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FORM FACTORS WITH RISING PHASE

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A B S T R A C T

We investigate form factors whose phase, $\delta(s)$, rises indefinitely (infinite number of resonances). We prove that, if $\delta(s) \sim s^\rho$ with $0 < \rho < 1$, no polynomial-bounded form factor exists ; if $\rho = 1$, we construct all possible form factors with given phase. This result shows, in particular, that, at least in the elastic approximation, if Regge trajectories rise at all, then they rise linearly.

As is well known, the Watson final state interaction theorem links the phase of a form factor $F(s)$ in the elastic timelike region to the corresponding phase shift, $\delta(s)$. If the form factor is then supposed to be analytic in the cut plane, polynomial bounded in all directions and one neglects inelastic effects, one can convert that information into an integral equation (Omnès equation) which is in turn solvable. The solution is unique up to an arbitrary polynomial of degree controlled by the growth of $|F(z)|$. As a matter of fact, the methods employed ^{1),2)} are only applicable if $\delta(s)$ tends to a constant at infinity, i.e., only a finite number of resonances exist.

However, recent analyses ³⁾ seem to indicate that, in a realistic situation, an infinite number of resonances may be present due to the rising of Regge trajectories, which poses the problem of generalizing the analysis of Omnès equations to the case where the phase, $\delta(s)$, grows for large s . We are aware of the fact that the problem is somewhat academic since the relation [phase of $F(s)$ = phase of partial wave] is only valid in the elastic region, so that the fact that [phase of partial wave $\rightarrow \infty$] for large s may be irrelevant in a realistic situation. Nevertheless, it is clear that the same difficulty occurs in the usual Omnès problem and, on the other hand, one might argue that the effect of inelasticity may be to distort the phase of $F(s)$ without altering its general features ; and also solutions to Omnès equations are useful in an N/D treatment of partial waves ⁴⁾, where the relation [phase of N = phase of partial wave] is exact by construction. Finally, at any rate, our results are interesting in themselves in that they permit the analysis of form factors with general phases.

As it is known ^{1),2)}, the Omnès equation is equivalent to the following problem. One is given the following properties of the function $F(z)$: (a) F is real analytic in the plane cut from (say) $+1$ to $+\infty$; (b) it is polynomial bounded everywhere, i.e.,

$$|F(z)| < a + b|z|^N, \quad N \text{ fixed} \geq 0; \quad (1)$$

(c) the phase of F is known ; with obvious notation we write

$$\text{ph } F(s) = \delta(s). \quad (2)$$

We reserve the letter s for the variable on the cut. Furthermore, we shall assume that (d) $\delta(s)$ is continuous everywhere and

$$\delta(s) \simeq \text{constant} \times (s-1)^{1+\eta}, \quad \eta > 0, \quad \text{for } s \simeq 1 \quad (3)$$

(it is not difficult to remove this condition, although we shall keep it here for simplicity) and, finally, (e) one has

$$\delta(s) \simeq s^\rho + s^{-\sigma}, \quad 0 < \rho \leq 1, \quad \sigma > 0, \quad \text{for } s \rightarrow \infty. \quad (4)$$

The coefficients in front of s^ρ , $s^{-\sigma}$ have been taken unity for convenience ; for our analysis, it is only necessary that the coefficient of s^ρ be positive, which is, of course, always true. Here again we can extend the proofs to more complicated asymptotic behaviours of δ (e.g., involving logarithmic terms) by using, for example, the techniques of Ref. ⁵⁾, but we think Eq. (4) is general enough for our purposes. Then, given (a) through (e), one is asked to find the most general form of the function $F(z)$. In the usual situation, ρ is zero or negative and the answer is known to be

$$F(z) = p(z) \exp \left\{ \frac{z}{\pi} \int_1^\infty ds \frac{\delta(s)}{s(s-z)} \right\}, \quad (5)$$

where $p(z)$ is a polynomial of degree $\leq N$; of course, the problem in our case ($\rho > 0$) is that the integral in (5) blows up for $|z| \rightarrow \infty$. Nevertheless, we shall be able to generalize the analysis to get that, if $\rho = 1$, the solutions are of the type ^{*)}

^{*)} One should not be too surprised if the Γ function appears in the expression of F , as Γ functions seem to appear somehow when one has rising trajectories ; see the second and third papers of Ref. ³⁾.

$$F(z) = \frac{\varphi(z)}{\psi(z)} e^{\frac{z}{\pi} (1 + \log \pi)} \left[\Gamma\left(-\frac{z}{\pi}\right) \right]^{-1} \times \quad (6)$$

$$\times \exp \left\{ \frac{z^2}{\pi} \int_1^\infty ds \frac{\delta(s)}{s^2(s-z)} + \frac{z}{\pi} \int_1^\infty ds \frac{\delta(s)-s^p}{s^2} \right\},$$

where φ, ψ are highly arbitrary entire functions. If, however, $\rho < 1$, no solution exists at all. Therefore, we have the following result : if the phase of the form factor increases, but slower than s , then the form factor is not bounded by any polynomial. If one believes the experimental results that seem to show that $|F(t)|$ decreases for $t < 0$, and is at most bounded for $t > 0$, then the Phragmén-Lindelöf theorem ⁶⁾ tells us that $|F(z)|$ must grow like $\sim e^{|z|}$ in some directions, which is very unlikely. We also have the interesting result that, assuming polynomial boundedness and at least in the elastic approximation, if Regge trajectories rise at all they have to rise linearly.

To prove our statements we begin by constructing the function

$$\alpha(z) \equiv \frac{z^2}{\pi} \int_1^\infty ds \frac{\delta(s)}{s^2(s-z)} + \frac{z}{\pi} \int_1^\infty ds \frac{\delta(s)-s^p}{s^2}, \quad (7a)$$

which can be rewritten as

$$\alpha(z) = \frac{z^2}{\pi} \int_1^\infty ds \frac{s^p}{s^2(s-z)} + \frac{z}{\pi} \int_1^\infty ds \frac{\delta(s)-s^p}{s(s-z)}. \quad (7b)$$

It is clear that $\alpha(z)$ is analytic except on the cut $(+1, \infty)$, and there $\text{Im } \alpha(s) = \delta(s)$. Furthermore, for large $|z|$ it behaves like ⁵⁾

$$\alpha(z) \simeq -\frac{z}{\pi} \log(-z) + O(|z|^{-\rho}), \quad \text{if } \rho = 1, \quad (8)$$

$$\alpha(z) \simeq (-z)^{\rho} / \sin \pi \rho + O(|z|^{-\sigma}), \text{ if } \rho < 1. \quad (9)$$

Therefore, if we set $\Phi(z) \equiv \exp \{ \alpha(z) \}$, we will have that $\Phi(z)$ is real analytic except in the cut $(1, \infty)$, finite everywhere, has no zeros and its phase on the cut is, precisely, $\text{ph } \Phi(s) = \text{Im } \alpha(s) = \delta(s)$.

Consequently, for any $F(z)$ solution of (a) through (e), $F(z)/\Phi(z)$ has to be entire, i.e., one can write quite generally, $F(z) = f(z)\Phi(z)$ with $f(z)$ entire. Of course, as remarked above, the problem for $\rho > 0$ is that, as (8), (9) show, $|\Phi(z)|$ grows exponentially in some directions so that $f(z)$ cannot be simply a polynomial. Some of its limitations are given by the following.

Lemma 1 : $f(z)$ must have an infinite number of zeros.

Proof : trivial.

Lemma 2 : $f(z)$ must be of order ρ and finite type equal to $|1/\sin \pi \rho|$ if $\rho < 1$, and of order 1 and infinite type if $\rho = 1$.

Proof : If $F(z)$ has to satisfy (1), it is clear that $|f(z)|$ has to be bounded by (polynomial) $\times |\Phi(z)|^{-1}$.

Inserting the estimates (8), (9), the result follows.

From Lemma 2, we can get at once the announced result that, when $\rho \neq 1$, there exist no solutions to the Omnès problem. In fact, from (9) it follows that $|\Phi(z)|$ blow up exponentially in the angle between $\theta_1 = \pi/2\rho$ and $\theta_2 = -\pi/2\rho$. Now, it is known ⁶⁾ that an entire function of order $\rho < 1$ must increase along one of the two directions of any straight line passing through the origin. Therefore along $z = iy$ or $z = -iy$, $|z| \rightarrow \infty$, $|F(z)| = |f(z)| |\Phi(z)|$ blows up exponentially. Q.E.D.

This result is not true if $\rho = 1$; in fact here, because of Stirling's formula

$$f_0(z) \equiv e^{\frac{z}{\pi}(1+\log \pi)} [\Gamma(-z/\pi)]^{-1} \simeq \frac{1}{2\pi} \sqrt{\frac{z}{\pi}} \exp \left\{ \frac{z}{\pi} \log(-z) \right\},$$

so that using the estimate (8), we see that $F_0(z) = f_0(z) \Phi(z)$ has the right properties. The most general solution is now easy to obtain : if F_1, F_2 are solutions, F_1/F_2 must be meromorphic so that, as anticipated, the general solution is Eq. (6) where, because of condition (1) the zeros of ψ cannot fall outside the positive integers, and also

$$|\varphi(z)/\psi(z)| < a_1 + b_1 |z|^{N+1/2}$$

for $\arg z \neq 0$.

It is difficult to understand (let alone get rid of) the arbitrariness here ⁷⁾. Clearly, it is related to the location of the zeros of $F(z)$, on which we have no control. We shall only remark that a similar arbitrariness occurs in a totally different approach ⁸⁾ where one relates the pion form factor to the pion scattering amplitude via current algebra, and where the arbitrariness is caused by so-called "satellite terms".

We would also like to stress that contrarily to what has been stated by some people ⁹⁾, our analysis shows that the infinite number of resonances of, say, the Veneziano model (or a unitarized version of it of Lovelace's ³⁾ or Martin's type ¹⁰⁾) do not necessarily mean that the form factors have to grow exponentially, as we have constructed an infinity of polynomial bounded solutions.

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$$\varphi(z) = \prod_{\nu=1}^m (z - r_{\nu}), \quad \psi(z) = \prod_{\mu=1}^m (z - n_{\mu})$$
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