Ion dynamics and the magnetorotational instability in weakly-ionized discs

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ABSTRACT
The magnetorotational instability (MRI) of a weakly ionized, differentially rotating, magnetized plasma disc is investigated in the multi-fluid framework. The disc is threaded by a uniform vertical magnetic field and charge is carried by electrons and ions only. The inclusion of ion inertia causes significant modification to the conductivity tensor in a weakly ionized disc. The parallel, Pedersen and Hall component of conductivity tensor becomes time dependent quantities resulting in ac and dc components of the conductivity. The time dependence of the conductivity causes significant modification to the parameter window of magnetorotational instability.

The effect of ambipolar and Hall diffusion on the linear growth of the magnetorotational instability is examined in the presence of time dependent conductivity tensor. We find that the growth rate in the ambipolar regime can become somewhat larger than the rotational frequency, especially when the departure from ideal MHD is significant. Further, the instability operates on large scale lengths. This has important implication for angular momentum transport in the disc.

When charged grains are the dominant ions, their inertia will play important role near the mid plane of the protoplanetary discs. Ion inertia could also be important in transporting angular momentum in accretion discs around compact objects, in cataclysmic variables. For example, in cataclysmic variables, where mass flows from a companion main sequence star on to a white dwarf, the ionization fraction in the disc can vary in a wide range. The ion inertial effect in such a disc could significantly modify the magnetorotational instability and therefore, this instability could be a possible driver of the observed turbulent motion.

Key words: magnetohydrodynamics, star formation, accretion discs, charged grains, magnetorotational instability.

1 INTRODUCTION

Angular momentum transport has long been recognised as a key issue in accretion disc theories (Lynden-Bell 1969; Sakura & Sunyaev 1973). However, until the 1990s, a viable physical mechanism necessary to facilitate this transport in the absence of tidal effects or gravitational instabilities was unknown. The Balbus-Hawley (or magnetorotational) instability (Velikhov 1959; Chandrasekhar 1961) was proposed (Balbus & Hawley 1991; Hawley & Balbus 1992; Balbus & Hawley 1998) as a viable mechanism that can efficiently drive MHD turbulence and transport angular momentum in the disc. This opened the door for its application to a wide variety of astrophysical discs. The requirement for the magnetorotational instability to operate in such a disc is that the ambient magnetic field is subthermal at the disc midplane and is well coupled to the disc matter. Although, the lower bound on the weak, subthermal field has never been specified, in recent work this issue has been addressed in the framework of fully ionized, collisionless cold electron-ion plasma (Krolik & Zweibel 2006). For a highly ionized disc, the requirement of a weak, subthermal field is easily satisfied and the magnetorotational instability grows at the rotation frequency \( \Omega \) of the disc as a low frequency Alfvén mode with \( k V_A \sim \Omega \), where \( k \) is the wavenumber and \( V_A \) is the Alfvén velocity. However, many astrophysical discs are not well coupled to the magnetic field. Circumstellar, Protoplanetary (PPD), Dwarf Novae (DN), and, proto-neutron-star discs are good examples of weakly ionized discs with very low to low (PPDs) and high (DNs) fractional ionization. In PPDs for example, the sources of ionization are limited to the disc surface and magnetorotational instability may operate only in the outer envelope of the disc (Gammie 1996) unless some nonthermal source of ionization viz. the collision of the energetic electrons with neutrals or magnetorotational instability induced turbulent convective homogeniza-
tation of the entire disc (Inutsuka & Sano 2005) is assumed. DN discs are thought to have both hot and fully ionized accretion state as well as cold and mostly neutral accretion states (Cannizzo 1993; Gammie & Menou 1998). Therefore, the direct application of Balbus & Hawley (1991) results are difficult in a weakly ionized disc.

The effect of non-ideal MHD on the magnetorotational instability has been investigated by several authors: in the ambipolar regime (Blaes & Balbus 1994), hereafter BB94, (Maclow et al. 1995; Hawley & Stone 1998; Kunz & Balbus 2004), the resistive regime (Jin 1996; Papaloizou & Terquem 1997; Balbus & Hawley 1998; Sano et al. 1998; Sano & Miyama 1999; Fleming et al. 2000; Sano et al. 2000; Stone & Fleming 2003) and the Hall regime (Wardle 1999) W99 hereafter; (Balbus & Terquem 2001), BT01 hereafter; (Sano & Stone 2002a,b; Salmeron & Wardle 2003, 2005; Desch 2004). At the densities relevant to cloud cores, ambipolar and Hall diffusion plays an important role in the transport of mass and angular momentum (Wardle & Ng 1999; Balbus & Terquem 2001). W99 and BT01 found that collision of neutrals with the ionized gas in a weakly ionized disc determines the relative importance of Ambipolar, Hall or Ohmic diffusion on the magnetorotational instability. The ambipolar and Hall effects are particularly important when the ionization in the disc is very low and the departure from ideal MHD is severe.

The dynamics of a weakly ionized disc was investigated in the limit of zero inertia of the ionized plasma components by W99 and BT01. This is usually an excellent approximation when the fractional ionization is low, and allows the ionized components of the fluid (viz electrons, ions and grains) to be treated on an equal footing. However, there are situations – even in the low fractional ionisation limit – where the inertia of the charged species is important in determining their drift with respect to the neutral component and hence the diffusion of the magnetic field. In the weakly-ionised limit this becomes important when the inertial terms in the ion equation of motion start to compete with the magnetic limit this becomes important when the inertial terms in the diffusion of the magnetic field. In the weakly-ionised components of the fluid (viz electrons, ions and grains) by W99 and BT01. This is usually an excellent approximation. This is the convective derivative), is retained in the momentum equation for the ionised component but its effect on Ohm’s law is ignored in BB94. This limits the applicability of the results to the ambipolar diffusion limit.

To better appreciate this point, let us briefly recast the two fluid formulation of BB94 starting with separate ion and electron fluid equations. The equation of motion for the ionised fluid is derived assuming that magnetic field is frozen in the electron fluid,

\[ 0 = -e n_e (E + v_e \times B/c) \]  

and summing the electron and ion momentum equations, to yield

\[ \frac{dv_i}{dt} + \frac{\nabla P_i}{\rho_i} + \nabla \Phi + \nu_{in} (v_i - v_n) = \frac{J \times B}{c \rho_i} \]

(2)
equation (15) of BB94. Here \( e \) is the electronic charge, \( n_e \) is the electron number density, \( \nu_{in} \) is the ion-neutral collision frequency, \( v_e, v_i \) and \( v_n \) are the electron, ion and neutral velocities, \( \Phi \) is the gravitational potential, \( P_i \) is the ion pressure, \( E, B \) are the electric and magnetic fields and \( c \) is the speed of light and \( J = e n_e (v_i - v_e) \) is the current density. Taking the curl of (1) and using Maxwell’s equation,
$e \nabla \times E = -\partial_t B$, will give $\partial_t B = \nabla \times (v_i \times B)$, i.e. the magnetic field is convected away by the electron fluid. If we want to express the right hand side of induction equation in terms of ion velocity, we obtain

$$\frac{\partial B}{\partial t} = \nabla \times (v_i \times B) - \nabla \times F_H,$$

(3)

where the Hall term is

$$F_H = \frac{J \times B}{c n_e}.$$

(4)

Since $J \times B/c \sim \rho_i (d_i v_i + \nu_{in} v_i)$ (here $\nu_{in}$ is the ion-neutral collision frequency), the Hall term can be dropped from the induction equation only if $(\nu_{in} \omega) \ll \omega_{ci} (= e B/(m_i c))$, i.e. the ion-gyration period $(\omega_{ci}^{-1})$ is smaller/faster than the dynamical time $(\omega^{-1})$ or the ion-neutral collision time $(\nu_{in}^{-1})$. We see that unlike a two component-ion-electron plasma, where the Hall term can be introduced only through the ion inertial term $(d_i v_i)$, in a weakly ionized multi-component plasma, the Hall effect appears either via ion-neutral collision or via the $d_i v_i$ term or both.

Replacing $\partial_t B$ by $\Delta B/\Delta t$ and $\nabla \times F_H$ by $e B \Delta B/(4 \pi n_e \epsilon \Delta x)$, one sees that the Hall term scales as $1/(\omega_{pi} L)$, (here $\omega_{pi} = (4 \pi e^2 n_i/m_i)^{0.5}$ is the ion plasma frequency and $L$ is the characteristic size of the system), i.e. the Hall term is important on a scale shorter than the ion-inertial scale. Clearly, Hall MHD introduces two disparate, interacting scales, a microscopic scale, i.e. the ion-skin depth $(\delta_i = c/\omega_{pi} \equiv V_A/\omega_{ci})$ and a macroscopic scale, the disc size. The Hall term can be dropped if ion-inertial effects are unimportant. Leaving out the effect of inertia in the induction equation but retaining them in dynamics is not consistent and, in such a scenario, one would expect that magnetorotational instability will merely shift towards long wavelength, as has already been noted by BB94.

If we start with the ion equation of motion (BB94 (15)), and express the electric field as

$$E = -\frac{v_i \times B}{c} + \frac{m_i}{e} \left[ \frac{d v_i}{dt} + \nabla \Phi \frac{\rho_i}{\rho_i} + \nu_{in} v_D \right],$$

(5)

where $v_D = v_i - v_n$, then taking the curl and using Maxwell’s equation, one arrives at the following induction equation

$$\frac{\partial B}{\partial t} - \nabla \times (v_i \times B) = \frac{m_i}{e} \left[ \nu_{in} \nabla \times v_D + \nabla \times \frac{d v_i}{dt} \right].$$

(6)

Here uniform density is assumed while operating with the curl on equation (5). This equation has an additional term in comparision with equation (16) of BB94, with important consequences on the magnetic diffusivity, since the rate of change of magnetic flux is given as

$$\frac{d}{dt} \int \int \mathbf{B} \cdot d\mathbf{s} = \int \int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} + \oint v_i \times d\mathbf{l} \cdot \mathbf{B}$$

$$= \int \int \frac{\partial \mathbf{B}}{\partial t} - \nabla \times (v_i \times B) \cdot d\mathbf{s}.$$ 

(7)

Making use of equation (6) in (7), we get

$$\frac{d}{dt} \int \int \mathbf{B} \cdot d\mathbf{s} = -\frac{1}{\beta_i} \oint v_D \cdot d\mathbf{l}$$

$$-\frac{1}{\omega_{ci}} \int \div \left[ \frac{\partial v_i}{\partial t} - v_i \times (\nabla \times v_i) \right] \cdot d\mathbf{s}.$$ 

(8)

Here $\mathbf{B} = B/B$ and use has been made of $d v_i/dt = \partial_t v_i - v_i \times (\nabla \times v_i) + \nabla v_i^2/2$. The ion Hall parameter $\beta_i = \omega_{ci}/\nu_{in}$ gives the ratio between the ion-cyclotron to ion-neutral collision frequencies. The above equation can be rewritten as

$$\frac{d}{dt} \int \int \left[ \mathbf{B} + \frac{1}{\omega_{ci}} \nabla \times v_i \right] \cdot d\mathbf{s} = -\frac{1}{\beta_i} \oint v_D \cdot d\mathbf{l}.$$ 

(9)

We see from equation (9) that the generalized flux that is a combination of magnetic flux and vorticity is not conserved. The rate at which this flux decays is directly related to the collisional coupling between ions and neutrals. If the ion magnetization is weak, i.e. the ion-cyclotron frequency is less than ion-neutral collision frequency (\beta \to 0), then the flux-decay rate could be very large for a finite ion-neutral drift speed $v_D$. However, if the relative ion-neutral drift is negligible, the generalized flux is conserved irrespective of the ion magnetization level. Thus it is the combination of the magnetic flux and the vorticity that is conserved in the absence of collision (\beta \to \infty). The BB94 formulation assumes that magnetic flux is frozen in the fluid-which is valid if apart from ignoring the right hand side of equation (9), we also assume that \omega_{ci} \to \infty. In this limit however, the role of ion inertia becomes increasingly unimportant and we approach W99 limit. Clearly, BB94 does not treat the effect of ion inertia in a consistent fashion and their results are not applicable in most of the weakly ionized parameter space where the ion Hall parameter is \sim 1. In Fig.1, we plot the range of applicability of BB94 and W99. We see that BB94 is applicable when \beta_i \gg 1 for \omega_{ci} > \Omega for arbitrary relation between \nu_{in} and \Omega. BT01 show that \beta_i \gg 1 implies that the Hall term dominates over ambipolar term. W99 is applicable for \omega_{ci} > \Omega and \nu_{in} > \Omega for arbitrary \beta_i. Therefore, \omega_{ci}/\Omega < 1 and \nu_{in}/\Omega < 1 is an unexplored region in BB94 and W99 framework. We see that in a PPD, for a mil-Gauss field, for a positive grains of mass 10^{-12} – 10^{-15} g. \omega_{ci}/\Omega < 1. As has been noted elsewhere, near the mid-plane of PPDs, dust grains can be the dominant charged constituent over extended regions (\sim 1 – 5 AU). For sub-micron sized grains 0.1 \mu m, negatively charged grains dominate whenever \nu_{in} \sim 10^{15} cm^{-3} and positively charged grains dominate for \nu_{in} \sim 10^{14} cm^{-3} (Wardle & Ng 1999). Depending upon neutral density in the disc, the ratio \nu_{in}/\Omega can have any value and thus, it is important to extend the BB94 analysis to the unexplored regions with full, Hall and ambipolar effects in the spirit of W99. We shall give a general derivation of the induction equation in a weakly ionized medium in the text though, we adopt the conductivity approach of W99 to investigate the effect of ion inertia on the magnetorotational instability and find a significant increase in the growth rate. Further, the parameter window in which instability operates expands considerably to large scale lengths. It will have an important bearing on the angular momentum transport and onset of turbulence.

### 3 FORMULATION - MHD EQUATIONS

The dynamics of a weakly ionized disc, consisting of electrons, ions, neutrals and charged and neutral dust grains, in the presence of a gravitational field $\Phi$ of a central mass point $M$ is described by a set of multifluid equations. A multi-fluid approach representing each and every particle species is not
very fruitful since depending upon the fractional ionization, the presence of some of the ionized component in the disc can be neglected. We shall assume that the ion density in the disc is mainly due to the presence of positively charged grains. Such a situation will correspond to a very dense region of PPDs (Wardle & Ng 1999).

As the disc matter is weakly ionized, generally the inertia of the charged species are neglected (W99). However, apart from pure scientific curiosity about the effect of inertial effect may compete with collisional and electromagnetic effects in the dense region of PPDs as well as in the ion-neutral and electron-neutral rate coefficients are (Draine et al. 1983)

\[
< \sigma v >_j = 1.9 \times 10^{-9} \text{ cm}^3 \text{s}^{-1}
\]

< \sigma v >_e = 4.5 \times 10^{-9} \frac{T}{30 \text{ K}}^{\frac{1}{2}} \text{ cm}^3 \text{s}^{-1}.

(16)

Adopting a value of \( m_i = 30 m_p \) for ion mass and \( m_n = 2.33 m_p \) for mean neutral mass where \( m_p = 1.67 \times 10^{-24} \text{ g} \) is the proton mass, the ion neutral collision frequency can be written as

\[
\nu_{in} = \rho_n \gamma \equiv \frac{m_n \nu_{in}}{m_i + m_n} \equiv 1.4 \times 10^{-10} \text{ s}^{-1}.
\]

(17)

This also gives the limiting value for very small grains (\( \sim 3 - 3000 \text{ Å} \)). For larger, micron sized grains, ion-neutral collision rate can vary between \( 10^{-10} \) to \( 10^{-5} \) for sizes ranging between a few Angstrom to a few microns. This can be seen if we write the collision rate as (Nakano & Umebayashi 1986)

\[
< \sigma v >_{in} = 2.8 \times 10^{-5} T_{30}^{\frac{1}{2}} a^{-3}_{-5},
\]

(18)

where \( T_{30} \) is the gas temperature and \( a_{-5} \) is the grain radius in units of 30 K and \( 10^{-5} \) cm respectively.

We shall rewrite equations (10)-(13) in the local Keplerian frame. Thus, velocity \( v \) represents the departure from the Keplerian motion; the fluid velocity in the laboratory frame is \( v + v_K \) and \( \partial_t = \partial_t + \Omega \partial_\phi \) in the laboratory frame, where \( v_K = \sqrt{GM/r} \phi \) is the Keplerian velocity in the canonical cylindrical coordinate system (r, \( \phi \), z).

Noting that near the disc midplane, on scales small compared with the disc thickness, the radial gradient in gravitational potential will be exactly cancelled by the centripetal term due to Keplerian motion, and, (r, \( \phi \)) component of the equation (12)-(13) in the absence of any tidal effects, can be rewritten as,

\[
Av_t = \frac{e}{m_i} \left( E^t + \frac{v_i \times B}{c} \right) - \nu_{in} v_i - A v_n,
\]

\[
Av_n = -\frac{\nabla P_n}{\rho_n} + \frac{J \times B}{c \rho_n} + O \left( \frac{\partial_i}{\rho_n} \right),
\]

(19) (20)
where operator \( A = \begin{pmatrix} \frac{d}{dt} & -2\Omega \\ 0.5\Omega & \frac{d}{dt} \end{pmatrix} \). The induction equation can be written as

\[
\partial_t B = \nabla \times (v \times B) - c\nabla \times E' - 1.5\Omega B \times \dot{\phi}.
\]

(21)

In (21), \( \nabla \times E' \) contains the effect of non-ideal MHD and the last term accounts for the generation of the toroidal field from the poloidal one due to differential rotation of the disc (W99).

Before describing the conductivity approach and giving a formulation to the problem at hand, we shall give a description of the ion-inertial effects in the accretion discs. Writing \( E' = -v_i \times B/c + \eta J + F_H \).

(22)

Here

\[
\eta = \frac{c^2 m_i e^2 n_i}{4\pi n_e e^2} = \frac{c^2 m_the electrical resistivity of the gas and \( F_H = J \times B/e n_e \) is the Hall term. Even in the absence of ion inertia - the so called strong coupling approximation, (Shu 1983), the ambipolar term modifies the induction equation, i.e. when \( \rho_i/v_i = J \times B/c \).

\[ E' = \eta J + F_H - \frac{(J \times B) \times B}{c \rho_i v_i}, \]

(24)

When ion inertial terms are also present,

\[ v_i = \frac{J \times B}{c \rho_i v_i} - \frac{1}{v_i} \frac{d v_i}{d t} + O \left( \frac{\rho_e}{\rho_i}, \frac{\rho_e}{\rho_n} \right), \]

(25)

and, the generalized Ohm’s law becomes

\[ E' = \eta J + F_H - \left[ \left( \frac{J \times B}{c} \right) + A_1 v_i \right] \times B. \]

(26)

where \( A_1 = v_i^{-1}A \). Now taking the curl of equation (26), one may write the generalized induction equation as

\[
\frac{\partial B}{\partial t} = \nabla \times \left[ v \times B - \frac{4\pi \eta j}{c} J \times B \right] + \nabla \times \left[ \left( \frac{J \times B}{c} \right) - (A_1 v_i) \times B \right].
\]

(27)

The induction equation (27) on the right hand side contains inductive, Ohmic diffusion, Hall, ambipolar, and ion inertial terms respectively. The set of equation (10), (12), (13) and (27) can be closed by an equation of state.

Assuming that ion-inertial time scale is of the order of disc rotation period, we may write \( A_1 \sim \Omega/v_i \). Then the ratio of the ion inertial term to the inductive term can be written as

\[
\frac{|v \times B|}{|A_1 v_i \times B|} \sim \frac{v_i}{\Omega}. \]

(28)

A different ratio that measures the coupling of the neutral to the ion with respect to the Keplerian frequency have been used by BB94 (Menou & Quataert 2001)

\[
Re_A \equiv \frac{v_i}{\Omega} = \frac{\alpha}{\Omega}, \]

(29)

where \( \alpha = \rho_i/\rho_n \). It is clear from equation (29) that magnetorotational instability can act on both the neutral as well as on the ion fluid simultaneously if \( \alpha \sim 1 \).

Assuming an equation of state or dropping the pressure gradient term in the neutral equation of motion, equations (10), (19), (20) and (27) with Maxwell’s equations

\[
\nabla \times B = \frac{4\pi}{c} J, \quad \nabla \cdot B = 0,
\]

(30)

form a complete set.

4 LINEARIZATION

We consider a thin disc implying that the radial scale over which physical quantities vary is much larger than the disc scale height, \( H = C_s/\Omega \). The initial steady state is assumed uniform and homogeneous with a vertical magnetic field \( B = Bz \) and zero \( v, v_n, \nabla \times E', \nabla \times J \).

We shall assume transverse fluctuations and denote resulting two-dimensional vectors by subscript \( \perp \), to investigate Alfvén modes in the disc. We seek plane wave solution of the form \( \exp(i \omega t - k z) \).

4.1 The conductivity tensor

The conductivity tensor \( \sigma \) can be found by considering the drifts of charged particles in response to the electromagnetic field (Cowling 1957; Norman & Heyvaerts 1985; Nakano & Umebayashi 1986; Wardle & Ng 1999). We shall first derive conductivity tensor from equation of motion for charged particles, from (11) and (19) by eliminating \( v_e, \rho_e, \) and then give the expressions for parallel, Hall and Pedersen component of this tensor \( \sigma \) in the generalized Ohm’s law, \( J = \sigma \times E' \).

Since we assume a homogeneous steady state, it implies that the inhomogeneity scale length (in this case vertical scale height \( H \) of the disk) is much larger than the characteristic wavelength of the normal modes in the disc. Thus, the contribution of the pressure gradient will be neglected while inverting equation (13). Furthermore, we shall also ignore terms of the order \( \rho_i/\rho_n \) in the conductivity tensor. Expressing velocities \( v_j \) in equations (11) and (19) in terms of electric field \( E' \) the relationship between \( J \) and \( E' \) can be written as

\[
J = \sigma \cdot E' = \sigma_{||} E'_|| + \sigma_{\perp} \hat{B} \times E'_\perp + \sigma_{\perp} \hat{E}'\.\]

(31)

where \( \sigma_{||}, \sigma_{\perp} \) and \( \sigma_{\perp} \) are the field-parallel, Hall and Pedersen components of the conductivity tensor \( \sigma \) and \( \hat{B} \) is the unit vector along the magnetic field. If charged species \( j \) has particle mass \( m_j \), charge \( Z_j e \), number density \( n_j \), then the Hall parameter is given as

\[
\beta_j = \frac{Z_j e B}{m_j e^2} \left( \frac{m + m_j}{\sigma^2 \nu > 0} \right) \]

(32)

where \( m \) is the mean neutral particle mass and have dropped subscript ‘n’ from neutral quantities. As noted in section 2, the Hall parameter determines the magnitude of the magnetic flux transport. The ratio of ion to electron Hall parameter suggest that in the protostellar discs, \( \beta_j/\beta_e \sim 10^{-3} \ll 1 \). Recall that BB94 formulation is valid when the ion-Hall parameter is large (\( \beta_i \gg 1 \)), i.e. the when Lorentz force
dominates the ion-neutral collisional momentum exchange. However, this limit implies strongly magnetized ions and infrequent collisions.

The conductivity tensor is frequency dependent in the presence of ion inertia. The parallel conductivity is

$$\sigma_{\parallel} = \frac{e c n_i}{B} \left[ \beta_e + \frac{\beta_i}{1 + \left( \frac{\omega}{v_{in}} \right)^2} - i \beta_i \frac{\omega}{v_{in}} \right] .$$ \hspace{1cm} (33)

Since the plasma is quasi-neutral, we have assumed $n_e = n_i$. The conductivity has become complex. In the low frequency limit, when ion inertial effect is unimportant, i.e. $\omega/v_{in} \to 0$, $\sigma_{\parallel}$ reduces to W99. In $\omega/v_{in} \to \infty$, $\sigma_{\parallel}$ has both a real and an imaginary component

$$Re[\sigma_{\parallel}] \approx -\frac{e c n_i}{B} \frac{\omega_{in}}{\omega} .$$ \hspace{1cm} (34)

and,

$$Im[\sigma_{\parallel}] \approx -\frac{e c n_i}{B} \frac{\omega_{in}}{\omega} .$$ \hspace{1cm} (35)

The Pedersen conductivity is

$$\sigma_P = \frac{e c n_i}{B} \left[ 1 + \frac{\Delta \sigma_P}{\sigma_P^0} \right] .$$ \hspace{1cm} (36)

Here frequency independent part, $\sigma_P^0$ is

$$\sigma_P^0 = \frac{e c n_i}{B} \sum_j n_j Z_j \beta_j^2 ,$$ \hspace{1cm} (37)

and frequency dependent part, $\Delta \sigma_P$ is

$$\Delta \sigma_P = \frac{e c n_i}{B} \left( \frac{n_i \beta_i}{1 + \beta_i^2} \right) (Q(\omega) - 1) .$$ \hspace{1cm} (38)

Here $Q(\omega) = (1 + \beta_i^2) D_1/D_2$, $D_1 = i \omega/v_{in} + 1$, $D_2 = D_i^2 + \Omega_1 \Omega_2$, $\Omega_1 = 2 \Omega + \beta_i$, $\Omega_2 = 0.5 \Omega + \beta_i$ and $\Omega = \omega/v_{in}$. In order to isolate real and imaginary part of $\sigma_P$ and investigate the low and high frequency limits, we write $Re[Q(\omega)]$ and $Im[Q(\omega)]$ as

$$Re[Q(\omega)] = \frac{1 + \left( \frac{\omega}{v_{in}} \right)^2 + \Omega_1 \Omega_2}{1 - \left( \frac{\omega}{v_{in}} \right)^2 + \hat{\Omega}_1 \hat{\Omega}_2 + 4 \left( \frac{\omega}{v_{in}} \right)^2} ,$$ \hspace{1cm} (39)

and,

$$Im[Q(\omega)] = \frac{- \left( \frac{\omega}{v_{in}} \right) \left[ 1 + \left( \frac{\omega}{v_{in}} \right)^2 - \Omega_1 \Omega_2 \right]}{1 - \left( \frac{\omega}{v_{in}} \right)^2 + \hat{\Omega}_1 \hat{\Omega}_2 + 4 \left( \frac{\omega}{v_{in}} \right)^2} .$$ \hspace{1cm} (40)

In the low frequency limit, $\omega/v_{in} \to 0$, and assuming $\omega \sim \Omega$, $Re[Q(\omega)] \approx 1$ and $Im[Q(\omega)] \approx 0$. Thus $\sigma_P = \sigma_P^0$. In the high frequency limit, when $\omega/v_{in} \to \infty$, $Re[Q(\omega)] \approx 0$ and $Im[Q(\omega)] \approx -\omega/v_{in}$. Thus,

$$Re[\sigma_P] = \frac{e c n_i \beta_i}{B} ,$$ \hspace{1cm} (41)

and,

$$Im[\sigma_P] = -\frac{e c n_i \beta_i}{B} \left( \frac{\omega}{v_{in}} \right) .$$ \hspace{1cm} (42)

Like the parallel conductivity, the real part of the Pedersen conductivity in the high frequency limit is mainly due to electron magnetization and imaginary part is due to ion magnetization. Complex resistivity is well known in LCR circuits where the resonance condition is found by setting imaginary part of the impedance to zero. In the present case, a similar resonance condition can be derived by setting numerator of equation (40) to zero, i.e. $\omega^2/\nu_{in}^2 = -1 + \Omega_1 \Omega_2$, we get

$$1 + \frac{\Delta \sigma_P}{\sigma_P^0} \approx \frac{\Omega_1 \Omega_2}{2 \left( 1 + \frac{\omega}{v_{in}} \right)^2} .$$ \hspace{1cm} (43)

It is important to note that the scale of frequency dependent conductivity associated with the resonance ($i \omega \sim \Omega$), can become larger than the dc part. We anticipate, therefore, that the frequency dependent Pedersen conductivity will significantly modify the magnetorotational instability.

The Hall conductivity is

$$\sigma_H = -\left( 1 + \frac{\Delta \sigma_H}{\sigma_H^0} \right) \left( \frac{\sigma_{H}^0}{\sigma_H^0} \right) ,$$ \hspace{1cm} (44)

where the dc part is given as

$$\sigma_H^0 = \frac{e c n_i}{B} \sum_j Z_j \left( \frac{1 + \beta_j^2}{1 + \beta_j^2} \right) .$$ \hspace{1cm} (45)

The frequency dependent parts, $\Delta \sigma_{Hr}$, and, $\Delta \sigma_{H\phi}$ are

$$\Delta \sigma_{Hr} = \frac{e c n_i \beta_i^2}{B} (H_r(\omega) - 1) ,$$ \hspace{1cm} (46)

and,

$$H_r(\omega) = \frac{(1 + \beta_i^2) \hat{\Omega}_j}{\beta_i D_2} ,$$ \hspace{1cm} (47)

for $j = 1, 2$. The radial and azimuthal component of the Hall conductivities are not equal due to the unequal radial and azimuthal coefficient in the ion momentum equation. The real and imaginary part of $H_r$ is given by equations (39)-(40) if we recognize that right hand side of equation (47) has a factor $1/D_2 = Q(\omega)/D_1 (1 + \beta_i^2)$. Thus, near resonance

$$\left( 1 + \frac{\Delta \sigma_{Hr}}{\sigma_H^0} \right) \approx \frac{\Omega_1 \left( 1 + \Omega_1 \Omega_2 \right)}{\beta_i \left( 1 + \Omega_1 \Omega_2 \right)^2 + 4 \left( \frac{\omega}{v_{in}} \right)^2} .$$ \hspace{1cm} (48)

Except for $\hat{\Omega}_1$ becoming $\hat{\Omega}_2$, the remainder of the expression for the azimuthal factor will be identical to equation (48).

The analogy to LCR resonance can be brought closer if we express Ohm’s law, equation (31) in diagonal form. To that end, we shall express $E_{i}^{\prime}$ in the eigen-basis vectors of the rotation operator $e_{\pm} = (e_x \pm e_y)/\sqrt{2}$. Then $B \times e_{\pm} = \mp i e_{\pm}$ and, Ohm’s law for the transverse component $J_{\pm}$ can be written as

$$J_{\pm} = (\sigma_P \mp i \sigma_H) E_{i}^{\prime} \pm .$$ \hspace{1cm} (49)

While writing equation (49), we have assumed $\sigma_{Hr} = \sigma_{H\phi} = \sigma_H$. We may write $J_{\pm} = Z_{\pm} (\omega) E_{i}^{\prime} \pm$. Here the impedance $Z_{\pm}(\omega) = R_{\pm} + i X_{\pm}(\omega)$, with $R_{\pm} = Re[\sigma_H^0] \pm Re[\sigma_H^0]$ and, $X_{\pm} = Im[\sigma_H^0] \mp Re[\sigma_H^0]$. Near resonance, $Z_{\pm} = R_{\pm}$. Thus
$J_\pm = R_{\perp} E_{\perp} \sim \frac{E'_{\perp}}{(1 + \frac{\omega}{\omega_c}) (1 - \frac{\omega}{\omega_c})}$. (50)

For a growing mode $\omega \sim 2\pi a$, and, oscillations in the current can take place in the absence of a neutral-frame electric field $E'_{\perp} \to 0$. In a weakly ionized disc, if $\beta \lesssim 1$, the oscillation in the current is set by the ions diffusing across the ambient magnetic field. For $\nu_i \sim i\omega$, near resonance, collision will act like a driver of the resonance.

We note that near resonance, the conductivity may change sign. From (43), when $\Omega < \nu_i$, the negative conductivity will play an important role. The ratio of the dynamical to the ion-neutral collision frequency determines whether negative conductivity is important. The DC conductivity becoming negative within certain frequency range, in the microwave irradiation is well known in the condensed matter literature, e.g. Ryzhii (2005).

4.2 Dispersion relation

The linearized neutral equation of motion (20) can be written as

$$
\left( \begin{array}{cc}
\omega & 2i\Omega \\
-i\Omega & \omega
\end{array} \right) \delta v_{\perp} = -k v_A^2 \left( \frac{\delta B_{\perp}}{B} \right),
$$

where subscript $\perp$ denotes two dimensional vector in the disc plane and $v_A = B/\sqrt{4\pi \rho}$ is the Alfvén velocity in the total fluid. The linearized induction equation, after substituting for $\delta v_{\perp}$ is given as

$$
\left( \begin{array}{cc}
\omega_A^2 + 3\Omega^2 & 2i\omega \Omega \\
-2i\omega \Omega & \omega_A^2
\end{array} \right) \delta B_{\perp} = ik c \left( \begin{array}{c}
2\Omega \\
-i\omega / 2
\end{array} \right) \delta E'_{\perp},
$$

where $\omega_A^2 = \omega^2 - k^2 v_A^2$.

In the ideal MHD limit, when $\delta E'_{\perp} = 0$, one recovers magnetorotational mode. In the absence of rotation, a dispersion relation for ideal MHD can be derived by setting determinant of left hand side matrix to zero. The departure from ideal MHD is due to the collisional effects. They will appear when electric field is eliminated in favour of magnetic field. In the ideal MHD limit, when $\Omega < \nu_i$, the negative conductivity will play an important role.

When ion inertial effects are ignored, the expressions for $a$, $b$ and $c$, in equations (57)-(59) reduces to W99. Ideal MHD is recovered in $\omega_i > \Omega$ limit.

5 ENERGETICS OF THE DISC

Before we discuss numerical results, let us examine the various factors (viz. Lorentz force, Joule heating) that may affect magnetorotational instability. The Lorentz force, $J \times B$, acts on the neutrals through collisions. Making use of equation (31), it can be written as

$$
\frac{J \times B}{c \rho \Omega} = \left( \frac{\chi_B V_B + \chi_H \hat{B} \times V_B}{\gamma_B} \right).
$$

Here $\gamma_B = (\sigma_{\perp}/\sigma_{\parallel}) \chi_H = (\sigma_{\parallel}/\sigma_{\parallel}) \chi$ and $V_B = cE' \times B/B^2$ is the drift velocity of $B$ through the neutrals. The first term on the right hand side is a measure of simultaneous acceleration and frictional drag; viewed from a neutral frame, this force accelerates the neutral towards $E' \times B$ velocity. The parameter $\chi_B$ provides the strength of the collisional coupling. With decreasing $\chi_B$, i.e. when the ionized medium is far from ideal MHD regime, this term may become increasingly unimportant. Thus the modification to the ideal MHD modes will be severe in the small $\chi_B$ limit since collision modifies the fluid response to the magnetic field. Note that in the ambipolar regime, this term is responsible for dissipation as well as feeding of energy to the neutrals. The second term will accelerate the neutral in the direction of $E'$.

In order to understand the implications for the magnetorotational instability, we need to identify the conditions under which energy is fed to the fluctuations. Recall that the electric field $E' = E + v_n \times B/c$ is given in the neutral...
frame and thus,
\[ J \cdot E = \sigma_i E_{\|}^2 + \sigma_P \left( E_{\perp}^2 + v_n \cdot E_{\perp} \times B/c \right) + \frac{B}{c} \sigma_H \left( v_n \cdot E_{\perp} \right). \]

(61)

Clearly then, the energy exchange consists of Joule heating and acceleration of the neutral medium. The term \( \sigma_i E_{\|}^2 + \sigma_P \) is the Joule heating. This term is always positive for positive \( \sigma_P \). However, since \( \sigma_P \) may become negative near resonance and the Ohmic term \( \sigma_P E_{\perp}^2 \) may feed rather than dissipate energy. Therefore, in the ambipolar regime, fluctuations may grow. The terms \( v_n \cdot E_{\|} \times B \) and \( v_n \cdot E_{\perp} \), for ambipolar and Hall respectively, corresponds to the feeding or, extraction of the kinetic energy by the Lorentz force. Therefore, depending upon the sign of the kinetic energy terms, the Lorentz force may facilitate either growth or damping of the magnetorotational instability.

6 RESULTS

We shall discuss the dispersion relation (56) numerically in various limiting cases and discuss modifications due to ion inertia. In the absence of ion inertia, various \( \chi \) limits and its effect on the magnetorotational instability have been discussed in detail in W99. We assume that electrons are frozen in the magnetic field, i.e. \( \beta_e = \infty \). In this limit, \( \sigma_P / \sigma_H = \beta_i \).

Therefore, we shall solve the dispersion relation (56) by varying key parameters \( \chi \), \( \beta_i \), and \( \nu = \nu_\infty / \Omega \). In accretion disc environment, the value of \( \beta_i \), may vary in a wide range. We shall choose \( \beta_i \) between 0.1 and 1. Although higher value of \( \beta_i \) can be chosen, the growth rate of magnetorotational instability will be very small. The parameter \( \nu_\infty / \Omega \) is similarly varied in a wide range.

6.1 Variation of \( \beta_i \) for \( \chi = 0.1 \), \( \nu = 1 \)

In Fig. 2(a) we plot the growth rate by varying \( \beta_i \) while keeping \( \nu = 1 \) and \( \chi = 0.1 \) fixed. With the decreasing \( \beta_i \), the parameter window of magnetorotational instability extends towards short wavelength and the growth rate exceeds ideal MHD limit for \( \beta_i = 0.1 \). With the decrease in \( \beta_i \) when \( \omega_{ci} < \nu_\infty \sim \Omega \), the mode grows up to 0.92\( \Omega \) for \( \beta_i = 0.1 \). This can be possibly attributed to the fact that owing to the faster collisional (and Keplerian) time scales in comparison with the gyration time, rotational free energy becomes available to the fluctuations at the collisional time scale. For further decrease in the value of \( \beta_i \) to 0.01, the maximum growth rate decreases. This behaviour indicates that if \( \beta_i \) is increased beyond some critical value, the Hall diffusion is dominated by the ambipolar diffusion. The growth rate for \( \beta_i = 1 \) curve is small. This regime correspond to \( \omega_{ci} \sim \nu_\infty \sim \Omega \), i.e. the rate of ion gyration and collision with the neutrals is comparable with the rotational frequency. With the increase of \( \beta_i \), the growth rate decreases and appears altogether for very large \( \beta_i \). Thus, Hall effect (caused by the ion inertial and collisional effects), starts dominating the ambipolar diffusion and the mode starts growing. Further decrease in \( \beta_i \) and increase in Hall conductivity reduces the growth rate to 0.82\( \Omega \). The collisional effect weakens with decreased \( \beta_i \) and thus, the parameter window operates in both small and large wavelength regimes.

6.2 Variation of \( \chi \) and \( \nu \) for fixed \( \beta_i \)

In Fig. 2(b) the growth rate is given for varying \( \chi \) and fixed \( \beta_i = 0.1 \), \( \nu = 1 \). With the decreasing \( \chi \) (i.e. increasing collisional coupling), the wavenumber window of the instability shifts towards long wavelength, consistent with the fact that non-ideal MHD effects start playing an increasingly important role for smaller \( \chi \). The result is similar to W99. This result is also in agreement with Blaes & Balbus (1994). We see that with the decreasing \( \chi \) the growth rate remains unchanged. Only change is in the wavenumber window that shifts towards the long wavelength consistent with the Hall dominated result of W99.

In fig. 2(c) we plot the growth rate for \( \chi = 1 \) and \( \beta_i = 0.1 \) for varying \( \nu \). The results are along the expected line. With the increase in collision, the growth rate decreases due to increased rate of dissipation and the parameter window of instability shifts towards long wavelength. When the ion-neutral collision rate is comparable (or smaller) to the rotational frequency, the MRI is unaffected by the collision. This is because the rate of dissipative loss of the energy is comparable or slower than the rotational time scale \( \nu_\infty^{-1} \sim \Omega^{-1} \). Thus dissipation stops affecting the growth rate and it saturates around \( \sim 0.75 \). Further decrease of \( \nu \) does not affect the growth rate.

6.3 General limit with \( \beta_i = 1 \)

In figure 3(a) we plot the growth rate for the positive orientation of the magnetic field with respect to the rotation axis, i.e. \( s = 1 \). The results are similar to the known results of W99. However, there are some interesting differences towards small \( \chi \) limit. Whereas, the wavenumber window of magnetorotational instability shifts towards longer wavelengths in small \( \chi \) limit, the growth rate of instability is not very sensitive. The ion inertia is able to provide the free energy to the fluctuations that can counterbalance the dissipative losses in small \( \chi \) regime where collision coupling between ion and neutral is very strong. Thus, with the decrease in the value of \( \chi \), small wavelength fluctuations are all suppressed leaving large wavelength modes to grow at \( \sim 0.82 \Omega \).

In figure 3(b) we plot the growth rate against \( \nu_\infty / \Omega \). We see that when \( \nu_\infty \geq \Omega \) i.e., when collision time \( t_c \equiv \nu_\infty^{-1} \) is smaller than the rotational time \( t_r \equiv \Omega^{-1} \), and the free energy is dissipated by the collision. The growth rate of fluctuations decreases. We see from the plot that with increasing \( \nu \) the magnetorotational instability growth rate reduces significantly. However, the growth rate becomes insensitive beyond \( \nu = 10 \). This indicates that in the large \( \nu \) limit, ion inertial effect, namely Hall effect starts cancelling dissipation and thus, growth rate becomes insensitive to any further increase of \( \nu \). In the opposite limit, i.e. when \( \nu \ll 1 \) (or, \( t_c > t_r \)), the energy available to the fluctuation is at the rotational time scale and, infrequent, slow ion-neutral collision is unable to influence the growth of the instability. At \( \nu = 0.01 \), the growth rate becomes maximum \( \sim 0.9 \Omega \) and any further decrease in \( \nu \) do not change the growth rate significantly.
Ion dynamics and the magnetorotational instability in weakly-ionized discs

Figure 2. The magnetorotational instability growth rate for $\sigma_P = 1$ for varying $\beta_i$ and $\sigma_H$ with fixed $\nu = 1$ and $\chi = 0.1$ is plotted against the wavenumber. The number against the curve is the value of ion Hall $\beta_i$ for $\sigma_H = 1, 10$ and 100. In figure 2(b) we hold $\nu$ fixed and give growth rate for various $\nu$. In Fig.2(c) we hold $\chi$ fixed and vary $\nu$.

Figure 3. As for Fig.2 but for $\sigma_P = \sigma_H = 1$

6.4 Weak ambipolar limit with $\beta_i = 0.1$, $s = -1$

In figure 4(a) and 4(b) we plot the growth rate for $B_2 < 0$ by varying $\chi$ and $\nu$ respectively. In figure 4(a) the growth rate is slightly smaller than for corresponding case in Fig. 3(a) for $s=-1$. For $\chi \leq 0.1$ the growth rate is insensitive to the changing value of $\chi$ implying that if collision frequency is an order of magnitude smaller than the Keplerian frequency, the ambipolar effects are entirely compensated by the Hall and the growth rate remains constant. The curves are similar to W99 except that the mode exists for much smaller value of $\chi$ than was the case in W99. Also, the growth rate is larger. The growth rate increases with increasing $\chi$ and attains maximum value for $\chi = 10$. If $\chi$ is increased further, there is no change in the growth rate. The rate at which rotational energy becomes available and dissipation operates become comparable. For $\chi > 1$, the ideal MHD limit is approached and thus the effect of collision diminishes. Thus for $\chi = \infty$ the growth rate is maximum.

In figure 4(b) the growth rate is not very sensitive to change in $\nu$ except when it becomes large. For large $\nu$ the growth rate reduces in comparison with the small $\nu$ values and parameter window extends towards smaller wavelength.
6.5 General limit for \( s = -1 \) and \( \beta_i = 1 \)

In Fig. 5(a) we plot the growth rate for \( s = -1 \) for varying \( \chi \). The growth rate is insensitive when \( \chi < 1 \). Small \( \chi \) is a measure of departure from ideal MHD and we note that Hall effects play important role for \( \chi < 1 \). Since for \( s = -1 \) the sign of the wave helicity \( \mathbf{\Omega} \cdot \delta \mathbf{B} \) is negative since Alfvén wave is propagating in the negative direction (??). Thus, the increase in the non-ideal effect, manifested through Hall terms, does not have any bearing on the growth rate. With the increase of \( \chi \) the magnetorotational instability growth rate approaches ideal limit.

In Fig. 5(b) the variation of growth rate with \( \nu \) is given. For \( \nu \leq 1 \) there is no change in the growth rate. For \( \nu > 1 \) the instability is damped due to dissipation. The sign of the helicity ensures that non ideal effect do not feed the energy to the fluctuations. Thus we see the decrease in the growth rate with increasing \( \nu \). Therefore, we see that in the general case when the ambipolar conductivity may change, the growth rate of the instability becomes larger than the rotation frequency and the instability operates at long wavelengths. Thus, the ion-inertial effect introduces entirely new feature to the dynamics of a weakly magnetized disc. Not only it changes the length scale over which the instability can operate but also, how fast it can operate. These feature makes ion inertial effect very important for the application to the protostellar discs.

7 APPLICATIONS

The modification to the magnetorotational instability by ion inertia may have wide ranging application in the astrophysical discs. Before discussing the application of the results, we shall note that the parameters \( \chi, \nu \) and \( \beta_i \) are not independent but are related by the following equation

\[
\frac{\rho_i}{\rho_n} = \sqrt{1 + \beta_i^{-2} \left( \frac{\chi}{\nu} \right)} ,
\]

(62)

where \( \nu \equiv \nu_{\text{collision}} / \Omega \) is the normalized collision frequency and \( \chi \) is the measure of non-ideal MHD effects. Therefore, the choice of \( \chi, \beta_i \), and \( \nu \) constrains the ratio \( \rho_i / \rho_n \) and hence, the level of fractional ionization.

At densities relevant to cloud cores and protostellar discs (densities \( \geq 10^{13} \text{cm}^{-3} \)), grains are the dominant charge carriers and their presence can significantly alter the dynamics of the disc. The ionization fraction is strongly affected by the abundance and size distribution of the grains through the recombination process on the grain surface. Near the midplane of the PPDs grains are the main charged
constituent (Wardle & Ng 1999). We shall assume a Keplerian frequency for the minimum mass solar nebula (0.1 $M_\odot$), $\Omega \sim 10^{-8}\,\text{s}^{-1}$. Then, from equations (15) and (18) we write
\[ \frac{\nu_{in}}{\Omega} \sim \frac{m_n}{m_i} 10^{14} a_2 \left( \frac{n_n}{10^{15}\,\text{cm}^{-3}} \right). \] (63)

Taking $m_n/m_i \sim 10^{-14} - 10^{-15}$, we see that $\nu_{in}$ is comparable to the dynamical frequency and ion-inertia becomes important. The collision of the energetic electron with neutrals or magnetorotational instability induced turbulent convective homogenization of the entire disc (Inutsuka & Sano 2005) may allow the magnetic field to couple to the disc matter near the midplane. Therefore, the ion-inertia may modify the parameter window of instability near the midplane of the disc.

In AGNs, for example NGC 4258, a thin disc of 0.2 pc diameter, bound by a central mass of $\sim 2.1 \times 10^7\,M_\odot$, is rotating with a velocity 900 km s$^{-1}$ (Greenhill et al. 1995). The observed emission emanates from an annulus of inner radius $\sim 0.13$ pc and outer radius of 0.25 pc. Taking $R = 0.1$ pc, we get $\Omega \sim 10^{-10}\,\text{s}^{-1}$. Thus the ratio $\nu_{in}/\Omega \sim n_n/\text{cm}^{-3}$. Taking ionization fraction $X_e = 10^{-5}$ (Menou & Quataert 2001) at 0.1 pc, we see from equation (29) that $Re_A = 10^{-3}\,n_n/\text{cm}^{-3}$. The neutral density $n_n \sim 10^7\,\text{cm}^{-3}$ and thus, both ion-neutral $\nu_{in}$ as well as neutral-ion collision frequencies are very large in comparison with the rotation frequency. The charged grains are negligible in such a disc since from $\rho_i/\rho_n = 10^{-2}$, we have $n_i/n_n \sim 10^{-14}$ for micron-sized grains. Therefore, the charged grains are absent in such a disk and grain inertia will have no effect on the disc dynamics.

Given the uncertain nature of the disc size, if we assume a disc of 100 pc with a temperature gradient towards the outer edge of the disc then the inertial effects in such a disc will be due to the charged grains near the core of the disc and due to the lighter ions near the surface region of the disc. For $\Omega \sim 10^{-8}\,\text{s}^{-1}$ (at 100 pc), $\nu_{in}/\Omega \sim 10^{-2}\, n_n/\text{cm}^{-3}$, $X_e \leq 1$, we see that in the surface region of an disc, both ion-neutral $\nu_{in}$ and neutral-neutral $\nu_n$ collision frequencies will be affected by the magnetorotational instability. Thus, ion-inertial effect may be important in exciting MHD turbulence in the whole disc.

Cataclysmic Variables (CV) are close binary systems with a white dwarf accreting material from a Roche-lobe lobe component and a companion low mass main sequence star. The typical orbital frequency of the CVs vary between $\Omega \sim 10^{-3} - 10^{-5}\,\text{s}^{-1}$. Then $\nu_{in}/\Omega \sim (10^{-5} - 10^{-7}) \times n_n/\text{cm}^{-3}$. For $n_n \sim (10^5 - 10^7)\,\text{cm}^{-3}$, $\nu_{in}/\Omega \sim O(1)$. The temperature in CVs may vary in a wide range and disc can be modeled either as a weakly ionized plasma (Gammie & Menou 1998) or a completely ionized plasma (Saxton et al. 2005). Clearly, both ion and neutral inertial effect operate on an equal footing in CVs.

The circumnuclear disc at the galactic centre has a typical constant rotation speed of 110 Km s$^{-1}$ (Genzel & Townes 1987) between 2 to 4 pc. The corresponding rotational frequency at 2 – 4 pc is, $\Omega \sim 10^{-12} - 10^{-13}\,\text{s}^{-1}$. Then the ratio between ion-neutral collision to the Keplerian rotation frequency is
\[ \frac{\nu_{in}}{\Omega} \approx (10^2 - 10^3) \times n_n, \] (64)
for given $\nu_{in}$ value (equation (17)). Hence at $\Omega = 2 - 4\,\text{pc}$, the ion inertial response time, $\Omega^{-1}$ is thousand times slower than the collisional momentum exchange time, $\nu_{in}^{-1}$. Therefore the ability of ion inertia to modify the magnetorotational instability at $\Omega = 2\,\text{pc}$ is unclear although $Re_A \sim 1$ for $n_i/n_n \sim 10^{-3}$. At 100 pc where $\Omega$ drops by two orders, $\nu_{in}$ becomes comparable to the rotational frequency and inertial effect on the magnetorotational instability may become important.

### 8 CONCLUSIONS

The paper examines the role of ion inertial effect on the magnetorotational instability in a weakly ionized, thin, magnetized Keplerian disc. The vertical stratification and radial and azimuthal variations were neglected - an approximation valid for the wavelengths small compared to the disc scaleheight. The conductivity tensor becomes time dependent in the presence of ion inertial terms. This may result in the conductivity becoming negative near resonance. Further, radial and azimuthal component of Hall conductivity will be different. The following results were found.

(i) The conductivities in a weakly ionized gas is in general complex in the presence of ion inertia and may become negative near the resonance points. The magnetorotational instability gets significantly modified in the presence of time dependent conductivities.

(ii) In weak ambipolar regime, the presence of ion-inertial effect substantially modifies the behaviour of the instability in the non-ideal ($\chi < 1$) limit. The maximum growth rate in the Hall dominated regime is $\sim 0.92$ (in the units of $\Omega$) and large wavelength fluctuations can grow due to the inertial effect.

For a fixed $\chi$, the growth rate is maximum ($\sim 0.85$) for $\nu = 1$ and starts to decrease with increasing $\nu$. Further, the parameter window shifts towards longer wavelength with increasing $\nu$.

(iii) When both ambipolar and hall diffusion are comparable, the maximum growth rate is $\sim 0.85$ and 0.95 for fixed $\nu$ and $\chi$ respectively. The increase in the ambipolar diffusion causes the fixed $\chi$ case to have larger growth rate than fixed $\nu$ case.

To summarize, it is a common feature that large scale fluctuations exhibit the maximum growth rate of the magnetorotational instability when the ion inertial effects are included in the dynamics. This may have important implication on the onset of turbulence in weakly ionized discs. For example, in PPDs, when grains are dominant ions, the grain inertial effect will significantly modify the magnetorotational instability growth rate and thus, will play an important role in the onset of hydromagnetic turbulence. In AGNs also, grain will play important role. In CVs grain will provide the ion inertia away from the surface of the dwarf novae whereas, lighter ionized elements will provide the inertial effect close to the surface of the disc. Therefore, in CV discs, inertial effect may be important all across the disk and inertia modified magnetorotational instability may effect the whole disc. In circumnuclear discs, the effect of ion inertia may be important far away from the centre of the disc. All in all, ion inertia seems to play an important role on the onset of magnetorotational instability.

Present work investigates the role of ion inertia in the presence of an axial magnetic magnetic field. It will be interesting to investigate the role of ion inertia on the mag-
netorotational instability for a more general field geometry, particularly in the context of profile independent destabiliz-
ing feature of ambipolar diffusion (Kunz & Balbus 2004). We shall leave this problem for future consideration.

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