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APPLICATION OF HARMONIC ANALYSIS
TO INELASTIC ELECTRON-PROTON SCATTERING

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1. - INTRODUCTION

The aim of harmonic analysis is to map a class of functions on another class by means of Fourier transformations and to characterize the classes related in this fashion. A typical example of such relation is expressed in the classical Paley-Wiener theorem. Let us consider the class of infinitely differentiable functions on $\mathbb{R}_1$ with support in the interval $[-\alpha, \alpha]$, $\alpha > 0$. We denote this class by $C^\infty[-\alpha, \alpha]$. Fourier transforms of functions of this class are entire analytic. In fact, the Paley-Wiener theorem asserts ^{1), 2)} : the restriction of the entire analytic function $g(s)$ to the real axis is the Fourier transform of a function $f(x) \in C^\infty[-\alpha, \alpha]$, if and only if for every $n = 0, 1, 2, \dots$ a constant C_n exists such that for all complex s

$$|s^n g(s)| \leq C_n e^{\alpha |\Im s|} \quad (1)$$

It follows that $g(s)$ is of exponential order one and type α if α or $-\alpha$ is really a boundary point of the support of $f(x)$. Another version of the Paley-Wiener theorem can be formulated for distributions with support in the interval $[-\alpha, \alpha]$ (α may also be zero now). We denote this class of distributions by $D'[-\alpha, \alpha]$. We have : the restriction of the entire analytic function $g(s)$ to the real axis is the Fourier transform of a distribution $f(x) \in D'[-\alpha, \alpha]$, if and only if a constant C and an integer N exist such that for all complex s

$$|g(s)| \leq C (1 + |s|)^N e^{\alpha |\Im s|} \quad (2)$$

Theorems which relate a class of functions that are characterized by their singularities in a compact domain with the Fourier transformed class specified by the asymptotic behaviour at infinity can be called "theorems of Paley-Wiener type". In most applications of harmonic analysis to elementary particle physics it is the correspondence between these two properties which is of interest.

As an explicit and very important example we study the distribution

$$f_{\varphi}(x) = \frac{x_{\pm}^{\varphi}}{\Gamma(\varphi+1)} \quad (3)$$

where the subscript plus or minus denotes for arbitrary $f(x)$

$$f(x)_{\pm} = |f(x)| \Theta(\pm f(x)), \quad \Theta(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$$

The distribution (3) is defined to be entire analytic, that is to say, we define the regularization procedure for $\operatorname{Re} \varphi \leq -1$ such that $f_{\varphi}(x)$ is entire analytic in φ . We denote such regularization "analytic regularization". In particular we have for $\varphi = -1, -2, -3, \dots$

$$f_{-n-1}(x) = \delta^{(n)}(x), \quad n = 0, 1, 2, \dots \quad (4)$$

Its Fourier transform is ³⁾

$$\begin{aligned} g_{\varphi}(s) &= \int_{-\infty}^{+\infty} dx e^{ixs} f_{\varphi}(x) \\ &= i \left[e^{i\frac{\pi}{2}\varphi} s_{+}^{-\varphi-1} - e^{-i\frac{\pi}{2}\varphi} s_{-}^{-\varphi-1} \right] \quad (5) \end{aligned}$$

We see that $g_{\varphi}(s)$ goes smoother to zero at infinity, the smoother $f_{\varphi}(x)$ behaves at $x = 0$.

Instead of Fourier transformations on R_n we can study harmonic analysis on simple Lie groups. Again there exist theorems of Paley-Wiener type. As an example we consider the homogeneous Lorentz group $SL(2, \mathbb{C})$.

The Fourier transform of a function $f(a)$ on $SL(2, \mathbb{C})$ is an operator function $T^\lambda(f)$ where λ varies over the imaginary axis (the principal series) and the operator acts in a direct sum Hilbert space

$$\sum_M \oplus H_M$$

In this sum M runs over all integers and half integers. For an element $a \in SL(2, \mathbb{C})$ we define

$$a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad |a|^2 = \sum_{i,j=1,2} |a_{ij}|^2 \quad (6)$$

The Fourier transform of the function

$$f_\sigma(a) = |a|^{2\sigma}, \quad \operatorname{Re} \sigma < -1 \quad (7)$$

has a non-vanishing component only in the subspace H_0 (for $M = 0$) and this operator in H_0 maps the spin-zero vector on itself and all vectors orthogonal to it on zero. The matrix element between the spin-zero vectors is ⁴⁾

$$K_\sigma(\lambda) = \frac{1}{32\pi(-\sigma-1)} B\left(\frac{1}{2}(\lambda-\sigma-1), -\frac{1}{2}(\lambda+\sigma+1)\right) \quad (8)$$

This Fourier transform (8) is meromorphic in λ and can be continued to an entire analytic function of σ . The asymptotic property of $f_\sigma(a)$ for $|a| \rightarrow \infty$ reflects itself in two sequences of poles in the complex λ plane. If $\operatorname{Re} \sigma < -1$, $f_\sigma(a)$ is square integrable on $SL(2, \mathbb{C})$ with respect to the Haar measure. In this case there exists a strip of holomorphy around the imaginary λ axis of width $\operatorname{Re} \sigma + 1$. If we

continue in σ beyond the boundary of the half plane $\text{Re } \sigma < -1$, $K_\sigma(\lambda)$ generates a linear functional which operates on entire functions of λ by means of contour integrals. These contour integrals are obtained by analytic continuation of a formula of Parseval's kind in σ .

2. - THE KINEMATICS OF ELECTRON-PROTON SCATTERING

We want to apply the ideas presented in the Introduction to electron-proton scattering. This forces us to deal first with the kinematics of this process. In the one-photon-exchange approximation the elastic plus inelastic differential cross-section can be deduced from the diagonal matrix element

$$W_{\mu\nu}(x) = \frac{1}{2} \sum_{s=\pm\frac{1}{2}} \langle p, s | [j_\mu(x), j_\nu(0)] | p, s \rangle \quad (9)$$

of the commutator of the electromagnetic current between proton states. We normalize these states as

$$\langle p_1, s_1 | p_2, s_2 \rangle = (2\pi)^3 \frac{p_0}{M} \delta^{(3)}(\vec{p}_1 - \vec{p}_2) \delta_{s_1 s_2}$$

From the definition (9) and time-reversal invariance we find the symmetries

$$W_{\mu\nu}(x) = -W_{\nu\mu}(-x) = W_{\nu\mu}(x) \quad (10)$$

and the causality restriction

$$W_{\mu\nu}(x) = 0 \quad \text{for} \quad x^2 < 0 \quad (11)$$

By a Fourier transformation we obtain

$$\begin{aligned}\tilde{W}_{\mu\nu}(k) &= \frac{1}{2\pi} \int d^4x \, e^{ikx} W_{\mu\nu}(x) \\ &= \left(\frac{k_\mu k_\nu}{k^2} - g_{\mu\nu} \right) W_1(k^2, \nu) \\ &\quad + \left(p_\mu - \frac{\nu}{k^2} k_\mu \right) \left(p_\nu - \frac{\nu}{k^2} k_\nu \right) W_2(k^2, \nu)\end{aligned}\quad (12)$$

with

$$\nu = kp, \quad p^2 = M^2 = 1$$

The whole dynamics is contained in the structure functions W_1 and W_2 . Only space-like momenta k are of interest in electron-proton scattering. For the expression of the differential cross-section for electron-proton scattering and the total cross-sections σ_t and σ_ℓ of virtual photoproduction in terms of W_1 and W_2 , see Ref. 5).

Another decomposition of $\tilde{W}_{\mu\nu}(k)$ into invariant functions is ⁶⁾

$$\begin{aligned}\tilde{W}_{\mu\nu}(k) &= (k_\mu k_\nu - g_{\mu\nu} k^2) \tilde{V}_1(k^2, \nu) \\ &\quad + \left[(p_\mu k_\nu + p_\nu k_\mu) \nu - p_\mu p_\nu k^2 - g_{\mu\nu} \nu^2 \right] \\ &\quad \times \tilde{V}_2(k^2, \nu)\end{aligned}\quad (13)$$

where $\tilde{V}_{1,2}(k^2, \nu)$ are Fourier transforms [with the same normalization as in (12)] of causal functions $V_{1,2}(x)$

$$V_{1,2}(x) = 0 \text{ for } x^2 < 0, \quad V_{1,2}(-x) = -V_{1,2}(x) \quad (14)$$

The functions $\tilde{V}_{1,2}$ can be expressed by $W_{1,2}$ and vice versa as

$$\begin{aligned} W_1 &= k^2 \tilde{V}_1 + \nu^2 \tilde{V}_2 \\ W_2 &= -k^2 \tilde{V}_2 \end{aligned} \quad (15)$$

In the domain of space-like vectors k the linear combinations

$$\begin{aligned} E(\nu) [\tilde{V}_1 + \tilde{V}_2] \\ E(\nu) [k^2 \tilde{V}_1 + \nu^2 \tilde{V}_2] \end{aligned}$$

are non-negative. The support of the functions $\tilde{V}_{1,2}$ or $W_{1,2}$ for space-like k is the union of an elastic part

$$2|\nu| + k^2 = 0$$

and an inelastic part

$$2|\nu| + k^2 \geq (1 + m_\pi)^2 - 1$$

3. - HARMONIC ANALYSIS IN R_4

3.1. - The Scaling Law

If the quantities k^2 and ν tend to infinity such that the ratio

$$\xi = \frac{-k^2}{2\nu} , \quad k^2 < 0$$

stays fixed we speak of the "Bjorken limit" ⁷⁾. We define this limit only for $\xi \neq 0$. The existence of the Bjorken limits

$$\begin{aligned} B\text{-}\lim W_1(k^2, \nu) &= F_1(\xi) \\ B\text{-}\lim \nu W_2(k^2, \nu) &= F_2(\xi) \end{aligned} \tag{16}$$

is called the scaling law. Due to the support restriction on $W_{1,2}$, the support of these limit functions $F_{1,2}(\xi)$ is restricted to the interval

$$-1 \leq \xi \leq +1$$

The functions $F_{1,2}(\xi)$ have the symmetry

$$\begin{aligned} F_1(\xi) &= -F_1(-\xi) \\ F_2(\xi) &= +F_2(-\xi) \end{aligned} \tag{17}$$

From (15) and (16) we have also

$$\begin{aligned} B\text{-}\lim \nu \tilde{V}_1 &= \frac{1}{2\xi} [-F_1(\xi) + \frac{1}{2\xi} F_2(\xi)] \\ B\text{-}\lim \nu^2 \tilde{V}_2 &= \frac{1}{2\xi} F_2(\xi) \end{aligned} \tag{18}$$

The elastic contributions to $F_{1,2}(\xi)$ are proportional to the delta function $\delta(1-\xi)$. If the proton form factors of Sachs type, $G_E(k^2)$ and $G_M(k^2)$, behave as

$$\lim_{-k^2 \rightarrow \infty} (-k^2)^{-\frac{1}{2}} G_E(k^2) = \lim_{-k^2 \rightarrow \infty} G_M(k^2) = 0$$

the elastic contributions are zero. In the sequel we shall therefore assume that the limit functions $F_{1,2}(\xi)$ are continuous for $\xi \neq 0$. The positivity constraints yield finally

$$F_2(\xi) \geq 2\xi F_1(\xi) \geq 0$$

We want to show now that the scaling law results if the causal functions $V_{1,2}(x)$ exhibit a certain kind of singularities on the light cone. For this purpose we Fourier transform the distributions

$$F_\varrho(x) = \frac{1}{2\pi i} \varepsilon(x_0) g(x_0) \frac{(x_+^2)^\varrho}{\Gamma(\varrho+1)} \quad (19)$$

where ϱ is arbitrary complex and $g(x_0)$ is even and infinitely differentiable and tentatively assumed to be rapidly decreasing. For physical reasons we shall later have to modify the last premise. With

$$g(x_0) = \int_{-\infty}^{+\infty} d\sigma e^{i\sigma x_0} G(\sigma) \quad (20)$$

we get

$$\begin{aligned}\tilde{F}_\varphi(k) &= \frac{1}{2\pi} \int d^4x \, e^{ikx} F_\varphi(x) \\ &= \int_{-\infty}^{+\infty} d\sigma \, G(\sigma) \mathcal{F}_\varphi(k_0 + \sigma, \vec{k})\end{aligned}\tag{21}$$

Here we used the notation

$$\mathcal{F}_\varphi(k) = 2^{2\varphi+2} \varepsilon(k_0) \frac{(k_+^2)^{-\varphi-2}}{\Gamma(-\varphi-1)}\tag{22}$$

If we denote

$$(k_0 + \sigma)^2 - \vec{k}^2 = (\sigma - \sigma_+)(\sigma - \sigma_-)\tag{23}$$

so that

$$\begin{aligned}\sigma_+ &= -k_0 + |\vec{k}| \\ \sigma_- &= -k_0 - |\vec{k}|\end{aligned}\tag{24}$$

we can write finally

$$\tilde{F}_\varphi(k) = \frac{2^{2\varphi+2}}{\Gamma(-\varphi-1)} \left(\int_{\sigma_+}^{\infty} - \int_{-\infty}^{\sigma_-} \right) d\sigma \, G(\sigma) [(\sigma - \sigma_+)(\sigma - \sigma_-)]^{-\varphi-2}\tag{25}$$

As usual we understand the integral to be regularized analytically if necessary. For the particular value $\varphi = -1$ we find

$$\begin{aligned}
 F_{-1}(x) &= \frac{1}{2\pi i} \varepsilon(x_0) g(x_0) \delta(x^2) \\
 \tilde{F}_{-1}(k) &= \frac{1}{2|\vec{k}|} [G(\sigma_+) - G(\sigma_-)]
 \end{aligned}
 \tag{26}$$

and for $g = 0$

$$\begin{aligned}
 F_0(x) &= \frac{1}{2\pi i} \varepsilon(x_0) g(x_0) \Theta(x^2) \\
 \tilde{F}_0(k) &= -\frac{1}{|\vec{k}|^2} [G'(\sigma_+) + G'(\sigma_-)] \\
 &\quad + \frac{1}{|\vec{k}|^3} [G(\sigma_+) - G(\sigma_-)]
 \end{aligned}
 \tag{27}$$

We shall call g the order of the light cone singularity.

In the rest system of the proton we have

$$k_0 = \nu, \quad |\vec{k}| = (\nu^2 - k^2)^{\frac{1}{2}}
 \tag{28}$$

so that the Bjorken limits yield (signs are correlated !)

$$\lim_{\nu \rightarrow \pm\infty} \sigma_{\pm} = \xi,$$

(29)

$$\lim_{\nu \rightarrow \pm\infty} \sigma_{\mp} = \mp\infty$$

In the Regge limit $\nu \rightarrow \pm\infty$, k^2 fixed, we have correspondingly

$$\text{R-lim}_{\nu \rightarrow \pm\infty} \sigma_{\pm} = 0 ,$$

(30)

$$\text{R-lim}_{\nu \rightarrow \pm\infty} \sigma_{\mp} = \mp\infty$$

If we identify the most singular part on the light cone of

$$V_1(x) \text{ with } F_1(x), g(x_0) = g_1(x_0)$$

and of

$$V_2(x) \text{ with } F_0(x), g(x_0) = g_2(x_0)$$

(always in the rest system of the proton) we obtain the scaling laws with the identifications

$$G_1(\xi) = \frac{1}{\xi} \left[-F_1(\xi) + \frac{1}{2\xi} F_2(\xi) \right]$$

$$G_2'(\xi) = -\frac{1}{2\xi} F_2(\xi)$$

(31)

First we recognize that the supports of $G_1(\xi)$ and $G_2'(\xi)$ are restricted to the interval

$$-1 \leq \xi \leq +1 \quad (32)$$

so that both $g_1(x_0)$ and $x_0 g_2(x_0)$ are entire functions. The second function being odd, we may assume that $g_2(x_0)$ is itself entire. We neglect this way a possible delta function $\delta(x_0)$ in $g_2(x_0)$ which would lead to the undefined expression $\delta(x_0) \epsilon(x_0)$ in (19). This is equivalent to setting $G_2(\xi)$ equal zero outside the interval (32). Finally we want to use physical information to determine the behaviour of $g_{1,2}(x_0)$ at infinity which by Paley-Wiener arguments is connected with the smoothness of the functions $G_1(\xi)$ and $G_2'(\xi)$. Since we want our arguments that led to the formulae (13) to hold true, we have to assume at least that both functions $G_1(\xi)$ and $G_2'(\xi)$ are continuous for $\xi \neq 0$. This means that $F_{1,2}(\xi)$ should be continuous for $\xi \neq 0$, which every physicist will certainly accept. At $\xi = 0$, however, we must allow for distribution singularities, which can be determined by means of the information supplied by the Regge limits.

The Regge limit of a distribution $\tilde{F}_\varrho(k)$ (25) is easily evaluated and solely determined by the singularity of $G(\sigma)$ at $\sigma = 0$ only if the support of the distribution $\tilde{F}_\varrho(k)$ shrinks to the light cone, i.e., if the order of the light cone singularity of $F_\varrho(x)$ is one of the integers $\varrho = -1, 0, 1, \dots$. The Regge limit in the non-integral case can best be treated with the method of harmonic analysis on $SL(2, \mathbb{C})$ due to Toller and we postpone it therefore till Section 4.3. We cannot expect that the most singular part of $V_{1,2}(x)$ on the light cone that is responsible for scaling in the Bjorken limit, contains simultaneously the most singular function $g_{1,2}(x_0)$ at infinity which is responsible for the Regge limit. Instead we must be aware of the possibility that $V_{1,2}(x)$ contain terms with lower order singularities on the light cone but with the same asymptotic property of their weight functions $g_{1,2}(x_0)$ at $x_0 \rightarrow \infty$. Having this in mind we make the simplest possible model ansatz

$$\begin{aligned}
 V_1(x) &= \sum_{n=0}^N F_{-1+n}(x) + R_{1,N}(x) \\
 V_2(x) &= \sum_{n=0}^N F_n(x) + R_{2,N}(x)
 \end{aligned}
 \tag{33}$$

as an expansion in a sequence with decreasing orders of light cone singularities. Each $F_{-1+n}(x)$ ($F_n(x)$) contains a weight function $g_{1,n}(x_0)$ ($g_{2,n}(x_0)$). If $G_{1,n}^{(n)}(\xi)$ and $G_{2,n}^{(n+1)}(\xi)$ are continuous for $\xi \neq 0$, we obtain from (25)

$$\begin{aligned}
 \tilde{V}_1(k) &= \sum_{n=0}^N \frac{2^{n-1} (-1)^n}{|\vec{k}|^{n+1}} G_{1,n}^{(n)}(\sigma_+) - \text{terms obtained by the replacement } k \rightarrow -k \\
 &\quad + \tilde{R}_{1,N}(k) \\
 \tilde{V}_2(k) &= \sum_{n=0}^N \frac{2^n (-1)^{n+1}}{|\vec{k}|^{n+2}} G_{2,n}^{(n+1)}(\sigma_+) - \text{terms obtained by the replacement } k \rightarrow -k \\
 &\quad + \tilde{R}_{2,N}(k)
 \end{aligned}
 \tag{34}$$

where we kept only the lowest powers of $|\vec{k}|^{-1}$ for each n . This expansion (34) has to be reconciled with the Regge limits ⁸⁾ (see also our Section 4.2)

$$\begin{aligned}
 W_1(k^2, \nu) &\cong \beta_1(k^2) \nu^{\alpha(0)} \\
 W_2(k^2, \nu) &\cong k^2 \beta_2(k^2) \nu^{\alpha(0)-2}
 \end{aligned}
 \tag{35}$$

where $\alpha(0) = 1$ for Pomeron exchange. For $k^2 \neq 0$ we have from (15) and (35)

$$\lim_{\nu \rightarrow \infty} \nu^{-1} \tilde{V}_1 = \frac{1}{k^2} [\beta_1(k^2) + \beta_2(k^2)]$$

$$\lim_{\nu \rightarrow \infty} \nu \tilde{V}_2 = -\beta_2(k^2) \quad (36)$$

Both (34) and (36) can be made to agree if we assume that

$$G_{1,n}(\xi) = \frac{\Delta_{1,n}}{(n+1)! \xi^2} + \text{less singular distribution at } \xi = 0$$

$$G'_{2,n}(\xi) = \frac{\Delta_{2,n}}{n! \xi} + \text{less singular distribution at } \xi = 0 \quad (37)$$

where the main terms are to be understood as a principal value and its distribution theoretic derivative ⁹⁾. From (37) we deduce

$$\begin{aligned} \lim_{\xi \rightarrow 0} \xi^{n+2} G_{1,n}^{(n)}(\xi) &= (-1)^n \Delta_{1,n} \\ \lim_{\xi \rightarrow 0} \xi^{n+1} G'_{2,n}^{(n+1)}(\xi) &= (-1)^n \Delta_{2,n} \end{aligned} \quad (38)$$

and finally

$$\begin{aligned} \beta_1(k^2) + \beta_2(k^2) &= -\frac{1}{2} \sum_{n=0}^N \Delta_{1,n} \left(\frac{4}{-k^2}\right)^{n+1} + \rho_{1,N}(k^2) \\ \beta_2(k^2) &= +\frac{1}{2} \sum_{n=0}^N \Delta_{2,n} \left(\frac{4}{-k^2}\right)^{n+1} + \rho_{2,N}(k^2) \end{aligned} \quad (39)$$

This result is still a rigorous consequence of our ansatz (33). It is, however, a new hypothesis to interpret these expansions as asymptotic expansions for $-k^2 \rightarrow \infty$, as has been done first by Abarbanel et al. ⁸⁾. From (37) we find for the behaviour of $g_{1,n}(x_0)$ and $g_{2,n}(x_0)$ at infinity

$$\begin{aligned} g_{1,n}(x_0) &= -\pi \Delta_{1,n} |x_0| + \text{less singular terms} \\ g_{2,n}(x_0) &= -\pi \Delta_{2,n} |x_0|^{-1} + \text{less singular terms} \end{aligned} \quad (40)$$

The light cone singularity of highest order which is responsible for scaling [the term $n = 0$ in (33)] and the additional hypothesis on the asymptotic character of the expansion (39) give the prediction

$$\lim_{-k^2 \rightarrow \infty} k^2 \beta_{1,2}(k^2) \text{ finite} \quad (41)$$

which is therefore independent of the specific structure of the ansatz (33).

Our presentation of the material of this Section is the result of an attempt to formulate the ideas of Refs. ^{10), 11)} in mathematically more concise form. The material presented in the sequel is another product of these entertainments on the field of mathematics. For a review of the experimental situation concerning the scaling law, see Refs. ^{5), 12)}.

3.2. - Time-Dependent Light Cone Singularities

The distribution $F_{\mathfrak{g}}(x)$ (19) considered in Section 3.1 possesses a time-independent order $\mathfrak{g}$ of the singularity on the light cone. We may ask ourselves whether any physical restrictions can exclude the possibility that a causal function exhibits a singularity of the type $(x_+^2)^{\mathfrak{g}(x_0)}$. Physical restrictions are the spectrum conditions which are normally used to derive spectral representations for the causal functions.

The constraints on the light cone singularities are then implicitly contained in these spectral representations. To make them explicit turns out to be another and very beautiful application of harmonic analysis indeed.

The best suited spectral representation for the investigation of the spin-averaged diagonal matrix element of the current commutator is the Deser-Gilbert-Sudarshan representation ¹³⁾. For a causal function $V(x)$ of the type $V_{1,2}(x)$ and in the rest system of the proton the most popular form of this spectral representation is

$$V(x) = -i \int ds \Delta(x, s) \varphi(x_0, s) \quad (42)$$

where $\Delta(x, s)$ is the Pauli-Jordan function

$$\Delta(x, s) = i(2\pi)^{-3} \int d^4k \varepsilon(k_0) \delta(k^2 - s) e^{-ikx} \quad (43)$$

$\varphi(x_0, s)$ has support on $s \geq 0$ and obeys itself the integral representation

$$\begin{aligned} \varphi(x_0, s) &= \int_{-1}^{+1} d\sigma e^{i\sigma x_0} \tilde{\varphi}(\sigma, s) \\ \tilde{\varphi}(\sigma, s) &= \tilde{\varphi}(-\sigma, s) \end{aligned} \quad (44)$$

which allows us to apply Paley-Wiener type arguments to it. Since (42) is an integral over the product of possibly two distributions it has to be handled with care. The alternative form

$$V(x) = (2\pi)^{-3} \int d^4x \varepsilon(k_0) e^{-ikx} \varphi(x_0, k^2) \quad (45)$$

is well defined whenever $\varepsilon(k_0) \varphi(x_0, k^2)$ is a distribution in k for fixed x_0 . Of course, $V(x)$ ought to be a distribution of a specific kind, too, say temperate or more general. For the distributions $F_g(x)$ (19) the DGS spectral representation is valid with

$$\varphi(x_0, s) = g(x_0) 2^{2g+2} \frac{s_+^{-g-2}}{\Gamma(-g-1)} \quad (46)$$

In this case $\tilde{\varphi}(\sigma, s)$ is proportional to $G(\sigma)$ (20) which explains the support restriction in (44). On the other hand we learn from this example that the DGS representation does not take full account of the support conditions in momentum space. In fact the support of $\tilde{F}_g(k)$ (25) is

$$2|v| + k^2 \geq -1 \quad \text{instead of} \quad 2|v| + k^2 \geq 0$$

Our aim is to construct a class of distributions $V(x)$ which have a similar form as $F_g(x)$ (19) but with a time-dependent order $g(x_0)$. In other words we want to replace (46) by

$$\varphi(x_0, s) \underset{s \rightarrow \infty}{\cong} \phi(x_0) s_+^{-g(x_0)-2} \quad (47)$$

where $\phi(x_0)$ is an appropriately chosen function which makes $\varphi(x_0, k^2)$ satisfy (44) and which has zeros whenever

$$g(x_0) = n-1, \quad n = 0, 1, 2, \dots \quad (48)$$

Even, entire functions in x_0 , which depend parametrically on s and in the limit $s \rightarrow \infty$ behave as required in (47), can be constructed by means of the Mittag-Leffler functions ¹⁴⁾

$$E_{\gamma}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1+\gamma n)} \quad (49)$$

These functions are entire, have exponential order γ^{-1} and are of type one. For $|z| \rightarrow \infty$ in the direction $\theta = \arg z$ they behave as

$$E_{\gamma}(z) = \gamma^{-1} \sum_k \exp[|z|^{1/\gamma} e^{i(2\pi k + \theta)\gamma^{-1}}] + O(|z|^{-1}) \quad (50)$$

k integral, $|k + \frac{\theta}{2\pi}| \leq \frac{1}{4\gamma}$

We introduce therefore the function

$$G(z, s) = \Theta(s) s^{\beta} E_{2/\gamma}([\alpha |\log s|]^{2/\gamma} z^2) \quad (51)$$

$\alpha \geq 0$, β arbitrary, $\gamma > 0$

which for real $z = x_0$ is asymptotically equal to

$$G(x_0, s) \underset{s \rightarrow \infty}{\approx} \frac{1}{2} \gamma s^{\alpha |x_0|^{\gamma} + \beta} \quad (52)$$

It is even, entire, has exponential order γ and the $\alpha |\log s|$ type. In order that it fulfils the Paley-Wiener theorem connected with the integral representation (44) we must restrict γ to $\gamma < 1$. Since we have then still an exponential increase along the real axis in z , $\tilde{\varphi}(\sigma, s)$ is not a distribution. This disease can, however, be cured by a multiplier $\phi(x_0)$ such that

$$\varphi(x_0, s) = \phi(x_0) G(x_0, s) \quad (53)$$

if $\phi(x_0)$ is chosen as an even, entire function of type and order one [i.e., it satisfies (44) itself] with a sufficient exponential fall-off along the real axis. Such a multiplier exists as a theorem by Mandelbrojt ¹⁵⁾ asserts (because of lack of space we give the theorem rather than an explicit example).

Given a non-decreasing function $C(u) \geq 0$ for $u \geq 0$ such that

$$\int_1^{\infty} \frac{C(u)}{u^2} du < \infty \quad (54)$$

Then there exists a non-trivial even, entire analytic function $\phi(z)$ such that

$$|\phi(z)| \leq e^{a|\operatorname{Im} z| - C(|z|)} \quad (55)$$

for any $a > 0$. We choose δ in the interval

$$\gamma < \delta < 1$$

and set

$$C(u) = u^{\delta}, \quad a = 1 \quad (56)$$

The resulting function $\phi(x_0)$ has all properties required by (47) and (53) after an eventual multiplication with a polynomial that gives us the finite number of zeros required in (48). The weight function $\tilde{\varphi}(\sigma, s)$ constructed in this fashion is even infinitely differentiable in σ . The resulting quantity $V(x)$ has a highest singularity on the light cone

$$V(x)_{h.s.} = \frac{\gamma}{16\pi i} \varepsilon(x_0) \phi(x_0) \frac{\Gamma(-\varphi(x_0)-1)}{\Gamma(\varphi(x_0)+1)} \left(\frac{1}{4} x_+^2\right)^{\varphi(x_0)} \quad (57)$$

with

$$-\varphi(x_0) = \alpha |x_0|^\gamma + \beta + 2 \quad (58)$$

It is a distribution but not temperate.

Replacing $\alpha |\log s|$ in (51) by any other positive function $f(s)$ which satisfies

$$\lim_{s \rightarrow \infty} f(s) = +\infty \quad (59)$$

yields other classes of distributions with time-dependent order of singularity on the light cone. Among them there exist also temperate distributions. It would be quite interesting to investigate the Bjorken and Regge limits of several such classes.

4. - HARMONIC ANALYSIS ON $SL(2, \mathbb{C})$

4.1. - Elementary Kinematics

The commutator (9) is the absorptive part of the virtual Compton forward scattering amplitude with the ingoing and outgoing photon lying on the same mass shell. As any elastic forward scattering amplitude it can be submitted to a Toller decomposition, i.e., a harmonic analysis on the little group $SL(2, \mathbb{C})$ of the inhomogeneous group $SL(2, \mathbb{C}) \times T_4$ for vanishing four-momentum.

On the mass shell

$$k^2 = -\mu^2, \mu > 0$$

we take as reference momentum

$$k^R = (0, 0, 0, \mu) \quad (60)$$

whose little group is $SU(1,1)$. Similarly the little group for the reference momentum

$$p^R = (1, 0, 0, 0) \quad (61)$$

on the proton mass shell is $SU(2)$. We denote elements of $SL(2, \mathbb{C})$ by a , elements of $SU(1,1)$ by v , and elements of $SU(2)$ by u . By definition we have

$$\begin{aligned} u^\dagger u &= e \\ v^\dagger \sigma_3 v &= \sigma_3 \end{aligned} \quad (62)$$

By

$$\begin{aligned} \underline{p}^R &= a(p) \underline{p} a(p)^\dagger, \quad a(p) \in SL(2, \mathbb{C}) \\ \underline{p} &= p_0 e + \vec{p} \vec{\sigma} \end{aligned} \quad (63)$$

and in the same fashion for k , we define cosets $ua(p)$ of $SU(2)$ respectively $va(k)$ of $SU(1,1)$ in $SL(2, \mathbb{C})$. Instead of p and k we can therefore use these cosets in $SL(2, \mathbb{C})$ as variables on which our scattering amplitude is defined, or equivalently replace each coset

by one element of itself, the boost $a(p)$ or $a(k)$. The physical meaning of the rotation u or v which distinguishes the elements within one coset, is that of a rotation applied to the particle in its reference state $p^R(61)$ or $k^R(60)$.

Next we consider the amplitude not as a function on the pair of cosets or boosts, but we want to extend its definition on all elements of the cosets, thereby from three-dimensional manifolds of momenta, boosts or cosets onto the six-dimensional group manifold of $SL(2, \mathbb{C})$. The simplest prescription would be to require constancy on each coset. It is, however, more convenient (though all prescriptions are basically equivalent) to require that the value of the amplitude at two different points of the same coset, say $u_1 a(p)$ and $u_2 a(p)$, differs by the rotation $u_1 u_2^{-1}$ applied to the rest state, such that in going from one element of the coset to another element, the spin indices are linearly transformed. We may summarize the substitution of arguments by

$$\begin{array}{ccccc}
 p & \xrightarrow{\text{one-to-one}} & a(p) & \xrightarrow{\text{one-to-}SU(2)} & ua(p) = a_1 \\
 k & \xrightarrow{\text{one-to-one}} & a(k) & \xrightarrow{\text{one-to-}SU(1,1)} & va(k) = a_2
 \end{array}$$

Because of Lorentz invariance we have as effective arguments only

$$a_2 a_1^{-1} = v a(k) a(p)^{-1} u^{-1} \quad (64)$$

in other words : the amplitude depends effectively only on the double cosets

$$a = v n u \quad (65)$$

of $SU(1,1)$ and $SU(2)$ in $SL(2,C)$. The representatives n of these double cosets form a one-dimensional manifold, which by definition is parametrized by the "Toller angle". In our case it can be shown that the parametrization

$$n = e^{\frac{1}{2}\eta\sigma_3}, \quad -\infty < \eta < \infty \quad (66)$$

is possible.

Let us now keep the proton in its rest state $p = p^R$. Comparison of (64) with (65) shows that the k boost must necessarily be decomposable as

$$a(k) = v n u \quad (67)$$

The rotation u is determined such that it rotates the three momentum $\vec{k}$ into the positive third axis

$$|\vec{k}| \sigma_3 = u(\vec{k} \cdot \vec{\sigma}) u^{-1} \quad (68)$$

Inserting (66) to (68) into (63) yields

$$\mu \sigma_3 = e^{\frac{1}{2}\eta\sigma_3} (k_0 e + |\vec{k}| \sigma_3) e^{\frac{1}{2}\eta\sigma_3} \quad (69)$$

or finally [see (28)]

$$k_0 = v = -\mu \sinh \eta, \quad |\vec{k}| = \mu \cosh \eta \quad (70)$$

The fact that the Toller angle varies over the whole real axis instead of over the positive half axis is obviously due to the one-shell form of the hyperboloid $k^2 = -\mu^2$. Finally we mention that the projection of the Haar measure onto the manifold of double cosets (67) gives the measure

$$\text{const} \cdot d\eta^2 d\eta$$

4.2. - Toller Analysis

The idea of harmonic analysis is to expand the amplitude into co-ordinate functions (matrix elements) of unitary irreducible representations of $SL(2, \mathbb{C})$. In order to accommodate the double coset splitting (67) it is convenient to use co-ordinate functions with respect to two different bases. One basis reduces the $SL(2, \mathbb{C})$ representation restricted to $SU(2)$ and is discrete, the other reduces the $SL(2, \mathbb{C})$ representation to $SU(1, 1)$ and in general has both a discrete and continuous part ^{16), 17)}. However, the spin of the photon does not transform as a unitary representation of the little group $SU(1, 1)$, but as a finite-dimensional non-unitary representation. This necessitates an analytic continuation in the photon spin, which gives rise to a "discrete series" of $SL(2, \mathbb{C})$ (see below). The whole formalism is too complicated to be presented here. Instead I quote therefore only the resulting formulae. We shall concentrate our interest on the function $\tilde{V}_2$ (or W_2), for which the formulae are most compact. The general investigation is contained in the literature ^{18), 19), 20)}.

We define a Fourier transform of $\tilde{V}_2(\eta, \mu^2)$

$$\tilde{V}_2(\eta, \mu^2) = \tilde{V}_2(k), \quad \tilde{V}_2(-\eta, \mu^2) = -\tilde{V}_2(\eta, \mu^2) \quad (71)$$

on $SL(2, \mathbb{C})$ by

$$H_2(\lambda) = \frac{2}{(\lambda^2-1)(\lambda^2-4)} \int_{-\infty}^{+\infty} d\eta \operatorname{ch} \eta e^{-\eta \lambda} \tilde{V}_2(\eta, \mu^2) K(\lambda, \eta) \quad (72)$$

with the kernel

$$K(\lambda, \eta) = (\lambda+1)(\lambda+2)e^{2\eta} + 2(\lambda^2-4) + (\lambda-1)(\lambda-2)e^{-2\eta} \quad (73)$$

This Fourier transformation (72) can be inverted by

$$\tilde{V}_2(\eta, \mu^2) = (2 \operatorname{ch} \eta)^{-5} \int_{C_{\pm}} d\lambda e^{-\eta \lambda} H_2(\lambda) K(\lambda, \eta) \quad (74)$$

The general proof of (74) was given in Ref. ¹⁸⁾, however, it is easy to establish a direct proof which we postpone to the Appendix. The group theoretical meaning of the decomposition (72), (74) cannot be inspected from such direct proof of formula (74) but only in the general context.

A function $\tilde{V}_2(\eta, \mu^2)$, which falls off faster than any power of $\operatorname{ch} \eta$ at $|\eta| \rightarrow \infty$, has a Fourier transform (72) for which $(\lambda^2-1)H_2(\lambda)$ is an entire analytic function of λ . Holomorphy at $\lambda = \pm 2$ follows from the odd parity of $\tilde{V}_2(\eta, \mu^2)$ (71). The contour $C_{\pm}$ consists of the upward oriented imaginary axis and a positively (negatively) oriented circle around $+1$ (-1). The contributions of the circles (the "discrete series") appear in the course of the analytic continuation of the representation of the little group $SU(1,1)$. It is easy to show that both contours are in fact equivalent (see Appendix).

If the function $\tilde{V}_2(\eta, \mu^2)$ falls off like a power of $\operatorname{ch} \eta$, say like

$$V_2(\eta, \mu^2) \cong c (\operatorname{ch} \eta)^{-3-\varepsilon}, \quad \varepsilon > 0 \quad (75)$$

then $(\lambda^2 - 1)H_2(\lambda)$ is holomorphic in a strip of half width ε around the imaginary axis. A physical amplitude can always be regularized by multiplication with an appropriate power of $\text{ch}\eta$, say $(2\text{ch}\eta)^\sigma$, such that a strip of holomorphy results. The Fourier transform can be expected to be analytic in σ . Inserting this Fourier transform of the regularized amplitude into the inversion formula (74) and continuing both sides in σ along an appropriately chosen path, forces us in general to deform the contours $C_\pm$ into $C'_\pm$.

The well-known relation

$$\lambda = \alpha(0) + 1 \quad (76)$$

between a Toller singularity and its parent Regge singularity $\alpha(t)$ at $t = 0$ gives us $\lambda = +2$ for the upper edge of the Pomeranchuk singularity (the contour is deformed such that this singularity still is on the left of it). There is also a mirror singularity at $\lambda = -2$ due to the symmetry

$$H_2(\lambda) = -H_2(-\lambda) \quad (77)$$

In order to obtain the contribution of the Pomeranchuk singularity in the limit $\nu \rightarrow +\infty$ or $\eta \rightarrow -\infty$ [see (70)] we move the contour $C'_\pm$ to the left. The highest order term $e^{-2\eta}$ in $K(\lambda, \eta)$ is, however, multiplied with the factor $\lambda - 2$. A single pole in $H_2(\lambda)$ at precisely $\lambda = 2$ can therefore not cause the expected Pomeranchuk behaviour [see (36)]

$$\tilde{V}_2(k) \cong -\beta_2(k^2)\nu^{-1} \quad (78)$$

This must be due to a more complicated singularity, a cut or a double pole. In the first case additional factors depending logarithmically on ν will occur. In the second case (78) is reproduced by

$$H_2(\lambda) = \frac{2\beta_2(k^2)}{\mu(\lambda-2)^2} + O((\lambda-2)^{-1}) \quad (79)$$

4.3. - The Connection Between Harmonic Analysis in R_4 and on $SL(2, \mathbb{C})$

Instead of the complicated integral transformation (72), we study now the singularity structure of the integral transform

$$\mathcal{J}(\lambda) = \int_{-\infty}^{+\infty} d\eta \sinh \eta \lambda V(\eta) \quad (80)$$

where $V(\eta)$ is an odd function of η , for example $V(\eta) = \tilde{V}_2(\eta, \mu^2)$. In the latter case we have

$$\begin{aligned} H_2(\lambda) = & - [(\lambda^2-1)(\lambda^2-4)]^{-1} \{ (\lambda+1)(\lambda+2) \mathcal{J}(\lambda-3) \\ & + 3(\lambda-1)(\lambda+2) \mathcal{J}(\lambda-1) + 3(\lambda+1)(\lambda-2) \mathcal{J}(\lambda+1) \\ & + (\lambda-1)(\lambda-2) \mathcal{J}(\lambda+3) \} \end{aligned} \quad (81)$$

which proves that knowledge of $\mathcal{J}(\lambda)$ is sufficient to determine $H_2(\lambda)$. In particular we want to identify $V(\lambda)$ with $\tilde{F}_g(k)$ (21). This gives

$$\begin{aligned} \mathcal{J}(\lambda) = & \frac{1}{8} \left(\frac{2}{\mu}\right)^{2g+4} \int_{-\infty}^{\infty} d\sigma G(\sigma) \\ & \times \left\{ \frac{\Gamma(\lambda+g+2)}{\Gamma(\lambda+1)} \left|\frac{\sigma}{\mu}\right|^\lambda {}_2F_1(g+2, \lambda+g+2; \lambda+1; -(\frac{\sigma}{\mu})^2) \right. \\ & \left. - \text{the same expression with the replacement by } \lambda \rightarrow -\lambda \right\} \end{aligned} \quad (82)$$

As in Section 3.1, we assume that $G(\sigma)$ is continuous for $\sigma \neq 0$ and has the support $-1 \leq \sigma \leq +1$. The singularities of $J(\lambda)$ can then have two origins. If $G(\sigma)$ is infinitely differentiable at $\sigma = 0$, then

$$J(\lambda) = \Gamma(\lambda + \vartheta + 2) f(\lambda) - \Gamma(-\lambda + \vartheta + 2) f(-\lambda) \quad (83)$$

where $f(\lambda)$ is entire in λ . Global properties of $G(\sigma)$ go into the residues of these poles. If $G(\sigma)$ is an arbitrary distribution at $\sigma = 0$ but ϑ assumes one of the values $-1 + n$, $n = 0, 1, 2, \dots$ [in which case the support of $\mathcal{F}_\vartheta(k)$ (22) shrinks to the light cone] it is alone the singularity of $G(\sigma)$ at $\sigma = 0$ which determines the singularity structure of $J(\lambda)$. This was the case of our model ansatz (33). In general both kinds of singularities are present.

Explicit expressions of $J(\lambda)$ (82) in terms of $g(x_0)$ are very complicated. They have the general form

$$J(\lambda) = - \frac{i \pi \frac{1}{2} \lambda}{4\pi} \left(\frac{2}{\mu}\right)^{2\vartheta+3} \int_{-\infty}^{+\infty} dx_0 g(x_0) M(\lambda, \vartheta, \mu x_0) \quad (84)$$

At least an asymptotic formula can be given for the kernel

$$M(\lambda, \vartheta, x_0) \underset{x_0 \rightarrow \infty}{\cong} \sum_{m=0}^{\infty} \frac{\Gamma(\vartheta+m+2) \Gamma(\lambda+\vartheta+m+2) \Gamma(\lambda+2m+1)}{m! \Gamma(\vartheta+2) \Gamma(\lambda+m+1)} |x_0|^{-\lambda-2m-1}$$

+ terms obtained by the replacement $\lambda \rightarrow -\lambda$ (85)

The singularities of $\mathcal{J}(\lambda)$ induced by the asymptotic property of $g(x_0)$ are clearly determined by the series (85). However, the poles at $\pm \lambda = -\varphi - 2-n$ [see (83)] can also be obtained from this asymptotic series (85), since only a finite number of terms of this series add to the residue, and these terms are exact. In that case $|x_0|^{\pm \lambda - 2n - 1}$ has to be regarded as a distribution with analytic regularization.

Finally we consider $V(\eta)$ as a superposition of distributions $\tilde{F}_\varphi(k)$ with $\varphi = 0, 1, 2, \dots$ as in Section 3.1, Eqs. (34) to (41). If the functions $g_{2,n}(x_0)$ have the asymptotic behaviour (40), (82) or (84), (85) yield a pole of first order in $\mathcal{J}(\lambda)$ at the position $\lambda = -1$. Its residue is

$$-\frac{1}{2} \sum_{n=0}^N \Delta_{2,n} \left(\frac{2}{\mu}\right)^{2n+3} + \text{remainder}$$

Due to (81) this induces a pole of second order in $H_2(\lambda)$ at $\lambda = +2$ and comparison with (79) gives the asymptotic expansion for $\mu \rightarrow \infty$

$$\beta_2(-\mu^2) = \frac{1}{2} \sum_{n=0}^N \left(\frac{2}{\mu}\right)^{2n+2} \Delta_{2,n} + \varphi_{2,N}(-\mu^2) \quad (86)$$

which we know already from (39). With the help of Eq. (82) we could now easily modify our ansatz (33) such that it allows for non-integral orders of light cone singularities.

4.4. - Conclusion

A harmonic analysis on $SL(2, \mathbb{C})$ which permits us to deal with the Bjorken limit can also be invented^{20), 21)}. Since it involves problems falling outside the scope of these lectures I will not discuss it here. The connection between the Fourier transforms of Refs. 19), 20), 21) and the notation $H_2(\lambda)$ used in these lectures is explained in the Appendix.

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APPENDIX

A.1. - Proof of the Inversion Formula

We have actually to prove the "completeness relation"

$$\begin{aligned} & \frac{i}{2\pi} \int_{C_{\pm}} d\lambda e^{-\eta\lambda} K(\lambda, \eta) \frac{e^{-\eta'\lambda} K(\lambda, \eta') - e^{\eta'\lambda} K(\lambda, -\eta')}{(\lambda^2 - 1)(\lambda^2 - 4)} \\ & = (2\cosh\eta)^2 (2\cosh\eta')^2 [\delta(\eta - \eta') - \delta(\eta + \eta')] \end{aligned} \quad (A.1)$$

We split the product of the two kernels as

$$\begin{aligned} & K(\lambda, \eta) K(\lambda, \eta') = \\ & = (\lambda^2 - 1)(\lambda^2 - 4) (2\cosh\eta)^2 (2\cosh\eta')^2 + R(\lambda, \eta, \eta') \end{aligned} \quad (A.2)$$

such that $R(\lambda, \eta, \eta')$ is a polynomial in λ of degree three. Inserting the first part of (A.2) into (A.1) gives already the desired result. We have therefore to prove that the remainder of (A.2) inserted into (A.1) gives zero. Whatever the result of inserting the functions $R(\lambda, \eta, \eta')$ into (A.1) is, it must be a measurable function of η' . It suffices therefore to prove that this function vanishes almost everywhere.

From (73) we find the identities

$$\begin{aligned} e^{-\eta-\eta'} R(1, \eta, \eta') &= e^{\eta+\eta'} R(-1, \eta, \eta') \\ e^{-2(\eta+\eta')} R(2, \eta, \eta') &= e^{2(\eta+\eta')} R(-2, \eta, \eta') \\ &= 144 \end{aligned} \quad (A.3)$$

The first identity implies the equivalence of the contours $C_{\pm}$. Let us consider then the first part of the integral (A.1) with $K(\lambda, \eta)K(\lambda, \eta')$ replaced by $R(\lambda, \eta, \eta')$ and antisymmetrize it in η' later. If $\eta + \eta' < 0$, we start from C_- and pull the contour to $-\infty$. There is at most a contribution from the pole at $\lambda = -2$. If $\eta + \eta' > 0$, we start from C_+ and shift the contour to $+\infty$. Again we get a contribution at most from $\lambda = +2$. Due to the second identity (A.3) both contributions of $\lambda = \pm 2$ are identical. Antisymmetrization in η' yields zero. This completes the proof.

A.2. - A Notation

Our Fourier transform $H_2(\lambda)$ (72) can be expressed in terms of the Fourier transforms $F_{\alpha}(\chi, \xi)$ of Ref. 19) as

$$H_2(\lambda) = \frac{1}{2^{\frac{1}{2}} \lambda (\lambda^2 - \mu) \mu^2} [3(\lambda^2 - 1) F_{(0,0)}(\lambda, +) + 2F(\lambda, +)]$$

(A.4)

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