Atomic entanglement generation with reduced decoherence via
four-wave mixing

C. Genes and P. R. Berman

Michigan Center for Theoretical Physics,
FOCUS Center, and Physics Department,
University of Michigan, Ann Arbor 48109-1040, USA

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Abstract

In most proposals for the generation of entanglement in large ensembles of atoms via projective measurements, the interaction with the vacuum is responsible for both the generation of the signal that is detected and the spin depolarization or decoherence. In consequence, one has to usually work in a regime where the information acquisition via detection is sufficiently slow (weak measurement regime) such as not to strongly disturb the system. We propose here a four-wave mixing scheme where, owing to the pumping of the atomic system into a dark state, the polarization of the ensemble is not critically affected by spontaneous emission, thus allowing one to work in a strong measurement regime.
I. INTRODUCTION

Considerable attention has been given in recent years to the generation of multi-particle entanglement in large ensembles of long-lived atomic spins. Some authors have proposed to achieve this goal using interactions between atoms and light, where a quantum state exchange can take place [1, 2, 3, 4, 5], leading to the preparation of a desired collective spin state. A different set of proposals make use of an appropriate measurement on one of the field’s observables that leads to the collapse of the ensemble onto the desired entangled spin state. In general, these schemes fall into two categories: conditional [6, 7, 8, 9, 10, 11, 12] and deterministic [13, 14, 15]. In a conditional entanglement generation scheme, the prepared state of the atoms is conditioned by the outcome of the measurement on the field state. The random character of the state resulting from the measurement back-action can be removed if one performs a continuous quantum nondemolition measurement of a spin observable and adding a feedback loop for \textit{a posteriori} quantum state correction based on the detection outcome. In this way a deterministic (unconditional) quantum state preparation scheme can be realized, where the uncertainty in the final state is removed.

In general, multilevel atoms with long-lived ground substates are used, where two of the ground sublevels form an effective two-level atom. The manipulation of the collective atomic spin, obtained by summation over the individual spins associated with each atom in the medium, is achieved by driving ground-excited state transitions using classical or quantized radiation fields. As a result of the interactions, a signal field is generated that reflects some quantum mechanical fluctuations in the atomic ensemble; its detection can give information about the atomic ensemble state. In the off-resonance regime, an effective Hamiltonian can be found with a coupling between atoms and signal field that is proportional to the strength of the atom’s coupling to the vacuum and also to the amplitude of the driving field. An increase in the control field’s amplitude, therefore, would seem to allow one to generate optimal entanglement. However, the downside of using coupling through excited levels is that spontaneous emission comes into play, leading to a rapid decoherence of the system. In consequence, it is necessary to limit spontaneous emission losses to a small value, which forces one to work in a regime of weak coupling of the atomic system to the field system to be measured, and only weak entanglement can be obtained.

The competition between spontaneous emission and measurement strength is best illus-
trated in the case of spin squeezed state generation. The challenge there is to reduce fluctuations in a spin component orthogonal to the mean spin, while keeping the average spin length large. The measurement strength controls the reduction of fluctuations, while spontaneous emission leads to a diminishing of the spin length. In a recent publication, it has been shown that, for the case of a pencil-shaped medium with Fresnel number close to unity, in the regime of small decay, optimal results are limited by the resonant optical depth of the sample.

Despite the interplay between measurement strength and spontaneous decay, it is possible to imagine a situation in which spontaneous decay does not limit the value of the measurement coupling strength. In one such scenario, the system is prepared in a dark state which is preserved during the interaction by a convenient choice of driving fields. This is the situation presented in this paper, where a quantized signal field, generated via four-wave mixing in a double Λ atomic system in a pencil-shaped medium, is entangled with the collective atomic state. The generated signal pulse reflects fluctuations in the population difference between the two ground substates. The measurement on the signal field photon number can, therefore, give information on the $z$ component of the atomic spin, projecting the system into either a spin squeezed or Schrodinger cat state. At the same time, as opposed to the situation illustrated in the previous publication, the collective $x$ polarized spin state in which the system is initially prepared, is very nearly a dark state for this combination of fields and is subject to minimal decay. No severe limitations on the measurement coupling strength are therefore necessary, and results similar to the ideal case presented in the previous publication are obtained. With the assumption of perfect detection, the Heisenberg limit is the ultimate limitation to the squeezing parameter.

The paper is organized as follows: in Sec. II the proposed scheme is described and an analytical expression for the atomic operator giving rise to the signal field is obtained; spontaneous emission effects are also discussed. In Sec. III, an expression for the signal field amplitude operator is obtained, which is shown to reflect atomic population fluctuations. An effective Hamiltonian necessary for a wave function description of the problem is derived in Sec. IV, while in Sec. V the generation of entanglement via conditional measurement of the signal field is discussed. Some conclusions are presented in Sec. VI.
II. SCHEME AND METHOD

A pencil-shaped atomic medium aligned along the z axis (left end situated at \( z = 0 \)), with transverse area \( A \), length \( L \), and density \( n_a \) is considered. The internal structure of an atom [Fig. II(b)] is a double \( \Lambda \) scheme with ground levels 1 and 2 and excited levels 3 and 4. Three classical laser pulses having duration \( T \gg L/c \) (where \( c \) is the speed of light), wave vectors \( k_1, k_2 \) (pumps) and \( k_p \) (probe), and frequencies \( \Omega_1, \Omega_2 \) and \( \Omega_p \) are simultaneously shined on the atoms. We consider an off-resonant regime with one-photon detuning \( \Delta \) for the pump fields, one-photon detuning \( \Delta_p \) for the probe field, and two-photon Raman detuning \( \delta \) [see Fig. II(b)]. The three classical waves mix inside the atomic medium to generate a multitude of secondary waves; the one which propagates along the positive \( z \) direction [see Fig. II(a)] and which can be viewed as a reflection of the first pump wave off a spatial grating produced by the second pump and the probe is of interest and denoted as the signal wave. This wave is radiated on the \( 2 \rightarrow 3 \) transition, and has frequency \( \Omega_s = \Omega_p - (\Omega_1 - \Omega_2) \) and phase matched wave vector \( k_s = (\Omega_s/c) \hat{z} = k_p - (k_1 - k_2) \). A few methods can be employed to separate the signal wave from the probe and pumps. The simplest one [illustrated in Fig. II(a)] requires an increase of the angles made by the three primary wave with the \( z \) axis sufficient to provide a clear angular resolution. The other two involve both the use of a polarization beam splitter (to distinguish between probe, pump 1 and signal) or a spectral filter (that can distinguish between pump 2 and signal). Finally, a photodetector (PD) is used to detect the photon number of the signal field. In the following, we consider the dynamics of an atom located inside the sample, at position \( r \).

The three incoming pulses are represented as \( c \) number traveling waves

\[
E^{(i)}(r, t) = \frac{1}{2} \left[ E_i(r, t)e^{i(k_i \cdot r - \Omega_i t)} + E^*_i(r, t)e^{-i(k_i \cdot r - \Omega_i t)} \right] \hat{e}_i, \tag{1}
\]

for \( i = 1, 2, p \), with polarization unit vectors \( \hat{e}_i \). The matrix elements of the atomic dipole \( d \) between atomic states \( j \) and \( j' \) (along \( \hat{e}_i \)) is denoted by \( d_{jj'} = \langle j | d | e^{i(k_i \cdot r)} | j' \rangle \). The interaction between fields and atom is described by the (spatially dependent) Rabi frequencies

\[
\Lambda_1(r, t) = \frac{d^*_{14} E_1(r, t)}{2\hbar} e^{i(k_1 \cdot r)} = \chi_1(r, t) e^{i(k_1 \cdot r)}, \tag{2a}
\]

\[
\Lambda_2(r, t) = \frac{d^*_{24} E_2(r, t)}{2\hbar} e^{i(k_2 \cdot r)} = \chi_2(r, t) e^{i(k_2 \cdot r)}, \tag{2b}
\]

\[
\Lambda_p(r, t) = \frac{d^*_{13} E_p(r, t)}{2\hbar} e^{i(k_p \cdot r)} = \chi_p(r, t) e^{i(k_p \cdot r)}. \tag{2c}
\]
FIG. 1: (a) The three classical pulses indexed by 1, 2 and \( p \) are incident on the sample at an angle giving rise to a signal field that propagates along the cylinder’s axis. Detection of photon number takes place at the photodetector (PD). (b) Illustration of the internal structure of an atom as a double \( \Lambda \) scheme. (c) The signal field is represented as a reflection of the probe \((p)\) off a spatial grating generated by the pumps \((1 \text{ and } 2)\)

Assuming that the pump and probe pulses are long \([cT \gg L]\) and wide (transverse area larger than \( A \)), we can neglect the slow variation with \( r \) in \( \chi_j(r, t) \); the field envelope is replaced with its value at \( r = 0 \), and \( \chi_j(0, t) \) is replaced with \( \chi_j(t) \).

Atomic operators are defined as \( \sigma_{ii} = |i\rangle \langle i| \) (population operators), for \( i = 1, 2, 3, 4 \), and \( \sigma_{ij} = |i\rangle \langle j| \) (coherences), for \( i, j = 1, 2, 3, 4 \) but with \( i \neq j \). Ignoring the coupling to the vacuum for the moment, the Hamiltonian that describes the evolution of the atom driven by the three fields is a sum of the free Hamiltonian \((H_0)\) and the classical field-atom interaction Hamiltonian \((V_c)\)

\[
H_0 = \hbar \omega_{21} \sigma_{22} + \hbar \omega_{31} \sigma_{33} + \hbar \omega_{41} \sigma_{44}, \tag{3a}
\]

\[
V_c = -\hbar \left[ \Lambda_1(r, t) e^{-i\Omega_1 t} \sigma_{41} + \Lambda_1^*(r, t) e^{i\Omega_1 t} \sigma_{14} \right] - \hbar \left[ \Lambda_2(r, t) e^{-i\Omega_2 t} \sigma_{42} + \Lambda_2^*(r, t) e^{i\Omega_2 t} \sigma_{24} \right] - \hbar \left[ \Lambda_p(r, t) e^{-i\Omega_p t} \sigma_{31} + \Lambda_p^*(r, t) e^{i\Omega_p t} \sigma_{13} \right]. \tag{3b}
\]

In the Heisenberg picture, the rapid time variation in the coherences is removed: \( \sigma_{14} = \)
\(\tilde{\sigma}_{14} e^{-i\Omega_1 t}, \sigma_{24} = \tilde{\sigma}_{24} e^{-i\Omega_2 t}, \sigma_{13} = \tilde{\sigma}_{13} e^{-i\Omega_2 t}, \sigma_{12} = \tilde{\sigma}_{12} e^{-i(\Omega_1 - \Omega_2)t}, \sigma_{34} = \tilde{\sigma}_{34} e^{-i(\Omega_1 - \Omega_2)t}\) and \(\sigma_{23} = \tilde{\sigma}_{23} e^{-i\Omega_p - (\Omega_1 - \Omega_2)t}\). Equations of motion for the slowly varying atomic operators \(\tilde{\sigma}_{ij}\) are obtained (dropping the tildes) as

\[
\begin{align*}
\frac{d}{dt}\sigma_{11}(t) &= -i\Lambda_1(r, t)\sigma_{11} + i\Lambda_1^*(r, t)\sigma_{14} - i\Lambda_p(r, t)\sigma_{31} + i\Lambda_p^*(r, t)\sigma_{13}, \\
\frac{d}{dt}\sigma_{22}(t) &= -i\Lambda_2(r, t)\sigma_{22} + i\Lambda_2^*(r, t)\sigma_{24}, \\
\frac{d}{dt}\sigma_{33}(t) &= i\Lambda_p(r, t)\sigma_{31} - i\Lambda_p^*(r, t)\sigma_{13}, \\
\frac{d}{dt}\sigma_{44}(t) &= i\Lambda_1(r, t)\sigma_{41} - i\Lambda_1^*(r, t)\sigma_{14} + i\Lambda_2(r, t)\sigma_{42} - i\Lambda_2^*(r, t)\sigma_{24}, \\
\frac{d}{dt}\sigma_{14}(t) &= -i\Delta_{14} + i\Lambda_1(r, t) [\sigma_{11} - \sigma_{44}] + i\Lambda_2(r, t)\sigma_{12} - i\Lambda_p(r, t)\sigma_{34}, \\
\frac{d}{dt}\sigma_{24}(t) &= -i(\Delta + \delta)\sigma_{24} + i\Lambda_2(r, t) [\sigma_{22} - \sigma_{44}] + i\Lambda_1(r, t)\sigma_{21}, \\
\frac{d}{dt}\sigma_{13}(t) &= -i\Delta_{p1}\sigma_{13} + i\Lambda_p(r, t) [\sigma_{11} - \sigma_{33}] - i\Lambda_1(r, t)\sigma_{43}, \\
\frac{d}{dt}\sigma_{34}(t) &= -i(\Delta - \Delta_p)\sigma_{34} + i\Lambda_1(r, t)\sigma_{31} + i\Lambda_2(r, t)\sigma_{32} - i\Lambda_p^*(r, t)\sigma_{14}, \\
\frac{d}{dt}\sigma_{23}(t) &= -i(\Delta_p + \delta)\sigma_{23} + i\Lambda_p(r, t)\sigma_{21} - i\Lambda_2(r, t)\sigma_{43}, \\
\frac{d}{dt}\sigma_{12}(t) &= i\delta\sigma_{12} - i\Lambda_1(r, t)\sigma_{42} + i\Lambda_2^*(r, t)\sigma_{14} - i\Lambda_p(r, t)\sigma_{32}.
\end{align*}
\]

We are interested in the time and spatial modulation of the \(\sigma_{23}\) coherence, which is responsible with the generation of the signal. The rapid time variation of \(\sigma_{23}\), at frequency \(\Omega_s = \Omega_p - (\Omega_1 - \Omega_2)\) has already been eliminated. The terms giving a spatial modulation at the correct, phase-matched wave vector \(k_s = k_p - (k_1 - k_2)\), have to be identified in Eq. (\text{II}(i)).

We proceed to describe the dynamics of the atom by using a perturbative approach to solve Eqs. (\text{II}), in which terms up to the third order in \(\chi/\Delta_i\) (\(\chi = \chi_1, \chi_2, \chi_3\) and \(\Delta_i = \Delta, \Delta_p, \delta\)) are kept. We start with an \(x\) polarized state of the two-level system formed by the ground sublevels 1 and 2, where \(\sigma_{11}(0) = \sigma_{11}^0 = 1/2, \sigma_{22}(0) = \sigma_{22}^0 = 1/2, \sigma_{12}(0) = \sigma_{12}^0 = \sigma_{21}(0) = \sigma_{21}^0 = -1/2\). Owing to the assumptions of large detunings and long duration pulses, the upper state population operators \(\sigma_{ii}\) can be neglected. The coherences between ground and excited states adiabatically follow the fields. Assuming \(\delta \ll \Delta, \Delta_p\), the first
order solution of Eqs. (4e,f,g,i) are given by:

\[ \sigma^{(1)}_{14}(t) \simeq \frac{\Lambda_1(r,t)}{\Delta} \sigma^0_{11} + \frac{\Lambda_2(r,t)}{\Delta} \sigma^0_{12}, \]  
\[ \sigma^{(1)}_{24}(t) \simeq \frac{\Lambda_2(r,t)}{\Delta} \sigma^0_{22} + \frac{\Lambda_1(r,t)}{\Delta} \sigma^0_{12}, \]  
\[ \sigma^{(1)}_{13}(t) \simeq \frac{\Lambda_p(r,t)}{\Delta_p} \sigma^0_{11}, \]  
\[ \sigma^{(1)}_{23}(t) \simeq \frac{\Lambda_p(r,t)}{\Delta_p} \sigma^0_{21}. \]

To second order, as observed before, the excited state populations have derivatives which are identically zero; however, the ground states rate equations are

\[ \frac{d}{dt}\sigma^{(2)}_{11}(t) \simeq i \left[ \frac{\Lambda_1^*(r,t)\Lambda_2(r,t)}{\Delta} \sigma^0_{12} - \frac{\Lambda_1(r,t)\Lambda_2^*(r,t)}{\Delta} \sigma^0_{21} \right], \]  
\[ \frac{d}{dt}\sigma^{(2)}_{22}(t) = -\frac{d}{dt}\sigma^{(2)}_{11}(t). \]

An important observation can be made at this point. The system formed by the two ground sublevels is driven by an effective field with a Rabi frequency \[ |\chi_1(t)\chi_2(t)/\Delta| \] multiplied by a spatial phase, dependent on the atom’s position inside the medium: \[ e^{-i(k_1-k_2) \cdot r}. \]

In the copropagating pumps geometry, this spatial modulation is negligibly small over the length of the medium since \[ |k_1 - k_2| L \simeq \omega_2 L/c \ll 1. \] In addition, assuming \[ \chi_1(t) \] and \[ \chi_2(t) \] are real, the resulting effective Rabi frequency is real. The \[ \sigma^{(2)}_{11} \] is, in consequence, vanishing for any atom inside the medium, independent on its location. Also, owing to the fact that equal the ground state populations are equal, the \[ \sigma^{(2)}_{12} \] vanishes to this order as well; in consequence the system stays in a dark state with equal populations and coherence along the \[ x \] axis.

The two coherences, \[ \sigma_{34} \] and \[ \sigma_{12} \], that act as sources in Eq. (4i), are not driven directly by the field; their change from initial values is a second order contribution. They can be derived from Eqs. (4h,j), which give

\[ \sigma^{(2)}_{34}(t) \simeq \frac{\Lambda_p^*(r,t)}{\Delta\Delta_p} \left[ \Lambda_1(r,t)\sigma^0_{11} + \Lambda_2(r,t)\sigma^0_{12} \right], \]  
\[ \sigma^{(2)}_{12}(t) = \sigma^0_{12} + \frac{\Lambda_1(r,t)\Lambda_2^*(r,t)}{\Delta\delta} \left[ \sigma^0_{11} - \sigma^0_{22} \right]. \]

We are now in position to evaluate the third order approximation of \[ \sigma_{23} \]. Replacing Eqs.
in Eq. \ref{eq:4i}, one obtains

\[
\sigma_{23}^{(3)}(t) \simeq \frac{\Lambda_p(r,t)}{\Delta_p} \sigma_{21}^0 + \frac{\Lambda_1^*(r,t)\Lambda_2(r,t)\Lambda_p(r,t)}{\Delta \Delta_p \delta} \sigma_{11}^0 - \frac{\Lambda_1^*(r,t)\Lambda_2(r,t)\Lambda_p(r,t)}{\Delta \Delta_p^2} \sigma_{11}^0.
\]

(8)

Two of the terms in the above expression [first and fourth in the right hand side of Eq. \ref{eq:8}] describe a field propagating in the direction of \(k_p\), which is not phase matched [\(k_p \neq \Omega_s/c\)]. Both other terms give rise to a phase matched signal field; however, in the limit \(\delta \ll \Delta_p\) the third term is negligible compared to the second one, and is dropped. With the notation \(\sigma_z = (\sigma_{22} - \sigma_{11})/2\), the expression of \(\sigma_{23}\) can be simplified

\[
\sigma_{23}(t) = \frac{\Lambda_1^*(r,t)\Lambda_2(r,t)\Lambda_p(r,t)}{\Delta \Delta_p \delta} \sigma_{11}^0 - \frac{\Lambda_1^*(r,t)\Lambda_2(r,t)\Lambda_p(r,t)}{\Delta \Delta_p^2} \sigma_{21}^0.
\]

(9)

The analysis is not complete before the role of spontaneous decay is properly identified. The conditions imposed on the fields guarantee that the effective driving pulse doesn’t remove the system from the dark state; spontaneous decay can still destroy the coherence, as is the case in \cite{17}. However, we start by making an observation on a simple system of a \(\Lambda\) atom driven by two equal fields, in which, on each of the two transitions, Raman and Rayleigh scatterings cancel each other, leading to a state not affected by spontaneous emission. Our case is similar to this, although not completely identical. In the absence of the probe field, an initially balanced state (equal populations in the ground substates) with \(-1/2\) coherence, would be preserved by choosing equal amplitude pumps. The probe field provides an imbalance in the system, which can be compensated by choosing a field strength on the \(2 \rightarrow 4\) transition larger than the one on the \(1 \rightarrow 4\) transition by an amount that cancels the effect of the probe. This condition is \(\gamma'(\chi_1^2 - \chi_2^2)/\Delta^2 = \gamma\chi_p^2(t)/\Delta_p^2\) and is obtained by imposing the condition that the decay terms in the rate equations for \(\sigma_{11}(t)\) and \(\sigma_{22}(t)\) are identically zero. In addition, the coherence between the ground sublevels follows the fields with a slowly varying value of \(-\chi_2/\chi_1/2\). By limiting the intensity of the probe field to small values compared to the pump field intensities, this coherence stays close to the maximal value of \(-1/2\) throughout the interaction. The maintainance of coherence is the key feature of this level scheme, allowing for much better spin squeezing than in other projection schemes. A similar idea was used to improve spin squeezing in cavity-field interactions \cite{18}.
III. EMITTED FIELD

A wave equation for the signal field can be written where the polarization of the medium acts as a source. Defining the positive frequency part of the polarization resulting from a single atom (denoted by $\alpha$) located at position $\mathbf{r}$ as $\hat{P}_\alpha^{(+)}(t) = \hat{P}_\alpha(t) e^{i(k_s z - \Omega_s t)}$, its envelope is given by

$$\hat{P}_\alpha(t) = \hbar \left[ d_{23} \sigma_{23}(t) + \text{h.c.} \right]$$ (10)

The assumption of Fresnel number close to unity for the pencil-shaped medium can be invoked now; this leads to a one-dimensional behavior of the propagation of the signal field. In consequence, as in Refs. [19, 20], $z$ dependent continuous operators can be defined by performing an average over infinitesimal slices in the transverse direction of the medium. The continuous polarization operator is thus defined as

$$\hat{P}(z,t) = \lim_{\Delta z \to 0} \frac{1}{\Delta V_z} \sum_{\alpha \in \Delta V_z} \hat{P}_\alpha(t),$$ (11)

where $\Delta V_z = A \Delta z$ is the volume of a slice and the sum is performed over all atoms in the slice (number of atoms in a slice $N_z = n_a A \Delta z$). Continuous atomic operators can also be defined as

$$\hat{O}(z,t) = \lim_{\Delta z \to 0} \frac{1}{N_z} \sum_{\alpha \in \Delta V_z} \hat{O}_\alpha(t).$$ (12)

Replacing the expression for $\sigma_{23}$ previously derived in Eq. (10) and making use of continuous atomic operators, Eq. (11) becomes:

$$\hat{P}(z,t) = \hbar n_a d_{23} f(t) [\sigma_{11}(z) - \sigma_{22}(z)].$$ (13)

where the notation $f(t) = \chi_1^*(t)\chi_2(t)\chi_\rho(t)/\Delta \Delta p \delta$ has been made. As in Ref. [19, 20], the quantized signal field amplitude (the positive frequency part) can be written as $\hat{E}_{s}^{(+)}(z,t) = \mathcal{E}_s \hat{E}_s(z,t) e^{i(k_s z - \Omega_s t)}$, where $\mathcal{E}_s = \sqrt{\hbar \Omega_s / 2 \epsilon_0 AL}$ and $\hat{E}_s(z,t)$ is a slowly varying envelope operator. The wave equation in terms of slowly varying field and polarization envelope operators can be written as

$$\left[ \partial_z + \frac{1}{c} \partial_t \right] \hat{E}_s(z,t) = -i \left[ \frac{k_s}{2 \epsilon_0} \right] \hat{P}(z,t).$$ (14)

Setting $K(t) = \hbar k_s d_{23} f(t) / 2 \epsilon_0$, we find that population fluctuations at each point in the medium are connected to the signal field by

$$\left[ \partial_z + \frac{1}{c} \partial_t \right] \hat{E}_s(z,t) = -i n_a K(t) [\sigma_{11}(z) - \sigma_{22}(z)].$$ (15)
If collective atomic operators are defined

\[ S_z = \frac{N_a}{2L} \int_0^L dz \left[ \sigma_{11}(z) - \sigma_{22}(z) \right], \]  

(16)

the solution for Eq. (15) is found [for derivation see Appendix A] to be

\[ \hat{E}_s(L, t) = \hat{E}_s(0, t) - i \frac{2K(t)}{A} S_z. \]  

(17)

The result states that the signal field amplitude exiting the sample is amplified by a quantity proportional to the collective population operator. The factor \( 2[K(t)/A]S_z \) can also be reexpressed as \( n_a[2S_z/N_a]L \), which shows a linear increase with the length and atomic number density of the sample. The intensity of the emitted field can be calculated as an expectation value

\[ I_s \sim \left\langle \hat{E}_s(L, t)\hat{E}_s^\dagger(L, t) \right\rangle = \frac{4|K(t)|^2}{A^2} \langle S_z^2 \rangle. \]  

(18)

An initial population imbalance in the ground substates gives rise to a signal quadratic in the number of atoms in the sample. However, when the collective state of the system is a coherent one, polarized along the \( x \) direction, for example, population fluctuations only are reflected in the emitted field. The variance of \( S_z \) is in this case \( N^2_a/4 \), which leads to a gain in the field intensity, linear in number of atoms.

IV. EFFECTIVE HAMILTONIAN AND ENTANGLEMENT GENERATION

The conditional atomic generation process is similar to an EPR-type experiment, where entanglement is created between two subsystems (medium and signal field, in our case) by means of an interaction that lasts a finite time (duration of pulses); a measurement (detection of signal photon number) is performed on one of the subsystems (field) long after the interaction has ceased. Consequently, the other subsystem (atoms) is projected onto the state entangled with the state indicated by the detection outcome. A wave function approach (or density matrix, when imperfect detection is accounted for) can be taken to describe the coupled evolution of the two subsystems, while a continuous measurement theory is particularized to this case to describe the nondeterministic evolution of the system during the detection process.
The results of Ref. [17] are used in what follows. The complete details of the derivation of an effective Hamiltonian are found in Appendix C of Ref. [17], where a similar calculation is described. The measurement process, both under the assumption of perfect detection and including imperfect detectors, is also presented in detail in Sec. V. of Ref. [17]. We are concerned here, rather with the main differences between our proposed scheme and the ones proposed elsewhere.

The interaction between the signal field amplitude operator and the atoms in the sample is written in the Heisenberg picture. An integration over the transverse wave vector components of the generated field (allowed by the assumption of Fresnel number close to unity) followed by one over \( x \) and \( y \) leads to a one-dimensional formulation of the problem, where the continuous atomic operators are specified only by their \( z \) spatial location, while the field has a transverse spatial extent \( A \) (matching the cross-sectional area of the medium) [see also [20]]. The coherence between levels 2 and 3 is thereafter replaced, using Eq. (9), to lead to an effective Hamiltonian

\[
H_{\text{eff}}(t) = \hbar b(t) \left( \int dk_z d_y^\dagger(k_z) e^{i(\omega_k - \Omega)t} \right) S_z + \text{h.c},
\]

(19)

where \( b(t) = \left[ 4\pi d_{23}E_s/\hbar\sqrt{A} \right] f(t) \), while \( d_y^\dagger(k_z) \) is a one-dimensional field operator defined as

\[
d_\lambda(k_z) = \frac{1}{2\pi\sqrt{A}} \int dx dy \int dk_z dk_y a_\lambda(k) e^{i k_x x} e^{i k_y y}
\]

(20)

and satisfying the following commutation relations \([d_\lambda(k_z), d_\lambda^\dagger(k_z')] = \delta(k_z - k_z')\delta_\lambda\lambda'\).

With the observation that the Hamiltonian commutes with itself at different times \([H_{\text{eff}}(t), H_{\text{eff}}(t')] = 0\), the evolution operator over the duration of the interaction can be expressed in a simple form:

\[
U(T) = \exp \left[ -\frac{i}{\hbar} T \int dt H_{\text{eff}}(t) \right].
\]

(21)

The time integral brings the Fourier components of \( f(t) \) the incident pulse field envelope \( \int_0^T dt e^{i(\omega_k - \Omega)t} f(0, t) \simeq F(\omega_k - \Omega) \). The integral over \( k_z \) in Eq. (19) can be now represented by an effective one-photon creation operator with carrier frequency \( \Omega_s \) and duration \((c\Delta k)^{-1} \simeq T\) defined as

\[
c_y^\dagger = \frac{\sqrt{T}}{\sqrt{\int_0^T |f(t)|^2}} \int dk_z F(\omega_k - \Omega) d_y^\dagger(k_z),
\]

(22)
and obeying the usual commutation relation \([c_y, c_y^\dagger] = 1\). This leads to a simple form for the evolution operator

\[
U(T) = \exp[-iC(c_y^\dagger - c_y)S_z]. \tag{23}
\]

with

\[
C = \left[4\pi d_{23} \mathcal{E}_s / \hbar \sqrt{A_c^{3/2}} \right] \left( \int_0^T dt |f(t)|^2 \right)^{1/2}. \tag{24}
\]

The atoms-signal field system starts in an initial state with state vector

\[
|\psi(0)\rangle = |S_x = S\rangle_a \otimes |0\rangle_f = \sum_{M=-S}^S A(S, M) |S, M\rangle_a \otimes |0\rangle_f,
\]

where the index \(a\) denotes states of the atoms, while the index \(f\) denotes states of the field. The initial state of the atoms is an eigenstate of \(S_x\) [operator which is defined similarly to Eq. (23)] with binomial coefficients \(A(S, M) = \frac{1}{2\sqrt{(2S)!/(S+M)!M!(S-M)!}}\) (where \(S = N_a/2\)). After a period of coherent evolution governed by the evolution operator \(U(T)\) [Eq. (23)], a collapse induced by a measurement with an outcome of \(n_m\) photons leads to the following state vector for the atoms (assuming 100% detection efficiency):

\[
|\psi_{n_m}\rangle_a = \sum_{M=-S}^S \frac{A(S, M)(iCM)^{n_m}e^{-(CM)^2/2}}{\sqrt{\sum_{X=-S}^S |A(S, M)|^2 (CM)^{2n_m}e^{-(CM)^2}}} |S, M\rangle_a, \tag{25}
\]

This state vector describes spin squeezed states for \(n_m = 0\) and Schrodinger cat states for \(n_m > 0\). The results are similar to the ones presented in [17] (the reader is referred to that publication for relevant discussions and graphs), with one important exception. In [17], the value of \(C\) is limited to small values \((C \leq \sqrt{N_a}L/2N_a)\) to insure that spontaneous emission does not lead to mean spin depolarization. Here, that restriction does not apply since the total coherence is not seriously affected by spontaneous emission. In consequence, the results obtained here overlap with the ideal case presented in [17], where spin squeezing close to the Heisenberg limit and well-resolved Schrodinger cat states can be obtained.

V. CONCLUSIONS

We presented a probabilistic scheme in which the detection of photon number induces the collapse of the quantum state of a collection of atoms onto either a spin squeezed or a Schrodinger cat state. The main result of the paper is that spontaneous decay does not play
a critical role in the decoherence of the system, therefore allowing one to obtain a squeezing parameter close to the Heisenberg limit (assuming perfect detection). This has been done by maintaining the system in a dark state by choosing the appropriate configuration of driving pulses.

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VII. APPENDIX: SOLUTION FOR THE FIELD EQUATION

Using the Laplace transform with respect with $z$ (defined as $o(s) = L\{O(z)\} = \int_0^\infty dzO(z)e^{-sz}$) and defining $e(s, t) = L\{\hat{E}_s(z, t)\}$ and $\sigma_z(s) = L\{\sigma_z(z)\}$ Eq. (15) becomes

$$\partial_t \hat{e}_s(s, t) + cs \hat{e}_s(s, t) = c\hat{E}_s(0, t) - icK(t)\sigma_z(s).$$

(A1)

A formal integration results in

$$\hat{e}_s(s, t) = \hat{e}_s(s, 0)e^{cst} + \int_0^t dt'e^{-cst(t-t')}\left[c\hat{E}_s(0, t') - icK(t')\sigma_z(s)\right].$$

(A2)

The time dependent term $K(t')$ contains the slow varying field envelope, which is evaluated at time $t$ leading to

$$\hat{e}_s(s, t) = \hat{e}_s(s, 0)e^{cst} + \frac{1}{s}\left[1 - e^{-sc(t-t')}\right]\left[c\hat{E}_s(0, t) - icK(t)\sigma_z(s)\right].$$

(A3)

Applying the inverse Laplace transform $L^{-1}$, a general solution of Eq. (15), for arbitrary $z$ and $t$, is found

$$\hat{E}_s^{(+)}(z, t) = \hat{E}_s^{(+)}(z - ct, t)\ln(h(z) - h(z - ct)) + \hat{E}_s^{(+)}(0, t)\left[h(z) - h(z - ct)\right] -$$

$$- iK(t)\left\{\int_0^z dz'\hat{\sigma}_z(z') - \int_0^{z - ct} dz'\hat{\sigma}_z(z')\right\} h(z - ct).$$

(A4)

We evaluate now the field at the sample exit ($z = L$). In the limit of long pulses ($T \gg L/c$), the Heaviside function $h(L - ct)$ is zero for most of the time and will therefore be ignored.
In terms of the collective operator defined in Eq. (16), the field becomes:

\[ \hat{E}^{(+)}(L, t) = \hat{E}^{(+)}(0, t) - i \frac{2K(t)}{n_0 A} S_z. \]  

(A5)