Classification of multipartite entanglement containing infinitely many kinds of states

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We give a further investigation of the range criterion and Low-to-High Rank Generating Mode (LHRGM) introduced in [13], which can be used for the classification of $2 \times M \times N$ states under reversible local filtering operations. By using of these techniques, we entirely classify the family of $2 \times 4 \times 4$ states, which actually contains infinitely many kinds of states. The classifications of true entanglement of $2 \times (M + 3) \times (2M + 3)$ and $2 \times (M + 4) \times (2M + 4)$ systems are briefly listed respectively.

I. INTRODUCTION

The rapid development of quantum information theory (QIT) requires a further understanding of the properties of entanglement [1]. Among many fundamental questions in QIT, it is essential to find out how many different ways there exist, in which several spatially distributed objects could be entangled under certain prior constraint of physical resource, e.g., local operations and classical communications (LOCC). This issue was first addressed by Bennett et al. [2]. They proved that the Bell pair $\langle 00 \rangle + |11\rangle/\sqrt{2}$ is unique in the pure settings when infinitely many copies of states are available since the equivalent states can be used for the same tasks in QIT.

The situation becomes complicated when only a single copy of state is given. Due to the celebrated result by [3], the bipartite pure states have been endowed with a nice classification under LOCC. Unfortunately, there does not exist a similar theory in the multipartite setting because the Schmidt polar form [1, 2, 3, 6] no longer acts here. In addition, it turned out that the LOCC classification is lowly efficient for the multipartite entanglement [2, 8]. For simplicity, Dür et al. [3, 11] has considered the LOCC classification just in a stochastic manner (SLOCC for short [11, 12], or local filtering operations), under which two states are equivalent if and only if (iff) they are interconvertible with a nonvanishing probability. By this criterion, they have explicitly shown that there exist two sorts of fully entangled states in the three-qubit space, the GHZ state $\langle 000 \rangle + |111\rangle/\sqrt{2}$ and the W state $\langle 001 \rangle + |010\rangle + |100\rangle/\sqrt{3}$ [13], which implies that the GHZ state is not the sole representative inferred as before. Notwithstanding, the existing results [4, 5, 14] concentrating on the system with low dimensions showed that the SLOCC classification is not a universally valid method for the multipartite entanglement. On the other hand, it is of importance to describe the structure of multipartite entanglement with higher dimensions. This issue has been addressed in our earlier work [17], where we have introduced the range criterion to judge whether two multiple entangled states are equivalent under SLOCC. Based on this criterion, we proposed the so-called Low-to-High Rank Generating Mode (LHRGM) for the classification of $2 \times M \times N$ states. Specifically, we described how to write out the essential classes of true entanglement in the $2 \times 3 \times 3$, $2 \times (M + 1) \times (2M + 1)$ and $2 \times (M + 2) \times (2M + 2)$ spaces respectively.

The main intention of this paper is to classify the family of $2 \times 4 \times 4$ states, which actually contains infinitely many entangled classes under SLOCC (we shall use $\sim$ to denote the equivalence under SLOCC in this paper and only concern the true entangled states whose local ranks do not change under SLOCC [10]). This helps analyze the structure of generally multipartite entanglement which usually contains parameters. In order to do this, we first make a further study of the techniques including the range criterion and LHRGM in section II. In particular, we will clarify that the range criterion is an effective method for distinguishing the multipartite entangled states, especially for those owning product states in some ranges of the reduced density operators of these states. We also exemplify its use in the more general cases such as the $3 \times 3 \times 3$ and 4-qubit systems. As for the technique of LHRGM, we focus on the analysis of the family of $\{\Omega_1\}$ which is a special branch in this method. The result shows that this branch concerns the classification of $2 \times M \times M$ states. In fact, we discuss here the classification of true entanglement with infinitely many kinds of states under SLOCC related with $2 \times M \times M$ states. Subsequently, we completely classify the $2 \times 4 \times 4$ states by LHRGM in section III. In section IV, we summarize the existing results and propose the so-called quasi-combinatorial character in any sequence of true $2 \times M \times N$ systems, $N = 1, 2, \ldots, 2M$, and briefly give the classifications of true entanglement of $2 \times (M + 3) \times (2M + 3)$ and $2 \times (M + 4) \times (2M + 4)$ systems respectively, whose detailed proofs are omitted. The conclusions are proposed in section V.

II. SOME RESULTS FROM THE RANGE CRITERION AND LHRGM

Let us firstly recall the range criterion and the method of LHRGM for the $2 \times M \times N$ states, and the proof of...
these techniques can be found in [17]. A general pure state can be expressed as \( \rho_{A_1 A_2 \ldots A_N} = |\Psi\rangle A_1 A_2 \ldots A_N \langle \Psi| \), and the reduced density operator \( \rho^A_{k+1,k+2: \ldots : A_N} \equiv \text{tr}_{A_1 A_2 \ldots A_k} (\rho_{\Psi}) t_{1,2, \ldots , k} \in \{1,2, \ldots , N\}, k \leq N - 1 \). Any state \( |\Psi\rangle A_1 A_2 \ldots A_N \) in the \( D_1 \times D_2 \times \ldots \times D_N \) space (sometimes also written \( |\Psi\rangle D_1 \times D_2 \times \ldots \times D_N \)) can always be transformed into the adjoint form,

\[
|\Phi\rangle = \sum_{i=0}^{D_j-1} |i\rangle A_j \otimes |i\rangle A_1 A_2 \ldots A_{j-1} A_{j+1} \ldots A_N,
\]

where the computational basis \( |i\rangle A_j = \delta_{ik} \) and \( \{ |i\rangle A_1 A_2 \ldots A_{j-1} A_{j+1} \ldots A_N, i = 0, 1, \ldots , D_j - 1 \} \) are a set of linearly independent vectors, each \( |i\rangle A_1 A_2 \ldots A_{j-1} A_{j+1} \ldots A_N \) is the adjoint state of \( |i\rangle A_j \). If \( \rho \) acts on the Hilbert space \( \mathcal{H} \), then the range of \( \rho \) is \( \text{R}(\rho) = \rho |\Phi\rangle \langle \Phi| \in \mathcal{H} \).

Then we can write out the range criterion as follows.

**Range Criterion.** For two multiple states \( |\Phi\rangle A_1 A_2 \ldots A_N \) and \( |\Psi\rangle A_1 A_2 \ldots A_N \) there exist certain ILO's \( V_i, i = 1, \ldots , N \) making \( |\Psi\rangle A_1 A_2 \ldots A_N = V_1 \otimes \ldots \otimes V_N |\Phi\rangle A_1 A_2 \ldots A_N \) iff there exist a series of numbers \( i_1, i_2, \ldots , i_{N-1} \in \{1,2, \ldots , N\} \), such that \( \text{R}(\rho^A_{i_1 i_2 \ldots i_{N-1}}) = V_{i_1} \otimes \ldots \otimes V_{i_{N-1}} \).

Let \( |a_1 a_2 a_3, \ldots , a_N\rangle \) represent a set of states, and a state \( |\Phi\rangle A_1 A_2 \ldots A_N \) is a number of product states in \( \text{R}(\rho^A_{i_1 i_2 \ldots i_{N-1}}) \) is \( a_i, \ i = 2, \ldots , N - 1 \).

The range criterion has been formed for the classification of several sorts of true entanglement in [17], and here we would like to further discuss how to generically distinguish the triply entangled states in \( 2 \times M \times N \) space. For triply qubit states, a method of practical identification has been given in [10], where the 3-tangle [19] (decided by concurrence [20]) is employed as a criterion. Recently, resulful progress in calculation of the concurrence of arbitrary bipartite states has been achieved by [21], but it is unclear that whether there exist a generally certain criterion determining the relation between a multiple entangled state and the concurrences of its reduced density operators in all bipartite subspaces, simply similar to that in [10]. In fact, it remains a formidable challenge to QIT [22]. While in [17], the range criterion provides a universal and effective method to distinguish the multipartite entanglement in \( 2 \times M \times N \) space. We summarize this procedure as follows. Given a set of states \( |\psi_i\rangle, i = 1, 2, \ldots , \) in the \( 2 \times M \times N \) space. One first finds out their adjoint forms respectively. Then the set can be split into several subsets of entangled states with different local ranks. Clearly, any two states from different subsets are inequivalent under SLOCC [10]. For each subset, one applies the range criterion to judge whether the states are equivalent. The concrete example has been given in [17], where the product states in the ranges greatly help the judgement. Moreover, one can find out the ILO’s between two equivalent states through a procedure similar to the arguments in [13], while the calculation therein is relatively succint.

It should be noted that there exist a special kind of state, i.e., \( |\Psi\rangle_{2 \times M \times M} \). For this case, one has to compare whether \( |\Psi\rangle_{ABC} \sim |\Psi\rangle_{ACB} \). One may guess that the above relation always holds, for the existing results for \( 2 \times 2 \times 2 \) and \( 2 \times 3 \times 3 \) systems support it. However as shown in the next section, it is not true in the case of \( 2 \times 4 \times 4 \) system [22]. In general, a difficult problem emerges when trying to classify the multiple entangled states, i.e., the permutation of the parties makes the situation much more sophisticated, when (part of) the local ranks of the state are identical. For example, the multiqubit system can be in distinct state by exchanging the parties therein, which may be the most trouble finding out the essential classes of multiqubit states. In this case, the range criterion can effectively help analyze the structure of multipartite entanglement.

For instance, consider a family of 4-qubit state

\[
|\Phi_{x/4}^{ABC} = |00\rangle \langle 00| + |11\rangle \langle 11| + |00\rangle \langle x|11| + \langle x |00\rangle \equiv \Phi_{x/4}\]

For simplicity we can generalize the definition of \( |a_1 a_2 a_3, a_4, a_5\rangle \) in the range criterion, e.g., a state \( |\Phi_{x/4}^{ABC} = |a_1 a_2 a_3, a_4, a_5, a_6\rangle \) iff the number of product states in \( R(\rho_{A_1 A_2 A_3}) = a_1 \), in \( R(\rho_{A_2 A_3}) = a_2 \), in \( R(\rho_{A_3}) = a_3 \), in \( R(\rho_{A_4}) = a_4 \), in \( R(\rho_{A_5}) = a_5 \), in \( R(\rho_{A_6}) = a_6 \). Thus we denote that \( \text{R}(\Phi_{x/4}^{ABC}) \in \{2, \infty, \infty, \infty, \infty\} \). Due to the range criterion, the possible equivalence by permutation is nothing but \( \Phi_{x/4}^{ABC} \sim \Phi_{x/4}^{CDB} \) (the two subsystems AB, and CD are symmetric respectively).

So we can easily dispose of this 4-qubit family in terms of the range criterion, and the result is \( \Phi_{x/4}^{ABC} \sim \Phi_{x/4}^{CBA} \sim \Phi_{x/4}^{BAC} \sim \Phi_{x/4}^{CDB} \). A more interesting result is about the 3-qrit state

\[
|\Psi_{ABC} = |001\rangle + |010\rangle + |100\rangle + |220\rangle + |202\rangle \in \{1, \infty, \infty, \infty\}. \]

Now the exchange of the parties indeed leads to inequivalent classes of entanglement,

\[
|\Psi_{BAC} = |010\rangle + |010\rangle + |220\rangle + |202\rangle \in \{\infty, 1, \infty, \infty\}, \]

and

\[
|\Psi_{CBA} = |010\rangle + |010\rangle + |100\rangle + |220\rangle + |202\rangle \in \{\infty, 1, \infty\} \]

Notice there is no other new classes derived from the permutation of parties by symmetry. So the range criterion is often practical to judge whether a state is symmetric or the exchange of the parties cause new classes of states whose local ranks are (partly) identical. Next, we explore the possibility applying the range criterion to the classification of more universal entangled states. For the systems with higher dimensions, one may still employ the range criterion to distinguish the given states. However, the situations therein are more complicated for there are not always product states in the ranges of reduced density operators in bipartite subspaces. For example, consider such a family of entangled states in the
\[\Psi_x\] is symmetric by the operations of \(|012\rangle + |120\rangle + |201\rangle + |000\rangle + |111\rangle + x |222\rangle,\]

where the parameter \(x \neq 0\) cannot be removed by ILO’s. One can readily check that there is no product state in \(R(\rho_{AB}^x), R(\rho_{AC}^x), \) and \(R(\rho_{BC}^x)\), for all adjoint states of entangled forms. It is then not easy to classify this family of states, we briefly make out the clue to this case. Clearly, \(|\Psi_x\rangle_{ABC}\) is symmetric by the operations of \(|1\rangle \leftrightarrow |0\rangle\) on each party. A primary observation tells \(|\Psi_x\rangle \sim |\Psi_{x-1}\rangle\), which can be realized by the ILO’s \(V_A^{3\times 3} = \text{diag}(1, 1, x^{-1})\) and two permutation transformations

\[V_B^{3\times 3} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, V_C^{3\times 3} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},\]

up to some permutation of the parties. In fact, there does not exist other equivalence relations in this 3-quirit family. To simplify the proof, we adopt the concept of normal form of a pure multipartite state [22]. That is, one can transform any \(|\Phi\rangle_{A_1A_2...A_N}\) into its normal form by the ILO’s, whose local density operators are all proportional to the identity and the normal form is unique up to local unitary transformations. In the light of this fact, by the ILO’s \(V_A^{3\times 3} = V_B^{3\times 3} = V_C^{3\times 3} = \text{diag}(1, 1, x^{-1/3})\), we obtain the normal form of \(|\Psi_x\rangle_{ABC}\),

\[|\Psi_x^\text{norm}\rangle_{ABC} = x^{-1/3} |012\rangle + x^{-1/3} |120\rangle + x^{-1/3} |201\rangle + |000\rangle + |111\rangle + |222\rangle,\]

whose local density operators are indeed proportional to the identity, so the possible ILO’s making \(|\Psi_x^\text{norm}\rangle_{ABC}\) into \(|\Psi_x^\text{norm}\rangle_{ABC}\) must be unitary. Moreover, one can readily check that the local rank of the states in \(R(\rho_{AB}^x)\text{norm} (\text{also } R(\rho_{AC}^x)\text{norm}, \text{and } R(\rho_{BC}^x)\text{norm})\) can be no more than two iff \(x = -1\), while in any other case the local rank of the combination of three adjoint states is always three, so there exist at least one zero in every column of the unitary transformations. Recall that a unitary matrix acting on a \(d\)-dimensional space can always be written as a product of several two-level unitary matrices [22]. With these techniques one can find out the form of the unitary transformations and get the above assertion. A striking feature of the state \(|\Psi_x\rangle_{ABC}\) is that it actually contains \(infinitely\) many classes of entanglement, which is the main topic in this paper. Besides this example, we will demonstrate the concrete techniques for such generic system by the classification of \(2 \times 4 \times 4\) states in next section. On the other hand, intuitively the cases with no product states in the ranges are more involved than those owning at least one product state. Unfortunately, finding out the normal form of a state is often difficult, or sometimes meaningless if it is identical to zero [24]. Although one can follow the formal procedure similar to that in the appendix in [17], it is now difficult to find a solution, or to prove there is no solution for the equations set. So the range criterion works more effectively when there is at least one product state in some range of the reduced density operator in bipartite subspace.

Let us move to the method of Low-to-High Rank Generating Mode (LHRGM), which can be expressed as follows.

**LHRGM.** For the classification of true tripartite entangled states under SLOCC, the following equivalence relation is true,

\[|\Psi\rangle_{2\times M \times N} \sim \begin{cases} |\Omega_0\rangle \equiv (a |0\rangle + b |1\rangle) |M - 1, N - 1\rangle + |\Psi\rangle_{2\times (M - 1) \times (N - 1)}, \\
|\Omega_1\rangle \equiv (0 |0\rangle, M - 1, N - 1) + |1, M - 1, N - 2\rangle + |\Psi\rangle_{2\times (M - 1) \times (N - 2)}, \\
|\Omega_2\rangle \equiv |\Omega_0\rangle + |0, M - 1\rangle |\chi\rangle, b \neq 0, \\
|\Omega_3\rangle \equiv |\Omega_0\rangle + |1, M - 1\rangle |\chi\rangle, a \neq 0. \\
\end{cases}\]

The condition \(a \neq 0\) or \(b \neq 0\) keeps \(|\Omega_2\rangle\) and \(|\Omega_3\rangle\) not becoming \(|\Omega_0\rangle\). \(|\chi\rangle \equiv \sum_{i=0}^{N-2} a_i |i\rangle\) and the arbitrary constants \(a_i\)’s do not equal zero simultaneously.

Notice that all related states \(|\Psi\rangle_{2\times M \times N}\) in the LHRGM are truly entangled. The LHRGM is indeed an iterated method, in the sense that the calculation of the essential classes of highly dimensional states requires that of low-level families. As shown in [17], one first calculates \(|\Omega_0\rangle\), which requires a relatively low amount of calculations. Based on it, the \(|\Omega_2\rangle\) and \(|\Omega_3\rangle\)’s family can be calculated economically, where several important invariance of ILO’s have been employed so that the calculation can be enormously simplified. The calculation of above three families are necessarily required for any classification of \(|\Psi\rangle_{2\times M \times N}\). Moreover, in the cases of \(2 \times (M + 1) \times (2M + 1)\) and \(2 \times (M + 2) \times (2M + 2)\) system in [17], the \(|\Omega_2\rangle\)’s family proved to be a subset of the above three families by induction.

We advance another problem on this special family. That is, does any \(|\Omega_1\rangle\)’s family always belong to corresponding \(|\Omega_0\rangle\), \(|\Omega_2\rangle\), and \(|\Omega_3\rangle\)’s family? The answer is negative, since we have found an exception in the \(2 \times 5 \times 3\) system, i.e., the state \(|001\rangle + |010\rangle + |112\rangle + |112\rangle\). By theorem 2 in [17], it does not belong to anyone of other three families, which indicates that the induction will not be available for every classification of entanglement in \(2 \times M \times N\) space. Apply the LHRGM technique to the term \(|\Psi\rangle_{2\times (M - 1) \times (N - 2)}\) in \(|\Omega_1\rangle\),

\[|\Omega_1\rangle_{2\times M \times N} \sim |0, M - 1, N - 1\rangle + |1, M - 1, N - 2\rangle + \begin{cases} (a |0\rangle + b |1\rangle) |M - 2, N - 3\rangle, \\
|\Psi\rangle_{2\times (M - 2) \times (N - 3)} + |1, M - 2, N - 3\rangle + |0, M - 2, N - 4\rangle + |\Psi\rangle_{2\times (M - 2) \times (N - 4)}, \\
(a |0\rangle + b |1\rangle) |M - 2, N - 3\rangle + |0, M - 2\rangle |\chi\rangle + |\Psi\rangle_{2\times (M - 2) \times (N - 3)}, b \neq 0, \\
(a |0\rangle + b |1\rangle) |M - 2, N - 3\rangle + |1, M - 2\rangle |\chi\rangle + |\Psi\rangle_{2\times (M - 2) \times (N - 3)}, a \neq 0, \\
\end{cases}\]

where \(|\chi\rangle \equiv \sum_{i=0}^{N-4} a_i |i\rangle\). As for expression (I), one performs \(|M - 1\rangle_B \leftrightarrow |M - 2\rangle_B\) and \(|N - 1\rangle_C \leftrightarrow |N - 3\rangle_C\), so that (I) \(\sim |\Omega_0\rangle_{2\times M \times N}\), which becomes one of other three families. The same ILO’s make (III) and (IV) identical to \(|\Omega_2\rangle_{2\times M \times N}\) or \(|\Omega_3\rangle_{2\times M \times N}\). Thus, the only exception is (II) \(\sim |0, M - 1, N - 1\rangle + \)
|1, M − 1, N − 2⟩ + |0, M − 2, N − 3⟩ + |1, M − 2, N − 4⟩ + |Ψ⟩_{2×(M−2)×(N−4)}^{exc}.\) Continue this procedure, which will cease when \(M−1−k=N−2−2k\), or \(k=N−M−1\). Thus the unique exception is

\[
|Ψ⟩_{2×M×N}^{exc} = \sum_{i=0}^{N−M−1} (|0, M−1−i, N−1−2i⟩ + |1, M−1−i, N−2−2i⟩) + |Ψ⟩_{2×(2M−N)×(2M−N)}^{′}.
\]

Notice there are no two identical terms in \(|N−1−2i⟩, |N−2−2i⟩⟩, i=0, 1, ..., N−M−1\). We investigate the above expression in several cases. (the states \(|\Omega_i⟩, i=0, 1, 2\) are defined in 17, their explicit forms are given in the section IV )

(i) \(N=2M\). The state \(|Ψ⟩_{2×M×2M}^{exc}\) is just the unique class \(|Ω_0⟩\) in \(2×M×2M\) space.

(ii) \(N=2M−1\). It is the case of theorem 3 in 17. Evidently, we have \(|Ψ⟩_{2×M×(2M−1)}^{exc} \sim |Τ_1⟩\), which belongs to the \(|Ω_1⟩\) family.

(iii) \(N=2M−2\). It is the case of theorem 4 in 17. There are two situations here corresponding to \(|GHZ⟩\) and \(|W⟩\) state respectively. If \(|Ψ⟩_{2×2×2}^{′} \sim |GHZ⟩\), by operations \(|N−1⟩_C \rightarrow |0⟩_C\) and \(|M−1⟩_B \rightarrow |0⟩_B\), we then transform \(|Ψ⟩_{2×M×(2M−2)}^{exc}\) to \(|Ω_0⟩\)’s family.

On the other hand if \(|Ψ⟩_{2×2×2}^{′} \sim |W⟩\), by the same ILO’s \(|Ψ⟩_{2×M×(2M−2)}^{exc}\) always becomes either \(|Ω_2⟩\) or \(|Ω_3⟩\). Therefore, (ii) and (iii) confirm the fact that \(|Ω_1⟩_{2×(M+1)×(2M+1)}\) and \(|Ω_1⟩_{2×(M+2)×(2M+2)}\) belong to other three relevant families respectively.

(iv) \(N=2M−3\), and we get \(|Ψ⟩_{2×3×3}^{′}\). By the result of \(2×3×3\) states in 17 and skills similar to that in (iii), we find the unique exception

\[
|Ψ⟩_{2×M×(2M−3)}^{exc} = \sum_{i=0}^{M−4} (|0, M−1−i, 2M−4−2i⟩ + |1, M−1−i, 2M−5−2i⟩) + |010⟩ + |001⟩ + |112⟩ + |121⟩.
\]

Thus we only need to calculate other three families when we classify the entanglement in \(2×M×(2M−3)\) space. In general, it is necessary to analyze the expression of \(|Ψ⟩_{2×M×N}^{exc}\) since the induction operates merely in the cases of (ii) and (iii). Conclusively, the LHLMG technique equals the calculation of \(|Ω_0⟩\), \(|Ω_2⟩\), \(|Ω_3⟩\) and \(|Ψ⟩_{2×M×N}^{exc}\).

So far we have managed to check several special cases of \(|Ψ⟩_{2×M×N}^{exc}\). Evidently, it is determined by \(|Ψ⟩_{2×(2M−N)×(2M−N)}^{exc}\) that whether \(|Ω_1⟩_{2×M×N}\) will produce exceptional states not belonging to the other families. Let \(Q=2M−N\). As for the case of \(Q=3\), we have found the essential classes of \(|Ψ⟩_{2×3×3}^{exc}\) in 17. With the help of existing results and techniques, one can step-by-step calculate each \(|Ψ⟩_{2×Q×Q}^{′}\)’s family by LHLMG, \(Q=4, 5, ...\). It is difficult to provide a restrict criterion determining whether the exceptional family are equivalent to those derived from \(|Ω_0⟩\), \(|Ω_2⟩\) and \(|Ω_3⟩\), for one has to use the LHRG technique in the ABC and ACB system in turn, since there will exist the cases of \(M>N\). In addition, there will more frequently be of infinitely many classes of entanglement in this family (see next section), which also enhances its complexity. On all accounts, the calculation of essential classes of \(|Ψ⟩_{2×(M−1)×(2M−Q)}^{exc}\) by LHLMG requires that one knows the classes of true entangled states of \(|Ψ⟩_{2×(M−1)×(2M−Q−1)}\) and \(|Ψ⟩_{2×Q×Q}^{′}\).

To summarize, we have reviewed the range criterion and applied it to the more general cases such as the 4-qubit and 3-qutrit systems, which can be entangled in infinitely many ways. For the technique of LHLMG, we have analyzed the possible structure of \(|Ω_1⟩\) and the method of its simplified calculation.

### III. CLASSIFICATION OF 2 × 4 × 4 STATES

As mentioned in the preceding section, a distributed system can be entangled in infinitely many ways with increasing of dimensions of Hilbert space. The simplest case is the family of 2 × 4 × 4 states, so it worth a further investigation. An early result by W. Dür et al. 10, has pointed out that the entangled systems with a finite number of entangled classes only potentially exist in the 2 × M × N space. This assertion has been confirmed in the four-qubit system 14. Here, we will prove that the family of 2 × 4 × 4 states also contains infinitely many essential classes. In addition, the 2 × 4 × 4 system can be regarded as a special case of the five-qubit system. Hence, the classification of 2 × 4 × 4 states helps get insight into the structure of the five-qubit states, which remains a sophisticated problem in QIT.

Before we go to the classification of 2 × 4 × 4 states, we shall make some useful preparations, mainly quoted from 17. Define two ILO’s \(O_{A}^{1}(|ϕ⟩, α) : |ϕ⟩_A \rightarrow α|ϕ⟩_A\) and \(O_{A}^{2}(|ψ⟩, α) : |ϕ⟩_A \rightarrow |ϕ⟩_A + |ψ⟩_A\), respectively. Some existing results are also available. They are the 2 × 3 × 2 classes of entanglement,

\[
|ϕ_0⟩ = |000⟩ + |011⟩ + |121⟩ ∈ |1,∞,1⟩,
|ϕ_1⟩ = |000⟩ + |011⟩ + |110⟩ + |121⟩ ∈ |0,∞,0⟩,
\]

and the 2 × 3 × 3 classes,

\[
|ϕ_0⟩ = |000⟩ + |111⟩ + |022⟩ ∈ |1,∞,∞⟩,
|ϕ_1⟩ = |000⟩ + |111⟩ + |010⟩ + |111⟩ + |022⟩ ∈ |0,3,3⟩,
|ϕ_2⟩ = |010⟩ + |001⟩ + |112⟩ + |121⟩ ∈ |0,∞,∞⟩,
|ϕ_3⟩ = |100⟩ + |010⟩ + |001⟩ + |112⟩ + |121⟩ ∈ |0,1,1⟩,
|ϕ_4⟩ = |100⟩ + |010⟩ + |001⟩ + |022⟩ ∈ |1,∞,∞⟩,
|ϕ_5⟩ = |100⟩ + |010⟩ + |001⟩ + |122⟩ ∈ |0,2,2⟩.
\]

All these states are inequivalent under SLOCC in terms of the range criterion. Notice there exist the so-called invariance of \(|ϕ_1⟩\) under the ILO’s \(|0⟩_A \rightarrow |1⟩_A\) on all parties, and that of \(|ϕ_2⟩\) under the ILO’s \(|0⟩_A \rightarrow |1⟩_A, |0⟩_B \rightarrow |2⟩_B\) and \(|0⟩_C \rightarrow |1⟩_C\). This feature will help simplify the
There are 16 essentially entangled classes in $2 \times 4 \times 4$ space,
\[ |\Phi_0\rangle = |133\rangle + |\varphi_0\rangle \in [0, \infty, \infty]; \]
\[ |\Phi_1\rangle = |033\rangle + |\varphi_0\rangle \in [1, \infty, \infty]; \]
\[ |\Phi_2\rangle = |133\rangle + |\varphi_1\rangle \in [0, \infty, \infty]; \]
\[ |\Phi_3\rangle_{x} = (|0\rangle + x |1\rangle) |33\rangle + |\varphi_3\rangle \in [0, 4, 4], x \neq 0, 1; \]
\[ |\Phi_4\rangle = |133\rangle + |\varphi_2\rangle \in [0, \infty, \infty]; \]
\[ |\Phi_5\rangle = |133\rangle + |\varphi_3\rangle \in [0, \infty, \infty]; \]
\[ |\Phi_6\rangle = |033\rangle + |\varphi_2\rangle \in [0, 2, 2]; \]
\[ |\Phi_7\rangle = |133\rangle + |\varphi_2\rangle \in [0, \infty, \infty]; \]
\[ |\Phi_8\rangle = |033\rangle + |\varphi_2\rangle \in [1, \infty, \infty]; \]
\[ |\Phi_9\rangle = |133\rangle + |\varphi_2\rangle \in [0, \infty, \infty]; \]
\[ |\Phi_{10}\rangle = (|0\rangle + |1\rangle) |33\rangle + |\varphi_3\rangle \in [0, 3, 3]; \]
\[ |\Phi_{11}\rangle = |133\rangle + |\varphi_2\rangle \in [0, \infty, \infty]; \]
\[ |\Phi_{12}\rangle = |033\rangle + |\varphi_2\rangle \in [0, 1, 1]; \]
\[ |\Phi_{13}\rangle = |033\rangle + |\varphi_4\rangle \in [0, \infty, \infty]; \]
\[ |\Phi_{14}\rangle = |133\rangle + |\varphi_2\rangle \in [0, 2, 2]; \]
\[ |\Phi_{15}\rangle = |133\rangle + |\varphi_2\rangle \in [0, \infty, \infty]. \] (11)

Notice the class $|\Phi_3\rangle$ contains a parameter $x$, which cannot be removed. So the parameter $x$ characterizes that the classes of true entangled states of $2 \times 4 \times 4$ system are infinite.

**Proof.** Due to the LHRGM we write out
\[ |\Psi_{2 \times 4 \times 4}\rangle \sim \begin{cases} 
|\Omega_0\rangle = (|a\rangle + |b\rangle) |1\rangle |33\rangle + |\varphi_1\rangle, \\
|\Omega_1\rangle = |033\rangle + |132\rangle + |\Psi_{2 \times 3 \times 2}\rangle, \\
|\Omega_2\rangle = |\Omega_0\rangle + |\Omega_1\rangle, \\
|\Omega_3\rangle = |\Omega_0\rangle + |\Omega_1\rangle, \\
|\Omega_4\rangle = |\Omega_0\rangle + |\Omega_1\rangle, \\
\end{cases} \]

Here, $|\chi\rangle = \sum_{i} a_i |i\rangle$ since $N = 4$. Let us begin with the calculation of $|\Omega_0\rangle$. The style is similar to (17).

(i) $|\Omega_{0,0}\rangle = (|a\rangle + |b\rangle) |1\rangle |33\rangle + |\varphi_1\rangle$. If $ab \neq 0$, then it leads to the entangled class $|\Phi_0\rangle$ or $|\Phi_1\rangle$. For the case of $ab \neq 0$, it holds that $|\Omega_{0,0}\rangle \sim (|0\rangle + |a\rangle) |33\rangle + |\varphi_1\rangle$. By operations $O_{2 \times 4 \times 1}^{(1)} (|1\rangle, \alpha^{-1}) \otimes O_{2 \times 4 \times 1}^{(1)} (|1\rangle, \alpha)$, and $|2\rangle_B \mapsto |3\rangle_B$, $|2\rangle_C \mapsto |3\rangle_C$, we obtain the class $|\Phi_2\rangle$.

(ii) $|\Omega_{0,1}\rangle = (|0\rangle + |b\rangle) |1\rangle |33\rangle + |\varphi_1\rangle$. Due to the invariance of $|\varphi_1\rangle$, for the case of $ab = 0$ it is transformed into $|\Phi_2\rangle$. If $ab \neq 0$, then $|\Omega_{0,1}\rangle \sim (|0\rangle + |a\rangle) |33\rangle + |\varphi_1\rangle$. By using of the ILO's $O_{2 \times 4 \times 1}^{(1)} (|1\rangle, \alpha^{-1}) \otimes O_{2 \times 4 \times 1}^{(1)} (|0\rangle, \alpha) \otimes O_{2 \times 4 \times 1}^{(1)} (|0\rangle, \alpha)\), we obtain that $|\Omega_{0,1}\rangle \sim (|0\rangle + |a\rangle) |33\rangle + |\varphi_1\rangle$. For the case of $\varphi_2$, choose $b \neq 0$. By using of the ILO's $O_{2 \times 4 \times 1}^{(1)} (|1\rangle, \alpha^{-1}) \otimes O_{2 \times 4 \times 1}^{(1)} (|0\rangle, \alpha) \otimes O_{2 \times 4 \times 1}^{(1)} (|0\rangle, \alpha)$, we obtain $|\Omega_{0,1}\rangle \sim (|0\rangle + |1\rangle) |33\rangle + |\varphi_1\rangle$. By using of the ILO's $O_{2 \times 4 \times 1}^{(1)} (|1\rangle, \alpha^{-1}) \otimes O_{2 \times 4 \times 1}^{(1)} (|0\rangle, \alpha) \otimes O_{2 \times 4 \times 1}^{(1)} (|0\rangle, \alpha)$, we obtain $|\Omega_{0,1}\rangle \sim (|0\rangle + |1\rangle) |33\rangle + |\varphi_1\rangle$. For the case of $a = 0$, then $|\Omega_{0,1}\rangle \sim |\Phi_2\rangle$. Local operations $O_{2 \times 4 \times 1}^{(1)} (|0\rangle, \alpha^{-1}) \otimes O_{2 \times 4 \times 1}^{(1)} (|2\rangle, \alpha) \otimes O_{2 \times 4 \times 1}^{(1)} (|2\rangle, \alpha)$ make $|\Omega_{0,1}\rangle \sim (|0\rangle + |1\rangle) |33\rangle + (100) + (010) + (001) + (112) + (121) \sim |\Phi_2\rangle$.
same way, let $|\Omega_{i}\rangle_{i} = |\Omega_{0}\rangle_{i} + |\Omega_{1}\rangle_{i}$ for $i = 0, 1$. The local rank of subsystem is invariant under ILO's, which can be taken into the form of $|\Omega_{0}\rangle_{i}$. The sole new class is derived from the calculation of $|\Omega_{3}\rangle_{i} = |(0, + \alpha [1]) \rangle_{i} | 32 \rangle + | 100 \rangle + | 010 \rangle + | 001 \rangle + | 022 \rangle + | 13 \rangle \sum_{i=0}^{2} a_{i} | i \rangle$, which can be taken into the form of $|\Omega_{3}\rangle_{i}$ if $\alpha \neq 0$, by the ILO's $O_{2}^{B}(1), -1/\alpha [0] \rangle \otimes O_{2}^{C}(3), -1/\alpha [0] \rangle \otimes \sum_{i=0}^{2} a_{i} | i \rangle$. On the other hand let $\alpha = 0$. If $a_{1} \neq 0$, in virtue of the operations $O_{2}^{B}(1), -a_{2}/a_{1} [2] \rangle \otimes O_{2}^{C}(2), a_{2} | a_{1} [0] \rangle, | 2 \rangle \otimes | 3 \rangle_{B} \iff | 2 \rangle \otimes | 3 \rangle_{C}$ we make that $|\Omega_{3}\rangle_{i} \sim |\Omega_{0}\rangle_{i}$.

In what follows, the task is to prove that the $16$ entangled classes are essentially inequivalent under SLOCC. In principle, we distinguish the classes by range criterion, and specially in terms of the notation $[*, *, *]$ defined in range criterion. As the situation is more complicated than those in [17], we will fully employ the fact that the local rank of subsystem is invariant under ILO's. For example, consider the two classes

$$
|\Phi_{0}\rangle = |133\rangle + |000\rangle + |111\rangle + |022\rangle,
|\Phi_{2}\rangle = |133\rangle + |000\rangle + |111\rangle + |0 + 1\rangle |22\rangle,
$$

which can be rewritten as

$$
|\Phi_{0}\rangle_{ABC} = |0\rangle (|00\rangle + |22\rangle) + |1\rangle (|11\rangle + |33\rangle),
|\Phi_{2}\rangle_{ABC} = |0\rangle (|00\rangle + |22\rangle) + |1\rangle (|11\rangle + |22\rangle + |33\rangle).
$$

(Equation 13)

Evidently, the local rank of the state in $R(\rho_{BC})$ is 2 or 4, while that in $R(\rho_{BC})$ can be 2, 3 or 4. Thus the two states are inequivalent. For simplicity we define the set $(a_{0}, a_{1}, ...)$, and a state $|\Psi\rangle_{ABC}$ belongs to this set iff all the possible values of local ranks of the states in $R(\rho_{BC})$ are $a_{0}, a_{1}, ...$. Combined with the notation $[*, *, *]$ we have

$$
|\Phi_{0}\rangle_{ABC} \in [0, \infty, \infty] \cap (2, 4),
|\Phi_{2}\rangle_{ABC} \in [0, \infty, \infty] \cap (2, 3, 4).
$$

(A. Discrimination of the classes in $[0, \infty, \infty]$, $[1, \infty, \infty]$ and $[0, 2, 2]$)

There are nine entangled classes in the set of $[0, \infty, \infty]$. We list them with the notation of local ranks $(a_{0}, a_{1}, ...)$,

$$
|\Phi_{0}\rangle = |133\rangle + |0\rangle \in (2, 4),
|\Phi_{2}\rangle = |133\rangle + |1\rangle \in (2, 3, 4),
|\Phi_{4}\rangle = |133\rangle + |2\rangle \in (2, 3),
|\Phi_{5}\rangle = |133\rangle + |3\rangle \in (2, 4),
|\Phi_{7}\rangle = |133\rangle + |4\rangle \in (2, 3, 4),
|\Phi_{9}\rangle = |133\rangle + |5\rangle \in (2, 3, 4),
|\Phi_{11}\rangle = |133\rangle + |032\rangle + |f\rangle \in (3),
|\Phi_{13}\rangle = |033\rangle + |32\rangle + |4\rangle \in (2, 4),
|\Phi_{15}\rangle = |033\rangle + |32\rangle + |5\rangle \in (3).
$$

(15)

Except the sole class $|\Phi_{4}\rangle$ with (2,3), the other eight classes can be divided into three groups with respect to the local ranks, i.e., (2,4), (2,3,4) and (3). I. the case of (2,4), including three classes $|\Phi_{0}\rangle$, $|\Phi_{2}\rangle$ and $|\Phi_{4}\rangle$. Notice that the local rank of BC states are 2 or 3, and $(1 + 11 + 33) \in R(\rho_{BC})$, while there is one such state in $R(\rho_{BC})$ and $R(\rho_{BC})$ respectively. Due to the range criterion, $|\Phi_{0}\rangle$ differs from the other two states. In order to compare $|\Phi_{2}\rangle$ and $|\Phi_{4}\rangle$, write out

$$
R(\rho_{BC}) = a (|13\rangle + b |11\rangle + c |00\rangle + |12\rangle) + d (|10\rangle + |01\rangle),
R(\rho_{BC}) = a' |03\rangle + b' |00\rangle + c' |13\rangle + |02\rangle) + d' (|10\rangle + |01\rangle).
$$

(16)

Observe these two expressions. Although there are infinitely many product states in either of them, it occurs if $c = d = 0$ and $c' = d' = 0$. This implies that the possible ILO's must transform $(a' |3\rangle + b' |0\rangle)_{\Phi_{15}}$ into $(a |3\rangle + b |1\rangle)_{\Phi_{13}}$, which means either $|3\rangle_{\Phi_{13}}$ or $|0\rangle_{\Phi_{13}}$ will be taken into $(3 + \alpha |1\rangle)_{\Phi_{13}}$. However, the adjoint state of $|3\rangle_{\Phi_{13}}$ is $|03\rangle + |12\rangle$ and that of $|0\rangle_{\Phi_{13}}$ is $|10\rangle + |01\rangle$, and either of these two adjoint states makes the range an entangled state. Since the adjoint state of $|3\rangle_{\Phi_{13}}$ is of product form, we get the outcome that $|\Phi_{2}\rangle$ and $|\Phi_{4}\rangle$ are not equivalent under SLOCC.

II. the case of (2,3,4), including three classes $|\Phi_{2}\rangle$, $|\Phi_{4}\rangle$ and $|\Phi_{7}\rangle$. Similar to the case of I. $|\Phi_{2}\rangle$ is a distinctive class containing two rank-3 states in $R(\rho_{BC})$, while there is only one such state in $R(\rho_{BC})$ and $R(\rho_{BC})$ respectively. In order to compare $|\Phi_{2}\rangle$ and $|\Phi_{7}\rangle$, write out

$$
R(\rho_{AB}) = a (|13\rangle + b |02\rangle + c |00\rangle + d |10\rangle + |01\rangle),
R(\rho_{AB}) = a' |13\rangle + b' |12\rangle + c' |00\rangle + d' (|10\rangle + |01\rangle).
$$

(17)

Notice the possible ILO's must make that $|0\rangle_{\Phi_{0}} \Rightarrow |1\rangle_{\Phi_{0}}^{A}$ and $|1\rangle_{\Phi_{0}}^{A} \Rightarrow |0\rangle_{\Phi_{0}}^{A}$ due to the local ranks of the adjoint states. By the expressions of $R(\rho_{AB})$ and $R(\rho_{AB})$ we have $|0\rangle_{\Phi_{0}}^{B} \Rightarrow |3\rangle_{\Phi_{7}}^{B}$. Since the adjoint state of $|0\rangle_{\Phi_{0}}^{B}$ is $|10\rangle + |01\rangle$ making the range entangled while the adjoint state of $|3\rangle_{\Phi_{7}}^{B}$ is of product form, the two states $|\Phi_{2}\rangle$ and $|\Phi_{7}\rangle$ are inequivalent.
III. the case of (3), including two classes $|\Phi_{11}\rangle$ and $|\Phi_{15}\rangle$. Write out

\[ R(\rho_{\Phi_{11}}^{AB}) = a \, |0\rangle \langle 0| + b \, |1\rangle \langle 1| + c \, (|0\rangle \langle 0| + |1\rangle \langle 1|) + d \, (|0\rangle \langle 0| + |1\rangle \langle 1|), \]

\[ R(\rho_{\Phi_{15}}^{AB}) = a' \, |0\rangle \langle 0| + b' \, |1\rangle \langle 1| + c' \, (|0\rangle \langle 0| + |1\rangle \langle 1|) + d' \, (|0\rangle \langle 0| + |1\rangle \langle 1|). \]

(Equation 18)

Evidently, the product states in $R(\rho_{\Phi_{11}}^{AB})$ are $a' \, |0\rangle \langle 0| + b' \, |1\rangle \langle 1|$, while there exist the product states $|0\rangle \langle 0|$ and $|1\rangle \langle 1|$ in $R(\rho_{\Phi_{15}}^{AB})$, and there does not exist an ILO making $|3\rangle_{\Phi_{15}}$ to $|\Phi_{11}\rangle$ and $|1\rangle_{\Phi_{15}}$ simultaneously. Consequently, $|\Phi_{11}\rangle$ and $|\Phi_{15}\rangle$ differs from each other.

Therefore we have accomplished the task of comparison of the entangled classes in $[0, \infty, \infty]$. With the similar techniques we can easily treat the residual business. There are two classes belonging to $[1, \infty, \infty]$ as follows,

\[ |\Phi_{1}\rangle_{ABC} = (|033\rangle + |000\rangle + |111\rangle + |022\rangle) \in (1, 3, 4), \]

\[ |\Phi_{8}\rangle_{ABC} = (|033\rangle + |100\rangle + |010\rangle + |001\rangle + |022\rangle) \in (1, 4), \]

and one can immediately recognize the inequivalence of them. For the case of $[0, 2, 2]$, we also write the concrete forms of two classes both of which belong to $(3, 4)$

\[ |\Phi_{6}\rangle_{ABC} = (|033\rangle + |100\rangle + |010\rangle + |001\rangle + |112\rangle + |121\rangle), \]

\[ |\Phi_{14}\rangle_{ABC} = (|133\rangle + |032\rangle + |100\rangle + |010\rangle + |001\rangle + |122\rangle). \]

(Equation 19)

It is easy to find out the two product states in $R(\rho_{\Phi_{6}}^{AC})$ are $|0\rangle_{\Phi_{14}}$ and $|2\rangle_{\Phi_{14}}$ into $|3\rangle_{\Phi_{4}}$. Since either of the two adjoint states of $|0\rangle_{\Phi_{14}}$ and $|2\rangle_{\Phi_{14}}$ makes the range entangled, while the adjoint state of $|3\rangle_{\Phi_{4}}$ is of product form, so $|\Phi_{6}\rangle$ and $|\Phi_{14}\rangle$ are inequivalent. As the other three classes $|\Phi_{4}\rangle_{x, y}$, $|\Phi_{10}\rangle$ and $|\Phi_{12}\rangle$ belong to different sets, we thus assert that there are indeed 16 classes of states in true $2 \times 4 \times 4$ space, under the SLOCC criterion.

B. Classification of the entangled class $|\Phi_{3}\rangle$

In this subsection, we investigate the structure of the class $|\Phi_{3}\rangle_{x} = (|0\rangle \langle 0| + x |1\rangle \langle 1|) |33\rangle + |000\rangle + |111\rangle + (|0\rangle \langle 0| + |1\rangle \langle 1|) |22\rangle$, which contains a parameter $x \neq 0, 1$. Evidently, the set of product states in $R(\rho_{\Phi_{3}}^{AC})$ is $S_{x} = \{ (|0\rangle \langle 0| + x |1\rangle \langle 1|) |3\rangle, |000\rangle, |111\rangle, (|0\rangle \langle 0| + |1\rangle \langle 1|) |2\rangle \}$. Thus, if $|\Phi_{3}\rangle_{x} \sim |\Phi_{3}\rangle_{y}$, then the possible ILO’s must bring $S_{x}$ into $S_{y}$, i.e., every element in $S_{x}$ will be transformed into some element in $S_{y}$. Due to the symmetry of system $BC$, it holds that every term in $|\Phi_{3}\rangle_{x}$ will be transformed into some term in $|\Phi_{3}\rangle_{y}$, e.g., $(|0\rangle \langle 0| + x |1\rangle \langle 1|) |33\rangle \rightarrow (|0\rangle \langle 0| + y |1\rangle \langle 1|) |33\rangle, |000\rangle \rightarrow |000\rangle, |111\rangle \rightarrow |111\rangle, (|0\rangle \langle 0| + |1\rangle \langle 1|) |2\rangle \rightarrow (|0\rangle \langle 0| + |1\rangle \langle 1|) |2\rangle$. It then seems that we have to treat 24 subcases with respect to all kinds of matches, but actually only three of them is worth a further investigation.

This can be seen as follows,

\[ V_{ABC}(\rho) = v_{I} + v_{II} + v_{III} + v_{IV} \]

\[ = (|0\rangle \langle 0| + x |1\rangle \langle 1|) |33\rangle + (|000\rangle + |111\rangle + (|0\rangle \langle 0| + |1\rangle \langle 1|) |22\rangle, \]

\[ = (|0\rangle \langle 0| + y |1\rangle \langle 1|) |33\rangle + (|000\rangle + |111\rangle + (|0\rangle \langle 0| + |1\rangle \langle 1|) |22\rangle. \]

(Equation 21)

where $V_{ABC} = V_{A} \otimes V_{B} \otimes V_{C}$. Another alternative expression of this equation is

\[ V_{ABC}(\rho) = v'_{I} + v'_{II} + v'_{III} + v'_{IV} \]

\[ = (|0\rangle \langle 0| + x^{-1} |1\rangle \langle 1|) |33\rangle + (|000\rangle + |111\rangle + (|0\rangle \langle 0| + |1\rangle \langle 1|) |22\rangle, \]

\[ = (|0\rangle \langle 0| + y^{-1} |1\rangle \langle 1|) |33\rangle + (|000\rangle + |111\rangle + (|0\rangle \langle 0| + |1\rangle \langle 1|) |22\rangle. \]

(Equation 22)

where $V_{ABC} = V_{A} \otimes V_{B} \otimes V_{C}$. Similarily one can choose $V_{ABC}^{\prime} = V_{A} \otimes V_{B} \otimes V_{C}$, $V_{A}^{\prime} \otimes V_{B} \otimes V_{C}$, and $V_{ABC}^{\prime} = V_{A} \otimes V_{B} \otimes V_{C}$. We regard the position of every term invariant under each of these four transformations, e.g., $I \rightarrow I, II \rightarrow 3, III \rightarrow 4, IV \rightarrow 1$, although the contents of them are changed. We can bring this match into the case of $000 |000\rangle \rightarrow (|0\rangle \langle 0| + |1\rangle \langle 1|) |22\rangle_{y-1}$ by $V_{ABC}^{\prime}$. In the same vein, by virtue of these four transformations and the invariance of the matches, all the 24 subcases can be taken into three scenarios: (i) $II, III \rightarrow 2, 3$; (ii) $II \rightarrow 2, 3$ and $II \rightarrow 1, 4$; (iii) $II, III \rightarrow 4$, where we only need to replace the parameter $x$ in the final result with $x^{-1}$ as well as the parameter $y$ with $y^{-1}$, e.g., the equation $x = y$ contains other three equations $x^{-1} = y, x = y^{-1}$ and $x^{-1} = y^{-1}$, so that all possible value of $y$ can be achieved. For the case of (i), we can get the following results after some ILO’s

\[ |\Phi_{3}\rangle_{x}^{0} = (a |0\rangle \langle 0| + x |1\rangle \langle 1|) |33\rangle + (|000\rangle + |111\rangle + (a |0\rangle \langle 0| + |1\rangle \langle 1|) |22\rangle, \]

\[ |\Phi_{3}\rangle_{x}^{1} = (b |0\rangle \langle 0| + |1\rangle \langle 1|) |33\rangle + (|000\rangle + |111\rangle + (b |0\rangle \langle 0| + |1\rangle \langle 1|) |22\rangle. \]

(Equation 23)

where some coefficients have been moved away by the ILO’s $O_{1}$ and $a, b$ are the residual parameters. Compare these two expressions with that of $|\Phi_{3}\rangle_{y}$ supplemented by the possible ILO’s $2_{B} \rightarrow |3\rangle_{B}$ and $2_{C} \rightarrow |3\rangle_{C}$, we can get $y = x, x^{-1}$. For the case of (ii), it is easy to choose a definite transformation $000 |000\rangle \rightarrow (|0\rangle \langle 0| + |1\rangle \langle 1|) |22\rangle_{y'}$, where $x' = x, x^{-1}$ and $y' = y, y^{-1}$. Consequently, the resulting state is

\[ |\Phi_{3}\rangle_{y'} = (a |0\rangle \langle 0| + x' (|0\rangle \langle 0| + |1\rangle \langle 1|)) |33\rangle + (|000\rangle + (|0\rangle \langle 0| + |1\rangle \langle 1|)) |11\rangle, \]

(Equation 24)
If \( a = -1 \) we have

\[
|\Phi_3\rangle_{y'} \sim \left( \frac{x' - 1}{x'} |0\rangle + |1\rangle \right) |33\rangle + |000\rangle \\
+ \left( |0\rangle + |1\rangle \right) |22\rangle + |111\rangle,
\]

and hence \( \frac{x' - 1}{x'} = y' \), or \( y = 1 - x, 1 - x^{-1}, (1 - x)^{-1}, (1 - x^{-1})^{-1} \). For the case of \( a = -x' \), do the exchange between \( |11\rangle_{BC} \) and \( |33\rangle_{BC} \) and compare the resulting expression with that of \( |\Phi_3\rangle_{y'} \), we get \( 1 - x' = y' \) which leads to the same result above. While in the case of (iii), choose \( |0000\rangle \rightarrow \left( |00\rangle + y' |1\rangle \right) |33\rangle_{y'} \) and \( |1111\rangle \rightarrow \left( |00\rangle + |1\rangle \right) |22\rangle_{y'} \), and the resulting state is

\[
|\Phi_3\rangle_{y'} = \left( a(|0\rangle + y' |1\rangle) + x'(|0\rangle + |1\rangle) \right) |00\rangle \\
+ \left( |0\rangle + y' |1\rangle \right) |33\rangle + \left( |0\rangle + |1\rangle \right) |22\rangle \\
+ \left( a(|0\rangle + y' |1\rangle) + (|0\rangle + |1\rangle) \right) |11\rangle.
\]

Follow the same technique in (ii), choose \( a = -1, -y'^{-1} \) such that \( y' = x', x'^{-1}, \) which again leads to \( y = x, x^{-1} \). In all, we find that \( |\Phi_3\rangle_{x} \sim |\Phi_3\rangle_{1-x} \sim |\Phi_3\rangle_{1-x^{-1}} \sim |\Phi_3\rangle_{(x^{-1})^{-1}} \sim |\Phi_3\rangle_{(x^{-1})^{-1}} \). Define the set \( C_x \equiv \{ x, x^{-1}, 1 - x, 1 - x^{-1}, 1 - x, (1 - x^{-1})^{-1}, x \neq 0, 1 \} \), then \( |\Phi_3\rangle \sim |\Phi_3\rangle_{y} \) iff \( y \in C_x \). It is easy to verify that the whole complex number field can be expressed as \( C = \{ 0, 1 \} \cup C_{x_1} \cup C_{x_2} \cup \cdots, x_i \neq x_j \) for different \( i, j \). In addition, any two elements chosen from distinct sets \( C_{x_i} \) and \( C_{x_j} \) are not identical. These characters state that the entangled class \( |\Phi_3\rangle \) can be divided into infinitely many kinds of states. Q.E.D.

**IV. HIERARCHY OF ENTANGLEMENT IN 2 x 3 x N SPACE AND CLASSIFICATION OF 2 x (M + 3) x (2M + 3) AND 2 x (M + 4) x (2M + 4) STATES**

In the preceding section, we have proved that there exist infinitely many entangled classes in the 2 x 4 x 4 space under SLOCC, for the existence of state \( |\Phi_3\rangle_{x} \). It is known that there is only a finite number of essential classes in the 2 x 2 x N, N = 2, 3, 4, ... and 2 x 3 x N, N = 3, 4, 5, 6, ... space \([14, 15, 17]\), so the 2 x 4 x 4 states are the simplest family containing an infinite number of entangled classes. For some trivial cases, i.e., the unentangled states and those product in one party and entangled with respect to the other two, we omit the deduction of classification. Subsequently, we generally list the classification of 2 x 3 x N states in the following table. true rank of system Class

| \( 2 \times 3 \times N (N \geq 6) \) | \( |000\rangle + |011\rangle + |022\rangle + |103\rangle + |114\rangle + |125\rangle \) |
| \( 2 \times 3 \times 5 \) | \( |024\rangle + |000\rangle + |011\rangle + |102\rangle + |113\rangle \) |

Here, both of the states consist of two parts, including one original 2 x 2 x 4 class and the other portion by the added dimension. According to the LHRGM, every product term of the added portion always contains at least a new high ket, such as \( |2\rangle_B \) or \( |4\rangle_C \) in the term \( |024\rangle \). This suggests that the added portion could be viewed as a distribution of all high kets, complemented by the lower kets existing in the original class. By simply writing all distributions, one can primarily realize the structure of a certain kind of entanglement. On the other hand, the number of inequivalent classes differs with the changing dimensions. Specifically, there are orderly 1, 2, 6, 5, 2, 1 kinds of states in \( 2 \times 3 \times N \) space, \( N = 1, 2, 3, 4, 5, 6 \), which implies the 2 x 3 x 3 states has the most classes of entanglement. One can understand this fact like this. As the A.B system is in the 2 x 3 space, in which there are generally six bases such that \( \left\{ |00\rangle, |01\rangle, |02\rangle, |10\rangle, |11\rangle, |12\rangle \right\} \). A, B need to select rank(\( \rho_{\Phi_{ABC}} \)) basis from the above set, so that the three cooperators can construct a purely triple state. Let \( n, m \) be integers, and \( [a] \) denotes the max-
imum integer not more than \(a\). Due to the combinatorial theory, the combination \(\binom{n}{m}\) is monotonically increasing when \(m \leq [n/2]\) and monotonically decreasing when \(m \geq [n/2]\). So \(\binom{n}{m}\) reaches its maximum when \(m = [n/2]\). We call this monotonicity quasi-combinatorial character. Surprisingly, although not explicitly coinciding with the combinations, the above numbers of entangled classes indeed reflect this character. Another witness to this character can be seen in the \(2 \times 4\times N\) system, in which the \(2 \times 4 \times 4\) system can be entangled in infinitely many ways. By virtue of the results in [17], there are orderly 1, 1, 5, \(\infty\), 6, 2, 1 classes of states in the \(2 \times 4 \times N\) space, \(N = 1, 2, 3, 4, 5, 6, 7, 8\), and we will provide the result of \(g = 12\) later. Besides, there is always a unique class in \(2 \times M \times 1\) and \(2 \times M \times 2M\) space respectively. So we expect that there exists the quasi-combinatorial character in any sequence of true \(2 \times M \times N\) systems, \(N = 1, 2, ..., 2M\). Due to the experience in both the paper and [17], we can infer that the classification of the \(2 \times M \times N\) states requires an increasing amount of calculation when \(N\) approaches \(M\). At last, the results with respect to the classification of \(2 \times (M + 3) \times (2M + 3)\) and \(2 \times (M + 4) \times (2M + 4)\) states are provided. For convenience, we list some existing results appeared in [17] in (i), (ii), (iii),

(i) \(2 \times M \times 2M, M \geq 2\):

\[
|\Upsilon_0\rangle \equiv |0\rangle \sum_{i=0}^{M-1} |ii\rangle + |1\rangle \sum_{i=0}^{M-1} |i, i+M\rangle \in [0, 0, \infty];
\]

(ii) \(2 \times (M + 1) \times (2M + 1), M \geq 1\):

\[
|\Upsilon_1\rangle \equiv |0, M, 2M\rangle + |\Upsilon_0\rangle \in [0, 1, \infty];
|\Upsilon_2\rangle \equiv |0, M, 2M\rangle + |1, M, M - 1\rangle + |\Upsilon_0\rangle \in [0, 0, \infty];
\]

(iii) \(2 \times (M + 2) \times (2M + 2), M \geq 2\):

\[
|\Theta_0\rangle \equiv |1, M + 1, 2M + 1\rangle + |\Upsilon_0\rangle \in [0, 2, \infty];
|\Theta_1\rangle \equiv |0, M + 1, 2M + 1\rangle + |\Upsilon_1\rangle \in [0, \infty, \infty];
|\Theta_2\rangle \equiv |1, M + 1, 2M + 1\rangle + |\Upsilon_2\rangle \in [0, 1, \infty];
|\Theta_3\rangle \equiv |0, M + 1, 2M + 1\rangle + |1, M + 1, 2M\rangle + |\Upsilon_1\rangle \in [0, 1, \infty];
|\Theta_4\rangle \equiv |0, M + 1, 2M + 1\rangle + |1, M + 1, 0\rangle + |\Upsilon_2\rangle \in [0, 0, \infty];
|\Theta_5\rangle \equiv |0, M + 1, 2M + 1\rangle + |1, M + 1, 2M\rangle + |\Upsilon_2\rangle \in [0, 0, \infty].
\]

Based on these results we can carry on the new classification. Recall that there exist the so-called invariance of \(|\Theta_0\rangle, |\Theta_4\rangle\) and \(|\Theta_5\rangle\), e.g., \(|\Theta_0\rangle\) remains unchanged under ILO’s \(|0\rangle_A \leftrightarrow |1\rangle_A; \ |M + 1\rangle_B \leftrightarrow |M\rangle_B; \ |2M + 1\rangle_C \leftrightarrow |2M\rangle_C\) and \(|i\rangle_M \leftrightarrow |i\rangle_M, i = 0, ..., M - 1. One can find these invariances in [17], which will be useful in the brief arguments for new outcomes. First we investigate the case of \(2 \times (M + 3) \times (2M + 3)\) system. Due to the method of LHGRM, the required condition is the above \(2 \times (M + 2) \times (2M + 2)\) classes for the calculation of \(|\Omega_i\rangle, i = 0, 2, 3\), while we have found out the sole class \(|\Gamma_{14}\rangle\) (see below) derived from \(|\Omega_1\rangle\) in (iv) in section II. As far as the concrete process is concerned, by virtue of the techniques in this paper and [17] one can derive the classes \(|\Gamma_i\rangle, i = 0, 1, 2, 3, 4, 5, 6, 7, 8\) from the family of \(|\Omega_0\rangle\), and \(|\Gamma_i\rangle, i = 9, 10, 11, 12, 13\) from \(|\Omega_2\rangle\) and \(|\Omega_3\rangle\), which are two equivalent families when the low rank entangled classes are \(|\Theta_0\rangle, |\Theta_4\rangle\) and \(|\Theta_5\rangle\) in terms of the invariances. In addition, one can check that any pair of these classes are inequivalent under SLOCC by the range criterion. Consequently, we assert that the \(2 \times (M + 3) \times (2M + 3)\) system \((M \geq 2)\) can be entangled in 15 ways in all under SLOCC. (iv) \(2 \times (M + 3) \times (2M + 3), M \geq 2\) : (For the case of \(M = 1, |\Gamma_7\rangle, |\Gamma_{11}\rangle, |\Gamma_{13}\rangle\) disappear due to the absence of \(|\Theta_4\rangle_{2 \times 3 \times 4}\))

\[
|\Gamma_0\rangle \equiv |1, M + 2, 2M + 2\rangle + |\Theta_0\rangle \in [0, \infty, \infty];
|\Gamma_1\rangle \equiv ([0,1\rangle + |1\rangle) |M + 2, 2M + 2\rangle + |\Theta_0\rangle \in [0, 3, \infty];
|\Gamma_2\rangle \equiv |0, M + 2, 2M + 2\rangle + |\Theta_1\rangle \in [0, \infty, \infty];
|\Gamma_3\rangle \equiv |1, M + 2, 2M + 2\rangle + |\Theta_2\rangle \in [0, \infty, \infty];
|\Gamma_4\rangle \equiv |0, M + 2, 2M + 2\rangle + |\Theta_2\rangle \in [0, 2, \infty];
|\Gamma_5\rangle \equiv |1, M + 2, 2M + 2\rangle + |\Theta_3\rangle \in [0, 2, \infty];
|\Gamma_6\rangle \equiv |0, M + 2, 2M + 2\rangle + |\Theta_3\rangle \in [0, \infty, \infty];
|\Gamma_7\rangle \equiv |1, M + 2, 2M + 2\rangle + |\Theta_4\rangle \in [0, 1, \infty];
|\Gamma_8\rangle \equiv |1, M + 2, 2M + 2\rangle + |\Theta_5\rangle \in [0, 1, \infty];
|\Gamma_9\rangle \equiv |1, M + 2, 2M + 2\rangle + |0, M + 2, 2M + 1\rangle + |\Theta_2\rangle \in [0, 1, \infty];
|\Gamma_{10}\rangle \equiv |0, M + 2, 2M + 2\rangle + |1, M + 2, 2M + 1\rangle + |\Theta_3\rangle \in [0, 1, \infty];
|\Gamma_{11}\rangle \equiv |1, M + 2, 2M + 2\rangle + |0, M + 2, 2M + 1\rangle + |\Theta_4\rangle \in [0, 0, \infty];
|\Gamma_{12}\rangle \equiv |1, M + 2, 2M + 2\rangle + |0, M + 2, 2M\rangle + |\Theta_5\rangle \in [0, 0, \infty];
|\Gamma_{13}\rangle \equiv |0, M + 2, 2M + 2\rangle + |1, M + 2, 2M + 1\rangle + |\Theta_5\rangle \in [0, 0, \infty];
|\Gamma_{14}\rangle \equiv \sum_{i=0}^{M-1} (|0, M + 2 - i, 2M + 2 - 2i\rangle + |\varphi_2\rangle \in [0, \infty, \infty].
\]

Based on the above results we can further classify the entanglement of \(2 \times (M + 4) \times (2M + 4)\) system. Notice the calculation of \(|\Omega_1\rangle\) requires the classification of the \(2 \times 4 \times 4\) states, which has been given in section III and only the class \(|\Lambda_{36}\rangle\) (see below) is derived. By using of the LHGRM one can obtain the classes \(|\Lambda_i\rangle, i \in [0, 23]\) from the family of \(|\Omega_0\rangle\), and the classes \(|\Lambda_i\rangle, i \in [24, 35]\) from the family of \(|\Omega_2\rangle\) and \(|\Omega_3\rangle\). In particular, \(|\Lambda_3\rangle\) contains a parameter which cannot be removed, and thus the \(2 \times (M + 4) \times (2M + 4)\) states are also a family of infinitely many classes. One can analyze the family of \(|\Lambda_3\rangle\) similar to the case of \(\Phi_{13}\rangle\) in \(2 \times 4 \times 4\) space, since there are only four product states in \(R(\rho_{\Phi_{13}}\rangle\), and the
possible transformations indeed make a term in the initial state into another term in the final state. We thus get the results of a new family containing infinitely many classes by the existing techniques, although the calculation here is more involved than the former cases.

\[(\nu)\ 2 \times (M + 4) \times (2M + 4), M \geq 2 : \text{(For the case of)} \ M = 1, \ |A_i\rangle, i = 12, 13, 20, 22, 25, 28, 29, 31, 32 \text{disappear and an added class is} \ |045\rangle + |142\rangle + |\Gamma_9\rangle^{M=1}\]

\[
\begin{align*}
|A_{10}\rangle &\equiv |1, M + 3, 2M + 3\rangle + |\Gamma_0\rangle \in [0, \infty, \infty]; \\
|A_{11}\rangle &\equiv |0, M + 3, 2M + 3\rangle + |\Gamma_0\rangle \in [0, \infty, \infty]; \\
|A_{12}\rangle &\equiv \langle (0) + |1\rangle |M + 3, 2M + 3\rangle + |\Gamma_0\rangle \in [0, \infty, \infty]; \\
|A_{13}\rangle &\equiv \langle (0) + x |1\rangle |M + 3, 2M + 3\rangle + |\Gamma_1\rangle \\
&\in [0, 4, \infty], x \neq 0, 1; \\
|A_{14}\rangle &\equiv |0, M + 3, 2M + 3\rangle + |\Gamma_2\rangle \in [0, \infty, \infty]; \\
|A_{15}\rangle &\equiv |1, M + 3, 2M + 3\rangle + |\Gamma_3\rangle \in [0, \infty, \infty]; \\
|A_{16}\rangle &\equiv |0, M + 3, 2M + 3\rangle + |\Gamma_3\rangle \in [0, \infty, \infty]; \\
|A_{17}\rangle &\equiv \langle (0) + |1\rangle |M + 3, 2M + 3\rangle + |\Gamma_4\rangle \in [0, 3, \infty]; \\
|A_{18}\rangle &\equiv |1, M + 3, 2M + 3\rangle + |\Gamma_5\rangle \in [0, \infty, \infty]; \\
|A_{19}\rangle &\equiv |0, M + 3, 2M + 3\rangle + |\Gamma_5\rangle \in [0, \infty, \infty]; \\
|A_{20}\rangle &\equiv \langle (0) + |1\rangle |M + 3, 2M + 3\rangle + |\Gamma_6\rangle \in [0, 3, \infty]; \\
|A_{21}\rangle &\equiv |0, M + 3, 2M + 3\rangle + |\Gamma_7\rangle \in [0, \infty, \infty]; \\
|A_{22}\rangle &\equiv |1, M + 3, 2M + 3\rangle + |\Gamma_7\rangle \in [0, 2, \infty]; \\
|A_{23}\rangle &\equiv |1, M + 3, 2M + 3\rangle + |\Gamma_8\rangle \in [0, \infty, \infty]; \\
|A_{24}\rangle &\equiv |1, M + 3, 2M + 3\rangle + |\Gamma_8\rangle \in [0, 2, \infty]; \\
|A_{25}\rangle &\equiv |0, M + 3, 2M + 3\rangle + |\Gamma_9\rangle \in [0, \infty, \infty]; \\
|A_{26}\rangle &\equiv |1, M + 3, 2M + 3\rangle + |\Gamma_9\rangle \in [0, 2, \infty]; \\
|A_{27}\rangle &\equiv |0, M + 3, 2M + 3\rangle + |\Gamma_9\rangle \in [0, 2, \infty]; \\
|A_{28}\rangle &\equiv |1, M + 3, 2M + 3\rangle + |\Gamma_{10}\rangle \in [0, \infty, \infty]; \\
|A_{29}\rangle &\equiv |0, M + 3, 2M + 3\rangle + |\Gamma_{10}\rangle \in [0, \infty, \infty]; \\
|A_{30}\rangle &\equiv |1, M + 3, 2M + 3\rangle + |0, M + 3, 2M + 2\rangle \\
+ |\Gamma_1\rangle \in [0, 0, \infty]; \\
|A_{31}\rangle &\equiv |1, M + 3, 2M + 3\rangle + |1, M + 3, 0\rangle \\
+ |\Gamma_13\rangle \in [0, 0, \infty]; \\
|A_{32}\rangle &\equiv |0, M + 3, 2M + 3\rangle + |1, M + 3, 2M + 2\rangle \\
+ |\Gamma_13\rangle \in [0, 0, \infty]; \\
|A_{33}\rangle &\equiv |0, M + 3, 2M + 3\rangle + |0, M + 3, 2M + 1\rangle \\
+ |\Gamma_14\rangle \in [0, 0, \infty]; \\
|A_{34}\rangle &\equiv |0, M + 3, 2M + 3\rangle + |1, M + 3, 2M + 2\rangle \\
+ |\Gamma_6\rangle \in [0, 0, \infty]; \\
|A_{35}\rangle &\equiv |0, M + 3, 2M + 3\rangle + |1, M + 3, 2M + 2\rangle \\
+ |\Gamma_{10}\rangle \in [0, 1, \infty]; \\
|A_{36}\rangle &\equiv \sum_{i=0}^{M-1} (|0, M + 3 - i, 2M + 3 - 2i\rangle \\
+ |1, M + 3 - i, 2M + 2 - 2i\rangle) \\
+ |\Phi_{15}\rangle \in [0, \infty, \infty].
\end{align*}

All these states are incomparable under SLOCC by the range criterion. The laconic structures show the regular generation of the higher level classes derived from the low-level classes.

V. CONCLUSIONS

In this paper, we mainly classified the entangled states of \(2 \times 4 \times 4\) system, which is the simplest one containing infinitely many classes. So the range criterion is indeed useful for the classification of generic entanglement which usually contains parameters, and the results of the \(2 \times 4 \times 4\) states helps explore the structure of 5-qubit entanglement. It turned out that the range criterion efficiently operates for the discrimination of multipartite entangled states, and in principle one can classify any family of true \(2 \times M \times N\) entanglement by virtue of the LHRGM. Finally we have managed to obtain the classification of \(2 \times (M+3) \times (2M+3)\) and \(2 \times (M+4) \times (2M+4)\) states. These results are helpful to the further study in this aspect.

There are two main points we can anticipate from this paper. First, it is worth considering the asymmetry arising from the \(2 \times 4 \times 4\) entanglement. This phenomenon should be an important feature of the \(2 \times M \times M\) entanglement, whose classification is more complicated than the \(2 \times M \times N\) cases where \(M \neq N\). In addition, the study of this property is propitious for the more generic cases such as the 3-qubit entanglement. Second, we are going to apply the range criterion to the classification of multi-qubit system, which remains a difficult and important problem in QIT. However, since there exist more symmetry in the general multipartite system, other techniques may be required for the further exploration.
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[23] As defined in section III, the states $|\Phi_{11}\rangle$ and $|\Phi_{15}\rangle$ are two essential classes, and it is easy to find that $|\Phi_{11}\rangle_{ABC} \sim |\Phi_{15}\rangle_{ACB}$ by some ILO’s. We infer that this property also exists in the systems with higher dimension, and this is an interesting phenomenon requiring a further investigation.