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SINGULARITIES OF CURRENT COMMUTATORS ON THE LIGHT CONE

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A B S T R A C T

We consider the forward proton matrix element of the electromagnetic current commutator and express its singularities on the light cone in terms of the scaling functions. This relation between the light cone and the deep inelastic region implies a family of sum rules for the electroproduction structure functions and connects the equal time commutators with the scaling functions. In particular the operator part of the Schwinger term and the related divergences of the electromagnetic mass of the proton are expressed in terms of measurable quantities.

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1. INTRODUCTION

The recent high energy inelastic electron scattering experiments ¹⁾ have stimulated a number of theoretical investigations ²⁾ of the structure functions that describe this process. In particular, the parton model ³⁾ was proposed as an explanation of the so-called scaling law ⁴⁾, the most striking feature of the experimental results. It was however soon realized by a number of authors ^{5),6)} that the parton model is only a very specific representative in a large class of models that all lead to structure functions obeying the scaling law. The purpose of the present paper is to investigate this class in some detail.

We consider the structure functions ⁷⁾

$$W_{\mu\nu}(x) = \langle p | [j_\mu(x), j_\nu(0)] | p \rangle_c \quad (1.1)$$

i.e., the connected, spin averaged matrix element of the current commutator between proton states of momentum p . The corresponding Fourier transform

$$\tilde{W}_{\mu\nu}(q) = \frac{1}{2\pi} \int dx e^{iqx} W_{\mu\nu}(x) \quad (1.2)$$

is usually decomposed into two invariant functions W_1 and W_2

$$\tilde{W}_{\mu\nu}(q) = \left(\frac{q_\mu q_\nu}{q^2} - g_{\mu\nu} \right) W_1 + \left(p_\mu - \frac{\nu}{q^2} q_\mu \right) \left(p_\nu - \frac{\nu}{q^2} q_\nu \right) W_2 \quad (1.3)$$

which depend only on the two scalar variables q^2 and $\nu = pq$. Our analysis is based on the observation that the behaviour of the structure functions in the deep inelastic region reflects the behaviour of the current commutator in the vicinity of the light cone. This relation does not involve only the tip of the light cone (related to the equal time commutator); the behaviour of the current commutator on the entire light cone is relevant. To analyze this behaviour, spacelike planes of the type $x^0 = \text{const.}$ are not particularly useful. We instead make extensive use of lightlike planes

$$n_\mu x^\mu = \tau \quad (1.4)$$

where n_μ is a lightlike vector, $n_\mu n^\mu = 0$. In particular, we consider the restrictions of the structure functions $W_{\mu\nu}(x)$ to a lightlike plane of this type:

$$J_{\mu\nu}(\tau) = \int dx W_{\mu\nu}(x) \delta(nx - \tau) f(x) \quad (1.5)$$

where $f(x)$ is a suitable test function. Our basic assumption is that certain components of $W_{\mu\nu}(x)$ are well behaved in the vicinity of the light cone in the sense that certain components of $J_{\mu\nu}(\tau)$ possess finite limits as $\tau \rightarrow \pm 0$. We show that this rather general assumption leads to the experimentally observed scaling laws. Moreover, the singularity structure of the current commutator on the light cone is completely determined by the scaling functions. This fact is shown to lead to a number of physically interesting results.

The material is presented as follows. In the next introductory section we demonstrate the connection between the light cone singularities of a particular component of the current commutator and the asymptotic behaviour of the structure functions W_1 and W_2 . In Sections 3 and 4 we discuss the general relation between the restrictions of any invariant local distribution to the lightlike surface (1.4), its leading singularity on the light cone and its behaviour in the Bjorken limit. In Section 5 these results are re-expressed in terms of the current commutator, using a local invariant decomposition of $W_{\mu\nu}(x)$. Sections 6 and 7 contain some corollaries concerning the equal-time commutators, q number Schwinger terms and the related question of convergence of the electromagnetic mass shifts. In Section 8 we present a new class of dispersion sum rules for the structure functions W_1 and W_2 . Section 9 contains a brief discussion of the extension of the present analysis to the isovector and hypercharge currents.

2. RESTRICTIONS OF CURRENTS TO LIGHTLIKE PLANES

In this Section we concentrate on the particular component $n_\mu j^\mu(x)$ which represents the charge density associated with the lightlike surface $n_\mu x^\mu = \tau$ and is the analogue of the component $j^0(x)$ describing the charge distribution in the surface $x^0 = \text{const.}$ In the infinite momentum terminology⁸⁾ the quantity $n_\mu j^\mu(x)$ is referred to as the "good" component of the current and we will see that this component is the least singular one on the light cone.

The general postulates of quantum field theory do not guarantee that the restriction

$$j(f, \tau) = \int dx n_\mu j^\mu(x) \delta(nx - \tau) f(x) = \int d\sigma_\mu j^\mu(x) f(x) \quad (2.1)$$

of the field $j_\mu(x)$ to the lightlike surface $nx = \tau$ makes sense⁹⁾. The assumption that it does exist contains non-trivial information about matrix elements involving this field, in particular about the structure functions $W_{\mu\nu}(x)$. We assume, more specifically, that the matrix element of the commutator

$$\mathcal{J}(\tau) = \langle p | [j(g, \tau), j(h, 0)] | p \rangle_c \quad (2.2)$$

possesses finite "equal-time" limits $\tau \rightarrow \pm 0$. This commutator may be expressed in terms of the structure functions by

$$\mathcal{J}(\tau) = \int dx n^\mu n^\nu W_{\mu\nu}(x) \delta(nx - \tau) f(x) \quad (2.3)$$

where $f(x)$ denotes a convolution of g and h . [In the notation introduced in the first section we are considering the "good-good" component $\mathcal{J}(\tau) = n^\mu n^\nu \mathcal{J}_{\mu\nu}(\tau)$.] The behaviour of $\mathcal{J}(\tau)$ at $\tau = 0$ reflects the behaviour of $W_{\mu\nu}(x)$ on the light cone; if the limits $\tau \rightarrow \pm 0$ are to exist then the structure functions

$W_{\mu\nu}(x)$ cannot be very singular on the light cone. As a prototype of the relation between the behaviour on the light cone and in the deep inelastic region we now wish to show that this information about the nature of the singularity on the light cone is equivalent to the statement that, in the deep inelastic region, the longitudinal structure function obeys the scaling law.

To verify this statement, let us suppose for the moment that the test function $f(x)$ vanishes in the backward cone. Since $W_{\mu\nu}(x)$ vanishes outside the light cone, the integral defining $\mathcal{J}(\tau)$ receives contributions only from the forward cone and therefore vanishes¹⁰⁾ for $\tau < 0$. On the other hand, $\mathcal{J}(\tau)$ may tend to a non-vanishing limit as τ approaches zero from the positive side. We therefore expect in general the function $\mathcal{J}(\tau)$ to exhibit a discontinuity at $\tau = 0$. The essential point is that our assumption implies that $\mathcal{J}(\tau)$ is finite for all τ and therefore excludes a more singular behaviour at $\tau = 0$. The Fourier transform

$$\tilde{\mathcal{J}}(E) = \int d\tau e^{iE\tau} \mathcal{J}(\tau) \quad (2.4)$$

must therefore behave like

$$\tilde{\mathcal{J}}(E) \rightarrow \frac{\Delta}{E} \quad (2.5)$$

as $E \rightarrow \pm\infty$ and the coefficient Δ is determined by the discontinuity at $\tau = 0$:

$$\Delta = i \{ \mathcal{J}(+0) - \mathcal{J}(-0) \} \quad (2.6)$$

In terms of the structure functions $W_{\mu\nu}(x)$ we have

$$\tilde{\mathcal{J}}(E) = \int dx n^\mu n^\nu W_{\mu\nu}(x) f(x) e^{iEnx} \quad (2.7)$$

or equivalently

$$\tilde{J}(E) = \int dk n^\mu n^\nu \tilde{W}_{\mu\nu}(k+En) \tilde{f}(-k) \quad (2.8)$$

In order to satisfy (2.5) the structure functions must behave like

$$n^\mu n^\nu \tilde{W}_{\mu\nu}(k+En) \rightarrow \frac{\Delta(k)}{E} \quad (2.9)$$

as $E \rightarrow \pm\infty$ and the discontinuity is given by

$$J(+0) - J(-0) = -i \int dk \Delta(k) \tilde{f}(-k) \quad (2.10)$$

Note that in the limit $E \rightarrow \pm\infty$ both invariants q^2 and ν

$$q^2 = (k+En)^2 = k^2 + 2Ekn$$

$$\nu = p(k+En) = pk + Epn$$

(2.11)

tend to infinity with their ratio approaching a constant

$$\frac{q^2}{2\nu} \rightarrow \frac{kn}{pn} = -\xi \quad (2.12)$$

In other words, the limit $E \rightarrow \pm\infty$ is the Bjorken limit ⁴⁾. In terms of the invariant structure functions W_1 and W_2 the asymptotic statement (2.9) reads

$$\frac{pn}{2} \left[\frac{q^2}{2\nu} W_1 + \frac{\nu}{2} W_2 \right] \rightarrow \Delta(k) \quad (2.13)$$

i.e., the square bracket must approach a constant, which, of course, may depend on the direction ξ along which the limit is taken in the (ν, q^2) plane

$$\frac{q^2}{2\nu} W_1 + \frac{\nu}{2} W_2 \rightarrow \xi \overline{F}_2(\xi) \quad (2.14)$$

This conclusion is in agreement with the scaling laws proposed by Bjorken ⁴⁾

$$\begin{aligned} W_1 &\rightarrow \overline{F}_1(\xi) \\ &(\nu \rightarrow \infty ; -\frac{q^2}{2\nu} = \xi \text{ fixed}) \\ W_2 &\rightarrow \frac{1}{\nu} \overline{F}_2(\xi) \end{aligned} \quad (2.15)$$

and in terms of these functions we have

$$\overline{F}_2(\xi) = -\overline{F}_1(\xi) + \frac{1}{2\xi} \overline{F}_2(\xi) \quad (2.16)$$

To conclude this section we work out some consequences of the asymptotic relation (2.14). We first remark that the limits for $E \rightarrow +\infty$ and $E \rightarrow -\infty$ have to coincide. This implies that the Bjorken limit along the line $q^2/2\nu = -\xi$ must give the same result in the direction $\nu \rightarrow +\infty$ as for $\nu \rightarrow -\infty$. This is a non-trivial consequence of the fact that $W_{\mu\nu}(x)$ is causal. It implies, in particular, that $\overline{F}_\ell(\xi)$ vanishes for $|\xi| > 1$ since on kinematical grounds the functions $W_1(\nu, q^2)$ and $W_2(\nu, q^2)$ vanish in the region $q^2 < -2|\nu|$ (see the Figure). Furthermore the crossing relations

$$W_\alpha(\nu, q^2) = -W_\alpha(-\nu, q^2) \quad (\alpha = 1, 2) \quad (2.17)$$

imply

$$\overline{F}_\ell(\xi) = - \overline{F}_\ell(-\xi) \quad (2.18)$$

Next, we observe that the coefficient $\Delta(k)$

$$\Delta(k) = \frac{1}{2} kn \overline{F}_\ell(kn/pn) \quad (2.19)$$

depends on the vector k only through the projection kn . As we will see shortly this is again related to the causal nature of the structure functions. Inserting the above expression for $\Delta(k)$ in (2.10) we obtain the following representation¹¹⁾ for the discontinuity in terms of the scaling function $F_\ell(\xi)$:

$$J(+0) - J(-0) = \frac{i}{2} \int_{-\infty}^{\infty} d\lambda \frac{d^2}{d\lambda^2} g_1(\lambda pn) f(\lambda n) \quad (2.20)$$

where the function g_1 is essentially the Fourier transform of F_ℓ :

$$g_1(z) = 2 \int_0^1 \frac{d\xi}{\xi} \cos \xi z \overline{F}_\ell(\xi) \quad (2.21)$$

The above expression for the discontinuity involves the test function $f(x)$ only through its values on the line $x = \lambda n$ along which the lightlike surface $nx=0$ touches the light cone. This was to be expected from the causal nature of $W_{\mu\nu}(x)$ and is reflected by the fact that $\Delta(k)$ involves only the projection kn of the vector k as mentioned above.

In this introductory section we have restricted our attention to the "good-good" component $n^\mu n^\nu W_{\mu\nu}(x)$ of the current commutator. In the following we shall discuss the singularities of the structure functions on the light cone in a more systematic fashion and will show in particular that the leading singularity of $W_{\mu\nu}(x)$ on the light cone is given by

$$W_{\mu\nu}(x) = \frac{1}{2\pi i} \{ g_{\mu\nu} \square - \partial_{\mu\nu} \} \{ \epsilon(x^0) \delta(x^2) g_1(px) \} \\ + \text{less singular terms} \quad (2.22)$$

This expression explicitly exhibits the relation of the singularity on the light cone to the behaviour of the structure functions in the deep inelastic region.

3. SINGULARITIES ON THE LIGHT CONE AND BJORKEN LIMIT

In order to analyze the connection between the singularities of the structure functions on the light cone and their behaviour in the Bjorken limit in a more systematic manner, it is convenient to use a different kinematical decomposition of the structure functions:

$$\tilde{W}_{\mu\nu}(q) = \{ q_\mu q_\nu - g_{\mu\nu} q^2 \} \tilde{V}_1 \\ + \{ (p_\mu q_\nu + p_\nu q_\mu) \nu - p_\mu p_\nu q^2 - g_{\mu\nu} \nu^2 \} \tilde{V}_2 \quad (3.1)$$

The functions $\tilde{V}_1$ and $\tilde{V}_2$ are related to W_1 and W_2 introduced in the previous section by

$$\tilde{V}_1 = \frac{1}{q^2} \{ W_1 + \frac{\nu^2}{q^2} W_2 \}$$

$$\tilde{V}_2 = - \frac{1}{q^2} W_2$$

(3.2)

These quantities satisfy the positivity conditions ¹²⁾

$$\begin{aligned} \epsilon(\nu) \{ \tilde{V}_1(\nu, q^2) + \tilde{V}_2(\nu, q^2) \} &\geq 0 \\ \epsilon(\nu) \{ q^2 \tilde{V}_1(\nu, q^2) + \nu^2 \tilde{V}_2(\nu, q^2) \} &\geq 0 \end{aligned} \quad (q^2 < 4m_\pi^2)$$

(3.3)

The virtue of the decomposition (3.1) lies in the fact that the expression for $W_{\mu\nu}(x)$ in terms of $V_1(x)$ and $V_2(x)$ is local:

$$\begin{aligned} W_{\mu\nu}(x) = & \{ -\partial_{\mu\nu} + g_{\mu\nu} \square \} V_1 \\ & + \{ -(p_\mu \partial_\nu + p_\nu \partial_\mu) p\partial + p_\mu p_\nu \square + g_{\mu\nu} (p\partial)^2 \} V_2 \end{aligned}$$

(3.4)

Furthermore, Meyer and Suura ¹³⁾ have shown that if $W_{\mu\nu}(x)$ is causal, then $V_1(x)$ and $V_2(x)$ are also causal, i.e., vanish for $x^2 < 0$.

In the following we shall be interested in the singularities of V_1 and V_2 on the light cone and their relation to the behaviour of these functions in the deep inelastic region. Note that, according to Bjorken, the scaling properties of V_1 and V_2 are different:

$$\begin{aligned} \tilde{V}_1(\nu, q^2) &\rightarrow \frac{1}{2\nu} \frac{\overline{F}_2(\xi)}{\xi} \\ \tilde{V}_2(\nu, q^2) &\rightarrow \frac{1}{2\nu^2} \frac{\overline{F}_2(\xi)}{\xi} \end{aligned} \quad (\nu \rightarrow \infty, -\frac{q^2}{2\nu} = \xi \text{ fixed})$$

(3.5)

We must correspondingly expect different types of singularities on the light cone. We shall in the present section use the symbol $V(x)$ for either one of the functions $V_1(x)$ and $V_2(x)$ and prove some

general statements about the relation between the behaviour of a causal function $V(x)$ on the light cone and the behaviour of its Fourier transform in the Bjorken limit. In addition to causality we shall also make use of Lorentz invariance which allows us to express V as a function of two invariants, $V(x) = V(px, x^2)$ or $\tilde{V}(q) = \tilde{V}(\nu, q^2)$. Note that $\tilde{V}(\nu, q^2)$ is odd in ν

$$\tilde{V}(\nu, q^2) = -\tilde{V}(-\nu, q^2) \quad (3.6)$$

and has a support as exhibited in the Figure.

Singularity proportional to $\delta(x^2)$

To start our analysis let us suppose that the leading singularity of $V(x)$ on the light cone is proportional to $\delta(x^2)$:

$$V(x) = \frac{1}{2\pi i} \epsilon(x^0) \delta(x^2) g(px) + R(x) \quad (3.7)$$

The remainder R may still be singular at $x^2 = 0$, but we require it to be less singular than $(x^2)^{-1}$. We wish to show that if the leading singularity is of the above type then $\tilde{V}(q)$ obeys the scaling law

$$\tilde{V}(\nu, q^2) \rightarrow \frac{G(\xi)}{2\nu} \quad (\nu \rightarrow \pm\infty; -\frac{q^2}{2\nu} = \xi) \quad (3.8)$$

where $G(\xi)$ is an even function of ξ with support in $-1 \leq \xi \leq 1$. Furthermore, $G(\xi)$ is the Fourier transform of the function $g(px)$ that characterizes the leading singularity:

$$g(z) = 2 \int_0^1 d\xi \cos \xi z G(\xi) \quad (3.9)$$

To prove these statements we again consider lightlike restrictions of $V(x)$:

$$\mathcal{V}(\tau) = \int dx V(x) \delta(nx - \tau) f(x) \quad (3.10)$$

where n is a lightlike vector and $f(x)$ is a tempered test function. Our assumption concerning the remainder $R(x)$ in the decomposition (3.7) implies that, in the limit $\tau \rightarrow \pm 0$ only the leading singularity contributes to $\mathcal{V}(\tau)$:

$$\mathcal{V}(\pm 0) = \pm \frac{1}{2i} \int_0^\infty d\lambda g(\lambda p n) f(\pm \lambda n) \quad (3.11)$$

In particular, the discontinuity of $\mathcal{V}(\tau)$ at $\tau = 0$ is determined by the leading singularity. As we have shown in Section 2, a discontinuity of $\mathcal{V}(\tau)$ at $\tau = 0$ does indeed imply the scaling law (3.8) and one easily verifies the relation (3.9) by repeating the calculation carried through in Section 2. As a side remark let us note at this point that the Fourier transform of the leading singularity may be worked out explicitly with the result

$$\tilde{V}_s(\nu, q^2) = \frac{1}{2\sqrt{\nu^2 - q^2}} \{ G(\nu - \sqrt{\nu^2 - q^2}) - G(\nu + \sqrt{\nu^2 - q^2}) \} \quad (3.12)$$

Since $G(\xi)$ vanishes for $|\xi| > 1$ only the first term contributes in the Bjorken limit ($\nu \rightarrow +\infty$) and one immediately verifies that $\tilde{V}_s(\nu, q^2)$ obeys the scaling law (3.8) in the Bjorken limit. Note however, that the support of the expression (3.12) reaches out of the shaded region in the Figure, i.e., the singularity must always be accompanied by a remainder $R(x)$ to ensure the correct support properties in momentum space.

Sum rule

The scaling law (3.8) does not exhaust the information contained in (3.7) about the leading singularity on the light cone.

In fact, in deriving the scaling law we have made use only of the expression for the difference $\mathcal{V}_{(+0)} - \mathcal{V}_{(-0)}$ that follows from (3.11). To make use of the additional information contained in (3.11) we consider the Fourier decomposition of $\mathcal{V}(\tau)$

$$\begin{aligned}\mathcal{V}(\tau) &= \frac{1}{2\pi} \int dE \tilde{\mathcal{V}}(E) e^{-iE\tau} \\ \tilde{\mathcal{V}}(E) &= \int dk \tilde{V}(k + En) \tilde{f}(-k)\end{aligned}\tag{3.13}$$

in the limit $\tau \rightarrow \pm 0$. From the previous discussion we know that $\tilde{\mathcal{V}}(E)$ behaves like

$$\tilde{\mathcal{V}}(E) \rightarrow \frac{\Delta}{E} \quad (E \rightarrow \pm\infty)\tag{3.14}$$

where Δ may be expressed in terms of the scaling function $G(\xi)$. This slow decrease of the integrand is of course responsible for the fact that the limits $\tau \rightarrow \pm 0$ are different. Dividing the integral into a finite interval from $-N$ to $+N$ and a remainder for which the asymptotic expression (3.14) may be used one obtains

$$\mathcal{V}(\pm 0) = \frac{1}{2\pi} \lim_{N \rightarrow \infty} \int_{-N}^N dE \tilde{\mathcal{V}}(E) \pm \frac{1}{2i} \Delta\tag{3.15}$$

and one easily verifies that the difference $\mathcal{V}_{(+0)} - \mathcal{V}_{(-0)}$ is in agreement with (3.11). On the other hand, in order that the sum $\mathcal{V}_{(+0)} + \mathcal{V}_{(-0)}$ agrees with (3.11) we must have

$$\int_{-\infty}^{\infty} dE \tilde{\mathcal{V}}(E) = \frac{\pi}{2i} \int_{-\infty}^{\infty} d\lambda \epsilon(\lambda) g(\lambda pn) f(\lambda n)\tag{3.16}$$

where it is understood that the integral on the left-hand side is carried out in a symmetric fashion. Inserting the expression (3.13) for $\tilde{\mathcal{V}}(E)$ and observing that the above result must be valid for all test functions $f(x)$ we get the sum rule

$$\int_{-\infty}^{\infty} dE \tilde{V}(k+En) = \frac{1}{2} P \int_{-1}^1 d\xi \frac{1}{kn - \xi pn} G(\xi) \quad (3.17)$$

which may be rewritten in terms of the invariant variables ν and q^2 as

$$\int_{-\infty}^{\infty} d\nu \tilde{V}(\nu, 2\alpha\nu - \beta) = \frac{1}{2} P \int_{-1}^1 \frac{d\xi}{\alpha - \xi} G(\xi) \quad (3.18)$$

It is again understood that the integral on the left-hand side is carried out in a symmetric fashion such that the asymptotic tail $\sim G(\xi)/2\nu$ does not contribute. In (3.18) the quantities α and β stand for

$$\begin{aligned} \alpha &= kn/pn \\ \beta &= 2\alpha pk - k^2 \end{aligned} \quad (3.19)$$

and may be considered as independent parameters varying in the range $-\infty < \alpha < \infty$; $\beta \geq \alpha^2$. The integration on the left-hand side of the above sum rule extends over a straight line in the (ν, q^2) plane as indicated in the Figure. Note that in the limiting case $\beta = \alpha^2$ the straight line becomes tangent to the kinematical boundary $q^2 = \nu^2$.

The origin of this sum rule is again the fact that $V(x)$ is causal. The sum rule guarantees that the quantity $\mathcal{V}(+0)$ as given by (3.15) depends on the test function $f(x)$ only through its values on the forward light cone.

Singularity proportional to $\theta(x^2)$

In the foregoing we started from the assumption that the leading singularity of the function $V(x)$ is proportional to $\delta(x^2)$. In the remainder of this section we analyze the situation for the

case where the singularity is actually smoother and corresponds to a step function rather than a δ function. More precisely we suppose that:

$$V(x) = \frac{1}{2\pi i} \epsilon(x^0) \Theta(x^2) h(px) + S(x) \quad (3.20)$$

with a remainder $S(x)$ that vanishes on the light cone. The analysis of this singularity closely parallels the previous discussion. In the present case the lightlike restriction $\tilde{V}(\tau)$ of $V(x)$ is continuous and vanishes at $\tau = 0$, but possesses a discontinuous first derivative. The Fourier transform therefore behaves like

$$\tilde{V}(E) \rightarrow \frac{\Delta'}{E^2} \quad (3.21)$$

as $E \rightarrow \pm\infty$. This implies the scaling law

$$\tilde{V}(v, q^2) \rightarrow \frac{H(\xi)}{2v^2} \quad (v \rightarrow \pm\infty; -\frac{q^2}{2v} = \xi) \quad (3.22)$$

where $H(\xi)$ is an odd function of ξ with support in $-1 \leq \xi \leq 1$ and is again related to the function $h(px)$ characterizing the leading singularity:

$$h(z) = \int_0^1 d\xi \frac{\sin \xi z}{z} H(\xi) \quad (3.23)$$

Finally we note that the function $\tilde{V}(v, q^2)$ satisfies, in addition to the sum rule (3.18) which, in the present case reads

$$\int_{-\infty}^{\infty} dv \tilde{V}(v, 2\alpha v - \beta) = 0 \quad (3.24)$$

a further sum rule which may be derived in an entirely analogous fashion:

$$\int dv v \tilde{V}(v, 2\alpha v - \beta) = -\frac{1}{2} P \int_{-1}^1 \frac{d\xi}{\alpha - \xi} H(\xi)$$

(3.25)

4. SPECTRAL REPRESENTATION

The results obtained in the last section rely heavily on the fact that the structure functions vanish outside the light cone and at the same time vanish in the region $q^2 < -2|v|$ of momentum space. It is therefore instructive to illustrate our analysis by making use of a spectral representation that incorporates these support properties. According to Dyson ¹⁴⁾ every tempered distribution $V(x)$ that is causal and possesses a support in momentum space characteristic of the commutator $\langle p | [j_\mu(x), j_\nu(0)] | p \rangle_c$ may be written as

$$V(x) = \frac{1}{i} \int_0^\infty ds \Delta(x, s) \varphi(x, s)$$

(4.1)

where $\Delta(x, s)$ denotes the invariant Pauli-Jordan function

$$\Delta(x, s) = \frac{i}{(2\pi)^3} \int dk E(k) \delta(k^2 - s) e^{-ikx}$$

(4.2)

For simplicity we will in the following use the Deser-Gilbert-Sudarshan ¹⁵⁾ representation for the spectral function $\varphi(x, s)$ which reads

$$\varphi(x, s) = \int_{-1}^1 d\lambda \tilde{\varphi}(\lambda, s) e^{i\lambda p x}$$

(4.3)

[The analysis may be carried through in essentially the same manner by using the representations of Jost and Lehmann ¹⁶⁾ or of Dyson ¹⁴⁾.]

In order to analyze the behaviour of $V(x)$ at $x^2=0$ we note that the Pauli-Jordan function is given explicitly by

$$\Delta(x, s) = \frac{\epsilon(x^0)}{2\pi} \left\{ \delta(x^2) - \Theta(x^2) \frac{s}{2} \frac{J_1(\sqrt{s}x^2)}{\sqrt{s}x^2} \right\} \quad (4.4)$$

The function $V(x)$ will therefore exhibit a singularity of the type $\delta(x^2)$ provided the integral

$$g(x) = \int_0^\infty ds \varphi(x, s) \quad (4.5)$$

converges, in which case we have

$$V(x) = \frac{1}{2\pi i} \epsilon(x^0) \delta(x^2) g(x) + R(x) \quad (4.6)$$

The convergence of the integral (4.5) assures us that the quantity $R(x)$ is less singular than $1/x^2$. Our assumption about the leading singularity in the last section is thus seen to be equivalent to the requirement that the integral (4.5) converges.

To discuss the behaviour of the lightlike restrictions $\mathcal{V}(\tau)$ of $V(x)$ we observe that the quantity

$$\mathcal{D}(\tau, s) = \int dx \Delta(x, s) \delta(nx - \tau) f(x) e^{i\lambda p x} \quad (4.7)$$

possesses finite limits $\tau \rightarrow \pm 0$ which are independent of s . This shows that the existence of the limits $\mathcal{V}(\pm 0)$ again requires the convergence of the integral (4.5). In momentum space, the representation (4.1) takes the form

$$\tilde{V}(q) = \int ds d\lambda \epsilon(q + \lambda p) \delta[(q + \lambda p)^2 - s] \tilde{\varphi}(\lambda, s) \quad (4.8)$$

Applying the Bjorken limit $q = k + En$, $E \rightarrow \pm\infty$, $n^2 = 0$ to this expression one verifies the scaling law

$$\tilde{V}(q) \rightarrow \frac{G(\xi)}{2\nu} \quad (4.9)$$

with

$$G(\xi) = \int_0^\infty ds \tilde{\varphi}(\xi, s) \quad (4.10)$$

again provided that the integral over s converges. Clearly, the scaling function $G(\xi)$ is related to the coefficient $g(x)$ of the leading singularity by

$$g(x) = \int_{-1}^1 d\xi G(\xi) e^{i\xi p x} \quad (4.11)$$

in agreement with the result (3.9) derived in the last section. Note that $G(\xi)$ vanishes for $|\xi| > 1$ on account of the support properties of the spectral function. Finally, the sum rule (3.18) may be verified in a straightforward fashion by evaluating the integral

$$\int_{-N}^N dE \tilde{V}(k + En)$$

in terms of the representation (4.8).

Let us now turn to the second, weaker type of singularity. If the term proportional to $\delta(x^2)$ is to be absent we must require

$$\int_0^\infty ds \varphi(x, s) = 0 \quad (4.12)$$

In view of (4.4) the leading singularity will in this case be given by

$$V(x) = \frac{1}{2\pi i} \epsilon(x^0) \Theta(x^2) h(x) + S(x) \quad (4.13)$$

where $S(x)$ vanishes on the light cone and

$$h(x) = -\frac{1}{4} \int_0^\infty ds s \varphi(x, s) \quad (4.14)$$

provided that this integral converges. This shows that the second type of singularity may be characterized by (4.12) and the convergence of the integral (4.14).

To discuss the behaviour of the lightlike restriction $\mathcal{V}(\tau)$ in the present case we first observe that the limits $\mathcal{V}(\pm 0)$ vanish in view of (4.12), since the limits $\tau \rightarrow \pm 0$ of the quantity $\mathcal{D}(\tau, s)$ defined in (4.7) are independent of s . Note that the "time" derivative $d/d\tau \mathcal{D}(\tau, s)$ exhibits a singularity proportional to $\delta(\tau)$. This singularity does not, however, contribute to $d/d\tau \mathcal{V}(\tau)$ since it is independent of s . In fact, one easily verifies that the difference $\mathcal{D}(\tau, s) - \mathcal{D}(\tau, 0)$ possesses a finite derivative with respect to τ ; at $\tau = 0$ this derivative exhibits a finite discontinuity proportional to s . The existence of the limits $\tau \rightarrow \pm 0$ of $d/d\tau \mathcal{V}(\tau)$ therefore requires the vanishing of (4.12) and the convergence of the integral (4.14).

Finally, one verifies that the leading term in the Bjorken limit of (4.8) is absent by virtue of (4.12). The scaling law therefore reads

$$\tilde{V}(q) \rightarrow \frac{H(\xi)}{2\nu^2} \quad (4.15)$$

with

$$H(\xi) = \frac{1}{2} \frac{d}{d\xi} \int_0^\infty ds s \tilde{\varphi}(\xi, s) \quad (4.16)$$

Comparison of this expression for the scaling function $H(\xi)$ with (4.14) immediately leads to the relation (3.23). In the same manner ¹⁷⁾ one may also check the sum rules (3.24) and (3.25).

To conclude this discussion of the light cone singularities in terms of the spectral representation let us call attention to the fact that the spectral "function" is actually a distribution which may exhibit singularities. A case of physical interest is a singular scaling function $H(\xi)$ behaving like $\text{const.}/\xi$ in the vicinity of $\xi = 0$. (Note that this corresponds to a coefficient of the leading singularity falling off like $h(x) \sim \text{const.}/px$ as one moves out on the light cone.) In this case the integral $\int ds s \tilde{\varphi}(\lambda, s)$ over the spectral function must exhibit a singularity; as we will not make further use of the spectral representation we shall not discuss the nature of this singularity here, but refer the reader to the papers of Brandt ¹⁸⁾ and Brown ⁵⁾ for an analysis of this problem.

5. LEADING SINGULARITIES OF CURRENT COMMUTATOR

We now return to the structure functions of the electromagnetic current and apply the general statements about light cone singularities of causal functions derived in Section 3. Our basic objects are the lightlike restrictions of the various components of $W_{\mu\nu}(x)$; we have already studied the particular component $n^\mu n^\nu W_{\mu\nu}(x)$ in Section 2. In order to analyze the remaining components in a systematic fashion it is convenient to introduce a set of three vectors $\bar{n}$, e^1 and e^2 which, together with the lightlike vector n constitute a basis in Minkowski space. We choose the first member $\bar{n}$ to be lightlike and to lie in the plane spanned by n and the proton momentum p . For convenience, the normalization of

n and $\bar{n}$ is fixed by the conditions $pn = p\bar{n} = 1/\sqrt{2}$ which implies $n\bar{n} = 1$. The remaining two elements e^1 and e^2 are spacelike, orthogonal to both n and $\bar{n}$ and normalized by

$$g^{\mu\nu} e_\mu^r e_\nu^s = -\delta^{rs} \quad (r, s = 1, 2) \quad (5.1)$$

The expansion of the current in this basis reads

$$j_\mu(x) = j_\parallel(x) \bar{n}_\mu + j_\perp(x) n_\mu + j_1(x) e_\mu^1 + j_2(x) e_\mu^2 \quad (5.2)$$

We shall also find it convenient to decompose the differential operator ∂_μ in an analogous fashion

$$\partial_\mu = \partial_\parallel \bar{n}_\mu + \partial_\perp n_\mu + \partial_1 e_\mu^1 + \partial_2 e_\mu^2 \quad (5.3)$$

Note that the derivatives $\partial_\parallel, \partial_1$ and ∂_2 operate within the plane $nx = \tau$, whereas the operator $\partial_\perp$ is the "time" derivative $d/d\tau$.

In this notation the expression (3.4) for the structure functions $W_{\mu\nu}(x)$ in terms of the invariants $V_1(x)$ and $V_2(x)$ reads:

$$W_{\parallel\parallel}(x) = -\partial_\parallel^2 V_1 - (\partial_\parallel^2 + \frac{1}{2} \partial_\perp^2) V_2 \quad (5.4)$$

$$W_{\parallel r}(x) = -\partial_\parallel \partial_r V_1 - \frac{1}{2} (\partial_\perp + \partial_\parallel) \partial_r V_2 \quad (5.5)$$

$$W_{\parallel\perp}(x) = (\partial_\parallel \partial_\perp - \partial_\perp^2) V_1 + (\partial_\parallel \partial_\perp - \frac{1}{2} \partial_\perp^2) V_2 \quad (5.6)$$

$$W_{r\perp}(x) = -\partial_r \partial_\perp V_1 - \frac{1}{2} (\partial_\perp + \partial_\parallel) \partial_r V_2 \quad (5.7)$$

$$W_{rs}(x) = (-2\delta_{rs}\partial_{||}\partial_{\perp} + \delta_{rs}\partial^2 - \partial_r\partial_s)V_1 - \frac{1}{2}\delta_{rs}(\partial_{\perp} + \partial_{||})^2 V_2 \quad (5.8)$$

$$W_{\perp\perp}(x) = -\partial_{\perp}^2 V_1 - (\partial_{\perp}^2 + \frac{1}{2}\partial^2) V_2 \quad (5.9)$$

where

$$\partial^2 = (\partial_1)^2 + (\partial_2)^2$$

In the Introduction we have restricted our attention to the component $W_{||||}(x)$. As may be seen from the above expression in terms of $V_1(x)$ and $V_2(x)$, the lightlike restrictions of $W_{||||}(x)$

$$J(\tau) = \int dx W_{||||}(x) \delta(nx - \tau) f(x)$$

may be expressed in terms of the lightlike restrictions of $V_1(x)$ and $V_2(x)$ since the derivatives $\partial_{||}$, ∂_1 and ∂_2 in front of these quantities act within the surface $nx = \tau$ and may therefore by partial integration be turned onto the test function $f(x)$. The assumption that the quantity $W_{||||}(x)$ admits lightlike restrictions is therefore satisfied if $V_1(x)$ and $V_2(x)$ have this property. The basic hypothesis used in the Introduction is thus seen to be equivalent to the requirement that both $V_1(x)$ and $V_2(x)$ can be restricted to a lightlike plane.

The situation is different for the components $W_{||r}(x)$ which possess lightlike restrictions only if in addition the quantity $\partial_{\perp}V_2(x)$ has this property, i.e., if the restriction of $V_2(x)$ possesses a derivative with respect to τ . In the case of $W_{||\perp}(x)$ and of $W_{r\perp}(x)$ both $\partial_{\perp}V_1(x)$ and $\partial_{\perp}V_2(x)$ have to exist when smeared over a lightlike plane, etc. The existence of the lightlike restrictions of the various components of $W_{\mu\nu}(x)$ as exhibited in (5.4)-(5.9) thus presents a hierarchy of conditions on the singularities

of $V_1(x)$ and $V_2(x)$. If all components of $j_\mu(x)$ were to exist when smeared over a lightlike plane then all six quantities V_1 , V_2 , $\partial_\perp V_1$, $\partial_\perp V_2$, $\partial_\perp^2 V_1$ and $\partial_\perp^2 V_2$ would have this property. It is clear from our analysis in Section 3 that this is a very stringent requirement. It would in fact imply that both $V_1(x)$ and $V_2(x)$ vanish on the light cone and their Fourier transform would behave like ν^{-3} in the Bjorken limit. This is, however, ruled out by experiment which has revealed that the structure functions scale according to (3.5). From our analysis in Section 3 we know that these scaling laws correspond to light cone singularities proportional to $\delta(x^2)$ and to $\theta(x^2)$ respectively:

$$V_1(x) = \frac{1}{2\pi i} \epsilon(x^0) \delta(x^2) g_1(px) + R_1(x)$$

$$V_2(x) = \frac{1}{2\pi i} \epsilon(x^0) \theta(x^2) h_2(px) + R_2(x)$$

(5.10)

where

$$g_1(z) = 2 \int_0^1 \frac{d\xi}{\xi} \cos \xi z \overline{F}_2(\xi)$$

$$h_2(z) = \int_0^1 d\xi \frac{\sin \xi z}{\xi z} \overline{F}_2(\xi)$$

(5.11)

The quantity $V_1(x)$ contains the leading singularity of the structure functions $W_{\mu\nu}(x)$, provided that the function $F_\ell(\xi)$ is different from zero. (Note that the experimental results are compatible with $F_\ell=0$; we will come back to this important question later.) In this case the lightlike restriction of $V_1(x)$ exists and has a discontinuity at $\tau=0$. The derivative $\partial_\perp V_1(x)$ can therefore not be restricted to the surface $\tau=0$. On the other hand the lightlike restrictions of $\partial_\perp V_2(x)$ do exist and possess a discon-

tinuity at $\tau = 0$ related to the function $F_2(\xi)$. Since this function is different from zero, the second derivative $\partial_1^2 V_2(x)$ cannot be restricted to the surface $\tau = 0$. Therefore, if F_ℓ turns out to be different from zero, both $W_{|| \quad ||}(x)$ and $W_{|| \quad r}(x)$ but none of the other components possess lightlike restrictions.

6. EQUAL TIME COMMUTATORS

The information contained in (5.10) about the singularities of the current commutator on the whole light cone includes in particular the tip of the cone which is relevant for the equal time commutation relations. To work out these commutation rules we consider spacelike restrictions of $V_1(x)$ and $V_2(x)$:

$$S_\alpha(t) = \int dx V_\alpha(x) \delta(x^0 - t) f(x) \quad (\alpha = 1, 2) \quad (6.1)$$

According to (5.10) these quantities behave like $S_1(t) \sim t$, $S_2(t) \sim t^3$ as $t \rightarrow 0$, more precisely:

$$\begin{aligned} S_1(t) &= -i t g_1(0) f(0) + r_1(t) \\ S_2(t) &= -\frac{2i}{3} t^3 h_2(0) f(0) + r_2(t) \end{aligned} \quad (6.2)$$

where $r_1(t)$ and $r_2(t)$ tend to zero faster than t and t^3 respectively. In other words, the equal time restrictions of the quantities $V_1, V_2, \partial_0 V_2, \partial_0^2 V_2$ vanish whereas

$$\begin{aligned} \delta(x^0) \partial_0 V_1(x) &= -i g_1(0) \delta^4(x) \\ \delta(x^0) \partial_0^3 V_2(x) &= -4i h_2(0) \delta^4(x) \end{aligned} \quad (6.3)$$

The equal time commutators of the current can now be read off from the expression (3.4) of the structure functions $W_{\mu\nu}(x)$ in terms

of the invariants $V_1(x)$ and $V_2(x)$. Since this expression involves only second derivatives, the quantity $V_2(x)$ does not contribute at equal times and we get

$$\begin{aligned} \langle p | [j_0(x), j_0(0)] | p \rangle_c &= 0 \\ \langle p | [j_0(x), j_i(0)] | p \rangle_c &= i g_i(0) \partial_i \delta^3(x) \end{aligned} \quad (6.4)$$

In view of (5.11) we therefore obtain the following expression for the Schwinger term:

$$g_i(0) = 2 \int_0^1 \frac{d\xi}{\xi} F_\ell(\xi) \quad (6.5)$$

An immediate consequence of this relation is that the electromagnetic mass shift of the proton diverges quadratically, if F_ℓ turns out to be different from zero, since the quadratic divergence is given by the Schwinger term¹⁹⁾. [Note that the quantity $F_\ell(\xi)/\xi$ is positive; therefore, absence of the quadratic divergence requires $F_\ell(\xi) \equiv 0$.] Of course, in order that the neutron-proton mass difference is not quadratically divergent, it suffices that

$$\int_0^1 \frac{d\xi}{\xi} \{ F_\ell^p(\xi) - F_\ell^n(\xi) \} = 0 \quad (6.6)$$

We add a remark concerning the equal time commutators of the space components. These commutators involve the quantity $\partial_0^2 V_1(x)$ which is not determined by the information about the singularity of $V_1(x)$ on the light cone as exhibited in (5.10). Note however that the equal time restriction of the Lorentz invariant quantity $\square V_1(x)$ can have an unambiguous meaning only if it vanishes. The equal time commutators of the space components therefore vanish if they exist at all.

7. ABSENCE OF THE LEADING SINGULARITY: $F_\ell(\xi) = 0$

In terms of the longitudinal and transverse cross-sections introduced by Hard ²⁰⁾ the ratio F_ℓ/F_2 is given by

$$\frac{\sigma_\ell}{\sigma_t + \sigma_\ell} \rightarrow 2\xi \frac{F_\ell}{F_2}$$

in the Bjorken limit. As we have pointed out previously, the experimental results ¹⁾ are compatible with $\sigma_\ell/\sigma_t \rightarrow 0$, i.e., $F_\ell(\xi) \equiv 0$. In view of the connection between this possibility and the finiteness of the electromagnetic mass shift the case $F_\ell = 0$ merits a separate discussion and we will now work out the corresponding singularity structure. If F_ℓ vanishes, both structure functions V_1 and V_2 are less singular on the light cone than $\delta(x^2)$ and we thus have a particularly regular current commutator. In fact, this extra regularity implies that the "charge density" $j_\parallel(x) = n_\mu j^\mu(x)$ is a rather smooth object: it not only exists when smeared over a lightlike plane, but actually the commutator $\langle p | [j_\parallel(x), j_\parallel(0)] | p \rangle_c$ is continuous at $\tau = nx = 0$. This suggests that the "equal time" commutator $\delta(x-x') [j_\parallel(x), j_\parallel(y)]$ might have an unambiguous meaning in general. We shall briefly come back to this interesting point later in connection with current algebra.

The vanishing of $F_\ell(\xi)$ allows us to go further down in the hierarchy (5.4)-(5.9) of conditions for the existence of lightlike restrictions of the various current components and to assume that the quantity $\partial_\perp V_1(x)$ exists when restricted to a lightlike plane. This hypothesis implies the scaling law

$$\tilde{V}_1(q) \rightarrow \frac{H_1(\xi)}{2\nu^2} \quad (7.1)$$

The corresponding singularity on the light cone is given by

$$V_1(x) = \frac{1}{2\pi i} \epsilon(x^0) \Theta(x^2) h_1(px) + R_1(x)$$

$$h_1(z) = \int_0^1 d\xi \frac{\sin \xi z}{z} H_1(\xi)$$
(7.2)

To determine the equal time commutators in the present case, we observe that according to (6.2) the spacelike restrictions of V_α , $\partial_0 V_\alpha$ and $\partial_0^2 V_\alpha$ vanish at $t=0$; therefore all components of the connected current commutator vanish at equal times:

$$\langle p | [j_\mu(x), j_\nu(0)] | p \rangle_c = 0$$
(7.3)

whereas the time derivatives obey the commutation rules

$$\langle p | [\partial_0 j_0(x), j_\mu(0)] | p \rangle_c = \langle p | [\partial_0 j_\mu(x), j_0(0)] | p \rangle_c = 0$$

$$\langle p | [\partial_0 j_i(x), j_k(0)] | p \rangle_c = 4i \{ \delta_{ik} h_1(0) + (\delta_{ik} p_0^2 - p_i p_k) h_2(0) \} \delta^3(\underline{x})$$
(7.4)

As shown by Boulware and Deser ¹⁹⁾, these commutators are related to the logarithmic divergence of the electromagnetic mass shifts. More precisely, the logarithmically divergent part of the electromagnetic mass shift of the proton is proportional to

$$\delta m_p \sim \alpha \{ 2h_1(0) + h_2(0) \}$$
(7.5)

The electromagnetic mass of the proton will therefore be finite, if in addition to $F_\ell(\underline{x}) \equiv 0$ we have $2h_1(0) + h_2(0) = 0$ which amounts to the sum rule

$$\int_0^1 d\xi \{ 2\xi H_1(\xi) + \overline{F}_2(\xi) \} = 0 \quad (7.6)$$

Note that this sum rule is compatible with the positivity conditions (3.3).

8. SUM RULES

We have shown in Section 3 that a causal function that behaves like ν^{-n} in the Bjorken limit must satisfy n sum rules of the type of superconvergence relations. If the scaling function $F_\ell(\xi)$ is different from zero, then the structure function $\tilde{V}_1$ behaves like ν^{-1} and we have the sum rule:

$$\int_{-\infty}^{\infty} d\nu \tilde{V}_1(\nu, 2\alpha\nu - \beta) = \frac{1}{2} \mathcal{P} \int_{-1}^1 \frac{d\xi}{\alpha - \xi} \frac{\overline{F}_2(\xi)}{\xi} \quad (8.1)$$

The quantity $\tilde{V}_2$ on the other hand scales like ν^{-2} and hence satisfies two superconvergence sum rules:

$$\int_{-\infty}^{\infty} d\nu \tilde{V}_2(\nu, 2\alpha\nu - \beta) = 0 \quad (8.2)$$

$$\int_{-\infty}^{\infty} d\nu \nu \tilde{V}_2(\nu, 2\alpha\nu - \beta) = -\frac{1}{2} \mathcal{P} \int_{-1}^1 \frac{d\xi}{\alpha - \xi} \frac{\overline{F}_2(\xi)}{\xi} \quad (8.3)$$

These relations are valid for all values of the parameters α and β in the range $-\infty < \alpha < \infty$; $\beta \geq \alpha^2$. Let us also recall that the integration on the left-hand sides of these sum rules which extend over a straight line in the (ν, q^2) plane are to be carried out in a symmetric fashion to avoid divergent contributions from the asymptotic tails $\nu \rightarrow \pm \infty$. In the event that $F_\ell(\xi)$ vanishes, and that $\tilde{V}_1$ scales according to (7.1) we get the further sum rule

$$\int_{-\infty}^{\infty} dv v \tilde{V}_1(v, 2\alpha v - \beta) = -\frac{1}{2} P \int_{-1}^1 \frac{d\xi}{\alpha - \xi} H_1(\xi) \quad (8.4)$$

The reader should perhaps be warned that these sum rules involve distributions and are strictly valid only after smearing over the parameters α and β . This remark does not merely have decorative value as we expect some of the relations to be meaningless at $\alpha = 0$ where the contribution of the Pomeron leads to divergent integrals, Refs. 18), 21).

In a recent paper ²²⁾ we have shown that these sum rules are intimately related to dispersion relations for the forward Compton scattering amplitude involving dispersion integrals along the straight line $q^2 = 2\alpha v - \beta$. This paper contains one particular application of the sum rules (8.2) and (8.3). A more detailed study of these sum rules, concerning in particular the limit $\alpha \rightarrow 0$ will be published elsewhere.

9. EXTENSION TO U(2)

Our analysis of the electromagnetic structure functions can easily be generalized to the isovector and hypercharge currents; the extension to non-conserved currents, however, is more involved and we shall not discuss this problem here. We denote the current associated with the hypercharge by $j_\mu^0(x)$ and use the notation $j_\mu^i(x)$ ($i=1,2,3$) for the isovector currents. The corresponding charges

$$Q^a = \int d\sigma^\mu j_\mu^a(x) \quad (a=0,1,2,3) \quad (9.1)$$

obey the structure relations of U(2):

$$[Q^a, Q^b] = i C_{abc} Q^c \quad (9.2)$$

The structure functions

$$W_{\mu\nu}^{ab}(x) = \langle p | [j_\mu^a(x), j_\nu^b(0)] | p \rangle \quad (9.3)$$

may again be decomposed into two invariants by writing

$$\begin{aligned} \tilde{W}_{\mu\nu}^{ab}(q) = & \{q_\mu q_\nu - g_{\mu\nu} q^2\} \tilde{V}_1^{ab} \\ & + i(p_\mu q_\nu + p_\nu q_\mu) \gamma - p_\mu p_\nu q^2 - g_{\mu\nu} p^2 \} \tilde{V}_2^{ab} \end{aligned} \quad (9.4)$$

In these expressions we have suppressed the dependence of the matrix elements on the isospin of the nucleon. The quantities $\tilde{V}_1^{ab}$ and $\tilde{V}_2^{ab}$ actually represent 2x2 matrices in isospin space which may be expressed in terms of isospin invariant structure functions as follows:

$$\begin{aligned} V_\alpha^{ik} &= \int^{ik} A_\alpha + \frac{i}{2} \epsilon^{ikl} \tau^l B_\alpha \\ V_\alpha^{i0} &= \frac{1}{2} \tau^i (C_\alpha + i D_\alpha) \\ V_\alpha^{0i} &= \frac{1}{2} \tau^i (C_\alpha - i D_\alpha) \\ V_\alpha^{00} &= E_\alpha \end{aligned} \quad (\alpha=1,2)$$

(9.5)

Separation of non-causal part

Suura and Meyer¹³⁾ have shown that the contribution of the one-nucleon intermediate state to the quantity V_2 exhibits a kinematical singularity at $\nu = q^2 = 0$, whose origin may be traced back to the fact that the charges do not commute. As a consequence of this kinematical singularity the Fourier transform $V_2^{ab}(x)$ does not

vanish for $x^2 < 0$ although, of course, the non-causal contribution is killed by the differential operator in front of V_2 :

$$W_{\mu\nu}^{ab}(x) = \{ -\partial_{\mu\nu} + g_{\mu\nu} \square \} V_1^{ab}(x) + \{ -(p_\mu \partial_\nu + p_\nu \partial_\mu) p\partial + p_\mu p_\nu \square + g_{\mu\nu} (p\partial)^2 \} V_2^{ab}(x) \quad (9.6)$$

Since our analysis makes essential use of the causal nature of the structure functions, it is convenient to separate the non-causal part of V_2 by writing

$$V_2^{ab} = V_c^{ab} + V_n^{ab} \quad (9.7)$$

This splitting of V_2 into a causal part V_c and a non-causal part V_n is not unique. A convenient expression for the non-causal part has been given by Suura and Meyer¹³⁾:

$$\tilde{V}_n^{ab}(q) = -\frac{i}{2} C_{abc} \tau^c \frac{1}{q^2} \{ \epsilon(1+\nu) \delta(q^2+2\nu) + \epsilon(1-\nu) \delta(q^2-2\nu) \} \quad (9.8)$$

The virtue of this choice lies in its simplicity. Its support in momentum space is concentrated on the lines $q^2 = \pm 2\nu$ which correspond to one-nucleon ($q^2 < 0$) and Z graph ($q^2 > 0$) contributions. It does not, however, coincide with the actual expression for these contributions which reads

$$\tilde{V}_B^{ab}(q) = -\frac{1}{q^2} \{ F^{ab}(q^2) \epsilon(1+\nu) \delta(q^2+2\nu) - F^{ba}(q^2) \epsilon(1-\nu) \delta(q^2-2\nu) \} \quad (9.9)$$

where the quantities $F^{ab}(q^2)$ may be expressed in terms of the nucleon form factors. The non-causal part is obtained from the Born term by replacing $F^{ab}(q^2)$ by $+\frac{1}{2} C_{abc} \tau^c$ and the causal remainder V_c must compensate for this difference.

Let us briefly discuss an alternative choice for V_n that does not share this disadvantage at the cost of being more complicated. We first observe that we cannot simply identify V_n with the Born term V_B since the remainder $V_2 - V_B$ is not causal. In fact, the nucleon is an extended object and its contribution to the current commutator reflects this non-locality through the appearance of the nucleon form factors in the Born term. Contributions from other intermediate states are necessary to compensate for this non-causal nature of the Born term. We exhibit a specific model for these additional contributions in the Appendix by constructing a local current commutator whose only contribution in the half plane $q^2 < 0$ is given by the Born term. In this sense the model may be considered as a minimal extension of the nucleon contribution leading to a local current commutator. This model in particular exhibits the kinematical singularity correctly and therefore provides an alternative choice for V_n .

To conclude this discussion of the non-causal part of V_2 we note the following properties of the expression (9.8). The contribution of V_n to the current commutator is given by

$$W_{\mu\nu}^{ab}(x)_n = \frac{i}{2} C_{abc} \tau^c \{ \cos px (p_\mu \partial_\nu + p_\nu \partial_\mu) \Delta(x, m^2) - g_{\mu\nu} p^\rho \partial_\rho (\cos px \Delta(x, m^2)) \}$$

(9.10)

where $\Delta(x, m^2)$ denotes the Pauli-Jordan function of the proton $m^2 = p^2 = 1$. This expression shows explicitly that the non-causal character of V_n does not show up in the current commutator.

The singularity of V_n on the light cone as exhibited explicitly in (9.10) is reflected by the behaviour of (9.8) in the Bjorken limit:

$$\tilde{V}_n^{ab}(q) \rightarrow \frac{i}{8\nu^2} C_{abc} \tau^c \{ \delta(1-\xi) + \delta(1+\xi) \}$$

(9.11)

The appearance of the distributions $\delta(1 \pm \xi)$ in the scaling law by no means implies that the quantity $V_2 = V_c + V_n$ exhibits such a singularity at $\xi = \pm 1$. It merely reflects the particular choice of the non-causal contribution. In fact, the alternative choice of V_n described in the Appendix provides a local model whose structure functions vanish faster than ν^{-2} in the Bjorken limit.

Restrictions imposed by positivity

We are now ready to discuss the light cone singularities of the hypercharge and isovector current commutators. In terms of the notation introduced above the electromagnetic current is given by

$$j_\mu = j_\mu^{(3)} + \frac{1}{2} j_\mu^{(0)} \quad (9.12)$$

In the case of the electromagnetic structure functions of the proton we were able to rely on the experimentally observed scaling laws. We base our discussion of the functions V_1^{ab} and V_2^{ab} on the assumption that the electromagnetic neutron structure functions scale analogously:

$$\begin{aligned} V_1^{p,n} &\rightarrow \frac{1}{2\nu\xi} F_2^{p,n}(\xi) \\ V_2^{p,n} &\rightarrow \frac{1}{2\nu^2\xi} F_2^{p,n}(\xi) \end{aligned} \quad (9.13)$$

We wish to show that this information about the electromagnetic structure functions of the nucleon allows us to put very stringent restrictions on the structure functions of the hypercharge and isovector currents. In fact, this information suffices to infer that the whole set of structure functions V_1^{ab} and V_2^{ab} obeys these scaling laws. This claim is based on the positivity conditions which in the present case read

$$\begin{aligned}
 u^\dagger \{ \tilde{V}_1^{ab} + \tilde{V}_2^{ab} \} u x^{a*} x^b &\geq 0 \\
 u^\dagger \{ q^2 \tilde{V}_1^{ab} + \nu^2 \tilde{V}_2^{ab} \} u x^{a*} x^b &\geq 0
 \end{aligned}
 \tag{9.14}$$

($\nu \geq 0, q^2 < 4m_\pi^2$)

where u is an arbitrary two component spinor and x^a stands for an arbitrary complex four-vector. It is straightforward to express these conditions in terms of the isospin invariants A_α , B_α etc., defined in (9.5). The resulting inequalities are rather involved; for our purpose the following subset is however sufficient:

$$\begin{aligned}
 a &\geq 0 \\
 e &\geq 0 \\
 (2a+e)^2 &\geq b^2 + c^2 + d^2
 \end{aligned}
 \tag{9.15}$$

where a stands for either

$$a = A_1 + A_2$$

or

$$a = q^2 A_1 + \nu^2 A_2$$

and analogously for b, c, d , and e . The essential point is that the "diagonal" quantities a and e dominate the "off-diagonal" elements b, c , and d . This is true, in particular in the Bjorken limit. On the other hand, one easily verifies that the electromagnetic structure functions of proton and neutron determine the combination $a + \frac{1}{4}e$. Since a and e are both positive this combination provides an upper bound for a and e which extends to b, c , and d by virtue of (9.15). This shows that the whole set of structure func-

tions V_1^{ab} must scale proportional to ν^{-1} whereas the quantities V_2^{ab} scale one power faster in the Bjorken limit. Of course, some of the off-diagonal components may actually tend to zero faster than the diagonal ones as $\nu \rightarrow \infty$, i.e., some of the off-diagonal scaling functions may vanish. In this sense the singularities of the electromagnetic current commutator dominate those of the hypercharge and isovector currents. A detailed discussion of the behaviour of these currents on the light cone would only duplicate our analysis of the electromagnetic current and we refrain from listing the corresponding expressions for the commutators on the light cone and at equal times. Needless to say that the sum rules derived for the electromagnetic structure functions must also be satisfied by their $U(2)$ extensions.

Let us, however, mention one interesting point. If it turns out that the longitudinal scaling functions $F_L(\xi)$ of both proton and neutron vanish and that the quantities V_1^p, V_1^n actually scale one power faster, then the same must be true of the whole set V_1^{ab} . This would imply, in particular, that the current commutator on a lightlike surface is given entirely by the non-causal part:

$$\delta(nx) \langle p | [j_{||}^a(x), j_{||}^b(0)] | p \rangle_c = i C_{abc} \langle p | j_{||}^c(0) | p \rangle \delta^4(x)$$

It might therefore be of interest to investigate the hypothesis that the commutation relation ²³⁾

$$\delta(nx - ny) [j_{||}^a(x), j_{||}^b(y)]_c = i C_{abc} j_{||}^c(x) \delta^4(x - y) \quad (9.16)$$

is valid in general. This hypothesis, which, by the way, is satisfied by the quark model, is stronger than the conventional equal time commutation relations of current algebra and may be regarded as a sort of superlocal current algebra.

10. CONCLUSION

To summarize the main results of the present investigation we note the following points:

- i) we have shown that the experimentally observed scaling laws reflect the singularities of the current commutator on the light cone. The leading singularity is related to the longitudinal scaling function $F_\ell(\xi)$; if this function does not vanish, i.e., if the ratio $R = \sigma_\ell / \sigma_t$ does not tend to zero in the deep inelastic region, then only the components $W_{\parallel\parallel}(x)$ (good-good) and $W_{\parallel r}(x)$ (good-bad) of the current commutator possess "equal-time" limits on lightlike planes.

The experimental situation concerning R is not conclusive. If it turns out that F_ℓ vanishes and that the longitudinal combination actually scales one power faster

$$v [W_1 + \frac{v^2}{q^2} W_2] \rightarrow -\xi H_1(\xi) \quad \left\{ \begin{array}{l} v \rightarrow \infty \\ -\frac{q^2}{2v} = \xi \end{array} \right.$$

then all good-good and good-bad components of the current commutator are well behaved on the light cone. Since F_2 is different from zero, the bad-bad current commutators do not possess equal-time limits on lightlike planes;

- ii) we have given the explicit relations between the light cone singularities and the scaling functions. These relations clarify Bjorken's expression for the scaling functions in terms of "almost equal-time commutators at infinite momentum". Let us emphasize one particular consequence of locality: the scaling laws must be valid independently of the sign of q^2 and the scaling functions in the upper and lower half (v, q^2) plane must be the same;

- iii) some equal time commutators may be expressed in terms of the scaling functions. In particular, the absence of the leading singularity, $F_\ell = 0$, is equivalent to the absence of operator Schwinger terms in the equal time commutator $[j_0, j_i]$ and, according to Boulware and Deser, to the absence of the quadratic divergence of the electromagnetic mass shifts. In this case, the equal time commutator $[\frac{d}{dt} j_i, j_k]$ may be expressed in terms of the scaling functions $F_2(\xi)$ and $H_1(\xi)$ [see i) above]. This permits us to formulate necessary and sufficient conditions for the absence of the logarithmic divergence in the mass shift in terms of these scaling functions. Consequently, by measuring the scaling functions with sufficient accuracy one measures the q number Schwinger term and the coefficients of the quadratic and logarithmic divergences of the electromagnetic mass shift;
- iv) the relations between the light cone singularities of current commutators and the scaling functions have an important corollary in momentum space. They imply a new family of sum rules relating the Hilbert transform of the scaling functions to integrals over the electroproduction structure functions, thus connecting the deep inelastic region to the low energy domain.

It is evident that the methods used in the present paper may be applied to other matrix elements of the current commutator. If the light cone singularities occurring in the forward proton matrix element are a characteristic feature of the current commutator rather than a special property of the proton then asymptotic scaling laws and sum rules for various inelastic processes involving virtual photons of high mass can be derived.

A P P E N D I X

In this Appendix we exhibit a model for a causal function $V(z)$ whose Fourier transform coincides with the expression

$$\tilde{V}_B(\nu, q^2) = G(q^2) \delta(q^2 + 2\nu) \quad (\text{A.1})$$

in the region $q^2 < s_0$ where s_0 is a positive number in the range $0 < s_0 < 4$. Once this problem is solved it is easy to extend the model to a representation for the Born term (9.9). We start from the ansatz

$$\tilde{V}(\nu, q^2) = \int_0^1 d\lambda (1-\lambda)^{-2} \epsilon(\nu+\lambda) h\left(\frac{q^2+2\lambda\nu}{1-\lambda}\right) \quad (\text{A.2})$$

which amounts to a special choice of the spectral function in the Deser-Gilbert-Sudarshan representation and is therefore causal. The support of the function $h(s)$ is restricted to $s > s_0$. This support property guarantees that $\tilde{V}(\nu, q^2)$ vanishes in the region $q^2 < s_0$, $\nu < 0$ whereas in the region $\nu > 0$ we have $\epsilon(\nu+\lambda) = +1$ and the substitution $s = (q^2 + 2\lambda\nu)/(1-\lambda)$ leads to the expression

$$\tilde{V}(\nu, q^2) = \frac{1}{q^2 + 2\nu} \int_{s_0}^{\infty} ds h(s) \quad (\text{A.3})$$

provided $q^2 + 2\nu \neq 0$. The function $\tilde{V}(\nu, q^2)$ therefore vanishes in the whole region $q^2 < s_0$ except for the line $q^2 + 2\nu = 0$ if we choose $h(s)$ to be given by

$$h(s) = \frac{d}{ds} H(s) \quad (\text{A.4})$$

where $H(s)$ vanishes for $s \leq s_0$ and for $s \rightarrow \infty$. On the other hand one verifies that the integral $\int dq^2 \tilde{V}(\nu, q^2)$ across the line $q^2 + 2\nu = 0$ is finite and corresponds to the expression

$$\tilde{V}(\nu, q^2) = \delta(q^2 + 2\nu) \int_{s_0}^{\infty} ds \frac{H(s)}{s - q^2} \quad (\text{A.5})$$

which is valid for all $q^2 < s_0$. The problem is therefore solved if the function $G(q^2)$ can be represented as

$$G(q^2) = \int_{s_0}^{\infty} ds \frac{H(s)}{s - q^2} \quad (\text{A.6})$$

Recall that $G(q^2)$ is essentially the square of a form factor and therefore satisfies a dispersion representation of the above type as a consequence of causality. It is interesting to observe that the use of the Deser-Gilbert-Sudarshan representation which incorporates causality automatically leads to this dispersion relation for the form factors. It is straightforward to construct a model for the entire Born term as given in (9.9). One first separates the kinematical singularity at $\nu = q^2 = 0$ and considers the quantity $V_B - V_n$ which is free of this singularity. The model described in the Appendix immediately provides a local representation which coincides with this difference in the region $q^2 < s_0$.

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- 6) After this work was essentially completed we learned that R. Jackiw, R. Van Royen and G. West (MIT preprint CTP 118 Jan. 1970) have independently obtained most of the results reported in the present paper. We are indebted to Professor J.S. Bell for drawing our attention to this work.

- 7) More precisely, the function $W_{\mu\nu}(x)$ is the spin average over the states $|p, s\rangle$, $s = \pm \frac{1}{2}$ which are normalized by $\langle p', s' | p, s \rangle = (2\pi)^3 p^0 \delta(\underline{p}' - \underline{p}) \delta_{s's}$. We use units such that the nucleon mass $m = 1$ and employ the metric $g_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$.
- 8) For a formulation of the infinite momentum method in terms of lightlike planes see, e.g.,
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- 10) For definiteness we take the lightlike vector n to point into the future.
- 11) This expression is a reformulation of the so-called "almost equal time limits at infinite momentum" introduced by Bjorken, Ref. 4).
- 12) The positivity conditions (3.3) are not valid in the region $q^2 > 4m_\pi^2$. This restriction has its origin in the fact that we are considering the connected part whereas the positivity conditions concern the full commutator. Since the disconnected part is related to the vacuum expectation value, it vanishes below the threshold at $q^2 = 4m_\pi^2$ and the connected part therefore coincides with the full commutator in the region $q^2 < 4m_\pi^2$.

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FIGURE CAPTION

Kinematics in the (ν, q^2) plane

The structure functions vanish outside the shaded region except for the contribution of the one-nucleon intermediate state which is concentrated on the lines $q^2 = \pm 2\nu$. The dashed line refers to the sum rules of Sections 3 and 8.