DUAL MULTIPARTICLE THEORY

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INTRODUCTION

These notes are concerned with recent developments in the Generalized Veneziano Model, and to a large extent correspond to the content of a series of lectures given at CERN in the winter of 1970. The Generalized Veneziano Model is certainly one particular realization of a dual resonance model, but it is the one in which the theory can be pushed further and in which some attempts at implementing unitarity are beginning to be made, with reasonable success.

The unitarization of the Veneziano Model is expected to be accomplished following the program originally suggested by Kikkawa, Sakita and Virasoro, who pointed out that the Veneziano formula violates unitarity in exactly the same way as the Born Approximation in Lagrangian Field Theory. The N-point Veneziano amplitude is considered as providing the tree-diagrams or "Born Approximation" to the scattering matrix; and unitarity will be implemented in a perturbative way -- as in Lagrangian field theory -- when one goes beyond the tree diagrams and includes here diagrams containing closed loops. What is usually meant by "diagram" and how they are defined is explained in the text. The KSV program is far from being finished; one can say that the theory is complete and unique up to the level of diagrams with only one closed loop, essentially after the successful renormalization of its well-known exponential divergence. It is up to this level that we shall describe it here.

These notes are almost self-contained, the only thing being assumed is knowledge of the Bardakçı-Ruegg generalization of the Veneziano formula for N-particles, and the theory is elaborated using the operator formalism. These notes also contain original material. In particular a general method to deal with operators in the Fock space of harmonic oscillators characteristic of the dual theory is discussed at length in Chapter 6 and then applied to the calculation of one-loop diagrams. In Chapter 8 we discuss the non-trivial problem of how the different planar and non-planar one-loop diagrams must be added in order to have a perturbative-unitary amplitude.
Even if, as we show, one-loop calculations can also be done by simpler methods, our techniques of diagonalization of infinite matrices can easily be generalized to the multiloop case. However, this is still in a rather preliminary stage, so we have not included in these notes the multi-loop problem, as well as the closely related problem of the three-reggeon vertex.
1. THE OPERATOR FORMALISM

1.1 The Bardakçi-Ruegg formula

The \((s,t)\) Veneziano four-point function is

\[
\mathcal{B}(-\alpha(s), -\alpha(t)) = \int_0^1 \alpha(x) x^{-\alpha(s)} (1-x)^{-\alpha(t)} - 1
\]

(1.1)

and it has an infinite number of poles in the \(s\) and \(t\)-channels such that the duality property is satisfied:

\[
\sum_{s} s \rightarrow \quad \quad = \quad \sum \quad \quad \text{Fig. 1.1}
\]

In the generalization to the many-point function \(^{3,4)}\), there is a term corresponding to each distinct ordering of the external particles:

\[
\mathcal{B}(-\alpha(s_i)) = \int \mu \prod u_i^{-\alpha(s_i)} - 1.
\]

(We shall not specify the form of the integration measure \(\int \mu\).) There are poles in all "planar channels" (clusters of adjacent particles, labelled \(i\) in the above formula). Duality now states the equality of all tree diagrams which are planar if the specific ordering of the external lines is maintained, e.g.

\[
\begin{array}{ccc}
6 & 5 & 4 \\
\quad & 3 & 2
\end{array} =
\begin{array}{ccc}
5 & 6 & 3 \\
\quad & 4 & 3
\end{array}
\]

\[\text{Fig. 1.2}\]

The first two configurations, called "multiperipheral", are particularly easy to analyse. For the configuration of Fig. 1.3:

\[\text{Fig. 1.3}\]
we have the expression

$$B = \prod_{i=1}^{N-1} \left( \int_{c} \prod_{i} x_{i}^{-\alpha(s_{i})-1} (1-x_{i})^{-\alpha+i} \right) \prod_{1 \leq i < j \leq N} (1-x_{ij})^{-p_{i} \cdot p_{j}}, \quad (1.2)$$

where $x_{i}$ and $s_{i}$ are internal variables and "Mandelstam" variables corresponding to the internal lines. $x_{ij}$ is the product of $x$ variables for lines between the vertices for $p_{i}$ and $p_{j}$ and so

$$x_{ij} = x_{i} \cdot x_{i+1} \ldots x_{j-1}.$$ 

$\alpha(s)$ is the Regge trajectory function, assumed to have the form

$$\alpha(s) = a + \frac{1}{2} s.$$ 

The above formula, the Bardakçi-Ruegg formula is the starting-point of the work described here. An understanding of its derivation is not necessary for this. The work leading up to it is described in Chan's Royal Society lectures$^{3}$).

1.2 Operator results

Consider an infinite set of annihilation and creation operators$^{5,6}$:

$$\left[ a_{\mu}^{(m)}, a_{\nu}^{(n)\dagger} \right] = -g_{\mu\nu} \delta_{mn}. \quad (1.3)$$

$\mu, \nu$ are Lorentz indices $n,m = 1,2 \ldots \infty$.

Since $g_{\mu\nu} = (1,-1,-1,-1)$ the space components $a_{i}^{(n)\dagger} (i = 1, 2, 3)$ are normal creation operators for a three-dimensional harmonic oscillator, whose level spacing we take to be $n$, so that acting on a state with energy $mn$, they yield a state with energy $(m+1)n$.

We assume that the time components act similarly, though we must recognize that since $[a_{0}^{(n)}, a_{0}^{(n)\dagger}] = -1$ the state $a_{0}^{\dagger}|0\rangle$ has negative length. Such ghost states are one of the difficulties of the theory. The total energy operator of the system is

![Fig. 1.4](image)
\[ H = - \sum_{n=1}^{\infty} n \ a_n^{(n)^*} a_n^{(n)} \]  

(1.4)

where, as usual, the Lorentz index is suppressed, and the minus sign is a consequence of our metric.

With these operators one can write down "Feynman-like rules" for the multiperipheral configuration of Fig. 1.5:

\[ B = \langle 0 | V(p_{N+1}) \cdots V(p_1) D(\alpha(s_2)) V(p_2) D(\alpha(s_1)) V(p_1) | 0 \rangle, \]

(1.5)

where we have the correspondences

\[ V(p_i) = e^{\sum_{n=1}^{\infty} \frac{p_i}{\sqrt{n}} a_n^{(n)^*} a_n^{(n)}}, \]

(1.6)

\[ D(\alpha(s_i)) \equiv \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 + \alpha(s_i) + \lambda^2} \]

(1.7)

\[ s_i \equiv \left( \sum_{j=0}^{N} p_j^2 \right) \]

\[ 0 \]

\[ <0 | \]

\[ f_{N+1} \]
Notice that since $H \geq 0$ and $\lambda_0 \geq 0$, $D(\alpha)$ has poles when $\alpha = 0, 1, 2, \ldots$. Thus the above expression does show immediately the expected poles corresponding to the internal lines drawn. Further, the residues factorize, at least in an operatorial sense, since the operators to the right of the propagator depend only on the momenta of particles to the right of that line, and similarly for the left. This factorization will be considered in more detail later.

1.3 The physical picture

Before proving the equivalence between these rules and the Bardakçi-Ruegg formula, we shall see that these operatorial rules are not just a mathematical device but already suggest an extraordinary physical picture of the particles involved.

Let us suppose $a = 1$, and ignore the time component oscillators for the time being. The condition $a = 1$ is quite unphysical since it means that the mass $m$ of the spin-zero particle on the trajectory satisfies

$$\alpha(m^2) = 0 = a + \frac{1}{2} m^2$$

with $m^2 = -2a = -2$. The particle having imaginary mass, this implies that it is a tachyon.

In this artificial case

$$D(\alpha) = \frac{1}{H - \alpha}$$

and this looks like the propagator of formal scattering theory with $H$ the "Hamiltonian" of the system. The objects which propagate are the eigenstates of this Hamiltonian, and these are the excitations of the harmonic oscillators. Consider first the excitations of the first oscillator, that is the spectrum of

$$H = -a^{(1)*} \cdot a^{(1)}.$$ 

These excitations occur at equally spaced intervals just as do the states of Veneziano theory. Further, at any given level there is just one state with the maximum possible angular momentum, and this angular momentum is equal to the number of the level. Thus the leading Regge trajectories
of the three-dimensional harmonic oscillator is linear and non-degenerate, just as at the starting point of Veneziano theory. Of course, since the well is infinitely deep there is no scattering, only an infinite system of bound states. This, of course, corresponds to the poles being on the real axis in Veneziano theory.

It is believed that the Veneziano amplitudes we have considered apply only to mesons built out of a quark and antiquark. One can think of the harmonic oscillator potential as being due to a force between the \( q \bar{q} \) which is proportional to their spatial separation. This is just as if they were joined by a light "rubber band"\(^5\).

So far we have cheated because there is more than one harmonic oscillator, that is there are more degrees of freedom to be excited. One way of adding a degree of freedom is to add an extra particle to the rubber band. This could be a "parton" of \( q \bar{q} \) pair. This gives a new mode of oscillation. To get an infinite number of new modes, we add an infinite number of "partons" and convert the light rubber band to a massive one. The oscillations of such a system are described by a wave equation and are like those of a violin string. The characteristic of a violin string is that its vibrations produce a musical note because any possible mode of vibration is composed of integral multiples of a fundamental note. When quantized its Hamiltonian is exactly of the form we have above for \( H \) from Veneziano theory. Thus an harmonic oscillator with unit spacing corresponds to the fundamental vibration, the oscillator with spacing of two units corresponds to vibrations of the first harmonic, and so on.

This physical picture was suggested by Nambu\(^5\) and Susskind\(^6\). A similar picture was derived by Nielssen without reference to the operator formalism. In the pictures, mesons interact by joining ends to form longer rubber bands which eventually split up again. These rubber bands trace "world sheets" in time, and duality corresponds to the fact that these sheets can be shredded in any direction to give back rubber bands.

Baryons can be pictured as three quarks embedded in rubber bands, and consequently have a much higher degeneracy.

If a \( \neq 1 \) there is another scalar mode of excitation. Its effect is to add more states (genuine daughters) which do not affect the maximum angular momentum.
1.4 Equivalence of the operator rules to the Bardakci–Ruegg form

We prove this equivalence by means of a lemma which is also useful for subsequent work, and which will be generalized later in Section 6. The statement of the lemma is:

\[
\mathcal{V}(p_N) x_{N-1}^H \ldots \mathcal{V}(p_2) x_1^H \mathcal{V}(p_1) = e^{-\frac{2}{N} \sum_{i=1}^{N} \alpha^{(n)}_{ij}} \frac{1}{n} \prod_{i \leq j \leq N} (1 - x_{ij})^{-p_i \cdot p_j},
\]

where

\[
\mathcal{P}^{(n)} = p_N + (x_{i-1})^n p_{N-1} + \ldots + (x_{2N})^n p_2 + (x_{1N})^n p_1
\]

\[
\mathcal{P}^{(n)} = p_i + (x_{i2})^n p_2 + \ldots + (x_{iN})^n p_N,
\]

and the other notation has been explained before.

To prove this we "normal order" the left-hand expression by moving factors \(\exp \left[-2p_j \alpha^{(n)} / \sqrt{n}\right]\) to the left and \(\exp \left[2p_j \alpha^{(n)} / \sqrt{n}\right]\) to the right.

In Appendix A it is proved that

\[
f(a^{(n)}) x^H = x^H f(x^n a^{(n)})
\]

\[
x^H f(a^{(n)}) = f(x^n a^{(n)}) x^H.
\]

In other words, \(a^{(n)}\) picks up a factor \(x^n\) each time it moves right through a factor \(x^H\) and \(a^{(n)}\) picks up the same factor when it moves left through \(x^H\). These facts explain the first three factors on the right-hand side of our lemma.

We get a c-number factor coming from the commutator

\[
e^{[A, B]} = e^B e^A [A, B],
\]

where \([A, B]\) is the c number. In our case it is
\[
\exp \sum_{n,m} \left[ \frac{p_i \cdot a^{(n)}_{ij}}{\sqrt{n}}, - p_j \cdot x_{ij} \frac{a^{(m)}_{ij}^+}{\sqrt{n}} \right] = \exp \left( 0 \cdot \sum_{n=1}^{\infty} k_{ij} \right)
\]

\[
= \exp - p_i \cdot p_j \ln (1 - x_{ij}) = (1 - x_{ij})^{-p_i \cdot p_j}.
\]

The product of all such factors gives the remaining factor on the right-hand side of the lemma.

Taking the vacuum expectation value of our lemma we have:

\[
\langle 0 \mid V(p_n) \cdots x_i^{th} V(p_j) \mid 0 \rangle = \prod_{1 \leq i < j \leq N} (1 - x_{ij})^{-p_i \cdot p_j} \tag{1.10}
\]

since the operators on the right-hand side reduce to 1. The right-hand side becomes the Bardakçi-Ruegg expression if we integrate with

\[
\prod_{i=1}^{N-1} \int_0^1 dx_i \cdot x_i^{\alpha(x_i) - 1} (1 - x_i)^{\alpha - 1},
\]

and the left-hand side becomes our operator expression.

1.5 Defects

In case it seems that the operatorial formalism described above solves everything, we must point out that our rules only apply to multi-peripheral graphs

![Multi-peripheral Graphs]

and not to graphs

![Non-Multi-peripheral Graphs]
Of course the Veneziano terms corresponding to the second two graphs can be written in multiperipheral graphs but it is desirable to write them in a form with propagators corresponding to the lines integrated. When this is done, duality can be checked, and factorization can be made which will be important for the construction of closed-loop graphs which will be needed to unitarize the theory.

The second graph above will be constructed later by means of the "twisting operator", but the third one needs a "three-particle vertex".
2. **FACTORIZATION PROPERTIES OF THE VENEZIANO MODEL AND SPECTRUM OF STATES**

In this section we wish to determine the factorization properties of the Veneziano N-point function, and to deduce from it the spectrum of states predicted by the theory. The method, due to Fubini and Veneziano\(^7\), and independently to Bardakçi and Mandelstam\(^8\), is as follows: consider the \((N+M+2)\)-point function \(B_{N+M+2}^{1}\), and separate the external particles into two clusters of particles with momenta \((p_1, \ldots, p_{N+1})\) and \((q_1, \ldots, q_{M+1})\), respectively (Fig. 2.1):

\[
\begin{align*}
q_M & \rightarrow q_{M-1} \rightarrow \cdots \rightarrow q_1 \rightarrow \sum p_i \rightarrow \sum q_j \rightarrow \cdots \rightarrow q_1 \rightarrow q_{M+1} \\
q_{M+1} & \rightarrow \sum p_i \rightarrow \sum q_j \rightarrow \cdots \rightarrow q_1 \rightarrow q_{M+1}
\end{align*}
\]

**Fig. 2.1**

These two clusters are joined by an intermediate line of momentum \(\pi\):

\[
\hat{\pi} = \sum_{i=1}^{N+1} p_i = - \sum_{j=1}^{M+1} q_j.
\]

\(B_{N+M+2}^{1}\) has poles in \(s = \pi^2\) each time \(\alpha(s) = J, \ J = 0, 1, 2, \ldots\). We are going to show that the residue of the pole at \(\alpha(s) = J\) can be written in the form

\[
\text{Res} \ B_{N+M+2}^{1} \bigg|_{\alpha(s) = J} = \sum_{i=1}^{d(J)} g^{(J)}_i (q) \ g^{(J)}_i (p), \quad (2.1)
\]

and the crucial point is that the integer \(d(J)\) is independent of the number of external particles. This is genuine factorization, and \(d(J)\) represents the degeneracy of the level with \(\alpha(s) = J\); in Eq. (2.1), \(q\) and \(p\)
are shorthand notations for \((q_1, \ldots, q_{M+1})\) and \((p_1, \ldots, p_{N+1})\), respectively. Indeed we shall even prove more than Eq. (2.1), since we shall show that the amplitude itself (and not only the pole residues) can be written in a factorized form:

\[
B_{N+M+2} = \sum_{J=0}^{\infty} \sum_{i=1}^{d(J)} g_i^{(J)}(q) g_i^{(J)}(p) \alpha(s) - J.
\]  

(2.2)

Of course, once we obtain an equation like (2.1), we have also to check that \(d(J)\) is the smallest integer which allows the residue to be written in a factorized form. This raises the problem of linear dependences among the vertex functions \(g_i^{(J)}(q)\), which will be examined in Section 4.

2.1 Factorization in an operator form

The first proofs of factorization\(^7\)\(^8\) were obtained directly from the Bardakci-Ruegg formula, but the operator formalism makes Eqs. (2.1) and (2.2) almost obvious. It is convenient to have the propagator of the intermediate line in the form:

\[
D(\alpha(s)) = \sum_{l_o=0}^{\infty} \left( \frac{l_o - q}{l_o} \right) \frac{1}{H + l_o - \alpha(s)},
\]

(2.3)

which is obtained by expanding the factor \((1 - z)^{a-1}\) in the definition (1.7) of \(D\). Then \(B_{N+M+2}\) is given by

\[
B_{N+M+2} = \sum_{l_o=0}^{\infty} \left( \frac{l_o - q}{l_o} \right) \langle 0 | V(q_1) D \ldots V(q_i) \frac{1}{H + l_o - \alpha(s)} V(p_i) \ldots D V(p_N) | 10\rangle
\]

or

\[
B_{N+M+2} = \sum_{l_o=0}^{\infty} \left( \frac{l_o - q}{l_o} \right) \langle 0 | \frac{1}{H + l_o - \alpha(s)} | p\rangle
\]

(2.4)

if we define the "physical ket" \(|p\rangle\) by:

\[
| p\rangle = V(p_1) D \ldots D V(p_N) | 10\rangle
\]

(2.5)
and similarly for \( |q| \). Now we can insert on each side of  
\[
[H + \lambda_0 - \alpha(s)]^{-1}
\]  
a complete set of occupation number states \( |\{\ell\}\rangle\):
\[
|\{\ell\}\rangle = \frac{1}{\sqrt{\ell_1! \ell_2! \cdots \ell_n!}} (\lambda_1^{(1)*})^{\ell_1} (\lambda_2^{(2)*})^{\ell_2} \cdots (\lambda_n^{(n)*})^{\ell_n} \cdots |0\rangle = I \ell_1, \ell_2, \ldots \ell_n, \ldots >
\]  
(only a finite number of \( \ell_n \)'s are different from zero!)
\[
\sum_{\{\ell\}} |\{\ell\}\rangle <\{\ell\}| = I
\]
\[
H |\{\ell\}\rangle = \ell |\{\ell\}\rangle, \quad \ell = \sum_{n=1}^\infty \ell_n \lambda_n^2.
\]  
(2.7)

Since the "propagator" \([H + \lambda_0 - \alpha(s)]^{-1}\) is diagonal in occupation number space, we get from Eq. (2.7)
\[
B_{N+m+2} = \sum_{\ell_0} \sum_{\ell_0} \left( \lambda_0^{(0)} \right)^{\ell_0-q} \langle q |\{\ell\}\rangle \frac{1}{\ell_0+\ell_0-\alpha(s)} \langle \{\ell\}|p\rangle
\]  
(2.8)

In order to have the residue at the pole \( \alpha(s) = J \) we only have to take all the integers \( \ell_0 \) and all the configurations \( \{\ell\} \) such that
\[
- \text{Res} \left. B_{N+m+2} \right|_{\alpha(s) = J} = \sum_{\ell_0, \ell = J} \left( \lambda_0^{(0)} \right)^{\ell_0-q} \langle q |\{\ell\}\rangle \langle \{\ell\}|p\rangle
\]  
(2.9)

This is the factorized form we required in Eq. (2.1). The intermediate particles are labelled by the occupation numbers \((\ell_1, \ell_2, \ldots)\) and by an integer \( \ell_0 \). This last integer can be interpreted as an occupation number for a scalar mode, to which we can associate annihilation and creation operators \( a^{(0)}, a^{(0)*} \). Then it is easy to include the scalar mode in the formalism.

The degeneracy \( d(J) \) of the level characterized by \( \alpha(s) = J \) is given by the number of ways one can partition \( J \) in the form:
with integers $\lambda_0, \lambda_n$. To be more correct, we should also take into account the reduction of the Lorentz tensors with respect to $O(3)$, but we shall neglect this complication here\footnote{1}. Then the number of states at $\alpha(s) = J$ is given by the degeneracy of the energy level $J$ of the Hamiltonian $H$:

$$H = a^{(0)} a^{(0)} + H. \quad (2.10)$$

The vertex function $\langle q | \{ k \} \rangle$ can be interpreted as the amplitude for the decay of an excited particle with quantum numbers $(\lambda_1, \lambda_2, \ldots)$ into $(M + 1)$ scalar particles

If we wish to include the quantum number $\lambda_0$, we have only to define:

$$\langle q | \{ k' \}, \lambda_0 \rangle = \sqrt{\frac{\lambda_0 - q}{\lambda_0}} \langle q | \{ k \} \rangle. \quad (2.11)$$

2.2 The Fubini-Veneziano factorization

From Eq. (2.9) one can easily derive the original factorized form of Fubini-Veneziano\footnote{1} and Bardakçî-Mandelstam\footnote{3}. We have just to transform the definition of the physical ket $|p\rangle$ by using the lemma proved in Section 1; with this lemma, and calling $d\phi_N$, the integrand of the Bardackçî-Ruegg expression for the $N$-point function, one immediately shows that
\[ |p> = \int \mathcal{D} \varphi_{N+2}(x, p) \ e^{-\sum_{i=1}^{N+1} \frac{P^{(n)}_i \varphi^{(n)}_i}{\sqrt{2}}} |0> \equiv \int \mathcal{D} \varphi_{N+2}(x, p) |p> \]

Now:

\[ \langle \ell_1, \ell_2, ... |p> = \langle \ell_1, \ell_2, ... | e^{-\sum_{i=1}^{N+1} \frac{P^{(n)}_i \varphi^{(n)}_i}{\sqrt{2}}} |0> = \]

\[ = \prod_{n=1}^{\infty} \frac{(-1)^{n}}{(V_n)^{L_n}} \frac{(P^{(n)})^{L_n}}{\sqrt{2}^n} \]

by expanding the exponential. Then we can cast Eq. (2.9) into the form:

\[ -\text{Re} \text{ } B \mid \alpha(s) = \sum \int_{\ell_0 = 0}^{\infty} \frac{Y_{\ell_0} Y_{\ell_n}}{\ell_0!} \prod_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{2}^n} \int \mathcal{D} \varphi_{N+2}(x, p) \left( \varphi^{(n)} \right)^{L_n} \]

\[ \langle Q^{(n)} \rangle = \sum_{j=1}^{N+1} \left( y_{j1} \right) q_j \]

This is (with slight changes in the notations) the factorized form of Fubini and Veneziano.

2.3 Miscellaneous remarks

In the course of the calculation we have forgotten about all the Lorentz indices, but a term in Eq. (2.13) with, for instance \( \ell_1 = 2, \ell_2 = 1, \ell_n = 0, n \geq 3 \), should be written

\[ \int \mathcal{D} \varphi_{N+2}(x, p) \varphi^{(1)} \varphi^{(2)} \varphi^{(3)} \int \mathcal{D} \varphi_{N+2}(x, p) \left( \varphi^{(1)} \varphi^{(2)} \varphi^{(3)} \right) \]
Let us now try to form the state with maximum angular momentum \( J \) inside the level \( \alpha(s) = J \). The maximum angular momentum \( S \) contained in a tensor like

\[
[p^{(1)}]^{\lambda_1} [p^{(2)}]^{\lambda_2} \ldots [p^{(n)}]^{\lambda_n}
\]

is of course:

\[
S = \lambda_1 + \lambda_2 + \ldots + \lambda_n + \ldots
\]

but we have also:

\[
J = \lambda_0 + \lambda_1 + 2\lambda_2 + \ldots + n\lambda_n + \ldots
\]

if the state belongs to the level \( \alpha(s) = J \). If we want \( S = J \), by subtracting the two equations we find that only \( \lambda_1 \neq 0 \). Then the angular momentum \( J \) will be formed by reducing the tensor \( [p^{(1)}]^J \) with respect to \( O(3) \); \( J \) appears only once, and the leading trajectory is thus non-degenerate.

The second remark will be about ghosts: let us take the level \( \alpha(s) = 1 \), which can be formed either with:

\[
\lambda_0 = 1, \quad \lambda_n = 0, \quad n \geq 1
\]

or:

\[
\lambda_0 = 0, \quad \lambda_1 = 1, \quad \lambda_n = 0, \quad n \geq 2.
\]

The first possibility corresponds to the exchange of a well-behaved, scalar particle, and the second one corresponds to a coupling of the form:

\[
\int d^4x \, \langle \mathcal{O}^{(i)} \rangle \int d^4x' \, \langle \mathcal{P}^{(i)} \rangle = \int d^4x \int d^4x' \, \delta^{(s)}(x, x') \left[ Q_0^{(i)} P_0^{(i)} - \Phi^{(i)} \Phi^{(i)} \right].
\]

This corresponds to the exchange of a well-behaved vector-particle, and of a scalar ghost. More generally, ghosts arise from the time-component of Lorentz tensors.
It is clear that there are no ghosts on the leading trajectory, and it would seem, from the previous example, that ghosts appear on the first daughter. However, one can show that ghosts on the first daughter disappear because of the Ward-like identities \(^7\), but unfortunately they remain on the second and further daughters. There is a general tendency to consider ghosts in the Veneziano model as a fact of life, and more or less to forget about them, but in fact they pose a serious problem which should not be underestimated.

The third and final remark concerns double factorization. The method given above allows one to obtain immediately the amplitude \( A_{\{\ell \}, \ell_0}^{\{\ell' \}, \ell_0'} \) for the process (Fig. 2.2):

\[
\text{excited particle } (\{\ell \}, \ell_0) \rightarrow \text{excited particle } (\{\ell' \}, \ell_0') + \text{N scalars}
\]

![Fig. 2.2](image)

Indeed, by making a double factorization of the Bardakçi-Ruegg formula in operator form, and dividing by the vertex functions one finds:

\[
A_{\{\ell \}, \ell_0}^{\{\ell' \}, \ell_0'} = \sqrt{(\ell_0 - q)(\ell_0' - q)} \langle \{\ell' \} | V(p_i) D \cdots D V(p_w) | \{\ell \} \rangle. \tag{2.14}
\]

This double factorization is the starting-point for loop calculations.

2.4 Degeneracy of the levels when \( s \to \infty \)

To close this section, we wish to derive an asymptotic formula for the degeneracy of the level \( \alpha(s) = J \) when \( J \) (or \( s \)) tends to infinity. It is possible to get the result by using the theorems of Hardy and Ramanujan\(^5\), but it is interesting to give a more elementary argument, using the fact that the degeneracy of the level \( \alpha(s) = J \) is the same as the degeneracy of the level with energy \( J \) of the Hamiltonian:

\[
\hat{H} = a^{(0)} \dagger a^{(0)} - \sum_{n} a^{(n)} \dagger a^{(n)}. \]
We ignore the first term in $H$ (although it is very easy to take it into account), and introduce the partition function $Q(T)$ of the system of harmonic oscillators:

$$Q(T) = \text{Tr} e^{-H/T} = \text{Tr} \left( \prod_{\kappa = 1}^{\infty} \frac{1}{i - e^{-\kappa T}} \right)^4,$$

where $T$ is the absolute temperature; the fourth power arises from the Lorentz indices. If the energy is large, $T \to \infty$, and we have to study the partition function in this limit. Introducing

$$x = e^{-1/T}$$

we have

$$Q(x) = \prod_{\kappa = 1}^{\infty} (1 - x^{i-1})^{-4} = x \prod_{\kappa = 1}^{\infty} \frac{b x^{\kappa}}{\kappa (i - x^\kappa)} \sim x^{b(1-x)}$$

with $b = 2\pi^2/3$.

From usual formulae of statistical mechanics, the entropy $S$ is given by

$$S = \frac{\partial}{\partial T} (T \ln Q) = \frac{b}{x} Q - x \frac{\partial \ln Q}{\partial x} \sim \frac{2b}{1-x},$$

while $J$ (which plays the role of the energy) is given by:

$$J = x \frac{\partial \ln Q}{\partial x} \sim \frac{b}{(1-x)^2}.$$

Eliminating $x$ in the last two equations, we have:

$$d(J) = e^{S(J)} = e^{2\sqrt{bJ}} = e^{2\pi \sqrt{(2/3)J}} \text{ when } J \to \infty. \quad (2.16)$$

The number of states increases as the exponential of the mass (since $\sqrt{J} \sim \sqrt{m} \sim m$). This exponential increase is a very interesting feature, which was obtained from quite different arguments by Hagedorn 10 in his statistical model. This large number of levels is somewhat reminiscent
of what happens in nuclear physics. Contrary to the case of ghosts, it should not be considered, at least in our opinion, as an unphysical feature of the theory.
3. TWISTS

3.1 Definition and first properties

For reasons which will become clear later, it is very important to introduce the so-called twisting operation. A "twisted internal line" is defined as follows: if a tree graph is drawn with such a line (represented graphically by putting a cross on it), one has to reverse the order of the external lines either on the right, or on the left of this line (both operations are equivalent, because of the invariance of the N-point function under anticyclic permutations), as follows

\[ \begin{align*}
(\text{a}) & \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \\
(\text{b}) & \quad 1 \quad 2 \quad 3 \quad 6 \quad 5 \\
(\text{c}) & \quad 1 \quad 4 \quad 5 \\
(\text{b'}) & \quad 1 \quad 2 \quad 3 \\
(\text{c'}) & \quad 1 \quad 4 \quad 5
\end{align*} \]

Fig. 3.1

The graphs (b') and (c') are drawn in a non-multiperipheral configuration, and comparing (a) with (b') and (c'), we can see why this is called the twisting operation.

In order to understand clearly the relevance of the twisting operation, let us consider the particularly simple case of the four-point function. Since we are working with scalar spinless particles with no internal quantum numbers, the full amplitude is obtained by adding (s,t), (s,u) and (t,u) Veneziano terms. Consider now an (s,t) Veneziano term and its
twisted counterpart:

\[ \begin{array}{c}
\frac{2}{3} \\
\downarrow \\
\frac{1}{4}
\end{array} \quad \frac{2}{3} \quad \begin{array}{c}
\frac{2}{3} \\
\downarrow \\
\frac{1}{4}
\end{array} \text{ (a)} \quad \text{ (b)}
\]

Fig. 3.2

Then, we see that the twisted \((s,t)\) Veneziano term is the \((s,u)\) Veneziano term. Both have to be added, and as far as the \(s\)-channel singularities are concerned this is precisely the signature operation. Indeed, it is trivial to show that half of the \(s\)-channel residues are even under \(t \leftrightarrow u\) exchange and half are odd; and only the poles with positive signature survive when the \((s,t)\) diagram and its twisted counterpart are added. This is still true whatever number of particles is concerned. Indeed, if we add the two graphs:

\[ \begin{array}{c}
q_1 \\
q_2 \\
q_{M+1}
\end{array} + \begin{array}{c}
p_1 \\
p_2 \\
p_{N+1}
\end{array} \text{ (a)} \quad \text{ (b)}
\]

Fig. 3.3

which have poles in the variable \(s\):

\[ s = (p_1 + p_2 + ... + p_{N+1})^2 = (q_1 + q_2 + ... + q_{M+1})^2 \]
at $\alpha(s) = J$, we can see that half of the intermediate particles disappear. In particular, on the leading trajectory all resonances with odd $J$ are cancelled. In a theory including isotopic spin, a leading trajectory with odd signature is obtained by subtracting the two graphs.

These remarks illustrate an important property of these Feynman-like diagrams which is not shared by ordinary Feynman diagrams. Figures 3.2 (a) and (b) represent the same amplitude -- a one-pole diagram -- from a field theory point of view, since the Feynman propagator does not "remember" in which order the particles in the final state are coupled. In our case, however, Fig. 3.2 (a) represents an infinite number of poles propagating in the $s$-channel, and the propagator must remember in which way it is coupled to the final state because this determines whether they are dual to an infinite number of $t$-channel or $u$-channel poles. The deep reason for this novel feature is that a given Veneziano term has crossing and duality properties in only two channels. Loop diagrams containing internal twisted lines are called "non-planar loop diagrams", and it is clear from the preceding discussion that this is a new kind of non-planarity which has nothing to do with the usual non-planar Feynman diagrams.

It may also be useful to draw duality diagrams in order to visualize twisting

\[ 1 \quad 2 \quad 3 \quad 4 = 1 \quad 2 \quad 4 \quad 3 \]

\[ 1 \quad 3 \quad 4 = \]

Fig. 3.4

The Veneziano term corresponding to Fig. 3.4 is indeed an $(s,u)$ term and not an $(s,t)$ one.

By twisting two neighbouring lines one obtains the so-called "twisted vertex"
or in terms of duality diagrams

Notice that two particles joined with a twisted line cannot resonate (e.g. particles 3 and 4 in Fig. 3.1). It is clear on the multiperipheral configuration (b) of Fig. 3.1 that 3 and 4 are not really adjacent.

The twisting operation may appear at first sight to be rather useless, since all tree graphs can be drawn in a multiperipheral configuration. However, this is no longer true when we consider tree graphs with excited particles.

Indeed, let us see how we could obtain an expression for the amplitude such as that of Fig. 3.7:
If we think that excited particles are defined by factorization, we see that the amplitude of Fig. 3.7 (a) can be obtained by factorizing the tree with twists of Fig. 3.8 (a) on the multiperipheral variable which corresponds to $\Sigma p_i$ and $\Sigma q_j$.

Alternatively, Fig. 3.7 (b) is obtained by factorizing the untwisted tree of Fig. 3.8 (b), but this time the variable which corresponds to $\Sigma p_i$ is not a multiperipheral variable but a variable dual to multiperipheral variables. Both procedures must of course give the same result. For the time being we prefer to avoid the procedure -- which we still have not mastered properly -- of factorizing on variables dual to multiperipheral ones by keeping multiperipherism with twists.

3.2 The twisting operator $\Omega$

Our task is now to build explicitly the twisting operator $^{13}$, which we shall call $\Omega$, in order to write down easily the amplitudes corresponding to tree graphs with twisted lines. If we call $|\bar{p}\rangle$ the physical state obtained from $|p\rangle$ by reversing the order of the particles
\[ |p\rangle = V(p_1) D \cdots D V(p_N) |0\rangle \]

(3.1)

we must have by definition

\[ \Omega |p\rangle = |\tilde{p}\rangle \]

(3.2)

or, graphically:

\[ \Omega |p\rangle = |\tilde{p}\rangle \]

(3.3)

It is convenient to define

\[ p_i = p_{N+2-i} \]

(3.4)

so that by Eq. (2.12) \( |\tilde{p}\rangle \) is given by

\[ |\tilde{p}\rangle = \int d\varphi_{N+2}(\tilde{x}, \tilde{p}) e^{-\tilde{p}^{\alpha} \ Alpha^{\alpha} \ \frac{\varphi_{N+2}}{\sqrt{N}}} |0\rangle = \int d\varphi_{N+2}(\tilde{x}, \tilde{p}) |\tilde{p}\rangle \]

(3.5)

The integrands \( d\varphi_{N+2}(x, p) \) and \( d\varphi_{N+2}(\tilde{x}, \tilde{p}) \) are the Veneziano integrands of the tree graphs drawn in (3.3); these tree graphs are given by the same integral, since the order of the external particles is the same (this is true even if \( \Pi^2 \neq \mu^2 \)). The change of variables which proves the equivalence of the two graphs is a duality transformation:

\[ \tilde{x}_{1i} = 1 - x_1, N+2-i \]

(3.6)
and with this change of variables one shows that

$$\int d \varphi_{N+2} (\tilde{z}, \tilde{p}) = \int d \varphi_{N+2} (x, p). \quad (3.7)$$

Let us now make the change of variables (3.6) in the definition of \( \tilde{P}^{(n)} \):

$$\tilde{P}^{(n)} = \sum_{i=1}^{N+1} (x_{n+i} - x_{n+i-1}) \tilde{p}_i = \sum_{i=1}^{N+1} (1 - x_{n+i-1}) \rho_{N+2-i}.$$  

$$\tilde{p}^{(n)} = \sum_{i=1}^{N} (1 - x_{n+i}) \rho_i. \quad (3.8)$$

Now:

$$\langle \tilde{p}^{(n)} \rangle = e^{-\sum \rho^{(n)} \cdot a^{(n)} / \sqrt{n}}$$

$$\langle \tilde{p}^{(n)} \rangle = e^{\sum \rho^{(n)} \cdot a^{(n)} / \sqrt{n}}$$

$$10 = \langle 1, \ldots, \rho_i, \ldots \rangle,$$

while:

$$10 = \langle 1, \ldots, \rho_i, \ldots \rangle,$$

where we have written down explicitly the component corresponding to the \( n \)th mode. From Eq. (A.7) it follows that

$$e^{-\sum \rho^{(n)} \cdot a^{(n)} / \sqrt{n}} e^{-\sum a^{(n)} \cdot (C_{nm} - \delta_{nm}) a_m} e = e^{-\sum \rho^{(n)} \cdot a^{(n)} / \sqrt{n}} \tilde{p} \rangle \langle \tilde{p} \rangle \quad (3.9)$$

if the matrix \( C_{nm} \) is given by
\[ C_{nm} = \sqrt{\frac{m}{n}} \binom{n}{m} (-)^m. \]

From Eqs. (3.7) and (3.9) we see that the twisting operator \( \Omega \) is given by

\[ \Omega = e^{\sum_n \frac{a_n^{(n)}}{\sqrt{n}}} \left( \sum_{m} a_{n,m}^{(m)} \left( \sqrt{\frac{m}{n}} \binom{n}{m} (-)^m - \delta_{nm} \right) a^{(m)} \right). \]  

(3.10)

The identity

\[ \Omega^2 = 1 \]  

(3.11)

follows immediately from \( 1 - (1 - x) = x. \)
4. GAUGE RELATIONS

We shall consider here again the diagram shown in Fig. 2.7 corresponding to a particular contribution to the \((N+M+2)\)-point function. Let us concentrate on the internal line with momentum \(\Pi\). If we twist this line we obtain a different contribution to the full amplitude; for example, by changing the ordering of particles in the state on the right of the line we are considering. However, if we twist once again and change the ordering of the particles on the state on the left, it is trivial to check that we recover the initial ordering of all external lines. Therefore, we can write

\[
\mathcal{B}_{N+M+2} = \langle q | D(\alpha(s),H) | p \rangle = \langle q | \Omega(\Pi) D(\alpha(s),H) \Omega(\Pi) | p \rangle
\]

(4.1)

for any arbitrary states \(|p\rangle\) and \(|q\rangle\). Equation (3.1) is sometimes referred to as "double twist invariance" of the generalized Veneziano amplitude. The remarkable fact is that this double twist invariance is only true for the matrix elements of the two operators \(D\) and \(\Omega^+ D\Omega\) between arbitrary physical states (this means: arbitrary number of scalar particles with arbitrary momenta). Indeed, as operators they are different

\[
D \neq \Omega^+ D\Omega \quad (4.2)
\]

or, using Eq. (3.11)

\[
D\Omega \neq \Omega^+ D \quad (4.2')
\]

We shall soon consider the proof of Eq. (4.2). However, at this stage it is important to notice that Eqs. (4.1) and (4.2) can hold simultaneously only if the "physical states" \(|p\rangle\) do not exhaust the Hilbert space in which the operators are defined; that is, the Hilbert space of our collection of harmonic oscillators. Therefore, there must exist in our Hilbert space, states \(|s\rangle\) (for "spurious") which are orthogonal to any physical state \(|p\rangle\):

\[
\langle s | p \rangle = 0 \quad (4.3)
\]

Expanding the spurious states in the occupation number basis \(|\ell\rangle\) we obtain relations of the form
\[ \sum_{\{e\}} C_{\{e\}} \langle \{e\} \mid p \rangle = 0, \quad (4.4) \]

that is, linear relations among the couplings obtained by factorization [see Eqs. (2.8) and (2.9)]. This means that our counting of states is not quite correct, because we shall see later on that these linear relations imply linear relations among the residues of the same pole of the Veneziano amplitude, and therefore the degeneracy is lower than the one obtained in the previous section.

4.1 Derivation of the gauge relations

The simplest way to derive these linear relations (or gauge conditions, or Ward-like identities) is to introduce the following set of operators

\[ L_0 = -\frac{i}{2} \Pi^2 + H, \quad (4.5) \]

\[ L_+ = -\Pi \cdot \alpha^{(4)} + \sum_{n=1}^{\infty} \sqrt{n(n+1)} \alpha^{(n+1)} \alpha^{(n)}, \quad (4.6) \]

\[ L_- = L_+^\dagger = -\Pi \cdot \alpha^{(4)} - \sum_{n=1}^{\infty} \sqrt{n(n+1)} \alpha^{(n)} \alpha^{(n+1)}. \quad (4.7) \]

Using the commutation relations of the creation and annihilation operators, one can easily show that they satisfy the commutation relations corresponding to the Lie algebra of O(2,1):

\[ [L_0, L_\pm] = \pm L_\pm \quad (4.8) \]

\[ [L_+, L_-] = -2L_0. \quad (4.9) \]

It is not a happy accident that this group structure appears in the theory. The group O(2,1) is isomorphic to the group of real projective transformations of the form
and it is well known that, when the Veneziano N-point amplitude is written in the Koba-Nielsen form, the integrand is a function of anharmonic ratios which are invariant under projective transformations. Indeed, starting from the Koba-Nielsen representation\(^1\), one can show\(^2\) that the operators we have introduced are precisely the generators of projective transformations. However, we shall not pursue this approach here.

It is important to notice that all the operators we have introduced up to now (except the vertex) can be written as group elements:

\[
\begin{align*}
Z^H & = \bar{z}^\Pi L c \ln \bar{z} \\
\Omega(\Pi) & = e^{L_+^H (\Pi)} (-)^H = (-)^H e^{-L_+^H (\Pi)}.
\end{align*}
\] (4.10)  (4.11)

The proof of Eq. (4.11) is left as an exercise. It is obtained by computing the normal order form of \((-1)^H e^{-L_+^H (\Pi)}\) and checking that this agrees with Eq. (3.10).

It is useful to use the 2 × 2 representation of O(2,1) to check relations among operators\(^3\). One can use the following representation of the O(2,1) algebra

\[
\begin{align*}
L_0 & = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
L_+ & = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
L_- & = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},
\end{align*}
\] (4.12)

and compute the 2 × 2 matrices corresponding to \(Z^H\) and \(\Omega\). Since \(\Omega\) is independent of the integration variable \(z\), the inequality \(\Omega^+ D\Omega \neq D\) is equivalent to \(\Omega^+ z^H \Omega \neq z^H\), and this can readily be checked with the 2 × 2 matrices.

The gauge relations are obtained by noticing that the operator

\[
W(\Pi) = L_0(\Pi) - L_-(\Pi)
\] (4.13)
annihilates an arbitrary physical state\textsuperscript{14,16}:

\[ W(\pi) | \rho \rangle = W(\pi) V(\rho) D \cdots D V(\rho_N) | \rho \rangle = 0 \quad (4.14) \]

The proof consists of two steps. First we commute \( W(\pi) \) with \( V(\rho)D \), and this can be done by using the following commutation relations:

\[ W(\pi) V(\rho) = V(\rho) \left[ W(\pi - \rho) - a \right] \quad (4.15) \]

\[ W(\pi - \rho) Z^H = Z^{H+1} W(\pi - \rho) + Z^H (1 - Z) L_\omega (\pi - \rho) \quad (4.16) \]

Then we get:

\[ W(\pi) V(\rho) D(\alpha(s), H) = V(\rho) D(\alpha'(s), H+1) W(\pi - \rho) + \]

\[ + V(\rho) \int_0^1 dZ Z^{H - \alpha'(s) - 1} (1 - Z) (1 - Z) L_\omega - a \]. \quad (4.17)

However, the integral is shown to be zero using \( H - \alpha(s) = L_\omega - a \) and the identity for beta functions \((\alpha + \beta)B(\alpha, \beta + 1) - \beta B(\alpha, \beta) = 0\). Therefore the final result is

\[ W(\pi) V(\rho) D(\alpha(s), H) = V(\rho) D(\alpha(s), H+1) W(\pi - \rho). \quad (4.18) \]

Using Eq. \( (4.18) \) we can keep commuting \( W \) along the chain until we end up with \( W \) acting on a two-particle state. Then one can easily show that \( W \) annihilates the two-particle state by explicit computation, making use of the identity

\[ - \frac{1}{\hbar} \left( \rho_N + \rho_{N+1} \right)^2 + \rho_N = 0 \quad (4.19) \]

which comes from \( \rho_{N+1}^2 = \mu^2 \).
Having shown that $W$ annihilates an arbitrary state, we get immediately the spurious states in the theory; because

$$
\langle \{\ell\} \mid W \mid p \rangle = 0 \quad \Rightarrow \quad |s\rangle = W^+ |\{\ell\}\rangle ,
$$

(4.20)

so $W^+$ transforms any occupation number state into a spurious state. Equation (4.20) are the gauge conditions of the theory. We can also ask: how different are the operators $z^H$ and $\Omega^+ z^H \Omega$? Using again $2 \times 2$ matrices one can easily prove that they are equal up to a "gauge transformation" $S(z)$

$$
S^+(z) \Omega^+ z^H \Omega S(z) = z^H
$$

(4.21)

where

$$
S(z) = (1 - z)^W.
$$

(4.22)

Equation (4.21) is the operatorial form of double twist invariance $^{17,18}$. The known properties of $W$ ensure that $S(z)$ leaves physical states invariant, i.e.

$$
S(z) |p\rangle = |p\rangle ,
$$

(4.23)

and therefore by taking matrix elements in Eq. (4.21) we recover the original form of double-twist invariance for matrix elements.

Using the commutation relations of the group, it can be shown that

$$
(\Omega S(z))^2 = 1 ,
$$

(4.24)

and therefore from Eq. (4.21) we get

$$
z^H \Omega S(z) = S^+(z) \Omega^+ z^H ,
$$

(4.25)

and this obviously means that $z^H \Omega S(z)$ is an Hermitian operator (whereas $z^H \Omega$ is not) $^{17}$.

### 4.2 Gauge relations and ghosts

Since $[W, H] \neq 0$, the gauge conditions give relations among couplings $\langle \{\ell\} \mid p \rangle$ corresponding to different eigenvalues of $H$. Indeed, $W^+ = L_0 - L_+$ and it is clear that $L_+$ will raise in one unit the eigenvalue of $H$. However,
the poles of the Veneziano amplitude are given, as discussed in Section 2.1, by the spectrum of the Hamiltonian $H = a(\sigma)^+ a(\sigma) + H$ with the "scalar mode" included. The couplings that correspond to the poles $\alpha(s) = 0$ are given by Eq. (2.11) with $l_0 + \sum_n n l_n = J$, and since the dependence of the coupling on the "scalar mode occupation number" $l_0$ is quite trivial, one can always use this extra mode to restore energy conservation and get relations between residues of the same pole.

We want to discuss in some detail how the gauge conditions work at the first excited level, $\alpha(s) = 1$. In this case, as discussed before, we have two possibilities:

\[ l_0 = 1, \quad l_n = 0; \quad n \geq 1 \quad \rightarrow \text{scalar particle} \]

\[ l_0 = 0, \quad l_1 = 1, \quad l_n = 0; \quad n \geq 2 \rightarrow \text{vector + scalar particles.} \]

Introducing the notation of Fubini and Veneziano\textsuperscript{7}:

\[ \langle p^{(n_1)}_{\mu_1} p^{(n_2)}_{\mu_2} \cdots p^{(n_0)}_{\mu_0} \rangle = \int d\Phi_{\mu_1} (x, \rho) \int d\Phi_{\mu_2} (x, \rho) \cdots \int d\Phi_{\mu_0} (x, \rho) \]

we obtain the residue from Eq. (2.13) (up to an over-all sign which is absorbed in the normalization of the amplitude):

\[ \text{Res } \left. B \right|_{\alpha(s) = 1} = \langle 1 \rangle \langle \gamma \rangle \langle \gamma \rangle - \langle Q_{\alpha(s)}^{(1)} \rangle \langle \gamma \rangle \]

\[ \langle \gamma \rangle \text{ (4.27) } \]

where it is obvious that the time-component of the vector coupling is a scalar ghost. In this case the gauge conditions imply:

\[ \langle 0 \mid W(\Pi) \mid \rho \rangle = \langle 0 \mid L_0 \mid \rho \rangle - \langle 0 \mid L_- \mid \rho \rangle \]

or

\[ \frac{-\lambda}{2} \Pi^2 \langle 0 \mid \rho \rangle = -\Pi^\mu \langle 0 \mid a^{(4)}_{\nu} \mid \rho \rangle. \text{ (4.28) } \]

From Eq. (2.12) for $|\rho\rangle$ we get

\[ \frac{-\lambda}{2} \Pi^2 \langle \gamma \rangle = -\Pi^\mu \langle \gamma \rangle. \text{ (4.29) } \]
Going to the centre-of-mass system ($\bar{\Pi} = 0$) we obtain a relation between the time-component and the scalar coupling:

$$\frac{1}{2} \Pi_0 \langle 1 \rangle = \langle \overrightarrow{P}_0^{(4)} \rangle .$$  \hspace{1cm} (4.30)

At the pole position $\alpha(s) = (\Pi_0/2) + a = 1$, then

$$\langle \overrightarrow{P}_0^{(4)} \rangle \langle \overrightarrow{Q}_0^{(4)} \rangle = \frac{1}{4} \Pi_0^2 \langle 1 \rangle \langle 1 \rangle = \frac{4-a}{2} \langle 1 \rangle \langle 1 \rangle ,$$ \hspace{1cm} (4.31)

and the residue is

$$\text{Res}_{\Pi_0} = \langle \overrightarrow{Q}^{(4)} \rangle \langle \overrightarrow{P}^{(4)} \rangle + \frac{4-a}{2} \langle 1 \rangle \langle 1 \rangle .$$ \hspace{1cm} (4.32)

Therefore, it is clear that the ghost is only apparent, the over-all scalar coupling being positive. Unfortunately, not all ghosts can be removed in this way\(^7\). These gauge conditions remove all ghosts along the first daughter, but ghosts at the level of the second or higher daughters are not removed by them.

4.3 Further gauge relations

In the unphysical case $a = 1$ -- unphysical because the ground state of the theory has $\mu^2 = -2$ -- it is possible to show\(^9\) that there are more gauge relations than the ones described before.

Let us introduce the operations

$$L^{(N)}_+ = -\sqrt{N} \cdot \Pi \cdot a^{(N)} - \frac{1}{2} \sum_{n=1}^{N-1} \sqrt{n(n-n)} \cdot a^{(n)} a^{(N-n)} - \sum_{n=1}^{\infty} \sqrt{n(n+n)} \cdot a^{(n)^\dagger} a^{(n+N)} ,$$  \hspace{1cm} (4.33)

$$L^{(N)}_- = L^{(N)}_+ ,$$

and the generalized gauge operator

$$\mathcal{W}_N (\Pi) = L_c (\Pi) = L^{(N)}_- (\Pi) = \frac{1}{2} (N-1) \mu^2 .$$ \hspace{1cm} (4.34)
The extra factor \((N - 1)\mu^2/2\) has been included in order to guarantee that \(W_N(\Pi)\) annihilates a two-particle state, as can easily be checked. Moreover, the commutation relation of the generalized gauge operator with the vertex is:

\[
W_N(\Pi) V(p) = V(p) \left[ W_N(\Pi - p) + \frac{N\mu^2}{2} \right]
\]  

(4.35)

and, using

\[
\left[ L_a, L^{(N)}_\pm \right] = \pm N L^{(N)}
\]  

(4.36)

one can show that

\[
W_N \mathcal{Z}^H \equiv \mathcal{Z}^{H-N} W_N + \mathcal{Z}^H \left( 1 - \mathcal{Z}^N \right) \left( L_a - \frac{1}{2}(N-1)\mu^2 \right).
\]  

(4.37)

Therefore, the complete commutation relation of \(W_N(\Pi)\) with a vertex and a propagator is

\[
W_N(\Pi) V(p) \mathcal{Z}^H = V(p) \mathcal{Z}^{H-N} W_N(\Pi)
\]  

(4.38)

\[
+ V(p) \mathcal{Z}^H \left[ \frac{N\mu^2}{2} + (1 - \mathcal{Z}^N) L_a - (1 - \mathcal{Z}^N)(N-1)\mu^2 \right].
\]

Integrating both sides of Eq. (4.38), it follows that

\[
W_N(\Pi) V(p) D(s, H) = V(p) D(s, H + N) W_N(\Pi - p)
\]  

\[
+ V(p)(1 - \alpha) \int (s, H) \mathcal{Z}^{H-N} \left( 1 - \mathcal{Z}^N \right)^{a-1} \left\{ N \mathcal{Z}^{N-1} - \frac{1 - \mathcal{Z}^N}{1 - \mathcal{Z}} \right\},
\]  

(4.39)

Therefore, if \(\alpha = 1\) the second term of the right-hand side of Eq. (4.39) vanishes and \(W_N(\Pi)\) "commutes" with \(V(p) D(s, H)\), changing \(H\) by \(H + N\) in \(D\). It follows that \(W_N(\Pi)\) annihilates an arbitrary physical state.

Since in this case there is an infinite number of gauge operators (for \(N = 0, 1, 2, \ldots\)) Virasoro was able to show that the time-components of all oscillators are redundant and hopefully all ghosts are compensated. However, we emphasize that this result is achieved only at the
expense of having tachyon as the ground state. There is no general
ghost-compensating mechanism in the usual Veneziano theory, since one can
explicitly calculate the residues of the Regge poles for the four-point
function\textsuperscript{20}) and check that the signs are not positive-definite\textsuperscript{21}). Either
the original Veneziano formula has to be modified to include extra exci-
tations that cancel the ghosts; or one has to expect that ghosts will be
killed by the unitarization procedure.
5. **TWISTED PROPAGATOR**

5.1 **Choice of a twisted propagator**

As explained before, the main reason for introducing the twist operator $\Omega$ is to factorize out amplitudes with excited particles as the one of Fig. 3.7. As discussed in Section 3, the twist operator allows us to write the tree of Fig. 3.8 (a) so as to obtain the one of Fig. 3.7 by factorization on a multiperipheral variable.

Although, due to the gauge conditions, the tree of Fig. 3.8 is the same whether we use $\Omega^+D$ or $D\Omega$ as a twisted propagator, this is no longer true for the factorized tree of Fig. 3.7. In other words, the two ways of writing the tree of Fig. 3.8 are equivalent, if we sum over a complete set of states on the lines we want to factorize, but they are not equivalent if we take two particular states $\alpha$ and $\beta$.

We have therefore an embarrassing choice, i.e. whether to choose $D\Omega$ or $\Omega^+D$ as a twisted propagator. Once a choice is made and the factorization performed, this choice is frozen owing to the fact that $D\Omega$ and $\Omega^+D$ are not operatorially equal.

It is easy to convince oneself that none of these twisted propagators are correct. Indeed, neither $D\Omega$ nor $\Omega^+D$ are Hermitian operators, so that a further factorization of Fig. 3.7 would show that we are lacking one of the fundamental ingredients of factorization. Indeed factorization implies that the "couplings" on the right and on the left of the factorized line are Hermitian conjugate; clearly, this cannot be the case with this choice of the twisted propagator.

The remedy is simple to find. We understood that the lack of double-twist invariance was related to gauge conditions, and we must take them into account in order to define a twisting operator which is able to yield an Hermitian twisted propagator; or, which is the same, a twisted propagator that satisfies double-twist invariance.

From the discussion in the previous section, it is clear that the twisting operator which ensures this property is:
\[ \Theta(x) = \Omega(1 - x)^W, \quad (5.1) \]

with
\[
\begin{align*}
\Theta(x) | p \rangle &= | \bar{p} \rangle \\
[\Theta(x)]^2 &= 1 \\
\Theta^+(x) x^L \Theta(x) &= x^L.
\end{align*}
\]  

(5.2)

Therefore a twisted propagator which satisfies all the requirements is:

\[ x^L \Theta(x) \equiv \Theta^+(x)x^L. \quad (5.3) \]

The operatorial expression, whose matrix elements between states \( \alpha \) and \( \beta \) gives the graph of Fig. 3.7 (a), is:

\[
\int \prod_{i=1}^{4} d\mu(x_i) \ V_1 x_1^L \Theta(x_1) V_2 x_2^L \Theta(x_2) V_3 x_3^L \Theta(x_3) V_4 x_4^L \Theta(x_4) V_5 \cdot
\]

\[ \mu(x) = x^{-a^{-1}} (1-x)^{a^{-1}} dx. \quad (5.4) \]

5.2 The signaturized propagator

The signaturized propagator is defined as the sum of the untwisted + the twisted propagators:

\[ x^L \left[ 1 + \Theta(x) \right] = \frac{1}{2} \left[ 1 + \Theta^+(x) \right] x^L \left[ 1 + \Theta(x) \right]. \quad (5.5) \]

The reason why Eq. (5.5) is called the signaturized propagator is easy to understand if we change to a representation which diagonalizes the twisting operator\(^{17}\). Indeed, by using the 2 \( \times \) 2 representation, it is easy to check that:

\[ \tau = (-1)^H e^{-i\frac{L}{2}} (1 - x)^{iW} \quad (5.6) \]
satisfies:
\[ \tau \otimes \tau^{-1} = (-1)^H \]  
(5.7)
and:
\[ \tau(4f)^L_0 \tau = xL_0 \]  
(5.8)

where
\[ f(x) = \frac{4 - \sqrt{4 - x}}{4 + \sqrt{4 - x}} \quad \text{or} \quad x = \frac{4f}{(1+f)^2}. \]  
(5.9)

Then, if we define a modified vertex (depending on the integration variables) \( V'(p_i) \) by:
\[ V'(p_i) = \tau_i \cdot V(p_i) \tau_i^+ \]
the chain in this new representation, also called the \( V \)-representation\(^\text{11})\), will look like\(^\text{17)}:\n\[
\int_0^1 \prod_{i=3}^{N-1} d\nu(f_i) \langle 0 | V'(p_2) (4f_3) \cdots L_0^c(\Pi_2) \cdots \cdots \langle 4f_{N-1} \rangle \cdots \cdots L_0^c(\Pi_{N-1}) \cdots \cdots V'(p_{N-1}) | 0 \rangle,
\]  
(5.10)

with
\[ d\nu(f) = df \, f^{-a} (1 + f)(1 - f)^{2a-1}, \]  
(5.11)
and
\[ (4f)^L_0 = (4f)^L_0 \quad \text{if the line is not twisted} \]
\[ = (4f)^L_0 (-1)^H \quad \text{if the line is twisted}. \]

Now, the complete residue at the pole contains the sum of the twisted and untwisted propagators, owing to the fact that both multi-Veneziano terms contribute to the same pole.
We see that in the V-representation, because of the \((-1)^{H}\) factor, both terms give either the same or the opposite contribution, depending on whether the eigenvalue of \(H\) which labels the particle at the pole is even or odd. In the first case the residues add up; in the second case they cancel out.

For the leading trajectory, which is characterized by the occupation numbers:

\[ \lambda_n = 0 \quad n = 0, 2, 3, \ldots \]
\[ \lambda_1 = J \]

the eigenvalues of \(H\) are:

\[ H = \sum_n n \lambda_n = J, \tag{5.12} \]

so that \((-1)^{H} = (-1)^{J}\) which shows that the leading trajectory has positive signature\(^{11,12}\).

5.3 Duality properties of the twisted propagator

Let us now discuss the duality problem. If in a scalar chain we replace a link in the following way:

\begin{center}
\begin{tikzpicture}
  \draw (0,0) -- (1,0);
  \draw (1,0) -- (2,0);
  \draw (1,-1) -- (2,-1);
  \draw (1,-2) -- (2,-2);
  \node at (0.5,0) {1};
  \node at (1.5,0) {2};
\end{tikzpicture}
\end{center}

with

\begin{center}
\begin{tikzpicture}
  \draw (0,0) -- (1,0);
  \draw (1,0) -- (2,0);
  \draw (1,-1) -- (2,-1);
  \draw (1,-2) -- (2,-2);
  \node at (0.5,0) {1};
  \node at (1.5,0) {2};
\end{tikzpicture}
\end{center}

Fig. 5.1

\begin{center}
\begin{tikzpicture}
  \draw (0,0) -- (1,0);
  \draw (1,0) -- (2,0);
  \draw (1,-1) -- (2,-1);
  \draw (1,-2) -- (2,-2);
  \node at (0.5,0) {1};
\end{tikzpicture}
\end{center}

or

\begin{center}
\begin{tikzpicture}
  \draw (0,0) -- (1,0);
  \draw (1,0) -- (2,0);
  \draw (1,-1) -- (2,-1);
  \draw (1,-2) -- (2,-2);
  \node at (0.5,0) {1};
\end{tikzpicture}
\end{center}

with

\begin{center}
\begin{tikzpicture}
  \draw (0,0) -- (1,0);
  \draw (1,0) -- (2,0);
  \draw (1,-1) -- (2,-1);
  \draw (1,-2) -- (2,-2);
  \node at (0.5,0) {2};
\end{tikzpicture}
\end{center}

Fig. 5.2

\(a)\)

\(b)\)

the result is unchanged, as is well known. However, this fact does not guarantee that this is still true if the wiggly lines are not part of a scalar chain but are arbitrary states. Nevertheless, we want this to be
true for arbitrary states in order to be able to classify arbitrary loop diagrams in the way introduced by Kikkawa, Sakita and Virasoro\textsuperscript{1).} Indeed, they found a very attractive way of classifying diagrams (planar, non-planar orientable, and non-planar non-orientable) on the assumption that duality holds also for internal lines in an arbitrary loop.

Concerning the duality property exhibited in Fig. 5.1, we shall only say for the moment that it is used to define a three-Reggeon vertex\textsuperscript{22). With respect to the duality relation exhibited in Fig. 5.2, we recognize that it relates two twisted multiperipheral diagrams. Indeed, in Fig. 5.2 we have drawn two diagrams, which are related in that they transform into each other when the internal line is dualized. According to the operator rules, we have, corresponding to the circled segments in Fig. 5.3 (a) and 5.3 (b):

\[ R_1 = x_1^{L_0} \Theta(x_1) V(p_1) y_1^{L_0} \Theta(y_1) V(p_2) z_1^{L_0} \mu(x_1) \mu(y_1) \mu(z_1) \]  
(5.13)

\[ R_2 = x_2^{L_0} V(p_2) y_2^{L_0} \Theta(y_2) V(p_1) z_2^{L_0} \Theta(z_2) \mu(x_2) \mu(y_2) \mu(z_2) \]  
(5.14)
Using a general method explained in the next section we can prove our operatorial duality relation which states that\textsuperscript{23})

\[ R_1 = R_2 \]  \hspace{1cm} (5.15)

provided

\[ x_2 = x_1 \; ; \; z_2 = z_1 \; ; \; y_2 = \frac{1 - y_1}{1 - y_1 (x_1 + z_1 - x_1 z_1)} \]  \hspace{1cm} (5.16)

Notice that the variables (x and z) corresponding to the legs of the dualized line do not transform, whereas the \( y_1, y_2 \) of the dualized line transform according to the law known to relate Chan variables\textsuperscript{3}) as represented in the dual graph of Fig. 5.4. [If \( u^{-\alpha(s)^{-1}} \) contains all the s-dependence in a multi-Veneziano integral, then \( u \) is the Chan variable for that channel.]

![Fig. 5.4](image)

In Fig. 5.3 the lines with variables \( x_1 \) and \( x_2 \) are twisted and untwisted, respectively. Because of the double-twist invariance equations

\[ \Theta^+(x) \; x^{L_0} \; \Theta^-(x) = x^{L_0} \; ; \; [\Theta(x)]^2 = \mathbb{1}, \]  \hspace{1cm} (5.17)

the operator expressions corresponding to the situations in which the \( x_1 \) line is untwisted and the \( x_2 \) one twisted are, respectively,

\[ \Theta^+(x_1) \; R_1 \]  \hspace{1cm} , \hspace{1cm} \Theta^+(x_2) \; R_2 \]
and these are still equal, by Eq. (5.15) above. Similar comments also apply to the twisting of the $z$ lines. By considering

\[
\left[ 1 + \Theta^t(x_1) \right] R_1 \left[ 1 + \Theta(z_1) \right] : \left[ 1 + \Theta^t(x_2) \right] R_2 \left[ 1 + \Theta(z_2) \right]
\]

we also obtain an identity corresponding to the $x$ and $z$ lines being sign-naturized.

Consider the sequence of duality transformations indicated in Fig. 5.5

![Diagram](image)

**Fig. 5.5**

The first and last steps follow from trivial identities of the type

\[
\Theta(x) V_1 |0\rangle = V_2 |0\rangle ,
\]

and the others from our duality equation (5.15).

In the original multiperipheral configuration, we have Chan variables in the propagators of each internal line. Because of the above comment about the change of variables (5.15), we must have Chan variables in the propagators in each subsequent configuration. Therefore we have learnt how to exhibit the unintegrated duality relation between two different multiperipheral configurations (the first and last in Fig. 5.5) by repeated...
application of our duality relation Eq. (5.15). More generally, we can find duality identities relating any two planar tree diagrams for the same cyclic ordering, providing these have no general three-particle vertices. In all cases our propagators have the Chan variables as integration variables.

We can also dualize twisted lines occurring in loops in order to find the duality and periodicity properties which are important in establishing that there is no double counting.

It is interesting to see which are the duality conditions if we use \( x^L \) \( \Omega \) as the twisted propagator instead of \( x^L \) \( \Theta(x) \). Then the operators \( Q_1 \) and \( Q_2 \) corresponding to Figs. 5.3 (a) and 5.3 (b), respectively, are given by

\[
Q_1 = \Omega V(p_1) y_1^L \Omega V(p_2) z_1^L \int d\mu(y_1) d\mu(z_1)
\]

(5.19)

\[
Q_2 = V(p_2) y_2^L \Omega V(p_1) z_2^L \Omega \int d\mu(y_2) d\mu(z_2)
\]

(5.20)

It is also easy to prove that

\[ Q_1 \equiv Q_2, \]

provided

\[
y_2 = 1 - y_1 (1 - z_1) ; \quad z_2 = \frac{y_1 z_1}{1 - y_1 (1 - z_1)}
\]

(5.21)

which is a different duality transformation. Notice that not only the variable of the dualized line is transformed as in Eq. (5.16), but also the variable of one neighbouring line.

In any case, the fact that duality is true operatorially guarantees, for instance, the cyclic and periodicity properties of non-planar diagrams\( ^{24,25} \). These properties of non-planar one-loop diagrams are discussed in Chapter 6.
5.4 Gauge properties of the twisted propagator

Either by using the general method explained at the end of this note, or by using the well-known identity

\[(1 - f) W x^{L_0} = \left[ \frac{x(1-f)}{1-xf} \right]^{L_0} (1 - xf) W \]  

(5.22)

where \( f \) is an arbitrary parameter \( |f| < 1 \), one can prove that:

\[(1 - f) W^\alpha x^\dagger \Theta(x) d\mu(x) = y^\dagger \Theta(y) d\mu(y) \]  

(5.23)

with

\[ y = \frac{x(1-f)}{1-xf} \]

We see, therefore, that the twisted propagator \( x^{L_0} \Theta(x) \) satisfies, as an operator, the same gauge condition as every physical scalar state (i.e. any state obtained by applying on the vacuum a chain of vertices and propagators): the gauge operator \( (1-f)^W \) leaves it invariant, and represents only a change of integration variable which maps the interval \([0,1]\) onto itself. This implies that spurious states, usually defined by:

\[ \langle s \rangle = \langle \Phi | W \]  

where \( \langle \Phi \rangle \) is an arbitrary bra, are not only uncoupled to physical states, but are not propagated by the twisted propagator.

Indeed, the integrated twisted propagator:

\[ T = \int_0^1 d\mu(x) x^{L_0} \Theta(x) \]  

(5.24)

satisfies

\[ WVT = V(W - a) T = 0 \]  

\[ T(W^+ - a) = 0 \]  

(5.25)
as can be seen by choosing \( f \) in Eq. (5.23) to be infinitesimal, and inte-
grating both sides of the equation.

However, one must be careful in the definition and manipulation of
integrated operators. The operators appearing in multi-Veneziano theory
are singular at \( x = 1 \), due in particular to the infinite number of oscil-
lation modes, and therefore the integrals may be meaningless.

The correct -- and safe -- prescription is to perform all matrix ele-
ments and trace operations in the Fock space of all the oscillators before
integrating over the parameters, and to perform the integration last. This
point is particularly relevant for loops, because of the fact that inte-
grated traces have no meaning. The renormalization procedure of Neveu
and Scherk\textsuperscript{29}, which will be discussed in Section 7, introduces a counter-
term (interpreted as mass renormalization) in the non-integrated part of
the trace.

Therefore, expressions involving integrated operators, such as
Eqs. (5.24) or (5.25), must be interpreted in the sense that they are
satisfied by the integral of every matrix element in the Fock space.

Let us allow, however, some of these formal manipulations; if the
projection operator \( \mathcal{P} \) that projects out spurious states is defined by\textsuperscript{14,27} \n
\[
\mathcal{P} = 1 - (W^+ - a) \frac{1}{W(W^+ - a)} W,
\]

it is clear that

\[
\mathcal{T} = T \mathcal{P} = \int_0^\lambda \mu(x) x^{L_0} \Theta(x) \mathcal{P} = \mathcal{D} \mathcal{O} \mathcal{P}.
\]

Therefore an integrated twisted propagator \( \mathcal{T} \) coincides with the one used
in Refs. 24 and 25. Indeed, as noted by Kaku and Thorn\textsuperscript{24}, \( \mathcal{D} \mathcal{O} \mathcal{P} \) formally
satisfies hermiticity and double-twist invariance.

However, double-twist invariance of \( \mathcal{D} \mathcal{O} \mathcal{P} \), as well as the commutation
relations which are needed in order to get rid of the \( \mathcal{P} \)'s, are correct
only for the integrated expressions, which in loop calculations are mean-
ingless. The general procedure adopted in recent literature\textsuperscript{24,25} has
been to allow for such manipulations, and then at some stage to go back
to the non-integrated trace in order to be able to perform a renormalization which could give a meaning to the integral. We shall show that both the non-planar and non-orientable one-loop diagrams calculated with the aforementioned prescription lead to the same result as the one obtained with our twisted propagator, and the clear-cut prescription that the integration is done only after the trace is calculated (and renormalized).
6. LOOP CALCULATIONS

6.1 Introduction

We shall discuss here the calculation and general features of one-loop Feynman-like diagrams in the Generalized Veneziano Model. This discussion will be based on a general method developed by us. As we will see later, the main difficulty arises from non-planar loop diagrams and is related to the evaluation of the trace of a non-diagonal operator. By non-diagonal here we mean an operator which has a non-vanishing matrix element between different harmonic oscillator modes.

Non-planar loop diagrams have been evaluated in general by Kaku and Thorn\textsuperscript{24}, and Gross, Neveu, Scherk and Schwarz\textsuperscript{25} by means of an ingenious trick which diagonalizes the operator whose trace has to be evaluated. This trick, however, is valid only for the particular operator under consideration. Our method, on the contrary, allows us to compute the trace of an arbitrary non-diagonal operator. We shall therefore describe it here and we shall use it to derive the integral representation of one-loop diagrams, because it is more systematic and because we believe it will become important in future developments of the theory. We have in mind, in particular, the calculation of multi-loop diagrams.

6.2 Canonical forms

We have seen in previous sections that all the operators that occur in the theory except the vertex -- propagators, twisting and gauge operators -- are elements of an O(2,1) group with generators $L_0(\Pi)$, $L_{\pm}(\Pi)$, $\Pi_{\mu}$ being the momentum of the internal line with which they are associated. Up to now, we have written these operators in the form $e^{\alpha L_+}$, $e^{\beta L_0}$. We shall now discuss how to handle them in a different way, by writing them as normal-ordered products of creations and annihilation operators. It is shown in Appendix A that

$$\chi^H = \mathcal{e}^{\sum_n a^{(n)}_+ (x^n - 1) a^{(n)}_-}$$

so that we can write
\[ e^{\alpha L_\pi(\pi)} = e^{\frac{\alpha \pi^2}{2}} \sum_{n=1}^{\infty} a^{(n)} \cdot (e^{\pi L_\pi} - 1) a^{(n)} \quad \text{.} \]  

(6.1)

Next we write in a similar way \( e^{\alpha L_\pi} \). It is shown in Appendix B that

\[ e^{\alpha L_\pi(\pi)} = e^{\frac{\alpha \pi^2}{2}} \sum_{n,m=1}^{\infty} \tau_{nm} a^{(n)} a^{(m)} + \sum_{n,m=1}^{\infty} \tau_{nm} a^{(n)} (e^{\pi L_\pi} - 1) a^{(m)} + \sum_{n} B_n a^{(n)} \quad \text{.} \]  

(6.2)

where \( \tau \) is a numerical matrix defined in the space of oscillator modes as

\[ \tau_{nm} = \sqrt{m(n + 1)} \delta_{n,m+1} \quad \text{.} \]  

(6.3)

and

\[ B_n = \frac{\alpha^n}{\sqrt{n}} \quad \text{.} \]  

(6.4)

We have suppressed here Lorentz indices; the Lorentz index of in \( B_n \) is contracted with that of \( a^{(n)} \), and the Lorentz indices of \( a^{(n)^T} \) and \( a^{(n)} \) are contracted together in the normal-ordered part; \( (e^{G_\pi} - 1) \) being a numerical matrix. Clearly we can write

\[ e^{\alpha L_\pi(\pi)} = e^{\frac{\alpha \pi^2}{2}} \sum_{n,m} B_n a^{(n)^T} \sum_{n,m} a^{(n)^T} [e^{\pi L_\pi} - 1] a^{(m)} \quad \text{.} \]  

(6.5)

Equations (6.1), (6.2), and (6.5) allow us to write an arbitrary \( O(2,1) \) group element in terms of normal-ordered products of \( \varepsilon, a^\dagger \). It is crucial to remark the general structure of the vector \( B_n \) given by Eq. (6.4): it is always the tensor product of a four-vector and an infinite-dimensional vector with components \( \alpha^n / \sqrt{n} \), \( \alpha \) being the parameter of the group transformation. The previous results suggest that we should consider a general operator written in the following way, which we call a "canonical form":
\[ 0 = e^{Aa^+} \cdot e^a^{(C-1)a} \cdot e^{Ba} \]

where we have simplified the notation in an obvious way: \( a_{(n)}^{(n)} \) and all the summations have been suppressed. These canonical forms are very useful because they have trivial matrix elements between coherent states. Introducing the coherent state \( |\lambda\rangle \)

\[ |\lambda\rangle = \prod_{n=1}^{\infty} \prod_{\mu=1}^{4} |\lambda_{n,\mu}\rangle \],

one finds

\[ \langle \rho | 0 | \lambda \rangle = e^{A_{\rho}^*} e^{C_{\rho} \lambda} e^{B_{\lambda}} \]  \hspace{1cm} (6.6)

An important point to remark is that the product of two canonical forms is again a canonical form. The multiplication law is:

\[ O_1 O_2 = e^{A_1 a^+} a^{(c_1-1)a} B_1 a \cdot e \cdot e^{A_2 a^+} a^{(c_2-1)a} B_2 a \cdot e \]

\[ = e^{\varphi} e^{Aa^+} a^{(C-1)a} B_a \cdot e \]  \hspace{1cm} (6.7)

where

\[ \varphi = B_1 A_2 = \sum_{n=1}^{\infty} B_{1n} A_{2n} \],

\[ A = A_1 + C_1 A_2 \]

\[ B = B_2 + B_1 C_2 \]

\[ C = C_1 C_2 \]  \hspace{1cm} (6.8)

This formula is easily proved by taking matrix elements of both sides of Eq. (6.7) between arbitrary coherent states. The matrix elements are the same, and since coherent states form a complete set, the result
follows. The general multiplication law is easily understood. Consider the product \( O_1 O_2 \). We want to send all \( a^+ \) to the left, all \( a \) to the right; then

1) crossing \( e^{A_2a^+} \) through \( e^{B_1a} \) gives the phase factor \( e^{B_1A_2} = e^\Phi \);

2) crossing \( e^{A_2a^+} \) through \( e^{a^+(C_1-1)a} \) : "rotates" the vector \( A_2 \) and yields \( e^{C_1A_2a} \). The coefficient of \( a^+ \) on the left is therefore \( A_1 + C_1A_2 \).

3) Next we push \( e^{B_1a} \) to the right, and in crossing \( e^{a^+(C_2-1)a} \) : the vector \( B_1 \) is again "rotated" and that factor becomes \( e^{B_1C_2a} \). That yields \( B = B_1C_2 + B_2 \).

4) Finally, C-matrices are multiplied together.

Since the group elements we have discussed before are particular cases of canonical forms, an arbitrary group element will be again a canonical form because of the multiplication law. But the important point is that all the momentum dependence is trivially exhibited in the "outside" factors \( e^{Aa^+} \), \( e^{Ba} \); \( C \) being a momentum-independent matrix. Moreover, the vertex is also a canonical form, with

\[
A_n = -p \frac{i}{ln} \quad B_n = p \frac{i}{ln} \quad C = I ,
\]

so we can treat all the operators on an equal footing. Because \( C_v = I \), when we multiply operators the vertex does not contribute to the "inside" momentum-independent matrix multiplication. It only contributes to the "outside" momentum-dependent vectors.

Finally, if we want to multiply operators, we must learn to multiply the corresponding infinite-dimensional C-matrices appearing in the corresponding canonical forms. Since all group elements can be written in canonical form, it follows from their multiplication law that their C-matrices form an infinite-dimensional representation of \( O(2,1) \). In Appendix C we prove an important theorem that allows us to reduce all the manipulations one needs to do with C-matrices -- and in particular their diagonalization -- to equivalent manipulations with \( 2 \times 2 \) matrices.
Theorem. The matrices \( C_{nm} \) corresponding to \( O(2,1) \) group elements, when acting on vectors of the form \( \xi^n/\sqrt{n} \), generate a projective transformation on the parameter \( \xi \). More precisely

\[
\sum_{m=1}^{\infty} C_{nm} \frac{\xi^m}{\sqrt{m}} = \frac{[\gamma(\xi)]^n}{\sqrt{n}} - \frac{[\gamma(0)]^n}{\sqrt{n}} \quad (6.10)
\]

\[
\gamma(\xi) = \frac{a\xi + b}{c\xi + d}. \quad (6.11)
\]

In other words, vectors of the form \( \xi^n/\sqrt{n} \) are mapped into vectors of the same form, \( \alpha^n/\sqrt{n} \), with \( \alpha \) given by Eq. (6.11). The \( 2 \times 2 \) matrix

\[
\tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

completely determines the \( C \)-matrix. Indeed, Eq. (6.10) is valid for arbitrary \( \xi < 1 \), and therefore:

\[
C_{nm} = \frac{1}{m!} \frac{\partial^n}{\partial \xi^m} \left( \frac{[\gamma(\xi)]^n}{\sqrt{n}} \right) \bigg|_{\xi=0}. \quad (6.12)
\]

The term \( \gamma^n(0)/\sqrt{n} \) guarantees that \( C \) maps the null vector (\( \xi = 0 \)) into the null vector, and it appears because sums run from 1 to \( \infty \) and the term \( m = 0 \) is missing (the "zero" mode recently introduced by Fubini and Veneziano)\(^{27a}\). The normalization of \( \tau \) is irrelevant and therefore we require \( \det C = 1 \).

Suppose we have the product of two infinite-dimensional matrices, \( C_1 \) and \( C_2 \); which generate projective transformations \( \alpha_1 \) and \( \alpha_2 \), respectively, with \( 2 \times 2 \) matrices \( \tau_1, \tau_2 \). Using the previous result it is quite straightforward to show (see Appendix C) that Eq. (6.10) is valid for \( C = C_1 C_2 \) with \( \gamma(\xi) \) determined by:

\[
\tau = \tau_1 \tau_2. \quad (6.13)
\]

Therefore, the product of \( C_1 \) and \( C_2 \) is completely determined by the product of the corresponding \( 2 \times 2 \) matrices.
We want to emphasize another important point. We have seen that, in general, the A and B vectors appearing in $e^{+\alpha L_0} e^{+\alpha L_4}$ or the vertex have the general form $\Pi(\alpha^n/\sqrt{n})$, where eventually $\alpha$ can be equal to zero ($e^{-\alpha L_0}$) or 1 (vertex). The most general operator is a product of these, and in taking the product we must apply C-matrices to A and B vectors. However, the theorem previously quoted implies that the structure of A and B vectors is preserved, and if we multiply an arbitrary number of operators the final canonical form will have A and B vectors that will be linear combinations of terms of the form $p_i(\alpha_i^n/\sqrt{n})$.

In Table 1 we list the canonical form of the most important operators we have to deal with. By using them together with the product law (6.7) one can easily prove operator identities like (5.15).

<table>
<thead>
<tr>
<th>Operator</th>
<th>$A_n$</th>
<th>$B_n$</th>
<th>$\tau$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z^H\Omega(\Pi)(1-z)\Omega(\Pi)$</td>
<td>$-\Pi\cdot\frac{1}{\sqrt{n}}$</td>
<td>$-\Pi\cdot\frac{z^n}{\sqrt{n}}$</td>
<td>$(-z\ 0\ 0\ 1)$</td>
</tr>
<tr>
<td>$V(p)$</td>
<td>$-p\cdot\frac{1}{\sqrt{n}}$</td>
<td>$p\cdot\frac{1}{\sqrt{n}}$</td>
<td>$(1\ 0\ 0\ 1)$</td>
</tr>
<tr>
<td>$\Omega(\Pi)$</td>
<td>$-\Pi\cdot\frac{1}{\sqrt{n}}$</td>
<td>0</td>
<td>$(-1\ 1\ 0\ 1)$</td>
</tr>
</tbody>
</table>
6.3 Trace calculations

In doing a loop calculation we must compute the trace of a product of a number of vertices and twisted and/or untwisted propagators. For a proof of this statement, we refer the reader to Appendix D. Such a product of operators can always be written in canonical form, and therefore we consider next the problem of computing the trace of a canonical form. This is most easily done by computing it in the coherent state basis, using the completeness relation given by Eq. (A.6) for each harmonic oscillator mode. Let us neglect, for a moment, Lorentz indices. Then, we have, using \( |z\rangle = \prod_{i=1}^{\infty} |z_i\rangle \):

\[
\mathcal{T}_\nu \left[ e^{Aa^+} e^{a^+(c-1)a} e^{B\alpha} \right] = \prod_{h=1}^{\infty} \left( \frac{i}{\pi} \int d^2z_n \right) e^{-|z_n|^2} \langle z | e^{Aa^+ a^+(c-1)a} e^{B\alpha} |z\rangle \\
= \prod_{h=1}^{\infty} \left( \frac{i}{\pi} \int d^2z_n \right) e^{\sum_{n,m} \left[ A_n z_n^* + B_m z_m + z_n^* \left( c_{nm} - \delta_{nm} \right) z_m \right]} \\
= \frac{e^{-\left( \mathcal{B}, \frac{1}{(1-c)^{\mathcal{A}}} \right)}}{\det(1-c)} \tag{6.14}
\]

If we now include the trace over Lorentz indices and remember that \( C \) does not depend on them, the only difference is that \( [\mathcal{B}, (1 - C)^{-1} \mathcal{A}] \) has to be understood as including also the scalar products of the four vectors contained in \( \mathcal{B} \) and \( \mathcal{A} \); and \( \det (1 - C) \) has to be replaced by \( \left[ \det (1 - C) \right]^n \).

The theorem we have proved in the previous section immediately allows us to calculate the eigenvalues of an arbitrary matrix \( C \). Suppose that \( C_{nm} \) is diagonal with eigenvalues \( \lambda_n \); then Eq. (6.10) tells us that
\[ \lambda_n \frac{s^n}{\ln} = \frac{\alpha(\xi)}{\ln} , \quad (6.15) \]

and for this to be true for arbitrary \( \xi \) the matrix \( \tau \) itself must be diagonal i.e. \( b = d = 0 \); and

\[ \lambda_n = \left( \frac{a}{d} \right)^n , \quad (6.16) \]

so the \( n^{th} \) eigenvalue of \( C \) is given by the ratio of the two eigenvalues of \( \tau \) to the \( n^{th} \) power. This almost determines the eigenvalues of \( C \), the only ambiguity being whether \( a \) is the largest or the smallest eigenvalue of \( \tau \). Therefore, the eigenvalues of \( C \) are either all larger than 1 or all smaller than 1. They are uniquely determined provided we have in independent criterion to choose between the two possibilities. We shall prove in the next section that in all cases of interest the eigenvalues of \( C \) are less than one. Then, \( C \) is a Fredholm operator because

\[ T_n \left[ C^+ C \right] = \sum_{n=1}^{\infty} \left( \frac{\lambda_-}{\lambda_+} \right)^{2n} = \frac{\left( \frac{\lambda_-}{\lambda_+} \right)}{1 - \left( \frac{\lambda_-}{\lambda_+} \right)} < \infty \quad (6.17) \]

since \( \lambda_-/\lambda_+ < 1 \), \( \lambda_\pm \) being the eigenvalues of \( \tau \). The Fredholm determinant is given therefore by

\[ \det (1 - C) = \prod_{n=1}^{\infty} \left[ 1 - \left( \frac{\lambda_-}{\lambda_+} \right)^n \right] . \quad (6.18) \]

In order to complete the trace calculation we need to compute \( [B, (1 - C)^{-1} A] \) and this is trivial if we change basis in order to have a diagonal \( C \). Let us call \( 0 \) the infinite-dimensional matrix that diagonalizes \( C \), i.e.

\[ C = 0^{-1} C_d 0 , \quad (6.19) \]

and let us call $\sigma$ the $2 \times 2$ matrix that diagonalizes $\tau$

$$\tau = \sigma^{-1} \tau_d \sigma.$$

Then all our previous results immediately show that

$$\sum_m O_{nm} \frac{\xi^m}{\sqrt{n}} = \frac{\alpha(\xi)}{\sqrt{n}} - \frac{\alpha(0)}{\sqrt{v}}, \quad (6.20)$$

where

$$\alpha(\xi) = \frac{A\xi + B}{C\xi + D},$$

and

$$\sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \quad (6.21)$$

Then, $\sigma$ acting on vectors with component $\xi^m/\sqrt{m}$ generates a projective transformation determined by $\sigma$, the matrix that diagonalizes the $2 \times 2$ representation of $C$.

We can use next the Born series to compute $(1 - C)^{-1}$. Since the eigenvalues of $C$ are less than 1, we know it converges. Using $\omega = \lambda_-/\lambda_+$ we have

$$\tau_n \left[ e^{\lambda^-} e^{A^+ (c-1)} e^{B \lambda} \right] = \frac{1}{\prod_{n=1}^\infty (1 - \omega^n)^4} \prod_{n=0}^\infty \left( B, C_{\text{diag}}^N OA \right)^N (0, B, C_{\text{diag}}^N OA).$$

We know the projection transformation generated by $\sigma$. It is quite straightforward to show also that:
\[
\sum_{m=1}^{\infty} (C_{\xi}^{m+1})_{nm} \xi_m = \frac{\beta(\xi)}{\xi} - \frac{\beta(c)}{\xi^n} \tag{6.23}
\]

\[
\beta(\xi) = \frac{D\xi + C}{B\xi + A}
\]

Then, if the vectors A and B have the general form

\[
A_n = \beta \frac{\xi^n}{\xi^n}, \quad B_n = \gamma \frac{\xi^n}{\xi^n}, \tag{6.24}
\]

we have our final result:

\[
T \left[ e^{A^{\alpha+}} e^{(C-1)A} e^{Ba} \right] = \frac{1}{\prod_{n=1}^{\infty} (1 - \omega^n)^4} \prod_{n=1}^{\infty} \left[ \frac{(1 - \omega^n \alpha(\xi)) \beta(\xi)}{(1 - \omega^n \alpha(c)) \beta(c)} \right]^{-p,q} \tag{6.25}
\]

To summarize, we list below the rules for computing the trace of a product of operators.

1) Determine A and B for the product.

2) Determine the 2 \times 2 matrix \( \tau \) corresponding to the product by multiplying the individual matrices (see Table 1).

3) Diagonalize \( \tau \). This determines \( \omega = \lambda_/-\lambda_+ \) and the matrix elements of \( \sigma \); so \( \alpha(\xi) \) and \( \beta(\xi) \) are known.

4) The trace is given by Eq. (6.25).

6.4 Fredholm properties of C-matrices and divergences of loop diagrams

We shall start this section by proving our previous statement that in all cases of interest the eigenvalues of C(\( \alpha \)) are less than one. We proceed as follows:
i) Consider first the untwisted propagator $x^H$, $0 \leq x \leq 1$. In this case $C_{nm} = x^n \delta_{nm}$, so clearly the eigenvalues are less than 1 unless $x = 1$. At this point $C_{nm}$ is no longer a Fredholm matrix. $C_{nm}$ for an untwisted propagator is bounded by the unit matrix, even at the point $x = 1$.

ii) Consider next the Hermitian twisted propagator $x^H \Omega(1 - x)^W$. Let us denote the C-matrix corresponding to $\Omega(1 - x)^W$ as $K_{nm}(x)$. Then we write

$$C_{nm}(x) = x^n K_{nm}(x) = (A(x) B(x))_{nm}, \quad (6.26)$$

where

$$A(x) = \sqrt{x^n} \delta_{nm},$$
$$B(x) = \sqrt{x^n} K_{nm}(x). \quad (6.27)$$

We compute next the norms of $A(x)$ and $B(x)$. The norm of an operator is equal to its largest eigenvalue, therefore

$$|A(x)| = \sqrt{x}. \quad (6.28)$$

Next we compute the norm of $B(x)$, by using

$$|B(x)|^2 = \max_{\{\psi\}} \frac{(B\psi, B\psi)}{(|\psi|^2, \psi)}. \quad (6.29)$$

If $\psi$ is expanded in terms of unit vectors $e_m$

$$\psi = \sum_{m=1}^\infty C_m e_m,$$

$$|B(x)|^2 = \sum_{m=1}^\infty |C_m|^2 \frac{\sum_{n=1}^\infty x^n K_{nm}(x)}{\sum_{n=1}^\infty |C_n|^2} = \frac{\sum_{n=1}^\infty |C_m|^2 x^n}{\sum_{n=1}^\infty |C_n|^2}. \quad (6.30)$$
where in the last step we have used the double-twist invariance of the twisting operator, i.e.

\[
(l - x)^W \overrightarrow{\Omega} x^H \overrightarrow{2} (l - x)^W = x^H .
\]

Clearly, \( |B(x)| \leq 1 \) if \( 0 \leq x \leq 1 \). Then using Eq. (6.28) we get

\[
|C(x)| \leq |A(x)| \quad |B(x)| \leq 1 \quad \text{if} \quad 0 \leq x \leq 1 .
\]

Therefore, the eigenvalues of \( C(x) \) are less than one in the range \( 0 \leq x < 1 \). At the point \( x = 1 \), \( C(x) \) is still bounded by the unit matrix.

Now we can calculate the eigenvalues of \( C \). The \( 2 \times 2 \) matrix \( \tau \) in this case is

\[
\tau = \begin{pmatrix}
-x & x \\
-x & 1
\end{pmatrix} , \tag{6.31}
\]

with eigenvalues

\[
\lambda_{\pm} = \left( \frac{1 - x}{2} \right) \left[ \begin{array}{c}
1 \pm \Delta \\
0
\end{array} \right] \tag{6.32}
\]

\[
\Delta = \sqrt{1 + \frac{4x}{1-x}} \geq 1 . \tag{6.33}
\]

Therefore, the eigenvalues of \( C_{nm}(x) \) are given by

\[
\omega^n = \left( - \frac{\Delta - 1}{\Delta + 1} \right)^n = (-1)^n \left( \frac{\Delta - 1}{\Delta + 1} \right)^n . \tag{6.34}
\]

They are all less than 1 as in the case of the untwisted propagator, and \( \omega \to 1 \) when \( x \to 1 \). Therefore, as in the untwisted case, we have an accumulation of eigenvalues at the point 1 when \( x \to 1 \) and the operator ceases to be Fredholm (for a Fredholm operator, only zero can be an accumulation point of eigenvalues). But is is still bounded by the unit matrix at this point.
iii) It can also be shown that the eigenvalues of the non-hermitian twisting operator $x^H \Omega$ are less than 1.

iv) Finally, consider the general case of the loop diagram with $N$ external lines and $M$ twisted lines. Since the vertex does not contribute to the C-matrix multiplication:

$$C(x) = C_1(x_1) \ C_2(x_2) \ C_3(x_3) \ldots \ C_N(x_N)$$

where $C_i(x_i)$ can represent either a twisted or an untwisted propagator. If all $x_i$ are less than 1, all $C_i(x_i)$ are Fredholm and therefore $C(x)$ is Fredholm. If at least $x_i \neq 1$, but $x_j = 1$ ($j \neq i$), then $C_i(x_i)$ is Fredholm and $C_j(x_j)$ ($j \neq i$) are bounded operators. $C(x)$ is also Fredholm because the product of a Fredholm operator and a bounded operator is a Fredholm operator. Therefore, $C(x)$ is not Fredholm only when all $x_i = 1$, $i = 1, \ldots, N$. Therefore $\det (1 - C)$ does not exist at this point, and this is the origin of the well-known divergence of one-loop diagrams (either planar or non-planar).

6.5 Examples

6.5.1 The planar loop

As a first example we consider the case of the planar loop of Fig. 6.1:
where $\Pi$ is the loop momentum. The trace which we have to evaluate is

$$T_{\Pi} \left[ V(p_u) x_{N}^H V(p_{u-1}) x_{N-1}^H \cdots V(p_i) x_i^H \cdots V(p_1) x_1^H \right]. \quad (6.36)$$

We put the operator in Eq. (6.26) into canonical form and find:

$$\sigma = \begin{pmatrix} \omega & 0 \\ 0 & 1 \end{pmatrix} \quad \omega = x_1 \cdots x_N \quad (6.37)$$

$$A = -\sum_{i=1}^{N} \frac{p_i}{\Pi} \left( x_{u \cdots x_{i+1}} \right)^\infty = -\sum_{i=1}^{N} \frac{p_i}{\Pi} \left( \frac{\omega}{u_i} \right)^\infty \quad (6.38a)$$

$$B = \sum_{i=1}^{N} \frac{p_i}{\Pi} \left( x_{i \cdots x_1} \right)^\infty = \sum_{i=1}^{N} \frac{p_i}{\Pi} u_i^\infty, \quad (6.38b)$$

where we have defined $u_i$ to be:

$$u_i = x_1 \cdots x_i. \quad (6.39)$$

The phase factor is given by

$$\prod_{i > j} \left( 1 - x_i \cdots x_{i+1} \right)^{-p_i \cdot p_j} = \prod_{i > j} \left( 1 - \frac{u_i}{u_j} \right)^{-p_i \cdot p_j}. \quad (6.40)$$

Then, from Eq. (6.25), the trace is immediately found to be:

$$\prod_{i=1}^{\infty} \left( 1 - \omega^4 \right)^{-\left(4 + N \cdot m \right)} \prod_{i > j} \left( 1 - u_i u_j^{-1} \right) \prod_{i=1}^{\infty} \left( 1 - \omega^{r_i} u_i u_j^{-1} \right)^{-p_i \cdot p_j} \left( 1 - \omega^{r_i} u_i u_j^{-1} \right)^{-p_i \cdot p_j}. \quad (6.41)$$
This result is easily seen to agree with the one of Refs. 9 and 28. Note that the infinite product has the form of an elliptic $\Theta_1$-function, a fact of crucial importance in the renormalization programme of Neveu and Scherk\(^{26}\).

6.5.2 Non-planar loop with one twist

Let us now consider the non-planar loop of Fig. 6.2, in which the line corresponding to the integration variable $x_1$ is twisted:

![Diagram](image)

Fig. 6.2

The trace to be evaluated is:

$$\tau_n \left[ x_1^H \Theta(x_1, \tau_n) V(p_N) x_N^H V(p_{N-1}) x_{N-1}^H \cdots V(p_2) x_2^H V(p_1) \right]. \quad (6.42)$$

We bring the operator in Eq. (6.42) into canonical form and find

$$\tau = \begin{pmatrix} -z & x_1 \\ -z & 1 \end{pmatrix}, \quad z = x_1 \cdots x_N, \quad (6.43)$$

(we wish to keep the letter $\omega$ to denote the ratio of the eigenvalues of $\tau$):
\[ A_n = -\frac{x_i^N \pi}{1^n} - \sum_{i=1}^N p_i x_i^n \sum_{m=1}^N \Theta_{mn}(t) \frac{(x_N x_{i+1})^m}{m} \]  
(6.44a)

\[ B_n = -\frac{\pi}{1^n} \left( x_1 x_2 \cdots x_N \right)^n + \sum_{i=1}^N \frac{p_i}{\pi^n} \left( x_i x_2 \right)^n \]  
(6.44b)

The phase factor is given by:

\[ (1-x_1)^{-\frac{n}{2}} \prod_{i=1}^N \left( 1 - x_i x_{i+1} \right)^{\frac{n}{2}} \prod_{i>j}^N \left( 1 - x_i x_j \right)^{-\frac{n}{2}} \left( 1 - x_i x_j \right)^{\frac{n}{2}} \]  
(6.45)

The first step is to diagonalize the matrix \( \tau \). The ratio of the eigenvalues \( \lambda_- \) and \( \lambda_+ \) (\( \lambda_- < \lambda_+ \)) turns out to be:

\[ \omega = \frac{\lambda_-}{\lambda_+} = \frac{(1-z) - (1+z) \Delta}{(1-z) + (1+z) \Delta} \]  
\[ \Delta = \left[ 1 - \frac{4xz}{(1+z)^2} \right]^{1/2} \]

and \( \tau \) can be written as:

\[ \tau = K \begin{pmatrix} 1 & 1 \\ \frac{z}{1-\lambda_-} & \frac{z}{1-\lambda_+} \end{pmatrix} \begin{pmatrix} \omega & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{z}{1-\lambda_+} & -1 \\ -1 & \frac{z}{1-\lambda_-} \end{pmatrix} = K \tau^{-1} \tau_0 \tau \]  
(6.46)

where \( K \) is a normalization constant which does not play any role in the problem.

We have now to evaluate the expression:

\[ e^{(B_i C_d^g O_A)} = \prod_{i=0}^{\infty} \left( 1 - \frac{1}{i-c} \right) \]  
(\( B_i C_d^g O_A \))
so that we must compute $O_A$ and $O^{-1\dagger}_B$. We find:

$$
\sum_{m=1}^{\infty} O_{hm} A_m = - \pi \frac{1}{\ln} \left[ (-\omega)^n - (-1)^n \right] - 
\sum_{i=1}^{N} \frac{p_i}{\ln} \left[ (-\omega \nu_i^{-1})^n - (-\omega)^n \right] \quad (6.47)
$$

$$
\sum_{m=1}^{\infty} (O^{-1\dagger})_{hm} B_m = \sum_{i=1}^{N} \frac{p_i}{\ln} \left[ (-\nu_i)^n - (\omega \nu_i)^n \right]
$$

where we have set:

$$
u_i = \frac{u_i (1 + \omega) - (z + \omega)}{u_i (1 + \omega) - (1 + \omega z)} \quad (6.48)
$$

Collecting all factors together and using momentum conservation ($\sum_{i=1}^{N} p_i = 0$) we get for the trace:

$$
\left( \begin{array}{c}
-\frac{\pi^2}{\omega} \\
-\frac{1}{\ln} \end{array} \right) \frac{u}{\prod_{i=1}^{\pi} \left[ (1 - \frac{u_i u_i}{\nu_i})(\frac{\nu_i (1 - \nu_i)}{\nu_i - \nu_i \omega_z}) \right]} \prod_{i=1}^{\pi} p_i 
$$

$$
\times \prod_{r=1}^{\pi^2} (1 - \omega^r) \times \prod_{i=1}^{\pi^2} \left\{ (1 - u_i u_i)^{-1} - p_i p_i \right\}
$$

$$
\times \prod_{r=0}^{\infty} \left[ (1 - \omega^{r+1}) \nu_j \nu_j \right] \left[ (1 - \omega^{r+1}) \frac{\nu_j}{\nu_i} \right]^{-p_i p_i} \quad (6.49)
$$
Notice that again in Eq. (6.42) we find an elliptic $\Theta_1$-function. In order to make a comparison with the result of Gross, Neveu, Scherk and Schwarz\textsuperscript{25}, we have to perform the following change of variables:

$$x_i \rightarrow \nu_i.$$ 

After a rather lengthy calculation we find:

$$M_{\text{loop}} = \int d\nu_1 \ldots d\nu_N (1-\nu_N)^2 (1-\nu_N \nu_{\lambda})^{a-1} (-\omega)^{-\frac{\pi^2}{2}} \prod_{\ell=2}^{N} (\nu_{\ell} - \nu_{\ell-1})^{a-1} \prod_{j=1}^{N} (-\nu_{\ell})^{n_i} \prod_{k=0}^{\infty} (1-\omega^{5+1})^{-\frac{4+N\pi^2}{(4+N\pi^2)}} \prod_{j=1}^{N} \left\{ \prod_{r=0}^{\infty} \left[ (1-\omega^{-1} \frac{\nu_j}{\nu_i})^{n_i} (1-\omega^{-1} \frac{\nu_i}{\nu_j})^{n_j} \right]^{-rac{1}{2}} \prod_{i,j} \right\}. \quad (6.50)$$

6.6 General rules for one-loop diagrams

The result [Eq. (6.50)] just obtained is easily seen to agree with the one of Ref. 25, if one makes the change of variables:

$$x_i \rightarrow x_1 \ldots x_i$$

on their result. It should be remembered that their method consisted in using $D\Omega \; \mathcal{P} \text{[cf. Eq. (5.27)]}$ as the twisted propagator.

We will now prove that this fact is not a coincidence, and that all one-loop non-planar diagrams are the same whether they are calculated with our prescription or with that of Ref. 25.

If the parameter $f$ which appears in the gauge condition for the twisted propagator [Eq. (5.23)] is itself a function of the variable $x$ (or $y$), the gauge condition reads

$$x^{L_0} \Theta(x) d\mu(x) = \left[ 1 - f(y) \right]^{a-W} y^{L_0} \Theta(y) \left[ 1 + \frac{\partial f}{\partial y} \right] d\mu(y)$$

$$= \left[ 1 - f(y) \right]^{a-W} y^{L_0} \Theta(y) \left[ 1 + \frac{y(1-y)}{1-f(y)} \frac{df}{dy} \right] d\mu(y), \quad (6.51)$$
where, as before,

\[ x = \frac{y}{1 - f + yf} \quad \text{or} \quad y = \frac{(1 - f)x}{1 - fx} \quad (6.52) \]

It is now possible to prove that

\[
\prod_{i=1}^{N} \mu(x_i) x_1^{L_1} \Theta(x_1) V_1 \ x_2^{L_2} \left[ 1, \Theta(x_2) \right] V_2 \ldots x_N^{L_N} \left[ 1, \Theta(x_N) \right] V_N =
\]

\[
= \prod_{i=1}^{N} \mu(y_i) \left[ 1 - (-1)^M \prod_{i=1}^{N} y_i \right] \left( 1 + \prod_{i=1}^{N} y_i \frac{y_i}{1 - y_i} \right)^W
\]

\[
\times y_1^{L_1} \left[ 1, \Omega \right] V_1 \ y_2^{L_2} \left[ 1, \Omega \right] V_2 \ldots y_N^{L_N} \left[ 1, \Omega \right] V_N \times
\]

\[
\left( 1 + \prod_{i=1}^{N} y_i \frac{y_i}{1 - y_i} \right)^{-W}, \quad (6.53)
\]

\([1, \Theta(x_i)]\) and \([1, \Omega]\) mean that we must choose either the 1 or the \(\Theta\) and \(\Omega\) according to whether the corresponding propagator is twisted or untwisted. \(M\) is the number of twisted propagators; \(M > 1\) owing to the fact that in Eq. (6.53) there is at least the first propagator that is twisted. \(y_i\) is the variable corresponding to the last twist in the chain. The \(y_i\)'s are functions of the \(x_i\)'s; we do not need the explicit functional dependence, which in any case maps the 0-1 interval into itself. For the proof we refer to Ref. 23. The method is to use Eq. (6.51) for the first propagator in order to produce a factor \((1 - f(y_1))^W\), and then to use the gauge conditions (5.22) and (5.23) in order to pull to the end of the chain all \((1 - x_1)^W\) which appear in the \(\Theta(x_1)\). In this manner all \(\Theta\) are converted into \(\Omega\), and we will encounter at the end of the chain a function of the variables raised to the \(W\). The identification of this function
with \(1 - f(y_1)\) determines \(f(y_1)\) and therefore allows the explicit calculation of \((df(y_1)/dy)_x_1 \ldots x_N\) which appears in the first gauge condition used.

We therefore see from Eq. (6.53) that apart from the factor 
\((1 - (-1)^M \Pi_{i=1}^N y_i)\), the chain with \(\theta\) is related to the chain with \(\Omega\) by a similarity transformation. Taking traces

\[
\prod_{i=1}^N d\mu(x_i) \left[ x_1^{L_0} \Theta(x_i) V_1 x_2^{L_0} (1, \Theta(x_i)) V_2 \ldots x_N^{L_0} (1, \Theta(x_N)) V_N \right] =
\]

\[
= \prod_{i=1}^N d\mu(y_i) \left[ 1 - (-1)^M \Pi_{i=1}^N y_i \right] T_{\eta} \left[ y_1^{L_0} \Omega_1 V_1 y_2^{L_0} (1, \Omega) V_2 \ldots y_N^{L_0} (1, \Omega) V_N \right] .
\]  

\[
(M \geq 1)
\]

Now, the integral over the right-hand side of Eq. (6.54) is what is obtained with the method of Refs. 24 and 25. We would say that Eq. (6.54) justifies their result. Indeed in the method of Refs. 24 and 25, the way of obtaining the result -- and in particular to commute through and then eliminate the projector \(P\) -- is correct only for the integrated expression. Now this integrated trace has no meaning (it is infinite) and, as we will discuss in the next section, the renormalization procedure imposes the necessity of having a well-defined trace before the integration is made (or, equivalently, the possibility of defining a cut-off integral over the trace). This is not possible with the \(P\) method, except with the illicit trick of performing the integration, then commuting and formally calculating the trace, and then to undo the integral. The result for the integrand of the one-loop diagram is, however, correct as shown by Eq. (6.54).

Let us stress that the procedure which we have followed in this proof can be used beyond the trace calculation, without encountering the difficulty of eliminating \(P\)'s.
The planar loop diagrams are the only ones that pick up contributions of states uncoupled to scalars (spurious states). To include them is as arbitrary as to exclude them. The only appealing feature in including them is simplicity. Indeed, as shown here, their exclusion is not needed in order to define twisted dual propagators, and, as recently shown by Bouchiat, Gervais and Sourlas, the conserved current defined by the theory is also conserved for matrix elements between spurious states.

Lastly, the theory without projection operators has an infinite parameter group structure -- discussed in Appendix C -- which can be a clue to a possible field theoretical formulation.

Having reduced the traces containing $\Theta$ to the ones with $\Omega$, we can refer to the work of Refs. 24 and 25 to eliminate the $\Omega$ from the traces. The rules for constructing the amplitude corresponding to a general one-loop diagram with $M$ arbitrarily placed twisted lines, are the following:

i) For every internal line write a factor $\alpha^i \alpha^{-\alpha(k_i)}$, where $k_i$ is the momentum of that line.

ii) Write the trace of the expression obtained by putting $V(p_i)$ for every vertex, $x_i^H$ for every untwisted line, and $(-x_i)^H$ for every twisted line.

iii) Write a factor $(1 - (-1)^M \prod_i x_i)$ for $M \geq 1$ and 1 or $(1 - \prod_i x_i)$ for $M = 0$ depending on whether spurious states are kept or are eliminated.

iv) For every factor $1 - c$ that we will discuss next, write $\Theta(1-c)(1-c)^{a-1}$. We have a factor $(1 - x_i)$ for each untwisted line and a factor $(1 - x_i x_{i+1} ... x_j)$, where $i$ and $j$ refer to two successive twisted lines with untwisted lines in the middle. In the particular case of just one twisted line, the factor becomes $(1 - \prod_i x_i)$.

v) The integration over the parameter $x_i$ between 0 and the limit implied by the $\Theta$ function is intended as well as the integral over the loop momentum. However, the renormalization must be performed before computing the integral over the parameters. We will discuss the renormalization for the planar loop in the next section.

*) $\Theta$ is the step function.
6.7 **Properties of orientable non-planar diagrams**

Non-planar loop diagrams are classified as orientable or non-orientable according to whether there is an even or odd number of internal twisted lines. The terminology "orientable" and "non-orientable" refers to the topology of the surface of the associated dual diagram\(^1\). Orientable non-planar diagrams have some novel features which we want to discuss now.

1) Consider a box diagram with two consecutive twisted lines, or, which is the same, with a "twisted vertex" of momentum \(k_4\). We shall call \(\Pi\) the loop integration momentum. The amplitude is written as

\[
M = \int d^4 \Pi \ M(\Pi), \quad (6.55)
\]

and from now on we shall consider the integrand \(M(\Pi)\), displayed in Fig. 6.3 (a). We have seen in Section 5 that it is possible to generate certain duality transformations in a diagram with two consecutive twisted lines which are explicitly displayed in Fig. (5.3) and which imply changes of variables given by Eq. (5.16). We shall therefore perform three successive duality transformations, as indicated in Fig. 6.3:

![Diagram](image)

**Fig. 6.3**
We have indicated in each box diagram the momentum of only one internal line, the momentum of the others being determined by momentum conservation. In going from (a) to (b) we dualized the line between 4 and 1; in going from (b) to (c) we dualized the line between 4 and 2; and finally, in going from (c) to (d) we dualized the line between 4 and 3. If we compare the diagrams (a) and (c) we find that they are the same (the order of the external lines is the same) except that $\Pi$ has been replaced by $\Pi - k_4$. Therefore, we conclude that

$$M(\Pi) = M(\Pi - k_4),$$

and it means that the integrand $M(\Pi)$ is a periodic function of $\Pi$ with period equal to the momentum of the vertex between the twisted lines. It can also be shown in a similar way that if there are several "twisted vertices", the period of $M(\Pi)$ is the sum of the momenta of all "twisted vertices". Since $M(\Pi)$ is periodic, the integral in Eq. (6.55) does not make sense; it diverges in a trivial way. Clearly in this case the naive application of the tree theorem (see Appendix D) does not make sense, and the orientable non-planar loop diagrams must be defined in a different way.

In order to understand this difficulty better, let us pause for a moment to consider the duality properties of the planar loop diagram. Dualizing different lines we get the set of duality transformations described below:

**Fig. 6.4**
From the point of view of Feynman diagrams, they are all different. They exhibit different singularities: the diagram Fig. 6.4 (a) has normal thresholds in the s- and t-channels, whereas the diagram of Fig. 6.4 (c) has a double pole in the s-channel, and so on. However, the duality properties of our diagrams mean that they are all the same in the sense that once we have obtained the amplitude of Fig. 6.4 (a), that amplitude also contains the singularities exhibited in Figs. 6.4 (b) ... (c) .... Contrary to Lagrangian field theory, diagrams connected by duality transformations must not be added (as in the original Veneziano formula).

Therefore, duality transformations on the planar box diagram exhibit always different singularities. However, in the orientable non-planar box diagram we have considered, a sequence of three duality transformations gives back the same diagram -- and therefore the same singularities -- at a different value of the momentum \( \Pi \). Since \( \Pi \) is an integration variable, the singularity is counted twice -- or better, an infinite number of times. We are facing a paradox here: duality leads to multiple-counting! The correct way out of this difficulty has been suggested in Ref. 24, and is the following: the amplitude has to be defined as:

\[
M = \int_{\text{one period}} d^{4} \Pi \, M(\Pi),
\]

so the integration is restricted to one period in \( \Pi \); duality takes care that the singularities occur for all the other values of \( \Pi \) which are not integrated over. Indeed, Kaku and Thorn\(^{24}\) have checked in detail that one gets in this case the correct imaginary parts. An alternative procedure suggested by Gross, Neveu, Scherk and Schwarz\(^{25}\) consists of keeping the infinite range of the loop-momentum integration and restricting the range of the integrals over the parameters \( x_{i} \). This is more desirable because in this case the loop momentum integration can be done analytically -- it is always of Gaussian form -- which in turn is a crucial step for the renormalization program. The fact that this procedure leads to the correct discontinuities for the one-loop diagrams is explicitly shown in Chapter 8.
2) Consider next the following orientable non-planar box diagram:

![Box Diagram](image)

Fig. 6.5

and let us look first at the singularities in the s-channel. If we cut the diagram along line A we find an infinite set of two-particle intermediate states (each internal line corresponds to an infinite tower of resonances). The discontinuities across the cuts are given by the product of two (s,u) Veneziano functions (see Section 3), which, from the point of view of the t-channel, are the ones that, loosely speaking, contain third-double spectral function effects. Therefore, the diagram we are considering contains second-order effects of the third-double spectral function in the t-channel.

When the λ-plane singularities of the (s,u) Veneziano amplitude are analysed in the t-channel, one finds fixed poles at the nonsense integers of the wrong-signature (−1, −3, ...) arising, as is well known, from t.d.s.f. effects. Fixed poles in the angular momentum plane are incompatible with unitarity, but this is not surprising since in any case the Born approximation is always incompatible with unitarity. When unitarity corrections are included, either they become moving poles, or Regge cuts appear with the property that they overlap the right-hand cut when λ reaches a wrong-signature nonsense integer in such a way that the usual unitarity equations break down and the existence of the fixed poles is therefore reconciled with unitarity. Kikkawa has shown that this is precisely the case of the diagram we are considering: in the t-channel
there are Regge cuts with precisely the right properties to shadow the fixed poles of the \((s,u)\) Veneziano amplitude.

But the most interesting peculiarity of this diagram is the \(s\)-channel singularities\(^{30}\). In terms of duality diagrams the diagram we are considering can be displayed as in Fig. 6.6 (a), whose content, as far as quantum numbers are concerned, is the same as that of Fig. 6.6 (b):

![Diagram](image)

**Fig. 6.6**

Therefore, one can see that there are no quark-antiquark pairs propagating in the \(s\)-channel, i.e. the \(s\)-channel is the one with the "vacuum" quantum numbers. We shall not make here a detailed analysis of this diagram, but rather refer to the paper of Gross, Neveu, Scherk and Schwarz\(^{25}\). It turns out that the diagram is convergent when \(s < -4/3\), and that the leading singularity is a logarithmic cut with a fixed intercept of \(1/3\) and a slope equal to half the slope of the input Regge trajectories\(^{31}\). By fixed intercept we mean that \(\alpha_c(0) = 1/3\) is independent of the intercept of the input Regge trajectories.

The features of this singularity make it very tempting to identify it with the Pomeranchuk singularity. It is present in the vacuum channel, its slope compares favourably with the current experimental
situation, and the fact that it is dual to a Regge cut in accordance with
the Harari-Freund conjecture\textsuperscript{32}) are certainly very encouraging results.
However, one must be rather cautious at this stage. The complete eluci-
dation of this phenomenon must await the solution of the problem of higher-
order corrections. Hopefully, they could shift the intercept to the right
one, but one must also remember that singularities that do not correspond
to any kind of normal threshold have begun to show up in the theory and
that it is not at all self-evident that they will occur only in the right
channels.
7. RENORMALIZATION OF ONE-LOOP DIAGRAMS

7.1 Divergence of the planar loop and introduction of a cut-off

When the correct expression for the one-loop planar diagram was first obtained, one immediately noticed that it was badly divergent\(^9,28\). Indeed, let us recall here for convenience the expression for the planar loop derived in the previous section (Fig. 7.1):

\[ L(\alpha, \omega) = \frac{g^N}{\pi} \int \prod_{i=1}^N dx_i \prod_{i=1}^N dx_i \cdot \frac{N^2}{(1-x_i)^{\alpha-1} - \alpha(\bar{n}_i^2)^{\alpha-1}} \]

\[ \prod_{\omega = 1}^\infty (1 - \omega^n)^{-\frac{1}{4}} \prod_{\omega = 0}^{\infty} (1 - \omega^n x_{i,j})^{-\frac{1}{4}} \cdot p_i \cdot p_j \]  

(7.1)

where

\[ \omega = \prod_{i=1}^N x_i \]

\[ x_{i,j} = x_{i+1} x_{i+2} \cdots x_j \]
and $g$ is the coupling constant. The loop amplitude $L(\alpha_{ij})$ is a function of the Regge trajectories $\alpha_{ij}$ in all planar channels (i.e. channels formed with a bunch of adjacent particles):

$$\alpha_{ij} = a + \frac{1}{2} \left( p_i + p_{i+1} + \ldots + p_{j-1} + p_j \right) = a - \frac{1}{2} S_{ij}$$

If we eliminate the so-called spurious states, we have to multiply the integrand in Eq. (7.1) by the factor $(1 - \omega)$. However, this factor is completely irrelevant to what follows.

The divergence arises when all $x_i$'s tend to one, which means that it comes from the point $\omega = 1$. It is easy to convince oneself that the terms raised to the power $-p_i \cdot p_j$ give a regular contribution, due to momentum conservation; but, on the other hand, the factor:

$$\prod_{n=1}^{\infty} (1 - \omega^n)^{-4}$$

behaves as

$$\exp \left( \frac{2 \pi^2}{3(1 - \omega)} \right)$$

when $\omega \to 1$, as was shown in Eq. (2.15). Because of this factor, which comes from the large degeneracy of levels, the integral (7.1) has an exponential divergence at the point $\omega = 1$.

On the other hand, the $d^n\Pi$ integration is perfectly well behaved, if we assume that it is defined only after the Wick rotation has been performed. Then it reduces to a trivial Gaussian integral.

If we want to give a meaning to the KSV programme, we are thus faced with the problem of renormalizing the divergence in the loop diagram: this important step was accomplished in a beautiful paper by Neveu and Scherk\textsuperscript{26}.

We first have to give some meaning to the integral in Eq. (7.1), and the obvious thing to do is to put a cut-off, by multiplying the integrand by the factor\textsuperscript{28}.
\[ \Theta(N - \varepsilon - \sum_{i=1}^{N} x_i) \quad 1 \gg \varepsilon > 0 \]

where \( \Theta \) is the step function.

The crucial point here is that this cut-off does not change the singularity structure of the loop, because there is no Feynman diagram with singularities arising from the point \( \omega = 1 \). Indeed, considering for the sake of definiteness the box diagram of Fig. 7.2 and its dual variations, we see that there is no singularity coming from the point \((1,1,1,1)\):

(a) \hspace{2cm} (b) \hspace{2cm} (c) \hspace{2cm} (d)

(a) = box \hspace{1cm} (b) = vertex \hspace{1cm} (c) = self-energy \hspace{1cm} (d) = tadpole
(0,0,0,0) \hspace{1cm} (1,0,0,0) \hspace{1cm} (1,0,1,0) \hspace{1cm} (1,1,1,0)

Fig. 7.2: \( \tilde{x}_i \) = line dual to line \( x_i \).

One sees from Fig. 7.2 (d) that there is no line dual to \( x_4 \). On the contrary, the point \((1,0,1,0)\), for instance, in Fig. 7.2 (c), remains in the region of integration, and thus the singularities coming from this graph are taken into account.
It is important to remark that once we have introduced a cut-off, we can no longer use the commutation relations of operators like $\hat{W}$ with the integrated form of the propagator. These commutation relations are needed if one uses the projection operator $P$ [defined in Eq. (5.26)] in order to eliminate spurious states. On the other hand, if we use (in non-planar loops) propagators like $x^{L_0} \Theta(x)$, everything is perfectly well defined, since we can perform all manipulations on the integrand, and do the integration only at the end.

Before we come to technical details, let us explain briefly the basic ideas of the renormalization procedure. We have defined a loop amplitude $L_\varepsilon(\alpha_{ij})$, depending on the cut-off $\varepsilon$, and we look for a counter-term $\tilde{L}_\varepsilon(\alpha_{ij})$. We would like this counter-term to satisfy the following properties:

1) First, we want of course to get a finite result when the cut-off is removed (i.e. $\varepsilon = 0$):

$$\lim_{\varepsilon \to 0} \left[ L_\varepsilon(\alpha_{ij}) - \tilde{L}_\varepsilon(\alpha_{ij}) \right] \text{ is finite}$$

2) Secondly, we want to be sure that the result is crossing symmetric and dual, so that we require $\tilde{L}_\varepsilon$ to be crossing symmetric and dual for $\varepsilon$ sufficiently small.

3) Since we want to preserve the unitarity cuts, we may expect that the counter-term can be written as a sum of poles in any planar channel (which ensures duality). Because the planar loop can be reduced to a self-energy graph in any planar channel $(i,j)$ (Fig. 7.3):

![Fig. 7.3](image-url)
the counter-term should indeed be a sum of single and double poles, and could then be interpreted as mass renormalization.

4) Finally, we want to have Regge behaviour:

\[
\lim_{\varepsilon \to 0} \left[ L_{\varepsilon} (\alpha_{ij}) - \tilde{L}_{\varepsilon} (\alpha_{ij}) \right]
\]

reggeizes when \( s_{ij} \to \pm \infty \).

We are now going to show that a counter-term with properties (1) to (4) can be explicitly constructed.

7.2 Expression of the loop in terms of elliptic functions

Let us first examine the term in Eq. (7.1) which comes from the trace calculation, namely:

\[
T = \prod_{n=1}^{\infty} (1 - \omega^n)^{-4} \prod_{n=0}^{\infty} \prod_{i,j} \left( 1 - \omega^n x_{ij} \right)^{-\rho_i \cdot \rho_j} \tag{7.2}
\]

We call \( P(\sqrt{\omega}) \) the partition function

\[
P(\sqrt{\omega}) = \prod_{n=1}^{\infty} \left( 1 - (\sqrt{\omega})^{2n} \right) = \prod_{n=1}^{\infty} (1 - \omega^n)
\]

and remark that:

\[
x_{ij} \equiv x_{i+1} \cdots x_j = \omega / x_{j+1} \cdots x_i = \omega / x_{j+1} \tag{7.3}
\]

We use Eq. (7.3) and momentum conservation in the form:

\[
\prod_{i < j} b_{\rho_i \cdot \rho_j} = b^{-N m^2/2}
\]

to rewrite Eq. (7.2) as:

\[
T = \left[ P(\sqrt{\omega}) \right]^{-4} \prod_{i < j} \left\{ (1 - x_{ij}) \cdot \prod_{n=1}^{\infty} (1 - \omega^n)^{-2} (1 - \omega^n x_{ij}) (1 - \omega^n / x_{ij}) \right\}^{-\rho_i \cdot \rho_j} \tag{7.4}
\]
One essential point in the analysis of Neveu and Scherk is to remark that the infinite product in Eq. (7.4) has the form of an elliptic function \( \Theta_i(z|\tau) \) defined by

\[
\Theta_i(z|\tau) = 2 P(\sqrt{\omega}) \omega^{\frac{1}{8}} \sin z \prod_{n=1}^{\infty} \left( 1 - \omega^n e^{2i\pi} \right) \left( 1 - \omega^n e^{-2i\pi} \right)
\]

(7.5)

where:

\[
\sqrt{\omega} = e^{i\pi \tau}
\]

Clearly, if we set

\[
\tau = \ell_n \omega / 2i \pi \quad z_{ij} = \ell_n x_{ij} / 2i
\]

(7.6)

we have:

\[
T = \left[ P(\sqrt{\omega}) \right]^4 \prod_{i \neq j} \left\{ -i \sqrt{x_{ij}} \Theta_i(z_{ij}|\tau) \right\} \frac{-i \sqrt{x_{ij}}}{\omega^{\frac{1}{8}} \left( P(\sqrt{\omega}) \right)^3}
\]

(7.7)

Finally, it is convenient to transform the denominator in Eq. (7.7) by using

\[
\Theta_i'(0|\tau) = 2 \omega^{\frac{1}{8}} \left( P(\sqrt{\omega}) \right)^3
\]

and we get the formula which is the starting point of the analysis:

\[
T = \left( P(\sqrt{\omega}) \right)^4 \prod_{i \neq j} \left\{ -2i \sqrt{x_{ij}} \Theta_i(z_{ij}|\tau) \right\} \frac{-i \sqrt{x_{ij}}}{\Theta_i'(0|\tau)}
\]
or

\[
\mathcal{T} = \left( \mathcal{P}(\sqrt{\omega}) \right)^{-4} \left( \frac{\omega}{x_1} \right)^a \left( \prod_{i=2}^{N-1} x_{i+1} \right) \prod_{i<j} \left\{ -\frac{\Theta_i(2z_{ij}|\tau)}{\Theta'_{ij}(0|\tau)} \right\}^{-\mathbf{p}_i \cdot \mathbf{p}_j}
\]

(7.8)

At this point it is convenient to perform the loop momentum integration

\[
\int d^4 \mathbf{n} \prod_{i=1}^{N} x_i^{-\omega(\mathbf{n}^2) - 1} = \frac{4n^2}{\omega \ln^2 \omega} x_1^{-a} \left( \prod_{i=2}^{N-1} x_{i+1} \right) \prod_{i<j} e^{-\frac{\mathbf{p}_i \cdot \mathbf{p}_j}{2 \ln \omega}}
\]

(7.9)

Combining Eqs. (7.8) and (7.9) we get for the loop amplitude *):

\[
\mathcal{L}^\omega(\mathbf{n}_i, \mathbf{n}_j) = 4n^2 \ln^2 \omega \int d^4 \mathbf{n} \prod_{i=1}^{N} \left\{ d\mathbf{x}_i \left( 1 - x_i \right)^{a-1} \right\} \frac{\omega^{a-1}}{\ln^2 \omega} \left( \mathcal{P}(\sqrt{\omega}) \right)^{-4} \prod_{i<j} \left\{ e^{-\frac{\mathbf{p}_i \cdot \mathbf{p}_j}{2 \ln \omega}} \frac{\Theta_i(2z_{ij}|\tau)}{\Theta'_{ij}(0|\tau)} \right\}^{-\mathbf{p}_i \cdot \mathbf{p}_j}
\]

(7.10)

7.3 Definition of the counter-term

The second step in the analysis is to use a transformation property of the elliptic function \( \Theta_1(z|\tau) \), in order to analyze its behaviour near the interesting point \( \omega = 1 \):

\[
\Theta_1(z|\tau) = e^{(-i\tau)^{1/2}} e^{-i\pi z^2/\tau} \Theta_1 \left( \frac{z}{\tau} \left| -\frac{1}{\tau} \right. \right)
\]

(7.11)

* In Eq. (7.10) and subsequently, every time we write a divergent integral, it must be understood that the integrand is multiplied by the cut-off factor.
It is convenient to introduce the variables $R$ and $\Theta_{ij}$ which occur naturally in the definition of $\Theta_{i}(\frac{z}{r} \mid - \frac{1}{r})$:

$$R = e^{-\eta / \tau} = e^{2\eta^2 / \eta \omega} \quad (7.12)$$

$$\Theta_{ij} = \frac{2\eta \ln x_{ij}}{\eta \omega} = \frac{2 \bar{z}_{ij}}{\tau}, \quad 0 \leq \Theta_{ij} \leq 2\pi \quad (7.13)$$

One notices that $R$ tends exponentially to zero when $\omega \to 1$, since

$$e^{2\eta^2 / \eta \omega} \sim e^{-2\eta^2 / (1-\omega)} \quad \omega \to 1$$

while the angles $\Theta_{ij}$ have a clear geometrical interpretation: put $N$ points $(1, \ldots, i, \ldots, j, \ldots, N)$ in natural ordering on the unit circle, then $\Theta_{ij}$ is the angle between the points $i$ and $j$ (Fig. 7.4)

![Fig. 7.4](image_url)

Indeed we can define the angle $\Theta_{i}$ as

$$\Theta_{i} = \Theta_{N,i} = \frac{2\eta \ln (x_{1} \ldots x_{i})}{\eta \omega} \begin{cases} 0 \leq \Theta_{i} < 2\pi \\ \Theta_{i} < \Theta_{j} \text{ if } i < j \end{cases}$$
Then:

$$\theta_{ij} = \theta_j - \theta_i$$

The variables \( \Theta \) and \( R \) have an interesting physical interpretation in the analogue models of Fairlie and Nielsen, and Susskind\(^{3+} \). For instance, in the first model, the loop amplitude is related to the current distribution in an annulus with external radius \( R \) and internal radius \( r \), the position of the electrodes being given by the angles \( \Theta_i \):

![Diagram](image)

Fig. 7.5

Let us now come back to Eq. (7.11). By differentiating both sides with respect to \( z \), we get a useful relation which allows a precise determination of the behaviour of the partition function near \( \omega = 1 \):

$$\Theta'(\omega; z) = 2 \left( -\frac{1}{\Theta \omega} \right)^{\frac{3}{2}} P^3(R) R^{\frac{N}{4}}$$

from which we derive:

$$\frac{4n^2}{e^{n^2 \omega}} P^{-n}(\sqrt{\omega}) = \omega^{\frac{N}{6}} R^{-\frac{N}{3}} P^{-4}(R) \quad (7.14)$$

This last equation shows that \( P^{-n}(\sqrt{\omega}) \) behaves like:
\[ R^{-\frac{1}{3}} \sim e^{2n^2/3(1-\omega)} \]

when \( \omega \to 1 \) (as was already known) and gives rise to the exponential divergence. Using Eqs. (7.11) and (7.14) we can bring the last factor in Eq. (7.10) into the form:

\[
\prod_{i<j} \left\{ -\frac{\ln \omega}{n} \sin \left( \frac{\Theta_{ij}}{2} \right) \prod_{n=1}^{\infty} \frac{(1 - R^{2n} e^{i \Theta_{ij}})(1 - R^{2n} e^{-i \Theta_{ij}})}{(1 - R^{2n})^2} \right\}^{-\rho_i \cdot \rho_j}
\]

\[ = \prod_{i<j} \psi_{ij}^{-\rho_i \cdot \rho_j} \] (7.15)

As a result of these manipulations, we can now obtain an extremely good estimate of Eq. (7.15) near \( \omega = 1 \), since the function \( \tilde{\psi}_{ij} \), defined by:

\[ \tilde{\psi}_{ij} = -\frac{\ln \omega}{n} \sin \left( \frac{\Theta_{ij}}{2} \right) \] (7.16)

differs from \( \psi_{ij} \) by terms of order \( R^2 \). Since \( L \) diverges as \( R^{-\frac{1}{3}} \), if we define a counter-term \( \tilde{L} \) by replacing in \( L \psi_{ij} \) by \( \tilde{\psi}_{ij} \), \( L - \tilde{L} \) will be convergent when the cut-off is removed. The first property we required for the counter-term is then satisfied.

7.4 Duality properties of the counter-terms and discussion of the renormalization

Let us write the counter-term explicitly, as a function of the trajectories \( \alpha_{ij} \):
where $F_N(x_i)$ is independent of the $\alpha_{ij}$'s. We remark that the term $T_{ij}$ which is raised to the power $-\alpha_{ij}$ in Eq. (7.17) is nothing but the anharmonic ratio of the points $(e^{i\theta_{ij}}, e^{i\theta_{i-1}}, e^{i\theta_{j}}, e^{i\theta_{j+1}})$ on the unit circle. From the work of Koba and Nielsen we know that the $T_{ij}$'s satisfy the duality conditions:

$$
\bar{T}_{ij} = 1 - \prod_{i,j} T_{ij}^{-\alpha_{ij}}
$$

(7.18)

where $\prod_{ij}$ means that we take the product over all channels dual to the channel $(ij)$. Equation (7.18) is crucial for what follows.

From now on we concentrate on the case $N = 4$, namely the box diagram with four incoming particles (Fig. 7.2). The amplitudes $L$ and $\tilde{L}$ will be functions of the trajectories $\alpha(s)$ and $\alpha(t)$:

$$
\alpha(s) = a + \frac{1}{2} (p_1 + p_2)^2 \quad \alpha(t) = a + \frac{1}{2} (p_2 + p_3)^2
$$

and we have for instance for $\tilde{L} [\alpha(s), \alpha(t)]$ (recalling that $\Theta_{ij} = \theta_j - \theta_i$):
\[ L(\alpha(s), \alpha(t)) = g^4 \sum_{i=1}^{4} \int_{0}^{\tilde{\alpha}(s)} \int_{0}^{\tilde{\alpha}(t)} dx_i \cdot F_i (x_i). \]

\[
\begin{pmatrix}
\sin(\theta_1 - \theta_i)/2 & \sin \theta_i/2 \\
\sin(\theta_1 - \theta_i)/2 & \sin \theta_i/2
\end{pmatrix}^{-\alpha(s)}
\begin{pmatrix}
\sin(\theta_1 - \theta_i)/2 & \sin \theta_i/2 \\
\sin(\theta_1 - \theta_i)/2 & \sin \theta_i/2
\end{pmatrix}^{-\alpha(t)}
\]

(7.19)

We call \( T_s \) and \( T_t \) the terms raised to the power \(-\alpha(s)\) and \(-\alpha(t)\), respectively. The duality relation (7.18) reduces to:

\[ T_s = 1 - T_t \]  

(7.20)

which can be checked quite easily. We now define new variables \( z_2 \) and \( z_4 \) by:

\[ z_2 = \frac{\sin(\theta_1 - \theta_i)/4 \cdot \sin \theta_i/4}{\sin(\theta_1 - \theta_i)/4 \cdot \sin \theta_i/4} \]

\[ z_4 = \frac{\cos(\theta_1 - \theta_i)/4 \cdot \cos \theta_i/4}{\cos(\theta_1 - \theta_i)/4 \cdot \cos \theta_i/4} \]  

(7.21)

so that obviously:

\[ T_s = z_2 z_4 \quad T_t = 1 - z_2 z_4. \]

One can also immediately check that:

\[ 0 \leq z_2 \leq 1, \quad 0 \leq z_4 \leq 1 \]

so that we can make in Eq. (7.19) the change of variables:

\[ (x_1, x_2, x_3, x_4) \rightarrow (x_1, z_2, x_3, z_4). \]

We finally expand the integrand in Eq. (7.19), and particularly \( T_t^{-\alpha(t)} \), in powers of \( z_2 \) and \( z_4 \), and integrate over these variables getting:
\[ \tilde{L}(\alpha(s), \alpha(t)) = g^6 \sum_{n, p, q > 0} \frac{1}{n + p - \alpha(s)} \frac{1}{n + q - \alpha(s)} \]

\[ \frac{n!(\alpha(t) + n + 1)}{n!(\alpha(t) + 1)} \int_0^1 dx_1 dx_3 \tilde{j}_{\rho q}(x_1, x_3) \]  

(7.22)

This last equation displays the duality and crossing symmetry properties of the counter-term \( \tilde{L} \), since it shows that \( \tilde{L} \) can be written as a sum of single and double poles in either the s- or t-channels, the residues being polynomials in the other variable.

The physical interpretation of the counter-term is as follows: consider the four-point function \( A \) to fourth order in the coupling constant \( g \) (Fig. 7.6):

![Fig. 7.6](image)

The first term in Fig. 7.6 is, of course, the usual four-point Veneziano function, written as a sum of s-channel poles. We add and subtract the counter term \( \tilde{L} \) (Fig. 7.7)
\[
A = \left\{ \begin{array}{c}
(g^2) \\
(g^4) \\
(g^4) \\
(g^4)
\end{array} \right\} + \left\{ \begin{array}{c}
(g^4) \\
(g^4) \\
(g^4) \\
(g^4)
\end{array} \right\} + O(g^6)
\]

Fig. 7.7

so that the second term between brackets in Fig. 7.7 is finite. Let us consider in Eq. (7.22) the term which contributes to the leading trajectory, and which corresponds to \( p = q = 0 \). This term \( \tilde{L}' \) can be written as:

\[
\tilde{L}' = g^4 K \sum_n \frac{1}{(\alpha(s) - n)^2} c_n P_n (c_\alpha \theta_\epsilon) \tag{7.23}
\]

where \( K \) is a (divergent) constant and \( c_n \) a numerical coefficient. Combining the Born term and \( \tilde{L}' \) we have:

\[
\sum_n \left( \frac{-g^2}{\alpha(s) - n} + \frac{g^4 K}{(\alpha(s) - n)^2} \right) c_n P_n (c_\alpha \theta_\epsilon)
\]

\[
\sum_n \frac{g^2 c_n P_n (c_\alpha \theta_\epsilon)}{\alpha(s) - g^2 \delta \alpha - n} + O(g^6) \tag{7.24}
\]

We see that the divergence can be eliminated to the order of \( g^2 \) if we interpret \( \alpha(s) - g^2 \delta \alpha \) as the renormalized Regge trajectory. Notice that \( \delta \alpha \) is independent of \( s \), since \( K \) is a constant, so that we subtract an infinite constant from the Regge trajectory. This renormalization procedure is quite similar to mass renormalization in quantum-electrodynamics.
What happens to the daughters is not yet clear; in any case this problem cannot be settled at present, because there are ambiguities in the choice of the counter-term, which become relevant at the daughter level.

Finally, one can show that the property (4) of the counter-term is also satisfied, namely that

\[
\lim_{\varepsilon \to 0} \left[ L_{\varepsilon} (\alpha_{i,j}) - \tilde{L}_{\varepsilon} (\alpha_{i,j}) \right]
\]

reggeizes when \( s \to \pm \infty \).

The renormalization of the planar loop with \( N \) external particles, as well as the renormalization of the non-orientable loops, can be performed along the same lines. There are some problems in interpreting the orientable non-planar loops, which were discussed in Section 6.
8. **UNITARITY**

In Section 6 we have seen how to calculate one-loop diagrams with an arbitrary number of twisted internal lines. The aim of this section is to prove that, in order to satisfy perturbative unitarity, one must simply add — with weight one — all inequivalent one-loop diagrams. Two diagrams are said to be inequivalent when they are not connected by any sequence of duality transformations. In order to prove this result, obtained independently by Gross et al.\(^{25}\), we shall carefully analyse the singularity structure of one-loop diagrams, partially repeating what was done in Section 6.7.

An arbitrary one-loop diagram can be expressed in the form:

\[
A_d = \int d^4 \prod \int d\alpha_1 \cdots \alpha_n \ x_1^{-\alpha_1-1} \ F_d \left( x, \prod \right) \tag{8.1}
\]

where \( F_d \) is the trace of an operator expression involving vertices and propagators \( x^H \theta(x) \) or \( x^H \) for each internal line according to whether it is twisted or not. Because of our operatorial double twist and duality conditions, we can write diagrammatically

\[
\begin{array}{c}
\begin{array}{c}
\times \\
1 \quad 2
\end{array}
\end{array}
\begin{array}{c}
\times \\
1 \quad 2
\end{array}
\end{array}
= \begin{array}{c}
\begin{array}{c}
\times \\
2 \quad 1
\end{array}
\end{array}
\begin{array}{c}
\times \\
2 \quad 1
\end{array}
\end{array}
\tag{8.2}
\]

We now want to examine the consequences of these properties for integrals of the form (8.1), and in particular to examine the question of double counting and unitarity: Is it possible to find a sum of terms like (8.1) corresponding to certain diagrams, such that the sum is
crossing symmetric and satisfies the correct sort of Cutkosky discontinuity formulae (which we regard as the perturbative form of unitarity)?

First we must explain what we mean by "correct sort" of Cutkosky discontinuity formulae. The poles of a given Veneziano term have residues which decompose into terms some of which are symmetric under twisting, and some of which are antisymmetric. The antisymmetric terms cancel when we add up all the terms required by the Bose statistics of the whole amplitudes and therefore correspond to an "unphysical particle" which appears in individual Veneziano terms but not in the complete amplitude. The symmetric terms correspond to physical particles, and it is only these which should appear in unitarity equation intermediate states and hence as intermediate particles in Cutkosky discontinuity formulae. This is the property we shall seek. In particular, we shall just look at the Landau singularities of the box diagram occurring in the four-particle amplitude.

As done in Section 6.7, we now repeatedly apply the duality relations (8.2) in order to pull the external line 1 of diagram (a) round the loop in an anti-clockwise sense and obtain

Thus the contributions of type (8.1) corresponding to three different diagrams are equal by duality. Furthermore, the first and last diagrams are the same. The only line not dualized is the one carrying momentum Π, and so its momentum is preserved. Notice that it joins different pairs of vertices in the initial and final configurations.
Since our duality relations are also true in unintegrated form, we also have relations between the integrands of the above diagrams. In particular we have

\[
\frac{1}{\alpha'(\Pi')-1} \int d^4 \Pi \frac{1}{\alpha'(\Pi)-1} \prod_i x_i \prod_i x_i' \mathcal{F}_d(x, \Pi) \prod_i x_i' \mathcal{F}_d(x', \Pi')
\]

where

\[
\Pi' = \Pi + p_a
\]

\[
\chi' = \chi'(\chi)
\]

(8.3)

The latter change of variables is found by the repeated application of our duality change of variables and is, of course, independent of \( \Pi \). Thus the integrand of diagram (a) satisfies an invariance property corresponding to the fact that diagram (a) is itself invariant under a sequence of dualizations.

Another perhaps clearer way of representing the result diagrammatically is via the diagrams

![Diagrams](image)

Fig. 8.2

The internal line forms a closed ribbon. The external lines are attached to one side or another of this ribbon and can be slid freely along until they collide.

This sort of argument can be generalized. Any single-loop diagram with an even non-zero number of twists can, because of duality, be put in a form with only two twists.
Fig. 8.3

By our duality transformations we can pull all q-lines past all p-lines without affecting the internal line with momentum \( \Pi \). This gives

Fig. 8.4

which can be rotated into the form of the original. Thus we obtain an invariance relation of the form (8.3), (8.4) with \( p_1 \) replaced by \( \sum_{i=1}^{N} p_i \).

In fact the only other kind of twisted loops has an odd number of twisted lines and can be dualized into a form with only one twist. Before discussing such diagrams we shall look at a consequence of the invariance properties (8.3) and (8.4).

If we carry out the x-integration in equation (8.3) we see that the resultant integral is periodic in \( \Pi \) with period \( p_1 \). The integral over \( \Pi \) therefore diverges. In particular its discontinuities in external momentum variables diverge. Since the original motivation of the loop was
to obtain an expression with these discontinuities correctly expressed, we must modify the integration range. The obvious suggestion is to restrict it to a "cell" in \( x, \Pi \) space, where by a cell we mean a region which leads to a covering of the whole original integration region precisely once when we repeatedly apply the maps (8.4). There are many choices of cell, and, of course, the integral is the same whatever the choice of cell. Since \( \Pi'(\Pi) \) is independent of \( x \) and \( x'(x) \) is independent of \( \Pi \), straight lines \( \Pi = \text{const} \) or \( x = \text{const} \) are mapped into similar straight lines by (8.4). In particular, it is therefore possible to choose cells with straight sides, for example

![Diagram](image)

**Fig. 8.5**

Choice (a) is made by Kaku and Thorn. The advantage of choice (b) made by Gross, Neveu, Scherk and Schwarz is that the \( \Pi \) integration can be done explicitly when the integral (8.1) is put in the form of an infinite range Gaussian.

Henceforth we shall understand that integrals of the form (8.1) involving an even number of twists are integrated over such a cell which includes the origin in \( x \). It is still possible to find the dual configurations as above, without moving out of the cell.

As a particular example of a diagram with an odd number of twists, we consider
and make the sequence of dualizations corresponding to pulling line 4 around the loop in a clockwise sense:

![Diagram](image)

Fig. 8.6

Notice that the final result differs from the original only in that the twist is on a different internal line. Now pull line 1 around similarly:

![Diagram](image)

Fig. 8.7

Now pull line 2 around and then line 3:

![Diagram](image)

Fig. 8.8

![Diagram](image)

Fig. 8.9
Thus we have 12 different dual equivalent configurations. Notice that the diagrams in the last column differ only in that the twist is on a different internal line each time. The very last diagram is in fact identical to the original one. This suggests another invariance condition like (8.3) and (8.4) but it turns out that the associated change of variables is the identity one

\[ \Pi' = \Pi \quad x' = x \]

So there is no invariance property and the integration region remains the complete one.

Similar results apply to all diagrams with an odd number of twists, which by the duality properties can always be written in the form with one twist.

We consider now the hypothesis that the single loop amplitudes for four-particle processes is the sum of all the distinct amplitudes of Figs. 8.3 and 8.10, i.e.

\[ 1 \quad 4 \quad 3 \quad 2 \quad 1 \quad 2 \quad 3 \quad 4 \]

\[ + \quad + \quad + \quad + \quad + \quad + \quad + \quad + \]

\[ + \quad + \quad + \quad + \quad + \quad + \quad + \quad + \]

Fig. 8.10

\[ + \quad + \quad + \quad + \quad + \quad + \quad + \quad + \]

Fig. 8.11
This sum obviously satisfies Bose statistics and crossing because it is symmetrized "by hand". We now check that this amplitude has the "right" sort of Cutkosky discontinuity formula for the Landau curve associated with the box diagram

![Box Diagram](image)

where the internal particles have masses $M_1, \ldots, M_4$ given by $\alpha(M_1^2) = J_1$. When $J_1 > 0$, several Landau curves coincide owing to the spectrum degeneracy and it is the total discontinuity which we shall then evaluate.

If $D$ is the first diagram in Fig. 8.11, the discontinuity of interest is given by

$$
\text{disc. } A_D = \int d^4 \Pi \prod_{i=1}^{4} \left[ - \frac{1}{2 \pi} \delta \left( q_i^2 - M_i^2 \right) \frac{1}{J_i} \left( \frac{2}{\partial \chi_i} \right) \frac{1}{F_D(x_i, \Pi)} \right] \bigg|_{x=0} \tag{8.5}
$$

since we first take the residue of the multiple pole due to the divergence at $x = 0$ and then use the usual Cutkosky formula. We shall write this diagrammatically as

![Discontinuity Diagram](image)

with the understanding that on the right-hand side of a discontinuity equation the diagram corresponds to a form like (8.5) rather than (8.1).

The contribution involving an odd number of twists must be treated carefully. We saw that by duality the integrals
\[ A_{d \beta} = \int d^4 \Pi \int_0^1 dx_{\beta} \, x_{\beta}^{-x \cdot 1} \, F_{d \beta}(x_{\beta}) \]

are all equal if \( d_1, d_2, d_3, d_4 \) are the diagrams

\[ \text{Fig. 8.14} \]

A divergence at the origin \( x_{\beta} = 0 \) in \( A_{d \beta} \) gives rise to the singularities of interest for \( \beta = 1, 2, 3, \) and 4. Since the variables \( x_{\beta} \) are related to each other by duality transformations, there must be other points besides the origin in each \( x_{\beta} \) which give rise to our singularity. Accordingly, when we work out the discontinuity of \( A_{d \beta} \) we must add the four contributions coming from the four points corresponding to the four origins \( x_1 = 0, x_2 = 0, x_3 = 0, \) and \( x_4 = 0. \) On relabelling the integration variables we can write the result as

\[ d_{sc} A_{d \beta} = \int d^4 \Pi \prod_{i=1}^4 \left[ -2i\pi \delta(q_i^2 - M_i^2) \frac{1}{2} \frac{2}{d_{\lambda_i}} \right] \left[ F_{d_1} + F_{d_2} + F_{d_3} + F_{d_4} \right] \]

which we write diagrammatically as

\[ \text{disc} \]

\[ \text{Fig. 8.15.} \]

This argument explains why the integration must be restricted to a cell for a diagram with an even number of twists. If we integrate over \( N \) cells we would get \( N \) equal contributions. If \( N \) is \( \infty \), this is disastrous.
It is crucial that $N = 1$ for the subsequent argument. Proceeding in this way we find, for each contribution of Fig. 8.11 in turn

\[ \text{disc} \quad \begin{array}{c}
\begin{array}{c}
1 \\
2 \\
3 \\
4 \\
\end{array}
\end{array} \quad = \quad \begin{array}{c}
\begin{array}{c}
\text{Fig. 8.16}
\end{array}
\end{array} \]

\[ \text{disc} \quad \begin{array}{c}
\begin{array}{c}
1 \\
2 \\
3 \\
4 \\
\end{array}
\end{array} \quad = \quad 0 \]

\[ \text{disc} \quad \begin{array}{c}
\begin{array}{c}
1 \\
2 \\
3 \\
4 \\
\end{array}
\end{array} \quad = \quad 0 \]

\[ \text{disc} \quad \begin{array}{c}
\begin{array}{c}
1 \\
2 \\
3 \\
4 \\
\end{array}
\end{array} \quad = \quad \begin{array}{c}
\begin{array}{c}
\text{Fig. 8.16}
\end{array}
\end{array} \]

\[ \text{disc} \quad \begin{array}{c}
\begin{array}{c}
1 \\
2 \\
3 \\
4 \\
\end{array}
\end{array} \quad = \quad \begin{array}{c}
\begin{array}{c}
\text{Fig. 8.16}
\end{array}
\end{array} \]

\[ \text{disc} \quad \begin{array}{c}
\begin{array}{c}
1 \\
2 \\
3 \\
4 \\
\end{array}
\end{array} \quad = \quad \begin{array}{c}
\begin{array}{c}
\text{Fig. 8.16}
\end{array}
\end{array} \]

\[ \text{disc} \quad \begin{array}{c}
\begin{array}{c}
1 \\
2 \\
3 \\
4 \\
\end{array}
\end{array} \quad = \quad \begin{array}{c}
\begin{array}{c}
\text{Fig. 8.16}
\end{array}
\end{array} \]

\[ \text{disc} \quad \begin{array}{c}
\begin{array}{c}
1 \\
2 \\
3 \\
4 \\
\end{array}
\end{array} \quad = \quad \begin{array}{c}
\begin{array}{c}
\text{Fig. 8.16}
\end{array}
\end{array} \]

\[ \text{disc} \quad \begin{array}{c}
\begin{array}{c}
1 \\
2 \\
3 \\
4 \\
\end{array}
\end{array} \quad = \quad \begin{array}{c}
\begin{array}{c}
\text{Fig. 8.16}
\end{array}
\end{array} \]

\[ \text{disc} \quad \begin{array}{c}
\begin{array}{c}
1 \\
2 \\
3 \\
4 \\
\end{array}
\end{array} \quad = \quad \begin{array}{c}
\begin{array}{c}
\text{Fig. 8.16}
\end{array}
\end{array} \]
The remarkable thing is that when we add up the right-hand side of these equations to get the total contribution to the discontinuity we get, diagrammatically

![Diagram]

where

![Equation]

Fig. 8.17

which corresponds to the integral

$$\int d^4 \eta \prod \left( - i \gamma \delta(q_i - 1) \right)^\frac{1}{j_i!} \left( \frac{2}{\partial \lambda_i} \right) \bar{G} \bigg|_{\lambda = 0}$$

where \( G \) is constructed from an operator loop with propagator \( x^H(1 + \Theta(x)) \) on each internal line.

The point to note is that this is not the same as

$$d\lambda \int d^4 \eta \int d\lambda \cdot x_i^{-\lambda_i - 1} G(x, \eta)$$

which gives a totally undesirable result. Thus it is essential to have the "unphysical particle" mentioned earlier in order to get the desired duality and unitarity propagator. Because of the crossing symmetric property we have the correct discontinuity properties across all the other box diagram singularities. Finally let us notice how essential it was to include diagrams with an odd number of twists.
SOME PROPERTIES OF COHERENT STATES

The operators which appear in the vertex \( V(p) \) and in the integrand of the propagator \( D(s) \)

\[
\int \epsilon \left( -p \cdot \alpha_n^{(n)+} \right) \int \epsilon \left( p \cdot \alpha_n \right) \int x^n \cdot \alpha_n^{(n)+} \cdot \alpha_n^{(n)}
\]

have a very simple action on the so-called coherent states.

Definition of a coherent state \( |f\rangle \) (one mode only):

\[
|f\rangle = e^{f\alpha^+} |0\rangle \quad (A.1)
\]

Properties

1) \( |f\rangle \) is an eigenstate of the annihilation operator \( a \):

\[
a |f\rangle = f |f\rangle \quad (A.2a)
\]

or

\[
e^{g^\dagger} |f\rangle = e^{g^\dagger f} |f\rangle \quad (A.2b)
\]

Proof:

\[
e^{g^\dagger} e^{f\alpha^+} e^{g} = e^{f\alpha^+} e^{g} e^{g^\dagger f}
\]

2)

\[
e^{g^\dagger} |f\rangle = |f + g\rangle \quad (A.3)
\]

3)

\[
\langle f | g \rangle = e^{f^* g} \quad (A.4)
\]

[follows from Eq. (A.2b)]

4)

\[
x^n a^+ |f\rangle = x^n f \quad (A.5)
\]
Since:

\[ [a^+a, a^{+m}] = m a^{+m} \]

\[ x a^+a a^{+m} x - a^+a = a^{+m} + (\frac{\hbar x}{\rho}) m a^{+m} + \cdots + (\frac{\hbar x}{\rho})^p m a^{+m} \cdots = (a^+x)^m \]

so that

\[ x a^+a \rho \int a^+ = \rho \int x a^+ x - a^+a \]

5)

\[ \mathbf{I} = \frac{1}{\mathbf{r}} \int d\mathbf{r} \mathbf{r} \cdot \int d\mathbf{f} \mathbf{e}^{-\frac{1}{2} \mathbf{f}^2} |f\rangle < f| \]

(A.6)

Proof: See, for example, Ref. 35.

6) We can define the tensor product of coherent states in the usual way:

\[ |f_1, f_2, \ldots, f_m, \ldots\rangle = \prod \rho \int a^{(m)} \cdot \langle 0 | \]

Then we have:

\[ \bigotimes_{n,m} a^{(m)} (c_{nm} - d_{nm}) a^{(m)} \]

\[ : e^{a^+ (c-1) a} : |f\rangle = |C f\rangle \]

(A.7a)

or, symbolically:

\[ e^{a^+ (c-1) a} : |f\rangle = |C f\rangle \]

(A.7b)

Proof: expand the exponential in Eq. (A.7a) taking into account the normal-order form, then use Eq. (A.2a) and Eq. (A.3).

Application: Since

\[ x a^+a |f\rangle = |x f\rangle \quad \text{and} \quad e^{a^+ (\chi - 1) a} |f\rangle = |x f\rangle \]
We have the identity

\[ X^{\alpha_+ \alpha} = : \exp \left[ \frac{i}{\hbar} \right] : \]

This identity shows that the operator part of the propagator \( X^H \) can be written as:

\[ X^H = : \exp \left[ \frac{i}{\hbar} \right] : \]
APPENDIX B

We shall prove here Eq. (6.2); that is to say, the canonical form of the operator \( e^{\alpha L_\alpha (\Pi)} \). This is done by letting it act on a coherent state \(|\psi\rangle = \prod_n |z_n\rangle\):

\[
e^{\sum_i \alpha_i L_i} |\psi\rangle = e^{\sum_i \alpha_i L_i} \prod_n |z_n\rangle = e^{\sum_i \alpha_i L_i} |0\rangle = e^{\sum_i \alpha_i L_i} (\prod_n |z_n\rangle) = 0
\]

\[e^{\sum_i \alpha_i L_i} \prod_n |z_n\rangle = e^{\sum_i \alpha_i L_i} |0\rangle = e^{\sum_i \alpha_i L_i} (\prod_n |z_n\rangle) = 0
\]

since \( e^{\sum_i \alpha_i L_i} |0\rangle = |0\rangle \) and \( e^{-e^{-A}} = e^{-e^{-A}} \). Next, using

\[
e^{\sum_i \alpha_i L_i} \prod_n |z_n\rangle = e^{\sum_i \alpha_i L_i} |0\rangle = e^{\sum_i \alpha_i L_i} (\prod_n |z_n\rangle) = 0
\]

we calculate the exponent in the right-hand side of Eq. (B.1). We get:

\[
e^{\sum_i \alpha_i L_i} (\prod_n |z_n\rangle) = e^{\sum_i \alpha_i L_i} (\prod_n |z_n\rangle) = 0
\]

and using the explicit form of \( L_\alpha (\Pi) \) and computing the iterated commutators, we can write the result as follows:

\[
e^{\sum_i \alpha_i L_i} (\prod_n |z_n\rangle) = e^{\sum_i \alpha_i L_i} (\prod_n |z_n\rangle) = 0
\]

where

\[
e^{\sum_i \alpha_i L_i} (\prod_n |z_n\rangle) = e^{\sum_i \alpha_i L_i} (\prod_n |z_n\rangle) = 0
\]
\begin{align}
\mathbf{c}_n &= \sum_{m=1}^{\infty} C_{nm} \mathbf{c}_m \\
C_{nm} &= \begin{cases} 
0 & \text{if } m < n \\
\frac{\lambda_{n-m}^m \pi^{m-n}}{(m-n)!} \prod_{i=1}^{m-n} \sqrt[n]{(m+n-i)} & \text{if } m \geq n
\end{cases}
\end{align}

(B.5)

The matrix \( C_{nm} \) can be written more compactly as

\begin{equation}
C_{nm} = (e^{\lambda \pi})_{nm} \\
\tau_{nm} = \sqrt[n]{(n+m+1)} \delta_{n,m+1}
\end{equation}

(B.6)

Carrying out some trivial simplifications, we can write the result as

\begin{equation}
\frac{\psi^{|\Psi\rangle}}{|n\rangle} = e^{\frac{\lambda \pi}{n} \sum_{n:m} \frac{\sum_{n:m} a^{(n)}_m a^{(n)}_m}{\sqrt{n}}} + \sum_{n:m} a^{(n)}_m a^{(n)}_m |0\rangle =

= e^{\sum_{n,m} a^{(n)}_m (e^{\lambda \pi} a^{(n)}_m - \delta_{n,m})} a^{(n)}_m |0\rangle = e^{\frac{\lambda \pi}{n} \sum_{n:m} \frac{\sum_{n:m} a^{(n)}_m a^{(n)}_m}{\sqrt{n}}} |\mathbf{z}\rangle
\end{equation}

(B.7)

and the identity between operators follows because of the completeness of coherent states.
APPENDIX C

We shall prove here the following theorem, quoted in Section 6.

**Theorem.** The matrices $C_{nm}$ corresponding to $O(2,1)$ group elements when acting on vectors of the form $\xi^n/\sqrt{n}$, generate a projective transformation on the parameter $\xi$. More precisely

$$
\sum_{\xi=1}^{\infty} C_{nm} \frac{\xi^m}{\sqrt{n}} = \left[ \gamma(\xi) \right]^n - \left[ \gamma(0) \right]^n
$$

(C.1)

with $\gamma(\xi)$ defined as

$$
\gamma(\xi) = \frac{a \xi + b}{c \xi + d}.
$$

(C.2)

We have called $\tau$ the $2 \times 2$ matrix characterizing the projective transformations

$$
\tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
$$

Proof: Since any $O(2,1)$ group element can be written as successive transformations around the three axes, it is sufficient to consider for the moment group elements of the form $e^{\alpha L_0}$, $e^{\alpha L_-}$, $e^{\alpha L_+}$

i) Consider $e^{\alpha L_0}$. In this case $C_{nm} = e^{n \alpha} \delta_{nm}$, and Eq. (B.1) is trivially obtained, the $\tau$-matrix being

$$
\tau(e^{\alpha L_0}) = \begin{pmatrix} e^{\alpha} & 0 \\ 0 & 1 \end{pmatrix}
$$

(C.3)

ii) Now we consider $e^{\alpha L_-}$. In Appendix B we have derived the $C$-matrix for this operator, given by

$$
C_{nm} = \alpha^m \frac{\Gamma{\left(\frac{n}{2}\right)}}{\Gamma{\left(\frac{n-2}{2}\right)}} \left( \frac{n-1}{n} \right)^{n-2} = \alpha^m \sqrt{n} \binom{n-2}{\alpha} \binom{n-1}{\alpha-1}
$$

(C.4)
Therefore
\[ \sum_{\mu \in \Lambda} C_{\mu \mu} \frac{\xi^\mu}{\sqrt{\xi}} = \frac{1}{\sqrt{\eta}} \sum_{\mu \in \Lambda} \alpha^{m-\eta} \binom{m-\eta}{\mu-1} \xi^\mu = \]
\[ = \frac{\xi^\eta}{\sqrt{\eta}} \sum_{\mu \in \Lambda} \binom{m-\eta}{\mu-1} \left( \alpha \xi \right)^{m-\mu} = \left( \frac{\xi}{1 - \alpha \xi} \right)^\eta \frac{1}{\sqrt{\eta}} \]  
(C.5)

Therefore, in this case Eq. (B.1) is also satisfied, with
\[ \tau(e^{\alpha L_-}) = \begin{pmatrix} 1 & 0 \\ -\alpha & 1 \end{pmatrix}. \]  
(C.6)

iii) Finally consider \( e^{\alpha L_+} \). Since \( e^{\alpha L_+} = e^{\alpha L_-} \) and the C-matrix for \( e^{\alpha L_-} \) it follows that
\[ C_{nm}(e^{\alpha L_+}) = C_{mn}(e^{\alpha L_-}). \]  
(C.7)

Then it is straightforward to prove that Eq. (B.1) is also valid for the C-matrix corresponding to \( e^{\alpha L_+} \), with
\[ \tau(e^{\alpha L_+}) = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}. \]  
(C.8)

We have therefore shown that the theorem is true for group elements of the form \( e^{\alpha L_0}, e^{\alpha L_+}, e^{\alpha L_-} \). From this result it follows that any group element will generate a projective transformation when acting on vectors of the form, because the repeated application of projective transformations is again a projective transformation. Indeed, let us consider two matrices \( C_1 \) and \( C_2 \) satisfying our theorem, i.e.
\[ \sum_{\mu \in \Lambda} C_{1 \mu} \mu \frac{\xi^\mu}{\sqrt{\xi}} = \frac{1}{\sqrt{\eta}} \left( \gamma_1(\xi) \right)^\eta - \frac{1}{\sqrt{\eta}} \left( \gamma_1(0) \right)^\eta \]  
(C.9)
\[ \sum_{\mu \in \Lambda} C_{2 \mu} \mu \frac{\xi^\mu}{\sqrt{\xi}} = \frac{1}{\sqrt{\eta}} \left( \gamma_2(\xi) \right)^\eta - \frac{1}{\sqrt{\eta}} \left( \gamma_2(0) \right)^\eta \]  
(C.10)
where $\gamma_1$ and $\gamma_2$ are determined by the $2 \times 2$ matrices:

$$
\varepsilon_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \quad ; \quad \varepsilon_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}
$$

Then, one can immediately show that

$$
\sum_{\nu = 1}^{2} \left( C_1 C_2 \right) \frac{\xi}{\sqrt{\kappa}} = - \frac{i}{\sqrt{\kappa}} \left( \mathcal{F}(\mathbf{s}) \right)^n - \frac{i}{\sqrt{\kappa}} \left( \mathcal{F}(\mathbf{o}) \right)^n
$$

with $\gamma(\xi)$ determined by the $2 \times 2$ matrix $\tau$ given by

$$
\tau = \tau_1 \tau_2 .
$$

As explained in the text, this property of $C$-matrices allows us to multiply and diagonalize them by simply multiplying $2 \times 2$ matrices.
APPENDIX D

In this appendix we prove that the amplitude for a closed loop can be written as a trace of a product of vertices and propagators\(^{11}\). For the sake of simplicity, we restrict ourselves to the case of a planar loop, but the procedure is easily extended to the case of non-planar loops.

To prove this property, we use the tree theorem and require that if the loop is cut somewhere, one finds again the corresponding tree graph (Fig. D.1)

![Diagram](image)

Fig. D.1

To be more precise, let us define the loop amplitude \( L \) as:

\[
L = \sum M \prod M(\Pi; p_i, \cdots, p_N)
\]  \hspace{1cm} (D.1)

We shall require that \( M \) has a pole at \( \alpha(\Pi^2) = J \), and that the residue \( M_J \) of this pole is obtained from the residue of the double pole which one gets by factorizing a tree graph twice:
\[ \text{Re}_0 \mathbf{M} \mid_{\mathcal{A}(n^2)} = \mathbf{M}_J = \sum_{\mathcal{P}_0 \in \mathcal{A}(n^2), \mathcal{P}_n = J} \left( \frac{\mathcal{P}_0 - \mathcal{Q}}{\mathcal{Q}_0} \right) \langle \{ \mathcal{Q} \} | U(p_1) D ... D U(p_n) | \{ \mathcal{P} \} \rangle \]

(D.2)

where we have used Eq. (2.14) in order to evaluate the residue.

Proceeding further we have:

\[
\mathbf{M} = \sum_J \frac{-1}{\alpha(n^2) - J} \mathbf{M}_J =
\]

\[
= \sum_J \frac{-1}{\alpha(n^2) - J} \sum_{\mathcal{P}_0 \in \mathcal{A}(n^2), \mathcal{P}_n = J} \left( \frac{\mathcal{P}_0 - \mathcal{Q}}{\mathcal{Q}_0} \right) \langle \{ \mathcal{Q} \} | U(p_1) D ... D U(p_n) | \{ \mathcal{P} \} \rangle
\]

\[
= \sum_{\mathcal{Q}} \sum_{\mathcal{P}_0} \left( \frac{\mathcal{P}_0 - \mathcal{Q}}{\mathcal{Q}_0} \right) \langle \{ \mathcal{Q} \} \left| \frac{-1}{d(n^2) - \mathcal{P}_0} \right. \rangle U(p_1) D ... D U(p_n) | \{ \mathcal{P} \} \rangle
\]

\[
= \text{Tr} \left\{ D(n^2) U(p_1) D(n^2)^2 ... D(n^2 + p_n)^2 U(p_n) \right\}
\]

In the last step we have made use of Eq. (2.3) for the propagator. We notice that the result is independent of the line which is sewn.
REFERENCES

   K. Kikkawa, S. Klein, B. Sakita and M.A. Virasoro, Wisconsin preprint
   COO-248 (1969), to be published.


   For a review of the work leading to the N-point function, see,
   for example, Chan's Royal Society lectures, CERN preprint
   TH 1057.


   Y. Nambu, unpublished.

6) L. Susskind, The physical interpretation of duality, Yeshiva
   preprint.


13) L. Caneschi, A. Schwimmer and G. Veneziano, Phys. Letters 30 B,

   C.B. Thorn, University of California Berkeley preprint, to be
   published.


    A. Neveu and J. Scherk, Nuclear Phys. in press.


    TH 1161, to be published.

24) M. Kaku and C.B. Thorn, Non-planar dual loop diagrams, Berkeley

25) D.J. Gross, A. Neveu, J. Scherk and J. Schwarz, Renormalization
    and unitarity in the dual resonance model, Princeton preprint

26) A. Neveu and J. Scherk, Phys. Rev. to be published.


28) K. Bardakçi, M.B. Halpern and J.A. Shapiro, Phys. Rev. 185, 1910

29) C. Bouchiat, J.L. Gervais and N. Sourlas, Orsay preprint 70/11.


31) H.B. Nielsen, private communication.
    L. Susskind, Phys. Letters, to be published.


33) We follow here the notations and conventions of E.T. Whittaker
    and G.N. Watson, Modern Analysis (Cambridge University Press,
    1963).

34) D.B. Fairlie and H.B. Nielsen, An analog model for KSV theory,
    preprint.

35) For a proof, see Klauder and Sudarshan, Fundamentals of Quantum