting the full spiral pattern from models with global rotation to the 4-year DMR data to constrain the allowed parameters of a Bianchi model of type VII\(h\) (Bunn et al. 1996; Kogut et al. 1997). Recently Bianchi VII\(h\) models were compared to the first-year WMAP data on large scales and it was shown that the best fit Bianchi model corresponds to a highly hyperbolic model \((\Omega_0 = 0.5)\) with a right-handed vorticity \((\frac{\omega_H}{H})_0 = 4.3 \times 10^{-10}\) (Jaffe et al. 2005, 2006). They also found that correcting the first-year WMAP data including the ILC map\(^1\) for the Bianchi template, makes the reported anomalies in the WMAP data such as alignment of quadrupole and octupole disappear and also boosts the low quadrupole and octupole. In a more recent work Land & Magueijo (2004) have performed a template fitting and found that although the “template” detections are not statistically significant they do correct the above anomalies. On the other hand, it was also found by Jaffe et al. (2005) that deviations from Gaussianity in the kurtosis of spherical Mexican hat wavelet coefficients of the WMAP first year data are eliminated once the data is corrected for the Bianchi template. In a recent work McEwen et al. (2005b) investigated the effect of this Bianchi correction on the detections of non-Gaussianity in the WMAP data. According to McEwen et al. (2005b) previous detections of non-Gaussianity observed in the skewness of spherical wavelet coefficients reported in McEwen et al. (2005a) are not reduced by the Bianchi correction and remain at a significant level. Cayon et al. (2006) argue that increasing the scaling of the template by a factor of 1.2 makes the spot vanish. But on the other hand it makes the resultant map more anisotropic.

In this paper we test the consistency of existence of a hidden Bianchi template in the WMAP data with our null detection of bipolar power spectrum (BiPS) in the WMAP first

\(^1\)The WMAP team warns that the original ILC map should not be used for detailed tests.
year maps (Hajian et al. 2004). The bipolar power spectrum is a measure of statistical isotropy in CMB anisotropy maps and is zero when statistical isotropy obtains. Properties of BiPS have been studied in great details in Hajian & Souradeep (2005) and Basak et al. (2006). We show that although correcting the WMAP first year maps for the Bianchi template may explain some features in the WMAP data\(^2\), this is done at the expense of introducing some anomalies such as preferred directions and the violation of statistical isotropy into the Bianchi corrected maps. This violation is stronger in the case of enhanced Bianchi templates proposed by Cayon et al. (2006). By testing statistical isotropy of Bianchi embedded CMB maps, we put a limit of \( \sigma_H^0 \geq 2.55 \times 10^{-10} (99.9\% CL) \) on Bianchi VII\(_h\) models. Here \( \sigma \) is the shear and \( H \) is the Hubble constant. Our result is marginally consistent with Jaffe et al. (2005) but not with the enhanced Bianchi templates of Cayon et al. (2006).

In summary, Bianchi templates have preferred directions which violate statistical isotropy even without considering a perturbed universe (Ferreira & Magueijo 1997).

To see this better, let’s assume that a CMB map is constructed out of two ingredients

\[
\Delta T(\hat{n}) = \Delta T^I(\hat{n}) + \Delta T^B(\hat{n}),
\]

where \( \Delta T^B \) is the Bianchi pattern and \( \Delta T^I \) represents the isotropic residual fluctuation.

\(^2\)Several methods have been used to look for possible anomalies in the WMAP data, here is a list of some of them: Eriksen et al. (2004a); Copi et al. (2004); Schwarz et al. (2004); Hansen et al. (2004); de Oliveira-Costa et al. (2003); Land & Magueijo (2005a,b,c); Bielewicz et al. (2004, 2005); Land & Magueijo (2005d); Copi et al. (2006); Land & Magueijo (2004); Naselsky et al. (2004); Hajian et al. (2004); Prunet et al. (2005); Gluck & Pisano (2005); Chen & Szapudi (2005); Freeman et al. (2006); Stannard & Coles (2005); Bernui et al. (2005, 2006); Movahed et al. (2006).
caused by variations in the density and gravitational potential of a FRW universe. A Bianchi template has preferred direction(s) and as it is shown in Fig. 1 will result in a non zero bipolar power spectrum of the CMB anisotropy map. This can be used to distinguish embedded Bianchi templates from a CMB anisotropy map of a purely FRW model.

The rest of this paper is organized as the following. Sections 2 to 8 are dedicated to review the classification, dynamics and properties of Bianchi models. A reader solely interested in the results of our research should skip them and jump to § 9. The first two sections (and also appendices) deal with the study of the symmetries of space and classification of Bianchi models. In § 4 field equations for Bianchi models are studied. Bianchi models which approach isotropy in late times are introduced in § 5. In section 6 we review the energy-momentum tensor and vorticity vector in these models. Null geodesics are derived in § 7 for all Bianchi models that approach isotropy. Section 8 is dedicated to deriving the CMB patterns of these models. In § 9 statistical isotropy of Bianchi VII$_h$ models are studied and compared to the null bipolar power spectrum of the WMAP data. Finally in § 10 we draw our conclusion.

2. Classification of Cosmological Models

We can classify cosmological models by their symmetries. For any space, the dimension of the group of symmetries of the space (group of isometries), is given by the sum of dimensions of groups of rotational and translational symmetries$^3$, i.e.,

$$\dim \text{group of isometries (r)} = \dim \text{group of rotational symmetries (s)} + \dim \text{group of translational symmetries (q)}.$$

In a 4-dimensional cosmological model, $r$ can pick up different values. These values

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$^3$For a detailed mathematical description of this see Appendix A.
Fig. 1.— Bipolar power spectrum (BiPS) of Bianchi VII\(_h\) models (arbitrary units); for a hyperbolic model \(x = 0.55, \Omega_0 = 0.5\) (left) and a nearly flat model \(x = 0.55, \Omega_0 = 0.99\) (right). The vertical axis is the bipolar power spectrum, \(\kappa_\ell\), plotted versus bipolar harmonic \(\ell\) (the horizontal axis). If the observed CMB map contains a pattern in it, it will violate the statistical isotropy which would be seen in the BiPS. A statistically isotropic CMB anisotropy map has a null bipolar power spectrum.

can be obtained by a variety of \(q\) and \(s\). The possibilities for the dimension of orbits, \(s\), are 0, 1, 2, 3 and 4. For isotropy of the spatial dimensions, \(q\) can be 0, 1 or 3. \(q = 2\) is excluded because in cosmology we consider non-empty perfect fluids in which \(\rho + p > 0\) and hence there will be uniquely defined notions of the average velocity of matter, and corresponding preferred world-lines. Their 4-velocity in each case is given by

\[
  u^\mu = \frac{dx^\mu}{d\tau},
\]

where \(\tau\) is proper time measured along the fundamental world-lines. We assume this 4-velocity is unique; that is, there is a well-defined preferred motion of matter at each space-time event. This \(u^\mu\) is therefore invariant and hence allowed rotations are those which act orthogonally on \(u^\mu\). There is no 2-dimensional subgroup of \(O(3)\) and hence \(q = 2\) is excluded from the allowed values that \(q\) can choose (Ellis & van Elst 1999). All over the
space, \( r \) should be fixed. But \( q \) can vary. For example right at the axis of symmetry, \( q \) can be bigger. But at that point the dimension of orbits is less and hence \( r \) remains fixed. Given these, we can have the following spaces:

(i) Possibilities for isotropy:

1. Isotropic: \( q = 3 \),
2. Local Rotational Symmetry: \( q = 1 \),
3. Anisotropic: \( q = 0 \);

(ii) Possibilities for homogeneity:

1. Space-time homogeneous models: \( s = 4 \)
2. Spatially homogeneous universes: \( s = 3 \)
3. Spatially inhomogeneous universes: \( s \leq 2 \).

Using the above classes, one can make every cosmological model with a given symmetry. Table 3 shows different varieties of cosmological models that may be constructed from different combinations of symmetries of the space. A complete list with more discussions is given in Ellis & van Elst (1999).

For example the family of FRW spaces that model the standard cosmology, are isotropic and spatially homogeneous universes \((q = 3, s = 3 \implies r = 6)\). Another interesting family is the spatially homogeneous but anisotropic case \((q = 0, s = 3 \implies r = 3)\) which is the family of Bianchi universes. This family has a simply transitive group of isometries \(G_3\) and hence no continuous isotropy group. In the rest of this paper we study these models in detail.
3. Modern Classification of Bianchi Models

Today, the Bianchi classification of 3-dimensional Lie algebras is no longer presented by the original Bianchi method (see Appendix B). Instead, a simpler scheme for classifying the equivalence classes of 3-dimensional Lie algebras is used. This scheme uses the irreducible parts of the structure constant tensor under linear transformations rather than the more complicated derived group approach of Lie and Bianchi. Following Ellis & MacCallum (1969), we decompose the spatial commutation structure constants $C^a_{bc}$ (eqn. A9) into a tensor, $n^{ab}$, and a vector, $a_b$

$$C^a_{bc} = \epsilon_{dabc} n^{ad} + \delta^a_c a_b - \delta^a_b a_c$$  \hspace{1cm} (4)

where $\epsilon_{abc}$ is the 3-dimensional antisymmetric tensor and $n^{ab}$ and $a_b$ are defined as

$$a_b = \frac{1}{2} C^a_{ba}$$ \hspace{1cm} (5)
$$n^{ab} = \frac{1}{2} C^{(a}_{cd} \epsilon^{b)cd}$$

The structure constants $C^a_{bc}$ expressed in this way, clearly satisfy the first Jacobi identity, eqn. (A10). Second Jacobi identity, eqn. (A11) shows that $a_b$ must have zero contraction with the symmetric 2-tensor $n^{ab}$;

$$C^a_{e[b} C^e_{cd]} = 0 \implies n^{ab} a_b = 0$$ \hspace{1cm} (6)

We can choose convenient basis to diagonalize $n^{ab}$ to attain $n^{ab} = \text{diag}(n_1, n_2, n_3)$ and to set $a_b = (a, 0, 0)$. The Jacobi identities are then simply equivalent to $n_1 a = 0$. Consequently we can define two major classes of structure constants

Class A : $a = 0$,
Class B : $a \neq 0$.

One can then classify further by the sign of the eigenvalues of $n^{ab}$ (signs of $n_1$, $n_2$ and $n_3$). This classification is given in Table 3. Parameter $h$ in class B, where $a$ is non-zero, is
defined by the scalar constant of proportionality in the following relation

\[ a_b a_c = \frac{h}{2} \epsilon_{bik} \epsilon_{cjl} n^{ij} n^{kl}. \]  
(7)

In the case of diagonal \( n^{ab} \), the \( h \) factor has a simple form \( h = a^2/(n_2 n_3) \).

In addition to these, we can have two different Bianchi models:

**Orthogonal models,** with the fluid flow lines orthogonal to the surfaces of homogeneity; which means the fluid 4-velocity \( U^\mu \) is parallel to the normal vectors \( n_\mu \). In this case, the matter variables will be just the fluid density and pressure.

**Tilted models,** with the fluid flow lines not orthogonal to the surfaces of homogeneity; i.e. the fluid 4-velocity is not parallel to the normals. Hence we also need the peculiar velocity of the fluid relative to the normal vectors. These components enter as further variables.

Rotating models must be tilted and have a more complex behavior (Ellis & van Elst 1999).

### 4. The Field Equations

We follow Collins & Hawking (1973a) and Collins & Hawking (1973b) to obtain the Einstein field equations for Bianchi models.

We saw that the spatially homogeneous models admit a group of symmetries, \( G \), which is transitive on a family of 3-dimensional space-like hyper-surfaces, \( S(t) \). We label these spaces by a time parameter \( t \) which is chosen to measure proper time along the geodesics normal to \( S(t) \). The normal to the hyper-surface is given by \( n_\alpha = t_\alpha \) such that
Table 1. Classification of Cosmological Models with Respect to Their Symmetries.

<table>
<thead>
<tr>
<th></th>
<th>inhomogeneous</th>
<th>spatially homogeneous</th>
<th>space-time homogeneous</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(s=2)</td>
<td>(s=3)</td>
<td>(s=4)</td>
</tr>
<tr>
<td>Anisotropic</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>((q = 0))</td>
<td>spatially self-similar, Bianchi Admitting Abelian or non-Abelian (G_2)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Local Rotational Symmetry</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>((q = 1))</td>
<td>Lemaître-Tolman-Bondi family Kantowski-Sachs</td>
<td>Godel stationary rotating universe</td>
<td></td>
</tr>
<tr>
<td>Isotropic</td>
<td>not allowed*</td>
<td>FLRW</td>
<td>Einstein static universe</td>
</tr>
</tbody>
</table>

Note. — Table reproduced from Ellis & van Elst (1999).

*inhomogeneous models cannot be isotropic about a general point because isotropy everywhere implies spatial homogeneity.

Table 2. Classification of Homogeneous Cosmological Models into Ten Equivalence Classes

<table>
<thead>
<tr>
<th>Group Class</th>
<th>Group Type</th>
<th>Eigenvalues of (n^{ab})</th>
<th>(a)</th>
<th>Dimension</th>
<th>FLRW Type</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>0,0,0</td>
<td>0</td>
<td>0</td>
<td>k=0</td>
<td>Abelian</td>
<td></td>
</tr>
<tr>
<td>II</td>
<td>+,0,0</td>
<td>0</td>
<td>3</td>
<td>—</td>
<td></td>
<td></td>
</tr>
<tr>
<td>VI(_b)</td>
<td>0,+,−</td>
<td>0</td>
<td>5</td>
<td>—</td>
<td></td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>VII(_b)</td>
<td>0,+,+</td>
<td>0</td>
<td>5</td>
<td>k=0</td>
<td></td>
</tr>
<tr>
<td>VIII</td>
<td>−,+,+</td>
<td>0</td>
<td>6</td>
<td>—</td>
<td></td>
<td></td>
</tr>
<tr>
<td>IX</td>
<td>+,+,+</td>
<td>0</td>
<td>6</td>
<td>k=+1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>V</td>
<td>0,0,0</td>
<td>+</td>
<td>3</td>
<td>k=-1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>IV</td>
<td>0,0,+</td>
<td>+</td>
<td>5</td>
<td>—</td>
<td></td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>VII(_h)</td>
<td>0,+,−</td>
<td>+</td>
<td>6</td>
<td>h &lt; 0</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>III</td>
<td>0,+,−</td>
<td>(n_2n_3)</td>
<td>6</td>
<td>VII(_h) with (h = -1)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>VII(_h)</td>
<td>0,+,+</td>
<td>+</td>
<td>6</td>
<td>k=-1</td>
<td>h &gt; 0</td>
</tr>
</tbody>
</table>

Note. — Table from Ellis & van Elst (1999) & Collins & Hawking (1973a).
\( g^{\alpha \beta} n_\alpha n_\beta = -1 \) and \( ; \) means covariant derivative\(^4\). We define three invariant vector fields \( E^\mu_A \) and its dual \( E^A_\mu \) on the surfaces of homogeneity by dragging the tangent vectors \( E^\mu_A \) at each point \( p \in S \) and covectors \( E^A_\mu \) (one-forms) under the group. The above dual basis satisfy

\[
E^A_\mu E^\mu_B = \delta^A_B
\]

and

\[
E^A_\mu E^\nu_A = \delta^\nu_\mu + n_\mu n^\nu.
\]

We drag \( E^\mu_A \) and \( E^A_\mu \) by the unit vector field \( n_\nu \) orthogonal to \( S(t) \), this is done by the Lie derivative of the fields

\[
\mathcal{L}_{n_\nu} E^A_\mu = \mathcal{L}_{n_\nu} E^\mu_A = 0.
\]

Now at every point \( p \in S(t) \), vector fields \( E^\mu_A \) and \( E^A_\mu \) are defined and are invariant under the 3-dimensional Lie group \( G \) (group of isometries of the space). Covector fields \( E^A_\mu \) satisfy the relation

\[
E^A_\mu \nabla_\nu - E^A_\nu \nabla_\mu = C^A_{BC} E^B_\mu E^C_\nu,
\]

where \( C^A_{BC} \) are constants given by the structure constants of the Lie group \( G \)

\[
C^A_{BC} = C^\mu_A E^A_\mu E^B_\nu E^C_\nu
\]

The metric of this model can be written in the following form

\[
g_{\mu \nu} = -n_\mu n_\nu + g_{AB} E^A_\mu E^B_\nu.
\]

We split the matrix \( g_{AB} \) into two parts: its volume and the distortion part (Misner 1968),

\[
g_{AB} = e^{2\alpha} \left(e^{2\beta}\right)_{AB}.
\]

\(^4\)Greek indices run from 0 to 3, Latin indices from 1 to 3. Latin indices may be lowered and raised by the \( 3 \times 3 \) matrix \( g_{AB} = g_{\mu \nu} E^\mu_A E^\nu_B \) and its inverse \( g^{AB} = g^{\mu \nu} E^A_\mu E^B_\nu \) which depend only on \( t \).
where the scalar $\alpha$ represents the volumetric expansion whilst $\beta$ is symmetric, trace free $3 \times 3$ matrix and hence $e^{2\beta}$ is given by the series

$$e^{2\beta} = \sum_{r=0}^{\infty} \frac{1}{r!}(2\beta)^r.$$ 

Now we define an orthonormal basis $X^\nu_\mu$ such that

$$X^0_\mu = -n_\mu, \ X^i_\mu = e^\alpha(e^{2\beta})_{iA}E^A_\mu.$$  \hfill (14)

where $i,j,k, \ldots = 1,2,3$. The 0-th component of these basis changes sign on raising or lowering, but rest of the components remain unchanged. Ricci rotation coefficients $\Gamma_{\alpha\beta\gamma}$ are obtained from the connection coefficients through $\delta_{\alpha\delta}\Gamma_{\beta\gamma}^\delta$. For this basis these coefficients satisfy $\Gamma_{\delta\epsilon\gamma} + \Gamma_{\gamma\epsilon\delta}$ and are given by

$$X^\gamma_\mu, X^\epsilon_\nu = \Gamma_{\gamma}^{\epsilon} X^\delta_\mu X^\nu_\delta$$  \hfill (15)

Using these properties we will obtain

$$\Gamma = \frac{1}{2} [X_{\gamma[\mu,\nu]}X^\mu_\delta X^\nu_\epsilon + X_{\epsilon[\mu,\nu]}X^\mu_\delta X^\nu_\gamma - X_{\delta[\mu,\nu]}X^\mu_\epsilon X^\nu_\gamma].$$  \hfill (16)

This with eqn. (10) will determine the components of Ricci rotation coefficients

$$\Gamma_{ij0} = -\Gamma_{0ij} = (\partial_0\alpha)\delta_{ij} + \sigma_{ij},$$

$$\Gamma_{i00} = -\Gamma_{00i} = 0,$$

$$\Gamma_{i0j} = -\nu_{ij}$$

$$\Gamma_{ijk} = \frac{1}{2}e^{-\alpha}(e^\beta)_{FA}(e^{-\beta})_{GB}(e^{-\beta})_{HC}C^A_{BC} [\delta_{Fi}\delta_{Gj}\delta_{Hk} + \delta_{Fk}\delta_{Gi}\delta_{Hj} - \delta_{Fj}\delta_{Gi}\delta_{Hi}] .$$

In the above expressions, $\partial_0$ denotes differentiation by $t$ and $\sigma_{ij}$ represents the shear of the normals $n_\mu$ and is given by anticommutator of $(\partial_0 e^\beta)$ and $e^{-\beta}$

$$\sigma_{ij} \equiv \frac{1}{2} \{(\partial_0 e^\beta), e^{-\beta}\}_{ij}$$

$$= \frac{1}{2} \left[(\partial_0 e^\beta)e^{-\beta} + e^{-\beta}(\partial_0 e^\beta)\right]_{ij}.$$
The extent of commutativity of $\partial_0 e^\beta$ and $e^{-\beta}$ is given by their commutator denoted by $\nu_{ij}$

$$\nu_{ij} \equiv \frac{1}{2} [(\partial_0 e^\beta), e^{-\beta}]_{ij}$$  \hspace{1cm} (19)

Ricci tensor of the space $S$ in the induced metric is

$$R^*_{ij} = -\frac{1}{4} e^{-2\alpha} \{ 2C^A_{BC}C^C_{DA}(e^{-\beta})_{Bi}(e^{-\beta})_{Dj}$$
$$+ C^A_{BC}C^D_{EF}(e^{-2\beta})_{BE}[2(e^{2\beta})_{DA}(e^{-\beta})_{Ci} - (e^{-2\beta})_{CF}(e^{\beta})_{Ai} (e^{\beta})_{Dj}]$$
$$+ C^A_{AB}C^C_{CE}(e^{-2\beta})_{BE}[(e^{\beta})_{Ci}(e^{-\beta})_{Di} + (e^{\beta})_{Cj}(e^{-\beta})_{Di}] \}.$$  \hspace{1cm} (20)

We can now write the field equations for this basis (Collins & Hawking 1973a). The field equations are

$$3(\partial_0 \alpha)^2 - \frac{1}{2} \sigma_{ij} \sigma_{ij} + \frac{1}{2} R^* = 8\pi T^{00},$$  \hspace{1cm} (21)

$$e^{-\alpha}[(e^{-\beta} \sigma e^{\beta})_{BA}C^A_{BC}C^{-\beta}_{Ci} - \sigma_{ij} (e^{-\beta})_{jC} C^A_{iA}] = 8\pi T^{0i},$$  \hspace{1cm} (22)

$$\partial_0 \sigma_{ij} + 3(\partial_0 \alpha) \sigma_{ij} + [\sigma, \nu]_{ij} + R^*_{ij} - \frac{1}{3} R^* \delta_{ij} = 8\pi(T_{ij} - \frac{1}{3} T_{kk} \delta_{ij})$$  \hspace{1cm} (23)

$$-6\partial_0^2 \alpha - 9(\partial_0 \alpha)^2 - \frac{3}{2} \sigma_{ij} \sigma_{ij} - \frac{1}{2} R^* = 8\pi T_{kk}.$$  \hspace{1cm} (24)

5. Approaching Isotropy

We say a model approaches isotropy if at arbitrary large time scales it has the following properties

(a) The universe should continue to expand forever, which means

$$t \rightarrow +\infty \implies \alpha \rightarrow +\infty$$

(b) Second condition is the isotropy condition. $T^{0i}/T^{00}$ represents an average velocity of the matter relative to the surface of homogeneity. The universe can only reach isotropy only if at large times this velocity vanishes

$$t \rightarrow +\infty \implies T^{00} > 0 \quad \text{and} \quad \frac{T^{0i}}{T^{00}} \rightarrow 0$$
(c) Anisotropy in the locally measured Hubble parameter should tend to zero. If we define $\sigma^2 = \frac{1}{2} \sigma_{ij} \sigma_{ij}$, this condition will then translate into

$$t \to +\infty \implies \frac{\sigma}{\partial_0 \alpha} \to 0$$

(d) If we notice that the anisotropy observed in CMB due to a specific Bianchi model is somehow related to the change of $\beta$ between the time of last scattering and the time it is observed, we will conclude that the amazing degree of isotropy of the CMB, will tell us that $\beta$ should be very small,

$$t \to +\infty \implies \beta \to \beta_0 \quad \text{where} \quad \beta_0 = \text{const.}$$

The following theorem which was proved by Collins & Hawking (1973a) shows that only 4 Bianchi types approach isotropy.

**Theorem 5.1** *If the dominant-energy condition$^5$ and the positive-pressure criterion$^6$ are satisfied, the universe can approach isotropy only if it is one of the types $I$, $V$, $VII_0$ and $VII_h^7$.*

Since we know that our universe is isotropic to a very high extent, from now on we will only be interested in these models.

$^5 T_{00} \geq |T^{\alpha \beta}|$

$^6 T_{kk} \geq 0$

$^7$Spatially homogeneous cosmological models with cosmological constant were investigated by Wald (1983) and Goliath & Ellis (1998) and more recently by Jaffe et al. (2005b).
6. Energy Momentum Tensor and Vorticity Vector

In a matter dominated universe the dominant contribution to the energy-momentum tensor comes from non-relativistic matter,

\[ T^{\mu \nu} = \rho_m U^\mu U^\nu, \]  

(25)

where \( \rho_m \) is the matter density and \( U^\mu \) is the flow-vector. The conservation equation tells us that flow-lines satisfy the geodesic equation, hence

\[ \partial_t (\rho_m U^0 e^{3\alpha}) = -\rho_m e^\alpha C^A_{AB} U^C (e^{-2\beta})_{BC}. \]  

(26)

in models of class A, right hand side of the above expression vanishes and will result in

\[ \text{Class A: } \rho_m = a_m (U^0 e^{3\alpha})^{-1} \]  

(27)

where \( a_m \) is a constant. Using this constant, \( T^{00} \) can be written as \( a_m e^{-3\alpha} (1 + E_m) \) where \( a_m \) is constant in class A models and hence the kinetic energy associated with the peculiar velocity of the matter relative to the surfaces of homogeneity will be

\[ E_m = U^0 - 1 = \left[ U_A U_B e^{-2\alpha} (e^{-2\beta})_{AB} + 1 \right]^{1/2} - 1. \]  

(28)

And the anisotropic part of the spatial components of the energy-momentum tensor will be

\[ T_{ij} - \frac{1}{3} T_{kk} \delta_{ij} = -a_m e^{-3\alpha} \frac{\partial E_m}{\partial \beta_{ij}}. \]  

(29)

In a radiation dominated universe, the energy momentum tensor would be

\[ T^{\mu \nu} = \frac{4}{3} \rho_r U^\mu U^\nu + \frac{1}{3} \rho_r g^{\mu \nu}, \]  

(30)

and the energy density of radiation obeys

\[ \partial_t \left[ \rho_r (U^0)^{4/3} e^{4\alpha} \right] = -\frac{4}{3} \rho_r (U^0)^{1/3} C^A_{AB} U^C e^{2\alpha} (e^{-2\beta})_{BC}. \]  

(31)
Right hand side will vanish in class A models and as a result

\[
\text{Class A : } \rho_\gamma(U^0)^{4/3}e^{4\alpha} = a_\gamma
\]  

(32)

Flow lines feel an acceleration caused by the radiation pressure

\[
\frac{4}{3}\rho_\gamma U_\nu U^\nu = -\frac{1}{3}\rho_\gamma h^{\mu\nu}, \text{ where } h^{\mu\nu} \equiv g^{\mu\nu} + U^\mu U^\nu.
\]  

(33)

Because of homogeneity \(\rho_{\gamma;\nu} = -\partial_t \rho_{\gamma} n_\nu\). Therefore

\[
\partial_t(\rho_\gamma^{1/4} U_A) = \rho_\gamma^{1/4} (U^0)^{-1} C_{CA}^B U_B U_D e^{-2\alpha} (e^{-2\beta})_{CD}.
\]  

(34)

The kinetic energy and the anisotropic part of the spatial components of the energy-momentum tensor are then given by (Hawking 1968);

\[
E_\gamma = (U^0)^{2/3} - 1 = [U_A U_B e^{-2\alpha} (e^{-2\beta})_{AB} + 1]^{1/3} - 1
\]  

(35)

\[
T_{ij} - \frac{1}{3} T_{kk} \delta_{ij} = -2 a_\gamma e^{-4\alpha} \frac{\partial E_\gamma}{\partial \beta_{ij}}.
\]  

In these models we are mainly interested in the vorticity

\[
\omega = (g_{AB} \omega^A \omega^B)^{1/2} = (\omega^i \omega^i)^{1/2}
\]  

(36)

The vorticity vector of the matter is defined as \(\omega^\mu = \frac{1}{2} \eta^{\mu\nu\lambda\rho} U_\nu U_\lambda n_\rho\). Thus

\[
\omega^A = \frac{1}{2} e^{-3\alpha} \epsilon_{ABC} \left[ \frac{1}{2} C_{BC}^D U_D U^0 + U_B \partial_0 U_C \right],
\]  

(37)

\[
\omega^0 = \frac{1}{2} e^{-3\alpha} \epsilon_{ABC} C_{BC}^D U_A U_D
\]

For small, non-relativistic velocities, \(U^0 \sim 1\) and \(U_A\) is small, so

\[
\omega^A \simeq \frac{1}{4} e^{-3\alpha} \epsilon_{ABC} C_{BC}^D U_D.
\]  

(38)
7. Null Geodesics

In general, all of our observations take place on our past light cone, which will develop many cusps and caustics at early times because of gravitational lensing, but is still locally generated by geodesic null rays. The information we receive comes to us along these null rays, with tangent vector

$$K^\mu K_\mu = 0$$  \hspace{1cm} (39)

where $K^\mu$ is the tangent vector to the null geodesic from the observer in a given direction. In the orthonormal basis $X^\alpha_\mu$ which was defined in the previous section, the null geodesic condition will become

$$K_0 K^0 + K^A K_A = 0$$  \hspace{1cm} (40)

As we saw, the 0-th component of these basis changes sign on raising or lowering. So,

$$K^0 = -K_0 \implies K^0 = (K^A K_A)^{1/2}$$  \hspace{1cm} (41)

and

$$K_A = g_{AB} K^B = e^{2\alpha} (e^{2\beta})_{AB} K^B$$  \hspace{1cm} (42)

with $K_A = (K \cos \theta, K \sin \theta \cos \phi, K \sin \theta \sin \phi)$ where $K$ is constant. For small anisotropies, i.e. $\beta_{ij} \ll 1$ (Barrow et al. 1985),

$$g_{AB} \simeq e^{2\alpha} \delta_{AB} \implies K^0 \simeq K e^{-\alpha}$$  \hspace{1cm} (43)

$$K_A \simeq e^{2\alpha} \delta_{AB} K^B$$

The null geodesics satisfy the geodesic equation

$$K_{\mu \nu} K^\nu = 0,$$  \hspace{1cm} (44)

In the orthonormal basis, the geodesic equations are

$$\partial_t K_A = C^B_{CA} \frac{K_B K^C}{K^0}.$$  \hspace{1cm} (45)
Using the small anisotropy condition, eqn. (43), the geodesic equation to first order will be

$$\partial_t K_A = \frac{1}{K} C_{B}^{[A} K_{B} K_{C]} e^{-\alpha}.$$  \hfill (46)

We introduce a new time parameter by $d\tau = e^{-\alpha} dt$ and we denote the differentiation with respect to this time by prime

$$K_A' = \frac{1}{K} C_{[A}^{B} K_{B} K_{C]}.$$ \hfill (47)

We are interested in those Bianchi types which are containing FRW models as special isotropic cases. These models are Bianchi types I, V, VII_0, VII_h and XI. Null geodesic equations and their solutions for these models were first derived by Barrow et al. (1985) and are given below

**Bianchi Type I**  All structure constants are zero,

$$\theta' = \phi' = 0 \hfill (48)$$

$$\Rightarrow \theta = \theta_0 = \text{const.}$$

$$\phi = \phi_0 = \text{const.}$$

**Bianchi Type V**  Non-zero structure constants:

$$C_{21}^{2} = -C_{12}^{2} = C_{31}^{3} = -C_{13}^{3} = 1.$$  

$$\theta' = -\sin \theta, \quad \phi' = 0,$$  \hfill (49)

$$\Rightarrow \tan \left(\frac{\theta}{2}\right) = \tan \left(\frac{\theta_0}{2}\right) \exp \left[-(\tau - \tau_0)\right]$$

$$\phi = \phi_0 = \text{const.}$$

**Bianchi Type VII_0**  Non-zero structure constants:

$$C_{31}^{2} = -C_{13}^{2} = C_{12}^{3} = -C_{21}^{3} = 1,$$
\[ \theta' = 0, \quad \phi' = -\cos \theta, \]  
\[ \implies \theta = \theta_0 = \text{const}. \]
\[ \phi = \phi_0 - (\tau - \tau_0) \cos \theta_0. \]  

**Bianchi Type VII**

Non-zero structure constants:

\[ C_{31}^2 = -C_{13}^2 = C_{12}^3 = -C_{21}^3 = 1, \]
\[ C_{12}^2 = -C_{21}^2 = C_{13}^3 = -C_{31}^3 = \sqrt{h}. \]

\[ \theta' = -\sqrt{h} \sin \theta, \quad \phi' = -\cos \theta, \]  
\[ \implies \tan \left( \frac{\theta}{2} \right) = \tan \left( \frac{\theta_0}{2} \right) \exp \left[ -\left( \tau - \tau_0 \right) \sqrt{h} \right] \]
\[ \phi = \phi_0 + (\tau - \tau_0) - \frac{1}{\sqrt{h}} \ln \left\{ \sin^2 \left( \frac{\theta_0}{2} \right) + \cos^2 \left( \frac{\theta_0}{2} \right) \exp \left[ 2(\tau - \tau_0) \sqrt{h} \right] \right\}. \]

**Bianchi Type IX**

\[ C_{\alpha BC}^A = \epsilon_{\alpha BC} \]

\[ \theta' = \phi' = 0 \]  
\[ \implies \theta = \theta_0 = \text{const}. \]
\[ \phi = \phi_0 = \text{const}. \]  

8. Patterns of Anisotropic Models on the CMB Sky

In the previous section, we found the null geodesic solutions in Bianchi types that include FRW model as an isotropic limit. Now we will compute the effect of these models
on the mean temperature of CMB sky. In this section, we do our calculations with the assumption of small anisotropy which means

$$\beta_{ij} \ll 1, \quad |U_i| \ll 1, \quad |\sigma_{ij}| \simeq |\partial_t \beta_{ij}| \ll \partial_t \alpha. \quad (53)$$

Relative to an observer with 4-velocity $U^\mu$, the null vector $K^\mu$ determines a redshift factor $(-K_\mu U^\mu)$. Hence the observed cosmological redshift $z$ for comoving matter is given by (Ellis & van Elst 1999)

$$1 + z_E = \frac{(K_\mu U^\mu)_E}{(K_\mu U^\mu)_0} \quad (54)$$

where subscript E and O refer to photon emission and reception respectively, $U_0^\mu$ is the velocity vector of the observer, $U_E^\mu$ is the velocity vector of the emitting surface. We want to study the effect of the redshift given in eqn. (54) on cosmic microwave background radiation which is considered to have a black body spectrum at a temperature $T_E$, coming from the surface of last scattering. The observed temperature $T_0$ is then

$$T_0 = \frac{T_E \Gamma}{1 + z_E(\theta_0, \phi_0)} \quad (55)$$

If we define $p^i$ to be the direction cosines of the null geodesic, $p_i = K_i/K = (\cos \theta, \sin \theta \cos \phi, \sin \theta \sin \phi)$, the above expression to first order will become

$$T_0 = T_E \Gamma^{-1} \left[1 + (p^i U_i)_0 - (p^i U_i)_E - \Sigma\right]. \quad (56)$$

where

$$\Gamma = e^{\alpha_E - \alpha_0}, \quad \Sigma = \int_E^0 p^i p^k \sigma_{ik} dt$$

The CMB pattern in eqn. (56) has an easy interpretation. The first term on the right hand side is a Doppler term and exhibits a dipole in the temperature anisotropy due to the motion of the observer relative to the hypersurfaces of constant time at the time of observation. This causes a net dipole. The second term represents a Doppler shift due to
the peculiar motion of the source of emitting photon on the surface of last scattering. This will cause a local redshift/blueshift but usually not a net dipole. These two terms are absent in orthogonal models but are present in tilted models. The third term, $\Sigma$, gives the temperature anisotropy caused by the shear of the Universe due to the anisotropic expansion. In types I and IX this causes a pure quadrupole pattern equal to $p_0^i p_0^j (\beta_0 - \beta_E)_{ij}$.

To obtain the signature of Bianchi models on CMB sky, we should solve equation (71). The time dependence of $U_i$ and $\sigma_{ij}$ are given by Einstein equations. And the variation of $\theta$ and $\phi$ with time are obtained from the (null-) geodesic equation which was discussed in section 7. Following is a very brief review of this matter.

### 8.1. CMB Patterns in Type I Bianchi Models

This is the simplest type for which the null geodesic direction cosines are all constant, $\pi^i = constant$. $\alpha$ and $\beta$ are given by

$$e^\alpha = t^{2/3}, \quad \beta = 0$$

(57)

The peculiar velocity is zero and the vorticity vanishes. For small perturbations, $\beta_{ij} = A_{ij} t^{-1}$ and the shear is $\sigma_{ij} = -A_{ij} t^{-2}$ where $A_{ij}$ is a constant matrix. Hence the only contribution to CMB pattern is a quadrupole which comes from the $\Sigma$ term and is equal to

$$p_0^i p_0^j A_{ij} (t_0^{-1} - t_E^{-1})$$

(58)

---

8In type I and IX, $p_E^i = p_0^i$, so the second term also gives a dipole variation (Collins & Hawking 1973b).
8.2. CMB Patterns in Type V Bianchi Models

This models is the simplest generalization of the $k = -1$ FRW models. $\alpha$ and $\beta$ are given by

$$e^\alpha = \frac{8\pi M}{3} \sinh^2 \frac{\tau}{2}, \quad \beta = 0$$

(59)

where $\tau$ is a new time variable defined by $\frac{dt}{d\tau} = e^\alpha$ with $\tau = 0$ when $e^\alpha = 0$, and $M = \rho_0 (H_0^2 - 8\pi \rho_0 / 3)^{-3/2} = \frac{\rho_0}{H_0 (1 - \Omega_0)^{1/2}}$. High density type V models behave like type I models, so here we consider low density models. The null geodesics for these models are calculated in the previous section. For small anisotropies, $\sigma_{ij} = A_{ij} e^{-3\alpha}$ where $A_{ij}$ is a constant matrix given by $A_{1j} = (8\pi M/3) U_j e^\alpha$ and $A_{23} = 0$ (one can choose the 2-3 axis such that this happens). The $\Sigma$ term will then cause an anisotropy in this form

$$-\frac{3}{2} A_{11} I_1 + \frac{1}{2} (A_{22} - A_{33}) I_1 \cos \phi + 2(A_{12} \cos \phi + A_{13} \sin \phi) I_2$$

(60)

where

$$I_1 = \int_{E} e^{-2\alpha} \cos^2 \theta d\tau, \quad \text{and} \quad I_2 = \int_{E} e^{-2\alpha} \sin \theta \cos \theta d\tau.$$  

(61)

8.3. CMB Patterns in Type VII$_0$ Bianchi Models

This is the most general family of models which include the $k = 0$ FRW solution. $\alpha$ and $\beta$ are given by

$$e^\alpha = \left( \frac{9t_0^2}{4H_0} \right)^{1/3} x, \quad \beta = 0$$

(62)

where $x$ is a constant. The parameter $x$ is equal to the characteristic length scale, $e^{\alpha_0}$, over which the invariant vectors change their orientation, multiplied to the expansion rate, $H_0$. The value of parameter $x$ does not make any physical difference to the FRW model but it governs the length scale of the anisotropy in an anisotropic model. For small anisotropies, $\beta$ ...
is not zero as given in Collins & Hawking (1973b). Null geodesic equations are calculated in
the previous section. The Doppler terms \((p^i U_i)_0\) and \((p^i U_i)_E\) arising from the 1-component
of the peculiar velocity give rise to a dipole anisotropy of magnitude \(U_0^{(1)}(\Gamma - 1) \cos \theta_0\). The
2 and 3 components of the peculiar velocity will cause the following pattern

\[
\left( \Gamma \cos \left[ 2 \left( 1 - \frac{1}{\Gamma^{1/2}} \right) \cos \theta \right] - 1 \right) (U_0^{(2)} \cos \phi + U_0^{(3)} \sin \phi) \sin \theta \quad (63)
\]

For \(x \ll 1\) this is a tightly wound spiral pattern which performs \(2/\pi x(1 - 1/\Gamma^{1/2})\)
revolutions and has spacing \(\pi x/\sin \theta_0[\Gamma^{1/2}/(\Gamma^{1/2} - 1)]\) between successive maxima (Collins
& Hawking 1973b). The \(\Sigma\) term gives a quadrupole pattern through \(\sigma_{11}\) which is like
\((\beta_0 - \beta_E)_{11} \cos^2 \theta\). The effect of the other components of \(\sigma_{ij}\) is complicated and some
approximate results near the equator have been given in Collins & Hawking (1973b).

For \(x \gg 1\) the second term vanishes and it becomes just a dipole pattern \((\Gamma - 1)(U_0^{(2)} \cos \phi + U_0^{(3)} \sin \phi) \sin \theta\). Also the \(p^i_0\) are almost constant and hence the \(\Sigma\) term gives
a quadrupole pattern \(p^i p^j (\beta_0 - \beta_E)_{ij}\).

8.4. CMB Patterns in Type VII\(_h\) Bianchi Models

This is the most general family of models which includes the \(k = -1\) FLRW solutions.
It has some of the features of both types V and type VII\(_0\). There is an adjustable parameter
\(h\) in this model which is given by the square of structure constants (see previous section).
In an unperturbed FRW model, the expansion scale factor, \(\alpha\), and \(\beta\) are given by

\[
e^\alpha = \frac{h^{1/2} \Omega_0}{H_0 (1 - \Omega_0)^{3/2}} \sinh \left( \frac{h^{1/2} \tau}{2} \right), \quad \beta = 0 \quad (64)
\]
As in type VII\textsubscript{0}, we introduce a factor \( x = H_0 e^{\alpha_0} \) which is related to the parameter \( h \) by
\[
x = \sqrt{\frac{h}{1 - \Omega_0}}.
\]
(65)

As we said before, \( x \) has no physical effect on the FRW models. It can be seen from the
fact that the present value of the scale factor can be arbitrarily chosen, but from eqn. (64)
we have
\[
e^{\alpha_0} = \frac{x}{H_0},
\]
(66)

And hence we see that parameter \( x \) can be scaled out of the solution. However, for large
values of \( x \) (or equally \( h \)), the models are similar to those of type V. As the present density
tends towards the critical density, \( \Omega_0 \rightarrow 1 \), and \( h \rightarrow 0 \) in such a way that \( x \) remains finite,
the behavior of the models tend to that of type VII\textsubscript{0}. Behavior of these models with
different parameters can be seen in figure 2.

Null geodesics have been given in the previous section and have a complicated form.
They start near the South Pole and spiral in the negative \( \phi \)-direction up towards the
equator, and then spiral in the positive \( \phi \)-direction up towards the North Pole. Small
anisotropies can be treated as perturbations to FRW model. \( \beta \) has been calculated for them
and is given in Collins & Hawking (1973b). Here we follow Barrow et al. (1985) who used
these to calculate the contribution to temperature anisotropies from the vorticity alone.
They argue that the inclusion of other pure shear distortions which are independent of
the rotation could only make the temperature anisotropies larger. And hence their results
would give the maximum level of vorticity consistent with a given value of temperature
anisotropy.

In type VII\textsubscript{h}, there are two independent vorticity components \( \omega_2 \) and \( \omega_3 \). They are
given in terms of the off-diagonal shear elements as
\[
\omega_2 = \frac{(3h - 1)\sigma_{13} - 4h^{1/2}\sigma_{12}}{3x^2\Omega_0},
\]
(67)
\[
\omega_3 = \frac{(1 - 3h)\sigma_{12} - 4h^{1/2}\sigma_{13}}{3x^2\Omega_0}.
\]

The observables in this model are two dimensionless amplitudes: \((\frac{\sigma_{12}}{H})_0\) and \((\frac{\sigma_{13}}{H})_0\). The vorticity is given by

\[
\omega = \frac{1}{2}e^{-\alpha}(1 + h)^{1/2} \left[(u_2)^2 + (u_3)^2\right]^{1/2},
\]

where \(u_2\) and \(u_3\) are the velocity components and their present values are given by

\[
(u_2)_0 = \frac{1}{3x\Omega_0} \left[3h^{1/2} \left(\frac{\sigma_{12}}{H}\right)_0 - \left(\frac{\sigma_{13}}{H}\right)_0\right],
\]

\[
(u_3)_0 = \frac{1}{3x\Omega_0} \left[\left(\frac{\sigma_{12}}{H}\right)_0 + 3h^{1/2} \left(\frac{\sigma_{13}}{H}\right)_0\right].
\]

Hence the present value of vorticity is

\[
\left(\frac{\omega}{H}\right)_0 = \frac{(1 + h)^{1/2}(1 + 9h)^{1/2}}{6x^2\Omega_0} \left[\left(\frac{\sigma_{12}}{H}\right)_0^2 + \left(\frac{\sigma_{13}}{H}\right)_0^2\right]^{1/2}.
\]

To first order, temperature fluctuations are given by

\[
\frac{\Delta T(\theta_0, \phi_0)}{T_0} \simeq (p^iU_i)_0 - (p^iU_i)_E - \int_E^0 p^i p^k \sigma_{ik} dt,
\]

where \(\theta_0\) and \(\phi_0\) are related to the actual observing angles by

\[
\theta = \pi - \theta_0, \quad \phi = \pi + \phi_0.
\]

Substituting for null geodesics from eqn. (51) and for velocities from eqn. (69) into eqn. (71), we will obtain

\[
\frac{\Delta T(\theta_0, \phi_0)}{T_0} = \left[\left(\frac{\sigma_{12}}{H}\right)_0 A(\theta_0) + \left(\frac{\sigma_{13}}{H}\right)_0 B(\theta_0)\right] \sin \phi_0
\]

\[
+ \left[\left(\frac{\sigma_{12}}{H}\right)_0 B(\theta_0) - \left(\frac{\sigma_{13}}{H}\right)_0 A(\theta_0)\right] \cos \phi_0,
\]

where coefficients \(A(\theta_0)\) and \(B(\theta_0)\) are defined by

\[
A(\theta_0) = C_1[\sin \theta_0 - C_2(\cos \psi_E - 3h^{1/2} \sin \psi_E)]
\]

\[
+ C_3 \int_{\tau_0}^{\tau} \frac{s(1 - s^2) \sin \phi d\tau}{(1 + s^2)^2 \sinh^4 h^{1/2} \tau/2},
\]

\[
B(\theta_0) = C_1[3h^{1/2} \sin \theta_0 - C_2(\sin \psi_E + 3h^{1/2} \cos \psi_E)]
\]

\[
- C_3 \int_{\tau_0}^{\tau} \frac{s(1 - s^2) \cos \phi d\tau}{(1 + s^2)^2 \sinh^4 h^{1/2} \tau/2}.
\]
The limits of integration are defined as

$$\tau_0 = 2h^{-1/2}\sinh^{-1}(\Omega_0^{-1} - 1)^{1/2},$$

$$\tau_E = 2h^{-1/2}\sinh^{-1}\left(\frac{\Omega_0^{-1} - 1}{1 + z_E}\right)^{1/2},$$

and constants \(C_1, C_2, C_3, s\) and \(\psi\) are defined by

\[
C_1 = (3\Omega_0^2)^{-1}; \\
C_2 = \frac{2sE(1 + z_E)}{1 + s_E^2}; \\
C_3 = 4h^{1/2}(1 - \Omega_0)^{3/2}\Omega_0^{-2}; \\
s = \tan\left(\frac{\theta}{2}\right) = \tan\left(\frac{\theta_0}{2}\right) \exp\left[-(\tau - \tau_0)\sqrt{h}\right], \\
\psi = \phi_0 + (\tau - \tau_0) - \frac{1}{\sqrt{h}} \ln\left\{\sin^2\left(\frac{\theta_0}{2}\right) + \cos^2\left(\frac{\theta_0}{2}\right) \exp\left[2(\tau - \tau_0)\sqrt{h}\right]\right\}.
\]

The expression for \(\frac{\Delta T(\theta_0, \phi_0)}{T_0}\) can be written in a compact form

\[
\frac{\Delta T}{T_0}(\theta_0, \phi_0) = (A^2 + B^2)^{1/2}\left(\frac{\sigma}{H}\right)_0 \cos\left(\phi_0 + \tilde{\phi}\right)
\]

where \(\tilde{\phi}\) is defined as

\[
\cos\tilde{\phi} = \left[\left(\frac{\sigma_{12}}{\sigma}\right) B - \left(\frac{\sigma_{13}}{\sigma}\right) A\right] (A^2 + B^2)^{-1/2}
\]

and \(\sigma\) is given by

\[
\sigma^2 = \sigma_{12}^2 + \sigma_{13}^2
\]

Eqn. (77) helps to understand the behavior of the CMB pattern in these models. If we look around any circle at a given \(\theta_0\) on the sky, the temperature variation will have a pure \(\cos\phi_0\) behavior. \(\tilde{\phi}\) determines the relative orientation of adjacent \(\theta_0\) =constant rings. In type V models, \(A(\theta_0) = 0\) and hence we have no spiraling in these models. On the other hand, \(B(\theta_0)\) determines the amplitude of \(\frac{\Delta T}{T_0}\) and the focusing into a hot spot (Barrow et al. 1985). Some of these maps have been shown in figure 2.
Fig. 2.— Bianchi patterns in type VII$_h$ models for $x = 0.1, 0.5, 1, 5$ (from top to bottom) and $\Omega_0 = 0.1, 0.5$ and 0.9 (from left to right). In low density models, the pattern is mostly focused in one hemisphere but for nearly flat models the spiral patterns widen and extend to both hemispheres.

8.5. CMB Patterns in Type IX Bianchi Models

This is a generalization of the $k = +1$ FRW models with

$$e^a = s^{00} \sin^{\tau/4}, \quad \beta = 0$$

(80)

For small anisotropies, $\beta_{ij}$ is given by a perturbation series derived by Collins & Hawking (1973b). The null geodesics have $p^i = constant$ and hence the Doppler terms $(p^i U_i)_0$ and $(p^i U_i)_E$ give rise to a dipole anisotropy equal to $(p^i U_i)_0(\Gamma - 1)$. The $\Sigma$ term causes a quadrupole anisotropy of the form $p^i p^j (\beta_0 - \beta_E)_{ij}$. 
9. Unveiling Hidden Patterns of Bianchi VII$_h$

Recently Bianchi VII$_h$ models were compared to the first-year WMAP data on large scales and it was shown that the best fit Bianchi model corresponds to a highly hyperbolic model ($\Omega_0 = 0.5$) with a right-handed vorticity ($\hat{\omega}^r_0 = 4.3 \times 10^{-10}$) (Jaffe et al. 2005). There are several aspects to this hidden pattern which should be mentioned. First of all, the proposed model is an extremely hyperbolic model and does not agree with the location of the first peak in the best fit power spectrum of the CMB. In the regime of VII$_h$ models, one should study the nearly flat models in order to have a consistent angular power spectrum. But the problem is that as it has been shown in Fig. 2, the spiral patterns in high density models are not concentrated in one hemisphere and can’t be used to cure the large-scale power asymmetry in the WMAP data reported by Eriksen et al. (2004a).

Second is that the Bianchi patterns we studied so far, are only valid for a matter dominated universe. Patterns must be recalculated in a universe with a dark energy component if one wants to compare them with the observed CMB data which is believed to be there in a $\Lambda$ dominated universe$^{10}$. These issues are addressed in great details by Jaffe et al. (2005b) and they conclude that the “best-fit Bianchi type VII$_h$ model is not compatible with measured cosmological parameters”.

For these reasons, we study the type VII$_h$ Bianchi models only as hidden patterns in the CMB anisotropy maps. We pay our attention to the specific model proposed by Jaffe et al. (2005) and in the next two sections, we address the question whether and to what extent one would be able to discover this pattern and other hidden anisotropic patterns in the CMB anisotropy maps.

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$^{10}$Bianchi-I patterns with dark energy have been studied by Ghosh (2006). Also Jaffe et al. (2005b) have considered Bianchi VII$_h$ models with dark energy.
Bipolar Power Spectrum Analysis

We choose a Bianchi VII\(_h\) model with \((\frac{a_{12}}{H})_0 = (\frac{a_{13}}{H})_0 = (\frac{a}{H})_0\). The temperature fluctuations induced by this model are given by eqn. (73) and will be

\[
\frac{\Delta T(\theta_0, \phi_0)}{T_0} = \left(\frac{\sigma}{H}\right)_0 [A(\theta_0) + B(\theta_0)] \sin \phi_0 \\
+ [B(\theta_0) - A(\theta_0)] \cos \phi_0,
\]

Power spectrum of the above Bianchi-induced fluctuations can be computed analytically (Barrow et al. 1985; McEwen et al. 2005b) and is given by

\[
C_l = \frac{4\pi^2}{2l + 1} \left(\frac{\sigma}{H}\right)_0^2 \left[ (I_A^l)^2 + (I_B^l)^2 \right],
\]

where

\[
I_A^l = \sqrt{\frac{2l + 1}{4\pi l(l+1)}} \int_0^{\pi} A(\theta) P_l^1(\cos \theta) \sin \theta d\theta
\]

\[
I_B^l = \sqrt{\frac{2l + 1}{4\pi l(l+1)}} \int_0^{\pi} B(\theta) P_l^1(\cos \theta) \sin \theta d\theta
\]

The majority of the power in this Bianchi template is contained in multipoles below \(l \sim 20\). Therefore to study this particular model we don’t need high resolution CMB anisotropy maps.

To do a statistical study of the Bianchi patterns, we generate the pattern for a given model. This pattern is given by the shear components, \(x\) parameter and the \(\Omega_0\). We will then simulate random CMB maps from the best fit \(C_l\) of the WMAP data (Spergel et al. 2003). We add these two maps with a strength factor \(\alpha\) which will let us control the relative strength of the pattern and the random map (see Fig. 3). The resultant map is then given by

\[
\Delta T(\hat{n}) = \Delta T^{CMB}(\hat{n}) + \alpha \Delta T^{Bianchi}(\hat{n}),
\]
Fig. 3.— Adding a pattern template with a strength $\alpha$ to a random realization of the CMB anisotropy map. This introduces a preferred direction in the map and violates the statistical isotropy. Maps are rotated to the Galactic center for illustration.

and hence the power spectrum of this map will be given by

$$C_l = C_{l}^{CMB} + \alpha^2 C_{l}^{Bianchi}$$

(85)

$\alpha$ is related to $(\frac{\sigma}{\Omega})_0$ through $(\frac{\sigma}{\Omega})_0 = \frac{\alpha}{T_0}$ where $T_0 = 2.73 \times 10^6 \mu K$.

We compute the bipolar power spectrum (BiPS) for the Bianchi added CMB anisotropy maps generated above. The unbiased estimator of BiPS is given by,

$$\kappa_\ell = \sum_{l'M'} \sum_{l,l'} W_{l} W_{l'} \left| a_{lm} a_{l'm'} C_{l'm'}^{\ell M} \right|^2 - B_\ell$$

(86)

where $W_l$ is the window function in harmonic space, $B_\ell$ is the bias that arises from the SI part of the map and is given by the angular power spectrum, $C_l$ and $C_{l'M'}^{\ell M}$ are Clebsch-Gordan coefficients. Bipolar power spectrum is zero for statistically isotropic maps and has been studied in great details in Hajian & Souradeep (2005) and Basak et al. (2006). Although BiPS is quartic in $a_{lm}$, it is designed to detect SI violation and not non-Gaussianity (Hajian & Souradeep 2003b, 2005, 2006). Since BiPS is orientation independent (Hajian & Souradeep 2005), we don’t need to worry about the relative orientation between the background map and the Bianchi template. For different strength factors we simulate 100 Bianchi added CMB maps\textsuperscript{11} at HEALPix resolution of $N_{\text{side}} = 32$ corresponding to

\textsuperscript{11}Simulation of background maps and the spherical harmonic expansion of the resultant map is done by HEALPix.
Fig. 4.— Four Bianchi added CMB maps with different strength factors, rotated to the Galactic center for illustration. The Bianchi template in the above maps has been computed with $x = 0.55$, $\Omega_0 = 0.5$ and strength factors $(\frac{\sigma_H}{H})_{0} = 1.83 \times 10^{-9}$ (top left), $(\frac{\sigma_H}{H})_{0} = 1.09 \times 10^{-9}$ (top right), $(\frac{\sigma_H}{H})_{0} = 7.3 \times 10^{-10}$ (bottom left), $(\frac{\sigma_H}{H})_{0} = 3.66 \times 10^{-10}$ (bottom right). Although the pattern can hardly be seen in the fourth map, it has a considerable non-zero bipolar power spectrum (see Fig 5).

$l_{\text{max}} = 95$. Some of these maps are shown in Figure 4. BiPS of each map is obtained from the total $a_{lm}$ which according to eqn. (84) are $a_{lm} = a_{lm}^{\text{CMB}} + \alpha a_{lm}^{\text{Bianchi}}$. We compute the BiPS for each map using the estimator given in eqn. (86). We average these 100 BiPS and compute the dispersion in them. The dispersion is an estimate of $1\sigma$ error bars. At last we use the total angular power spectrum given in eqn. (85) to estimate the bias. We use the best fit theoretical power spectrum from the WMAP analysis, Spergel et al. (2003), as $C_{l}^{\text{CMB}}$ in computing the bias. Some results of BiPS for different strength factors are shown in Figure 5. As it has been discussed in Hajian & Souradeep (2003b, 2005), we can use different window functions in harmonic space, $W_l$, in order to concentrate on a particular
$l$-range. This is proved to be a useful and strong tool to detect deviations from SI using BiPS method. In other words, multipole space windows that weigh down the contribution from the SI region of the multipole space will enhance the signal relative to the cosmic error. We use simple filter functions in $l$ space to isolate different ranges of angular scales; low pass, Gaussian filters

$$W^G_l = N^G \exp \left\{ - \left( \frac{2l + 1}{2l_s + 1} \right)^2 \right\}$$

(87)

that cut power on scales ($l \leq l_s$) and band pass filters of the form

$$W^S_l = 2 N^S \left[ 1 - J_0 \left( \frac{2l + 1}{2l_s + 1} \right) \right] \exp \left\{ - \left( \frac{2l + 1}{2l_s + 1} \right)^2 \right\},$$

(88)

where $J_0$ is the spherical Bessel function and $N^G$ and $N^S$ are normalization constants chosen such that,

$$\sum_l \frac{(2l + 1)W_l}{2l(l+1)} = 1$$

(89)

i.e., unit rms for unit flat band angular power spectrum $C^{XX'}_l = \frac{2\pi}{l(l+1)}$.

We use a simple $\chi^2$ statistics to compare our BiPS results with zero,

$$\chi^2 = \sum_{l=0}^{l_{\text{max}}} \left( \frac{\kappa_l}{\sigma_{kl}} \right)^2.$$ 

(90)

The probability of detecting a map with a given BiPS is then given by the probability distribution of the above $\chi^2$. Probability of $\chi^2$ versus the strength factor is plotted in Figure 6. Being on the conservative side we can constrain the strength factor to be $(\sigma_0)$ = 2.55 × $10^{-10}$ at a 99.9% confidence level.

We repeat the above exercise for nearly flat ($\Omega_0 = 0.99$) Bianchi VII$\alpha$ templates. Some of these Bianchi added CMB maps are shown in Figure 7. As we expect, the spiral patterns of these models are wider than the hyperbolic models and are not focused in one hemisphere. As it can be seen in Figure 8 they have a weaker BiPS signal. And hence we get a weaker limit of $(\sigma_0)$ = 4.7 × $10^{-10}$ at a 99.9% confidence level on them (see Figure 9).
Fig. 5.— Bipolar power spectrum of Bianchi added CMB maps with strength factors of 
\((\sigma_H) = 4.76 \times 10^{-10}, 3.66 \times 10^{-10}, 2.56 \times 10^{-10}, 1.25 \times 10^{-10}\). We have used \(W^S_I(4, 13)\) to filter
the maps in order to focus on the low-\(l\) (large angular scale) properties of the maps.

10. Conclusion

Bianchi templates which are imprinted by the unperturbed anisotropic expansion, have
preferred directions which violate statistical isotropy of the CMB anisotropy maps. In this
paper we study the consistency of the existence of a hidden Bianchi template in the WMAP
data with the previous detection of null bipolar power spectrum in the WMAP first year
maps (Hajian et al. 2004). We compute the bipolar power spectrum for low density Bianchi
VII\(_h\) models embedded in background CMB anisotropy maps with the power spectrum that
Fig. 6.— Probability distribution of $\chi^2$ versus the strength factor for $\Omega_0 = 0.5$ Bianchi template. Results of two different window functions, $W^G_i(25)$ (blue line) and $W^S_i(4, 13)$ (red, solid line) are shown. By testing statistical isotropy of Bianchi embedded CMB maps, we can put a limit of $(\sigma/H)_0 \geq 2.55 \times 10^{-10}$ ($99.9\% CL$) on Bianchi VII$_h$ models.

best fits the first year data of WMAP. We find non-zero bipolar power spectrum for models with $(\sigma/H)_0 \geq 2.55 \times 10^{-10}$. This is inconsistent with the null bipolar power spectrum of the WMAP data at (99.9% CL). We conclude that correcting the WMAP first year maps for the Bianchi template may make some anomalies in the WMAP data vanish but this will be done at the expense of introducing other anomalies such as preferred directions and violation of statistical isotropy into the Bianchi corrected maps.
Fig. 7.— Two Bianchi added CMB maps, $\Omega_0 = 0.99$, with different strength factors, $(\frac{\sigma}{H})_0 = 1.09 \times 10^{-9}$ and $3.66 \times 10^{-10}$, rotated to the Galactic center for illustration.

Fig. 8.— Bipolar power spectrum of Bianchi added CMB maps with strength factors of $(\frac{\sigma}{H})_0 = 7.33 \times 10^{-10}$ (left) and $(\frac{\sigma}{H})_0 = 4.76 \times 10^{-10}$ (right).

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Fig. 9.— Probability distribution of $\chi^2$ versus the strength factor for $\Omega_0 = 0.99$ Bianchi template. Results of two different window functions, $W_i^G(25)$ (red line) and $W_i^S(4, 13)$ (cyan line) are shown.

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A. Symmetries of Space

By symmetries of the space, we mean those transformations of the space into itself under which metric and every other physical and geometrical properties of the space remain invariant. Here we only consider continuous symmetries. These symmetries as we shall see, form continuous groups of symmetries and are generated by vector fields which are
Mathematical preliminaries

Let $M$ be an $m$-dimensional differentiable manifold and $X$ be a vector field in $M$. An integral curve $x(t)$ of $X$ is a curve in $M$, whose tangent vector at $x(t)$ is $X|_x$

$$\frac{dx^\mu(t)}{dt} = X^\mu(x(t)) \quad (A1)$$

Let $\sigma(t, x_0)$ be an integral curve of $X$ which passes a point $x_0$ at $t = 0$ and denote the coordinate by $\sigma^\mu(t, x_0)$. Eqn. (A1) then becomes

$$\frac{d}{dt}\sigma^\mu(t, x_0) = X^\mu(\sigma(t, x_0)) \quad (A2)$$

with the initial condition

$$\sigma^\mu(0, x_0) = x_0^\mu. \quad (A3)$$

The map $\sigma : \mathbb{R} \times M \rightarrow M$ is called a flow generated by $X \in \mathfrak{X}(M)^{12}$. For fixed $t \in \mathbb{R}$ a flow is a diffeomorphism from $M$ to $M$, denoted by $\sigma_t : M \rightarrow M$ and is made into a commutative group by the following rules

1. $\sigma_t(\sigma_s(x)) = \sigma_{t+s}(x),$

2. $\sigma_0$ = the identity map,

3. $\sigma_{-t} = (\sigma_t)^{-1}$. This group is called the one-parameter group of transformations.

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$^{12}\mathfrak{X}(M)$ denotes the set of vector fields on $M$. 
**Diffeomorphism:** A $C^\infty$ map is called a diffeomorphism if it is one-to-one and onto (and hence invertible) and its inverse is $C^\infty$ too.

Let $(M, g)$ be a (pseudo-) Riemannian manifold. A diffeomorphism $f: M \rightarrow M$ is an **isometry** if it preserves the metric

$$f^*g_{f(p)} = g_p. \quad (A4)$$

This means that $f$ is an isometry if for each $X, Y \in T_pM$ we have $g_{f(p)}(f \ast X, f \ast Y) = g_p(X, Y)$. In components, this condition becomes

$$\frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu} g_{\alpha\beta}(f(p)) = g_{\mu\nu}(p), \quad (A5)$$

where $x$ and $y$ are coordinates of $p$ and $f(p)$ respectively. The isometries of a space of dimension $n$ form a group, because the identity map, the composition of two isometries and the inverse of an isometry are all isometries too. For example in $\mathbb{R}^n$, the Euclidean group $E^n$ is the isometry group, that is $f: x \rightarrow Ax + T$, where $A \in SO(n)$ are rotations and $T \in \mathbb{R}^n$ causes the translations.

Let $(M, g)$ be a Riemannian manifold and $X \in \mathfrak{X}(M)$. If an infinitesimal displacement given by $\epsilon X$ generates an isometry ($\epsilon$ is an infinitesimal parameter), the vector field $X$ is called a **Killing vector fields**. The coordinates $x^\mu$ of a point $p \in M$ change to $x^\mu + \epsilon X^\mu(p)$ under this displacement$^{13}$. If $f: x^\mu \mapsto x^\mu + \epsilon X^\mu$ is an isometry, it satisfies (A5),

$$\frac{\partial(x^\alpha + \epsilon X^\alpha)}{\partial x^\mu} \frac{\partial(x^\beta + \epsilon X^\beta)}{\partial x^\nu} g_{\alpha\beta}(x + \epsilon X) = g_{\mu\nu}(x) \quad (A6)$$

The above expression tells us that $g_{\mu\nu}$ and $X^\mu$ satisfy the **Killing equation**

$$X^\eta \partial_\eta g_{\mu\nu} + \partial_\mu X^\eta g_{\eta\nu} + \partial_\nu X^\eta g_{\mu\eta} = 0. \quad (A7)$$

$^{13} \sigma^\mu_\epsilon(x) = \sigma^\mu(\epsilon, x) = x^\mu + \epsilon X^\mu(x)$ is a one-parameter group of transformations and the vector field $X$ is called the **infinitesimal generator** of this transformation.
And from the definition of the Lie derivative, this is written as

\[(\mathcal{L}_X g)_{\mu\nu} = 0.\]  \hspace{1cm} (A8)

This has a nice interpretation. Let \(\phi_t : M \rightarrow M\) be a one-parameter group of transformations which generate the Killing vector field \(X\). Eqn. (A8) then shows that the local geometry does not change as we move along \(\phi_t\). In this sense, the Killing vector fields represent the direction of the symmetry of a manifold.

A set of Killing vector fields is said to be linearly dependent if one of them is expressed as a linear combination of others with constant coefficients\(^{14}\). The maximum number of symmetries is related to \(\text{dim} M\) and in an \(m\)-dimensional manifold is \(m(m + 1)/2\). Spaces which admit \(m(m + 1)/2\) Killing vector fields are called \textbf{maximally symmetric spaces}. Let \(X_a\) and \(X_b\) be two Killing vector fields. It can be seen that

(a) a linear combination \(\alpha X_a + \beta X_b\) is a killing vector field \((\alpha, \beta \in \mathbb{R})\),

(b) the commutator \([X_a, X_b]\) is a Killing vector field\(^{15}\).

Thus all Killing vector fields form a Lie algebra of the symmetric operations on the manifold \(M\), with structure constants \(C_{abc}^{c}\)

\[[X_a, X_b] = C_{abc}^{c} X_c\]  \hspace{1cm} (A9)

where \(a, b, c = 1, 2, ..., r\) and \(r \leq m(m + 1)/2\). Structure constants satisfy

(a) \textit{Skew-symmetry}

\[C_{ab}^{c} = -C_{ba}^{c}\]  \hspace{1cm} (A10)

\(^{14}\)There may be more Killing vector fields than the dimension of the manifold. The number of independent symmetries has no direct connection with the dimension of \(M\).

\(^{15}\)Because \([X_a, X_b]\) \(g = \mathcal{L}_{X_a}(\mathcal{L}_{X_b}g) - \mathcal{L}_{X_b}(\mathcal{L}_{X_a}g) = 0\)
(b) Jacobi identity

\[ C^a_{[b} C^e_{cd]} = 0 \]  \hspace{1cm} (A11)

These are the integrability conditions that must be satisfied in order that the Lie algebra exist in a consistent way. The transformations generated by the Lie algebra form a Lie group of the same dimension.

In the case of our interest, a Lie group often appears as a set of transformations acting on a manifold. We need to study the action of a Lie group \( G \) on a manifold \( M \) in more details. Let \( G \) be a Lie group and \( M \) a manifold. The action of \( G \) on \( M \) is a differentiable map \( \sigma : G \times M \rightarrow M \) which satisfies the conditions

(i) \( \sigma(e, p) = p \) for any \( p \in M \)

(ii) \( \sigma(g_1, \sigma(g_2, p)) = \sigma(g_1 g_2, p) \).

The action \( \sigma \) is said to be transitive if, for any \( p_1, p_2 \in M \), there exists an element \( g \in G \) such that \( \sigma(g, p_1) = p_2 \). Which means it can move any point of \( M \) into any other point of \( M \). Given a point \( p \in M \), the action of \( G \) on \( p \) takes \( p \) to various points in \( M \). The orbit of \( p \) is the set of all points into which \( p \) can be moved by the action of the isometries of a space. This is

\[ Gp = \{ \sigma(g, p) | g \in G \}. \]  \hspace{1cm} (A12)

Any orbit \( Gp \) is clearly a subset of \( M \) and obviously the action of \( G \) on any orbit \( Gp \) is transitive. Orbits are in fact the largest surfaces through each point on which the group is transitive. If the action of \( G \) on \( M \) is transitive, the orbit of any \( p \in M \) is \( M \) itself. So, the maximum dimension of orbits is \( \dim M \).

\[ \dim \text{orbits} = s, \text{ where in a space of dimension } n \ (\dim M = n), \ s \leq n. \]
A subgroup $H(p)$ of a Lie group $G$ that acts on $M$ is called an isotropy group if it leaves the point $p \in M$ fixed,

$$H(p) = \{ g \in G | \sigma(g, p) = p \} \quad (A13)$$

$H(p)$ is also called the little group or stabilizer. We have\[\text{dim of isotropy group} = q, \text{ where } q \leq \frac{1}{2}n(n - 1).\]

Interestingly, the cosets of the isotropy group correspond to the elements in the orbit,

$$G_p \sim G/H(p) \quad (A14)$$

$G/H$ has a very interesting property indeed. There is a theorem which says Nakahara (1984): for any subgroup $H$ of a Lie group $G$, the coset space $G/H$ admits a differentiable structure and becomes a manifold called homogeneous space. Dimension of this coset space is given by

$$\text{dim}G/H = \text{dim}G - \text{dim}H \quad (A15)$$

In our case, $G$ is a Lie group which acts on a manifold $M$ transitively, and $H(p)$ is an isotropy group of $p \in M$. This theorem tells us that $H(p)$ is a Lie group and the coset $G/H(p)$ is a homogeneous space.

The dimension $r$ of the group of symmetries of the space (group of isometries), is then given by

$$r = s + q \implies 0 \leq r \leq n + \frac{n}{2}(n - 1) = \frac{1}{2}n(n + 1) \quad (A16)$$

which is what we had seen before. The above relation can be viewed as

\[\text{dim group of isometries} = \text{dim group of rotational symmetries} + \text{dim group of translational symmetries}.\]

If $q = 0$ then $r = s$ which means that dimension of group of isometries is just enough to move each point in an orbit into any other point. This is called a simply transitive group. There is no continuous isotropy group in this case.
B. Bianchi Models: Original Classification by Bianchi

In his historical paper *On the 3-dimensional spaces which admit a continuous group of motions*, Luigi Bianchi begins with finite-dimensional continuous Lie group $G_r$ generated by $r$ infinitesimal transformations $X_1, X_2, ..., X_r$ of an $n$-dimensional Riemannian manifold. Then his problem of determining which spaces possess a continuous group of motions gets reduced to the classification of all possible forms of metrics which possess a Lie group $G_r = \{X_1, \cdots, X_n\}$ which transforms the metric into itself. Lie’s classification of equivalence classes of 3-dimensional Lie algebras over the complex numbers was then refined by him to the real case by adding several types, giving Bianchi’s canonical form for the generating Lie algebra commutation relations for each type designated by consecutive Roman numerals I through IX, now known as the Bianchi classification.

This classification of all possible spaces which admit continuous groups of motions was first carried out by Bianchi. They are divided into six categories according to the type of their complete group $G$ of motions. Spaces whose group of motions is a $G_1$ are class (A), class (B) are spaces with group of motions $G_2$, they are class (C) when it is an intransitive $G_3$. The other two categories (D), (E) contain the spaces whose group of motions is transitive, (D) those with a $G_3$, and (E) those with a $G_4$. The sixth category (F) includes the spaces of constant curvature which are maximally symmetric spaces which admit a group $G_6$ of motions. He proves the impossibility of other spaces with continuous groups of motions. For example that there does not exist any 3-dimensional space whose group of motions

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motions contains a real 5-parameter subgroup.

**Category A**

*Groups* $G_1$

*Type I*

$ds^2 = \Sigma a_{ik} \, dx_i dx_k$

with coefficients $a_{ik}$ independent of $x_1$

The group:

$$X_1 f = \frac{\partial f}{\partial x_1}$$

Or, the group $G_3$ is Abelian and offers the first and simplest composition

$$[X_1, X_2] f = [X_1, X_3] f = [X_2, X_3] f = 0.$$  

**Category B**

*Groups* $G_2$

$$ds^2 = dx_1^2 + \alpha \, dx_2^2 + 2\beta \, dx_2 dx_3 + \gamma \, dx_3^2$$

with $\alpha, \beta, \gamma$ functions only of $x_1$

The group:

$$X_1 f = \frac{\partial f}{\partial x_2}, \quad X_2 f = \frac{\partial f}{\partial x_3}$$

composition:

$$[X_1, X_2] = 0$$
\[ ds^2 = dx_1^2 + \alpha \, dx_2^2 + 2(\beta - \alpha x_2) \, dx_2 dx_3 + (\alpha x_2^2 - 2\beta x_2 + \gamma) \, dx_3^2 \]

with \( \alpha, \beta, \gamma \) functions of \( x_1 \)

group:
\[
X_1 f = \frac{\partial f}{\partial x_3}, \quad X_2 f = e^{x_3} \frac{\partial f}{\partial x_2}
\]

composition:
\[
[X_1, X_2] f = X_2 f
\]

**Category C**

*Groups G_3 intransitive*

\( \alpha \)
\[ ds^2 = dx_1^2 + \varphi^2(x_1)(dx_2^2 + dx_3^2) \]

with \( \varphi(x_1) \) an arbitrary function of \( x_1 \)

group:
\[
X_1 f = \frac{\partial f}{\partial x_2}, \quad X_2 f = \frac{\partial f}{\partial x_3}, \quad X_3 f = x_3 \frac{\partial f}{\partial x_2} - x_2 \frac{\partial f}{\partial x_3}
\]

composition:
\[
[X_1, X_2] f = 0, \quad [X_1, X_3] f = -X_2 f, \quad [X_2, X_3] f = X_1 f
\]

\( \beta \)
\[ ds^2 = dx_1^2 + \varphi^2(x_1)(dx_2^2 + \sin^2 x_2 \, dx_3)^2 \]
group:

\[ X_1 f = \frac{\partial f}{\partial x_3}, \quad X_2 f = \sin x_3 \frac{\partial f}{\partial x_2} + \cot x_2 \cos x_3 \frac{\partial f}{\partial x_3}, \]

\[ X_3 f = \cos x_3 \frac{\partial f}{\partial x_2} - \cot x_2 \sin x_3 \frac{\partial f}{\partial x_3} \]

composition:

\[ [X_1, X_2] f = X_3 f, \quad [X_2, X_3] f = X_1 f, \quad [X_3, X_1] f = X_2 f \]

\[ \gamma) \quad ds^2 = dx_1^2 + \varphi^2(x_1) \left( dx_2^2 + e^{2x_2} dx_3 \right)^2 \]

group:

\[ X_1 f = \frac{\partial f}{\partial x_3}, \quad X_2 f = \frac{\partial f}{\partial x_2} - x_3 \frac{\partial f}{\partial x_3}, \]

\[ X_3 f = x_3 \frac{\partial f}{\partial x_2} + \frac{1}{2} (e^{-2x_2} - x_3^2) \frac{\partial f}{\partial x_3} \]

composition:

\[ [X_1, X_2] f = -X_1 f, \quad [X_1, X_3] f = X_2 f, \quad [X_2, X_3] f = -X_3 f \]

**Category D**

*Groups $G_3$ transitive*

**Type IV**

\[ ds^2 = dx_1^2 + e^{x_1} [dx_2^2 + 2x_1 dx_2 dx_3 + (x_1^2 + n^2) dx_3^2] \]

\[ \begin{align*}
X_1 f &= 2 \frac{\partial f}{\partial x_3}, \\
X_2 f &= \frac{\partial f}{\partial x_3}, \\
X_3 f &= -2 \frac{\partial f}{\partial x_1} + (x_2 + 2x_3) \frac{\partial f}{\partial x_2} + x_3 \frac{\partial f}{\partial x_3}
\end{align*} \]
composition:

\[[X_1, X_2] = 0, [X_1, X_3]f = X_1f, [X_2, X_3]f = X_1f + X_2f\]

**Type VI**

\[ds^2 = dx_1^2 + e^{2x_1} dx_2^2 + 2ne^{(h+1)x_1} dx_2 dx_3 + e^{2hx_1} dx_3^2\]

**group:**

\[X_1f = \frac{\partial f}{\partial x_2}, X_2f = \frac{\partial f}{\partial x_3}, X_3f = -\frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + hx_3 \frac{\partial f}{\partial x_3}\]

composition:

\[[X_1, X_2] = 0, [X_1, X_3]f = X_1f, [X_2, X_3]f = hX_2f\]

**Type VII\_1**

\[ds^2 = dx_1^2 + (n + \cos x_1) dx_2^2 + 2 \sin x_1 dx_2 dx_3 + (n - \cos x_1) dx_3^2\]

**group:**

\[X_1f = \frac{\partial f}{\partial x_2}, X_2f = \frac{\partial f}{\partial x_3}, X_3f = 2 \frac{\partial f}{\partial x_1} - x_3 \frac{\partial f}{\partial x_2} + x_2 \frac{\partial f}{\partial x_3}\],

with the composition

\[[X_1, X_2]f = 0, [X_1, X_3]f = X_2f, [X_2, X_3]f = -X_1f .\]

**Type VII\_2**

\[ds^2 = dx_1^2 + e^{-hx_1} \left\{ (n + \cos vx_1) dx_2^2 + (h \cos vx_1 + v \sin x_1 + hn) dx_2 dx_3 + \left( \frac{2 - v^2}{2} \cos vx_1 + \frac{hv}{2} \sin vx_1 + n \right) dx_3^2 \right\}, h \neq 0 (0 < h < 2)\]

where \(v = \sqrt{4 - h^2}\)
group:

\[ X_1 f = \frac{\partial f}{\partial x_2}, \quad X_2 f = \frac{\partial f}{\partial x_3}, \]
\[ X_3 f = \frac{\partial f}{\partial x_1} - x_3 \frac{\partial f}{\partial x_2} + (x_2 + h x_3) \frac{\partial f}{\partial x_3} \]

composition:

\[ [X_1, X_2] = 0, \quad [X_1, X_3] f = X_2 f, \quad [X_2, X_3] f = -X_1 f + h X_2 f \]

Type VIII

\[
\begin{align*}
ds^2 &= \frac{Q^{(4)}(x_1)}{24} \, dx_1^2 + Q(x_1) \, dx_2^2 + \left( Q(x_1) x_2^2 - \frac{Q'(x_1)}{2} x_2 + \frac{Q''(x_1)}{4} - \frac{h}{2} \right) \, dx_3^2 \\
&\quad + 2 \left( \frac{Q'''(x_1)}{12} + h \right) \, dx_1 dx_2 + 2 \left\{ \frac{Q''(x_1)}{24} - \left( \frac{Q'(x_1)}{12} + h \right) x_2 \right\} \, dx_1 dx_3 \\
&\quad + 2 \left( \frac{Q'(x_1)}{4} - Q(x_1) x_2 \right) \, dx_2 dx_3,
\end{align*}
\]

with \( Q(x_1) \) a fourth degree polynomial in \( x_1 \) with its first coefficient positive (or zero), and \( h \) a constant

group:

\[ X_1 f = e^{-x_3} \frac{\partial f}{\partial x_1} - x_2 e^{-x_3} \frac{\partial f}{\partial x_2} - 2 x_2 e^{-x_3} \frac{\partial f}{\partial x_3}; \]
\[ X_2 f = \frac{\partial f}{\partial x_3}, \quad X_3 f = \frac{\partial f}{\partial x_2} \]

composition:

\[ [X_1, X_2] f = X_1 f, \quad [X_1, X_3] f = 2 X_2 f, \quad [X_2, X_3] f = X_3 f \]

\[
ds^2 = \Sigma_{i,k} a_{ik} \, dx_i dx_k
\]
\[ a_{11} = 2e \cos 2x_3 + 2f \sin 2x_3 + (a^2 + d^2)/2 , \]
\[ a_{22} = 2 \sin x_1 \cos x_1 (b \sin x_3 - c \cos x_3) - a_{11} \sin^2 x_1 + a^2 + d \sin^2 x_1 , \]
\[ a_{33} = a^2 , \quad a_{13} = b \cos x_3 + c \sin x_3 , \]
\[ a_{12} = \cos x_1 (b \cos x_3 + c \sin x_3) + 2 \sin x_1 (e \sin 2x_3 - f \cos 2x_3) , \]
\[ a_{23} = a^2 \cos x_1 + \sin x_1 (b \sin x_3 - c \cos x_3) \]

\text{group:}
\[ X_1 f = \frac{\partial f}{\partial x_2} , \quad X_2 f = \cos x_2 \frac{\partial f}{\partial x_1} - \cot x_1 \sin x_2 \frac{\partial f}{\partial x_2} + \frac{\sin x_2}{\sin x_1} \frac{\partial f}{\partial x_3} , \]
\[ X_3 f = -\sin x_2 \frac{\partial f}{\partial x_1} - \cot x_1 \cos x_2 \frac{\partial f}{\partial x_2} + \frac{\cos x_2}{\sin x_1} \frac{\partial f}{\partial x_3} \]

\text{composition:}
\[ [X_1, X_2]f = X_3 f , \quad [X_2, X_3]f = X_1 f , \quad [X_3, X_1]f = X_2 f \]

\textbf{Category E}

\textit{Groups G}_4

a) \textit{Type II}
\[ ds^2 = dx_1^2 + dx_2^2 + 2x_1 \, dx_2 \, dx_3 + (x_1^2 + 1) \, dx_3^2 \]

\text{group:}
\[ X_1 f = \frac{\partial f}{\partial x_2} , \quad X_2 f = \frac{\partial f}{\partial x_3} , \quad X_3 f = -\frac{\partial f}{\partial x_1} + x_3 \frac{\partial f}{\partial x_2} , \]
\[ X_4 f = x_3 \frac{\partial f}{\partial x_1} + \frac{1}{2} (x_1^2 - x_3^2) \frac{\partial f}{\partial x_2} - x_1 \frac{\partial f}{\partial x_3} \]

\text{composition:}
\[ [X_1, X_2] = [X_1, X_3] = [X_1, X_4] = 0 , \]
\[ [X_2, X_3]f = X_1f , \quad [X_2, X_4]f = -X_3f , \quad [X_3, X_4]f = X_2f \]

b) \textit{Types III, VIII}

\[ ds^2 = dx_1^2 + e^{2x_1} dx_2^2 + 2ne^{x_1} dx_2 dx_3 + dx_3^2 \]

\textbf{group:}

\[ X_1f = \frac{\partial f}{\partial x_2} , \quad X_2f = \frac{\partial f}{\partial x_3} , \quad X_3f = \frac{\partial f}{\partial x_1} - x_2 \frac{\partial f}{\partial x_2} , \]
\[ X_4f = x_2 \frac{\partial f}{\partial x_1} + \frac{1}{2} \left( \frac{e^{-2x_1}}{1-n^2} - x_2^2 \right) \frac{\partial f}{\partial x_2} - \frac{ne^{-x_1} \partial f}{1-n^2} \partial x_3 \]

\textbf{composition:}

\[ [X_1, X_2] = 0 , \quad [X_1, X_3]f = -X_1f , \quad [X_1, X_4]f = X_3f , \]
\[ [X_2, X_3] = 0 , \quad [X_2, X_4] = 0 , \quad [X_3, X_4]f = -X_4f \]

c) \textit{Type IX}

\[ ds^2 = dx_1^2 + (\sin^2 x_1 + n^2 \cos^2 x_1) dx_2^2 + 2n \cos x_1 dx_2 dx_3 + dx_3^2 \]

\textbf{group:}

\[ X_1f = \frac{\partial f}{\partial x_2} , \quad X_2f = \cos x_2 \frac{\partial f}{\partial x_1} - \cot x_1 \sin x_2 \frac{\partial f}{\partial x_2} + \frac{n \sin x_2 \partial f}{\sin x_1 \partial x_3} , \]
\[ X_3f = -\sin x_2 \frac{\partial f}{\partial x_1} - \cot x_1 \cos x_2 \frac{\partial f}{\partial x_2} + \frac{n \cos x_2 \partial f}{\sin x_1 \partial x_3} , \quad X_4f = \frac{\partial f}{\partial x_3} \]

\textbf{composition:}

\[ [X_1, X_2]f = X_3f , \quad [X_2, X_3]f = X_1f , \quad [X_3, X_1]f = X_2f , \]
\[ [X_1, X_4] = [X_2, X_4] = [X_3, X_4] = 0 \]
Category F

Groups $G_6$ — spaces of constant curvature

These are the maximally symmetric space that foliate the equal time section of the standard homogeneous and isotropic FLRW cosmological model.
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