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AN S MATRIX DESCRIPTION OF STATE MIXING
PART I: GENERAL FORMALISM, WITH RHO-OMEGA MIXING
AS AN ILLUSTRATION *)

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A B S T R A C T

The notion of state mixing is formulated in terms of S matrix language alone. The object is to work with quantities which are closely related to experiment, while avoiding the particularities of Hamiltonian theory or field theory. The scheme applies to particles of any spin, it is explicitly unitary, and useful in phenomenology as well as in dynamical S matrix models. Unlike the mass matrix method, it avoids the assumption that the states to be mixed evolve from discrete eigenstates of an unperturbed Hamiltonian. The main ingredient is a perturbation method by which the inverse of the K matrix is related to a corresponding unperturbed matrix. In some cases the latter can be defined directly from physical scattering matrices, without the device of turning off interactions. In this and following papers the method is applied to $\rho - \omega$, $\varphi - \omega$, and $K_0 - \bar{K}_0$ mixing. The results amount to more than a simple rephrasing of earlier treatments. As the discussion of $\rho - \omega$ mixing in this paper illustrates, the physical arguments take quite a different form, and some new aspects of mixing questions are brought out.

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1. INTRODUCTION

One speaks of "mixing" in the following situation: there are two states of equal or nearly equal energy which are initially separated by a selection rule; they are then allowed to communicate through a perturbation which does not respect the selection rule. The theoretical apparatus for handling this problem as it occurs in atomic or nuclear physics has been available for a long time. In the theory of elementary particles the problem is not so clearly defined, since we must be cautious about analogies with non-relativistic quantum mechanics. We are not certain that Hamiltonian quantum mechanics, or even quantum field theory, is correct for the weak and strong interactions. Also, some of the problems are qualitatively different in particle theory. We are more likely to encounter states with extremely short life-times, or mixing forces which are so strong they cannot be treated by perturbation theory.

The recognition that the theory of mixing should be improved has led to quite a number of papers during the past two years ¹⁾⁻¹⁷⁾. Part of the work was stimulated by questions concerning unitarity in the neutral K meson system ¹⁾⁻⁹⁾, and part by the experimental observation of $\rho - \omega$ mixing ^{10)-17), 18)}. In these and earlier papers, one can distinguish three main approaches. The first is the mass matrix method ^{3), 4), 9), 19)}, which is based on Hamiltonian quantum theory. One supposes that the states to be mixed are discrete eigenstates $|\alpha\rangle$ of an unperturbed Hamiltonian H_0 . Attention is focused on the finite matrix $\langle\alpha|G(E)|\alpha'\rangle$, where $G(E)$ is the total Green function of the perturbed system. The inverse of this matrix is $\langle\alpha|E-M(E)|\alpha'\rangle$, where $M(E)$ is called the mass operator. The role of the states $|\alpha\rangle$ in the time evolution of an arbitrary state ^{9), 19)}, or in the scattering matrix ³⁾, may be partially parametrized in terms of $\langle\alpha|M|\alpha'\rangle$. A unitarity constraint (the "Bell-Steinberger sum rule") can be stated in terms of the overlap of mass matrix eigenvectors. The scheme is quite neat and appealing, and it is useful in phenomenology, but there could be some doubt about its generality. The assumption of underlying discrete eigenstates of H_0 seems rather

strong, even if Hamiltonian theory is accepted. For the purposes of unbiased phenomenology we should not be compelled to think of the ρ meson or an N^* resonance in that way. These could be simply dynamical resonances which do not evolve from discrete eigenstates of H_0 . The same could be said of K_0 and $\bar{K}_0$.

A second approach makes use of the language of field theory, Refs. 20)-23). The states to be mixed are assigned field operators (or currents), and a corresponding matrix propagator is introduced. Except for the initial reference to fields and symmetry operations on fields, the scheme resembles a phenomenological S matrix theory. The propagator is sandwiched between vertex functions to give the pole contribution to the S matrix, which is assumed to be the dominant part of S in appropriate circumstances. Since the same propagator appears in various scattering amplitudes, it represents an invariant characterization of the mixing phenomenon, just as the mass matrix did in the Hamiltonian method.

In a third approach ^{1),2),24)}, one scrupulously avoids reference to Hamiltonians or field operators. This proposal was sketched out some time ago ²⁴⁾, but was not worked out in sufficient detail. In this series of papers we hope to develop the method to the point where its advantages and disadvantages will become clear. We were encouraged by the papers of McGlinn and Polis ¹⁾, and McVoy and Durand ²⁾, who discussed the S matrix description of neutral K meson decays. These authors did not look into the definition of unmixed states in S matrix language, however. That question will now be one of our main concerns, as it was in Ref. 24).

It turns out that the S matrix viewpoint is more rewarding than one might have expected. Besides avoiding some of the conceptual difficulties of the other methods, it provides a rather clear view of mixing problems which is not simply in one-to-one correspondence with the other approaches. This is illustrated by our treatment of $\rho - \omega$ mixing in Section 4. The physical arguments look quite different from those of the mass matrix method.

It seems useful to have available alternative methods, since certain mixing questions (e.g., that of the neutral K mesons) are fundamental and probably rather subtle.

We shall not give much detailed comparison with other work in the present paper since it will be more appropriate after we have given several examples. We just mention the following points about the S matrix scheme:

- i) The number of free parameters is reduced to a minimum. There seems to be less freedom to introduce parameters than in the propagator method. For instance, we cannot invoke "vector mixing" to get additional parameters in the $\rho - \omega$ problem^{17),21)}.
- ii) The method applies to particles of any spin, and it is not necessary to assume conserved currents to simplify the expression for the S matrix¹⁷⁾.
- iii) In some cases there is considerable freedom in choosing the unmixed system. It is not always necessary to turn off a coupling constant to obtain it. For instance, in our treatment of the $\rho - \omega$ problem, the unmixed system is described by submatrices of the physical S matrix.
- iv) The scheme is explicitly unitary, even though many-body channels are included. This is possible because of the assumption of pole dominance (or dominance of poles plus a constant background).

In Section 2 we define the "complete" K matrix, which may refer to closed as well as open channels. We then examine in detail the connection between K and T, in the case where K has two poles with non-orthogonal residue vectors. It is hoped that this discussion will clear up some confusion that has arisen in connection with Ref. 1).

In Section 3, we introduce the unperturbed and perturbed inverse K matrices, denoted by H_0 and H. Three kinds of perturbation theory are described for relating the eigenvectors and energies of vanishing eigenvalues of H_0 and H: degenerate, almost degenerate, and non-degenerate.

In Section 4, the $\rho - \omega$ mixing in $e_+ e_- \rightarrow \pi_+ \pi_-$ is treated by non-degenerate perturbation theory on H . The reader interested only in applications can look at Section 4, separately.

In later papers, we shall treat $\varphi - \omega$ and $K_0 - \bar{K}_0$ mixing. The former is a relatively difficult dynamical problem, we believe, in which non-linear effects are important. We discuss it in general, and in terms of an S matrix dynamical model. In the $K_0 - \bar{K}_0$ problem we shall give some additional formal developments, including the S matrix description of conservation laws, and various forms of the unitarity sum rules.

2. THE K MATRIX DESCRIPTION OF NEIGHBOURING STATES

We work with the reaction matrix $K(w)$ in a particular angular momentum state. The energy in the centre-of-mass frame is denoted by w . For many of our purposes there will be an advantage in defining $K(w)$ so that it refers to channels which are closed as well as channels which are open at the energy w . This means that K is usually infinite-dimensional, since many-particle channels are included. We shall not introduce any explicit description of asymptotic many-particle states of definite angular momentum, but merely note that such descriptions are available ²⁵⁾. In order to explain what is meant by a K matrix for closed channels, we first note that the corresponding concept for the T matrix is clear. We start at any physical energy where a certain set of channels is open, and continue analytically through the physical Riemann sheet to some lower physical energy where fewer channels are open, or even to a real energy below all thresholds. At any energy the submatrix of T referring to a given set of channels is uniquely determined, being independent of the energy at which the continuation was begun. We shall begin the continuation at some high energy W , and define the "complete" T matrix as that referring to all channels which are open at energy W . Henceforth, T will always denote the complete T . The corresponding complete K is defined by

$$K(w) = [1 + iT(w+i0)\rho(w)]^{-1}T(w+i0). \quad (2.1)$$

Here $\rho(w)$ is a diagonal phase space matrix with diagonal elements

$$\rho_i(w) = \Theta(w-w_i)q_i(w), \quad (2.2)$$

where w_i is the threshold of channel i , and Θ is the unit step function. The exact choice of q_i will depend on the specific problem and does not concern us here. It is often convenient to include the dominant threshold momentum dependence in q_i , in order to give K the slowest possible variation with energy near thresholds.

The unitarity condition is assumed to be the following:

$$T(w+i0) - T^\dagger(w+i0) = 2i T(w+i0) \rho(w) T^\dagger(w+i0), \quad (2.3)$$

This statement goes beyond mere open-channel unitarity. It implies, but is not implied by, the unitarity of the physical S matrix referring only to open channels. The latter is

$$S(w) = 1 + 2i \rho^{1/2}(w) T(w+i0) \rho^{1/2}(w). \quad (2.4)$$

Through multiplication by T^{-1} on the left and $T^{\dagger-1}$ on the right, Eq. (2.3) becomes

$$\frac{1}{2i} [T^{-1}(w+i0) - T^{\dagger-1}(w+i0)] = -\rho(w). \quad (2.5)$$

Since T^{-1} can fail to exist only at isolated energies, (2.5) and (2.3) are equivalent. Now, the definition (2.1) gives

$$\frac{1}{2i} [T^{-1}(w+i0) - T^{\dagger-1}(w+i0)] = -\rho(w) + \frac{1}{2i} [K^{-1}(w) - K^{\dagger-1}(w)]. \quad (2.6)$$

Thus, the Hermiticity of K is necessary and sufficient for the generalized unitarity (2.3).

What properties of analyticity and smoothness can we expect for K ? According to Eq. (2.1), $K(w)$ is the boundary value of a meromorphic function between any two thresholds, but the meromorphic function is different for different pairs of thresholds. [Here it is assumed that $q_i(z)$ is meromorphic in the cut physical plane of T , with $q_i(w) \equiv q_i(w+i0)$.] Near a threshold K will not be analytic but it will normally have a continuous derivative with respect to w . It can have poles at real w , of course, which are associated with resonances. It can also fail to have a derivative at s wave, two-body thresholds w_i , where dq_i/dw has an inverse square-root singularity. At all other w we expect at least a continuous first

derivative, and between thresholds, K should be infinitely differentiable. In the general case, we simply assume these properties as reasonable. If all channels have two bodies, and T has an ND^{-1} representation, the properties emerge rather clearly. Then we have

$$T(z) = N(z)D(z)^{-1}, \quad (2.7)$$

$$K(w) = N(w) [Re D(w+i0)]^{-1}, \quad (2.8)$$

$$D(z) = 1 + \sum_{n=1}^{N-1} a_n z^n - \frac{z^N}{\pi} \int_{w_0}^{\infty} \frac{\rho(w) N(w) dw}{w^N (w-z)}, \quad (2.9)$$

where $N(z)$ is analytic in a neighbourhood of the right cut $[\underline{w}_0, \infty)$. The principal-value integral in $Re D$ has a continuous first derivative at all w if the threshold branch points in $\rho(w)$ are of the type $(w-w_i)^{\frac{1}{2}+\ell}$, $\ell \geq 1$. It is infinitely differentiable between thresholds with $\ell \geq 0$.

The discussion of the preceding paragraph applies as well to $K^{-1}(w)$, with the difference that poles of K^{-1} , which cannot be excluded on general grounds, do not correspond to resonances. Since $T^{-1} = K^{-1} - i\rho$, they are also poles of T^{-1} , which is to say vanishing eigenvalues of T . Poles of K^{-1} , which might be called Castillejo, Dalitz, Dyson (CDD) poles, would complicate the perturbation theory of Section 3. In the present paper we assume that any such poles are sufficiently far away from the energy region of interest. This assumption is not special to our scheme. Effectively, the same assumption occurs sub rosa in other approaches. For instance, the inverse propagator can have poles, which are usually omitted in the propagator phenomenology. In a later paper of this series we will explore the phenomenological consequences of CDD poles.

If we stay away from poles and s wave, two-body thresholds, we can approximate $K^{-1}(w)$ locally by a linear function, since it has

a continuous derivative. This will be helpful in the following. We note in passing that another definition of K is possible, in which K is the boundary value of the same analytic function all along the physical cut. One simply omits the step functions in $\rho(w)$, defining $q_i(w)$ so that it is pure imaginary for $w < w_i$. Hermiticity of this K is still necessary and sufficient for (2.3), but it differs in not having the representation (2.8). It can involve the phase space factors q_i both outside and inside of Cauchy integrals, while in (2.7) the q_i are all inside. Thus, (2.8) is likely to have a slower energy variation, since the integration has a smoothing effect. This has been observed in model calculations.

We now assume that there is a one-to-one correspondence between bound states or resonances and real energies $\hat{w}$ at which K^{-1} has a vanishing eigenvalue:

$$K^{-1}(\hat{w})\Phi = 0. \quad (2.10)$$

As usual, a bound state or resonance is taken to mean a pole of the T matrix at some energy E . In general, $\hat{w} \neq E$, unless E is a bound state energy. We make the important assumption that E is close, in some appropriate sense, to $\hat{w}$. The "appropriate sense" is roughly that the pole approximation for K should be sufficiently good all the way from $\hat{w}$ to E .

We suppose that K^{-1} is continuously differentiable and that it varies linearly with w in a sufficiently small neighbourhood of a resonance or bound state energy $\hat{w}$. We suppose also that the zero eigenvalue of $K^{-1}(\hat{w})$ is non-degenerate. This rules out the possibility of second order poles in K , which we do not wish to bother with here. The behaviour of the vanishing eigenvalue of $K^{-1}(w)$ near $w = \hat{w}$ is obtained by perturbing Eq. (2.10) with respect to w . Let $\Delta w = w - \hat{w}$, so that the perturbed equation reads

$$[K^{-1}(\hat{w}) + K_w^{-1}(\hat{w})\Delta w + O([\Delta w]^2)][\Phi + \Delta\Phi] = \Delta\lambda[\Phi + \Delta\Phi]. \quad (2.11)$$

where $K_w^{-1} = (\partial/\partial w)K^{-1}$. The matrices appearing on the left side are Hermitian, so that when we take the Hermitian scalar product of (2.11) with Φ , and drop terms beyond the first order, we have

$$\Delta\lambda = \Delta w (\Phi, K_w^{-1}(\hat{w})\Phi). \quad (2.12)$$

Now $K(w)$ has the eigenvalue $1/\Delta\lambda$ near $\hat{w}$, and hence a pole at $\hat{w}$. By our assumption of no degeneracy this is a simple pole. When w is sufficiently close to $\hat{w}$, the K matrix is approximated by the formula

$$K = \frac{\gamma/2}{\hat{w} - w} \Phi \Phi^\dagger, \quad \frac{\gamma}{2} = -(\Phi, K_w^{-1}(\hat{w})\Phi)^{-1} \quad (2.13)$$

where $\Phi^\dagger$ is the transposed complex conjugate of the column matrix Φ . Note that γ is real since K_w^{-1} is Hermitian.

If K consists exactly of one pole, as in (2.13), then it is very easy to find the corresponding T . We have

$$T = \frac{\gamma/2}{\hat{w} - i\frac{\gamma}{2}(\Phi, \rho(w)\Phi) - w} \bar{\Phi} \Phi^\dagger. \quad (2.14)$$

Since the one-pole K is analytic (rather than sectionally analytic as K should be), the T comes out to be sectionally analytic. This situation is avoided with the alternative definition of K mentioned above. In any case, (2.14) is a good approximation for w near $\hat{w}$, when (2.13) is good near $\hat{w}$. The quantity

$$\Gamma(w) = \gamma (\Phi, \rho(w)\Phi) \quad (2.15)$$

may be called the "energy-dependent" width, and $\Gamma(\hat{w})$ is roughly the Breit-Wigner width. The factor γ is sometimes called the "reduced width". The pole is at the appropriate solution E of the equation

$$\hat{w} - i\frac{\gamma}{2}(\Phi, \rho(E)\Phi) = E, \quad (2.16)$$

where $(\Phi, \rho(E)\Phi)$ actually means

$$\sum_i |\Phi_i|^2 g_i(E) , \quad (2.17)$$

the sum running over all channels which are open at $w = \hat{w}$, say ²⁶⁾.
The exact Breit-Wigner width is $-2 \text{Im } E$.

If K consists of several simple poles, with residue vectors which are orthogonal in the sense

$$(\Phi_i, \rho(w)\Phi_j) \equiv 0, \quad i \neq j, \quad (2.18)$$

then the poles contribute additively to T :

$$T = \sum_i \frac{\gamma_i/2}{i\hat{w}_i - i\frac{\gamma_i}{2}(\Phi_i, \rho\Phi_i) - w} (\Phi\Phi^\dagger)_i. \quad (2.19)$$

In general, one does not have the orthogonality (2.18) and (2.19) does not hold. It is easy to show, if the energy dependence of $\rho(w)$ is neglected, that an expression of the form (2.19) cannot yield a unitary S matrix in the non-orthogonal case, except in degenerate cases where the vectors $\rho^{\frac{1}{2}}\bar{\Phi}_i$ are linearly dependent. This fact, which is noted by McGlinn and Polis ¹⁾, sometimes causes confusion because it is possible to write a unitary formula which is similar in appearance but actually quite different from Eq. (2.19). The correct unitary formula, which we shall find later, has residues of the form $\psi\hat{\psi}^\dagger$, where $\psi \neq \hat{\psi}$.

We shall now discuss the case with two poles and non-orthogonal residue vectors in some detail. Three different expressions for the T matrix which goes with the two-pole K matrix will be derived. The first expression entails only the open channel parts of T and K . It follows from the observation that $\hat{T}$ and $\hat{K}$ have common eigenvectors, where

$$\hat{T} = \rho^{1/2} T \rho^{1/2}, \quad \hat{K} = \rho^{1/2} K \rho^{1/2}, \quad (2.20)$$

$$\hat{T} = \hat{K} (1 - i \hat{K})^{-1}. \quad (2.21)$$

Since $\hat{K}$ is Hermitian, it may be written in terms of its orthonormal eigenvectors $\bar{\Psi}_i$ and real eigenvalues k_i as

$$\hat{K} = \sum_i k_i (\bar{\Psi} \bar{\Psi}^\dagger)_i, \quad (\bar{\Psi}_i, \bar{\Psi}_j) = \delta_{ij}. \quad (2.22)$$

It follows that

$$\hat{T} = \sum_i \frac{k_i}{1 - i k_i} (\bar{\Psi} \bar{\Psi}^\dagger)_i. \quad (2.23)$$

Since $\hat{K}$ has the representation

$$\hat{K} = \sum_{i=1}^2 \frac{g_i}{w_i - w} \rho^{1/2} (\bar{\Phi} \bar{\Phi}^\dagger)_i \rho^{1/2}, \quad g_i = \gamma_i / 2, \quad (2.24)$$

the eigenvectors of $\hat{K}$ (those with non-zero eigenvalues) lie in the subspace spanned by $\rho^{1/2} \bar{\Phi}_1$ and $\rho^{1/2} \bar{\Phi}_2$. We put

$$\bar{\Psi} = \rho^{1/2} (a_1 \bar{\Phi}_1 + a_2 \bar{\Phi}_2), \quad (2.25)$$

and assume that the $\rho^{1/2} \bar{\Phi}_i$ are linearly independent. Then, because of the independence, the Hermitian eigenvalue problem

$$\hat{K} \bar{\Psi} = k \bar{\Psi} \quad (2.26)$$

is equivalent to the following two-dimensional non-Hermitian eigenvalue problem:

$$\begin{bmatrix} \frac{g_1 \rho_1}{w_1 - w} & \frac{g_1 \rho_{12}}{w_1 - w} \\ \frac{g_2 \rho_{21}}{w_2 - w} & \frac{g_2 \rho_2}{w_2 - w} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = k \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \quad (2.27)$$

$$\rho_{ij}(w) = (\bar{\Phi}_i, \rho(w) \Phi_j), \quad \rho_i = \rho_{ii}. \quad (2.28)$$

We obtain two energy dependent eigenvectors $\bar{\Psi}_i = \rho^{\frac{1}{2}}(a_1^i \bar{\Phi}_1 + a_2^i \bar{\Phi}_2)$ from (2.27), and these are necessarily orthogonal, since they correspond to two different eigenvalues k_i of the same Hermitian matrix $\hat{K}$.

For w sufficiently close to either w_1 or w_2 , a square root may be expanded to give the following approximation for the eigenvalues

$$k_i = \frac{g_i \rho_i}{w_i - w} + \frac{g_i g_j \rho_{12} \rho_{21}}{g_i \rho_i (w_j - w) - g_j \rho_j (w_i - w)}. \quad (2.29)$$

It is understood that $i \neq j$, here and in similar formulas to follow. This approximation is not necessarily good at all points between w_1 and w_2 . The resonant eigenvalue k_i becomes infinite at $w = w_i$, while the corresponding t_i takes on the value i of the unitarity limit. At this energy the eigenvector $\bar{\Psi}_i$ becomes parallel to the residue vector $\rho^{\frac{1}{2}} \bar{\Phi}_i$ of the matrix $\hat{K}$. This is seen by computing the coefficient ratio a_1/a_2 in the approximation (2.29). The result is

$$\left(\frac{a_j}{a_i} \right)^i = \frac{g_j \rho_{ji} (w_i - w)}{g_i \rho_i (w_j - w) - g_j \rho_j (w_i - w)}, \quad (2.30)$$

which yields $(a_j/a_i)^{1/2} \rightarrow 0$, $w \rightarrow w_i$. At $w = w_i$ the non-resonant eigenvalue k_j takes on the value

$$k_j = \frac{g_j \rho_j}{w_j - w_i} \left(1 - \frac{\rho_{12} \rho_{21}}{\rho_i} \right). \quad (2.31)$$

The non-resonant eigenvector ψ_j at $w = w_i$ is a mixture of $\rho^{1/2} \Phi_i$ and $\rho^{1/2} \Phi_j$, with the latter component dominating when the overlap ρ_{ij} is small:

$$\left(\frac{a_j}{a_i} \right)^j \rightarrow - \frac{\rho_i}{\rho_{ij}}, \quad w \rightarrow w_i. \quad (2.32)$$

If we now go to a high physical energy where all channels are open, we may obtain T from Eq. (2.23) for $\hat{T}$, since $\rho^{-1/2}$ exists when all channels are open. The resulting expression for T holds at all energies, if one is careful to interpret square roots so that the analytic continuation through the physical sheet is obtained. We have

$$T = \sum_{i=1}^2 \frac{k_i}{1 - i k_i} (a_i^i \Phi_i + a_i^i \Phi_i) (a_i^i \Phi_i + a_i^i \Phi_i)^\dagger, \quad (2.33)$$

where one must choose the correct branch of the somewhat complicated functions k_i and a_i^j .

The advantage of the formula (2.23) is that it takes a simple form at the energies $w = w_i$. The disadvantage of both (2.23) and (2.33) is that analytic continuation into the complex plane is obscured by the presence of several square roots. To find a form in which analyticity properties are clearly displayed, we make use of an observation of McGlinn and Polis¹⁾. With their method we obtain T directly, rather than $\hat{T}$. We substitute the following ansatz in the equation $T(1 - i \rho K) = K$:

$$T = \alpha (\Phi \Phi^\dagger)_1 + \beta (\Phi \Phi^\dagger)_2 + \gamma (\rho_{12} \Phi_1 \Phi_2^\dagger + \rho_{21} \Phi_2 \Phi_1^\dagger). \quad (2.34)$$

The coefficients α , β , γ are determined so that the equation is satisfied and the result is

$$T = [(W_1 - z)(W_2 - z) + g_1 g_2 \rho_{12} \rho_{21}]^{-1} \times \\ \times [g_1 (W_2 - z)(\Phi \Phi^\dagger)_1 + g_2 (W_1 - z)(\Phi \Phi^\dagger)_2 + i g_1 g_2 (\rho_{12} \Phi_1 \Phi_2^\dagger + \rho_{21} \Phi_2 \Phi_1^\dagger)] ,$$

(2.35)

$$\bar{W}_i = w_i - i g_i \rho_i(z) .$$

(2.36)

If the variation of the $\rho_{ij}(z)$ with energy is not too extreme, there will be two points $z = E_1$ and $z = E_2$ such that

$$(W_1 - E_i)(W_2 - E_i) + g_1 g_2 \rho_{12}(E_i) \rho_{21}(E_i) = 0 .$$

(2.37)

Thus, we expect that two poles in K will imply, in general, two poles in T .

We shall now show that the residues of the poles of T actually factorize; i.e., they are matrices of rank one. That this is well hidden in formula (2.35) seems to have caused some doubts about the McGlinn-Polis method. We determine the eigenvectors of the residue matrices

$$M_i = \lim_{z \rightarrow E_i} (E_i - z) T(z) ,$$

(2.38)

those which lie in the subspace spanned by Φ_1 and Φ_2 . At each pole we shall find that one of the two eigenvalues is zero, which is equivalent to factorization. In general, M_i is not symmetric (i.e., if time reversal invariance fails), so the left and right eigenvectors of M_i need not be equal. We then have to solve the two eigenvalue problems

$$M_i \psi = \mu \psi, \quad (2.39)$$

$$\hat{\psi}^\dagger M_i = \mu \hat{\psi}^\dagger. \quad (2.40)$$

Now M_i acting to the right projects into the subspace spanned by $\bar{\Phi}_1$ and $\bar{\Phi}_2$, while acting to the left it projects into the subspace of $\Phi_1^\dagger$ and $\Phi_2^\dagger$. The eigenvectors, therefore, have the forms

$$\psi = b_1 \bar{\Phi}_1 + b_2 \bar{\Phi}_2 \quad ; \quad \hat{\psi}^\dagger = \hat{b}_1 \Phi_1^\dagger + \hat{b}_2 \Phi_2^\dagger. \quad (2.41)$$

We assume that $\bar{\Phi}_1$ and $\bar{\Phi}_2$ are linearly independent so that (2.40), (2.41) and (2.36) imply the following relations for the coefficients:

$$\begin{bmatrix} g_1 [W_2 - E_i + i g_2 \rho_{12} \chi_{21}] & g_1 [(W_2 - E_i) \chi_{12} + i g_2 \rho_{12}] \\ g_2 [(W_1 - E_i) \chi_{21} + i g_1 \rho_{21}] & g_2 [W_1 - E_i + i g_1 \rho_{21} \chi_{12}] \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \mu \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad (2.42)$$

$$\chi_{12} = (\bar{\Phi}_1, \bar{\Phi}_2) = \chi_{21}^*. \quad (2.43)$$

The equations for the $\hat{b}_i$ are the same as (2.42), except that χ_{ij} , ρ_{ij} replace χ_{ji} , ρ_{ji} . Suppose that the secular equation of Eqs. (2.42) is written as

$$\mu^2 + B_i \mu + C_i = 0. \quad (2.44)$$

We substitute the following solution of (2.38) for E_i in (2.44):

$$E_i = \frac{1}{2} (W_i + \bar{W}_i + S_i), \quad (2.45)$$

$$S_i = (W_i - W_j) \left[1 - \frac{4g_1 g_2 \rho_{12}(E_i) \rho_{21}(E_i)}{(W_i - W_j)^2} \right]^{1/2}. \quad (2.46)$$

Some computation then shows that $C_i = 0$, so that indeed one of the two eigenvalues is zero. The other eigenvalue is

$$\mu_i = g_1(W_2 - E_i) + g_2(W_1 - E_i) + i g_1 g_2 (\rho_{12} \chi_{21} + \rho_{21} \chi_{12}). \quad (2.47)$$

The left eigenvalues are, of course, the same as the right eigenvalues. The solution of Eq. (2.42) turns out to be

$$\left(\frac{b_i}{b_j} \right)^i = \frac{g_i [(W_j - E_i) \chi_{ij} + i g_j \rho_{ij}]}{g_j [W_i - E_i + i g_i \rho_{ji} \chi_{ij}]} \quad (2.48)$$

Near the pole at $z = E_i$, the T matrix has the behaviour

$$T \cong \frac{h_i}{E_i - w} (\psi \hat{\psi}^\dagger)_i, \quad h_i = \frac{\mu_i}{(\hat{\psi}, \psi)_i (E_j - E_i)}. \quad (2.49)$$

There is still an arbitrary complex factor in the definition of each of the vectors ψ and $\hat{\psi}$. We choose the factors (without using up all of the arbitrariness) in such a way that

$$h_i = -\text{Im } E_i. \quad (2.50)$$

The real and imaginary parts of the particle mass will be denoted by M_i and $-\Gamma_i/2$:

$$E_i = M_i - i\Gamma_i/2. \quad (2.51)$$

If the energy dependence of $\rho_{ij}(w)$ is neglected, then $T(z)$ is analytic except for the two poles. With that assumption we arrive at the representation mentioned following Eq. (2.19):

$$T(z) = \sum_{i=1}^2 \frac{\rho_i/2}{E_i - z} (\psi \hat{\psi}^\dagger)_i. \quad (2.52)$$

When the theory is invariant under time reversal, one may define the phases of channel states so that K is real and symmetric. Then the Φ_i can be made real, so that $\chi_{12} = \chi_{21}$ and $\rho_{12}(z) = \rho_{21}(z)$, even at complex z . This means that we may take the b_i to be equal to the $\hat{b}_i$, in which case $\psi^T = \hat{\psi}^\dagger$, and expression (2.52) is symmetric. The b_i continue to be complex, in general, so $\psi \neq \hat{\psi}$, $\psi^T \neq \psi^\dagger$.

The formulas simplify a great deal if the overlap of the two resonances (as measured by ρ_{12} and χ_{12}) is not too large. We define the following dimensionless parameters:

$$\Delta = \frac{g_1 g_2 \rho_{12} \rho_{21}}{(W_1 - W_2)^2}, \quad \delta_i = g_i \frac{\rho_{12} \chi_{21} + \rho_{21} \chi_{12}}{W_i - W_j}. \quad (2.53)$$

"Small overlap" means that $|\Delta|$ and $|\delta_i|$ are small compared to one. For the pole positions E_i we have

$$E_i = W_i + (W_j - W_i)(\Delta + O(\Delta^2)), \quad (2.54)$$

whereas the residue eigenvalues have the expansion

$$h_i(\hat{\psi}, \psi)_i = g_i (1 + i\delta_j + [\Delta + O(\Delta^2)][1 - g_j/g_i]). \quad (2.55)$$

The ratio of the eigenvector coefficients is

$$\left(\frac{b_i}{b_j}\right)' = \frac{\left[\frac{W_j - W_i}{ig_j \rho_{ji}}\right] \left[\frac{(W_j - W_i) \chi_{ij} (1 + \Delta) + ig_j \rho_{ij}}{(W_j - W_i) \chi_{ij} + ig_j \rho_{ij} (1 + \Delta)} \right]}{1}, \quad (2.56)$$

where Δ^2 has been neglected in comparison with Δ . If Δ may be neglected in comparison with 1 in the second square bracket ²⁷⁾, we obtain the simple result

$$\left(\frac{b_i}{b_j}\right)' = \frac{W_j - W_i}{ig_j \rho_{ji}}. \quad (2.57)$$

This expression should agree with Eq. (18) of McVoy ²⁾, and it does agree after a trivial error in the first paper of Ref. 2) is corrected.

It is convenient to know the normalization of the b_i and $\hat{b}_i$, in the case where the Φ_i are normalized so that $(\Phi_i, \Phi_i) = 1$. Then with small overlap we have

$$\rho_i g_i = h_i = \Gamma_i / 2 \quad ; \quad \rho_i (\hat{\psi}, \psi)_i = 1,$$

according to (2.36), (2.54) and (2.55). Define normalization constants a_i , $\hat{a}_i$ as follows :

$$\underline{b}^1 = a_1 \begin{bmatrix} W_2 - W_1 \\ ig_2 \rho_{21} \end{bmatrix}, \quad \hat{\underline{b}}^1 = \hat{a}_1 \begin{bmatrix} W_2 - W_1 \\ ig_2 \rho_{12} \end{bmatrix}, \quad (2.58)$$

$$\underline{b}^2 = a_2 \begin{bmatrix} ig_1 \rho_{12} \\ W_1 - W_2 \end{bmatrix}, \quad \hat{\underline{b}}^2 = \hat{a}_2 \begin{bmatrix} ig_1 \rho_{21} \\ W_1 - W_2 \end{bmatrix}. \quad (2.59)$$

One then finds that

$$\rho_i(\hat{\psi}, \psi)_i = \rho_i \hat{a}_i a_i (W_1 - W_2)^2 [1 + i\delta_j - (g_j/g_i)\Delta] \cong \rho_i \hat{a}_i a_i (W_1 - W_2)^2 = 1. \quad (2.60)$$

Only the product $\hat{a}_i a_i$ is determined. In the case of time reversal invariance, it makes sense to adopt the symmetric convention

$$a_i = \hat{a}_i = \frac{\pm 1}{\rho_i^{1/2} (W_1 - W_2)}. \quad (2.61)$$

It may sometimes happen that g_j/g_i is large compared to 1, in which case it is possible to have Δ and δ_i small, but δ_j uncomfortably large. Fortunately, the normalization (2.61) is still correct, since by (2.55) the condition $\rho_i(\hat{\psi}, \psi)_i = 1$ is changed to

$$h_i(\hat{\psi}, \psi)_i = g_i [1 + i\delta_j - (g_j/g_i)\Delta]. \quad (2.62)$$

The square bracket in (2.60) cancels out.

We have now obtained three different expressions for the unitary T matrix that goes with the two-pole K matrix; namely, formulas (2.23) and (2.35), which are exact, and formula (2.52), in which the energy dependence of ρ_{ij} is neglected. The explicit inclusion of unitarity is, of course, the advantage of the K matrix approach. McVoy and Durand²⁾ have also worked out the implications of unitarity in the two-pole case, but by means of auxiliary constraints rather than by the K matrix. Their method is restricted to the case where ρ_{ij} is independent of energy, and that may be a disadvantage in problems such as $g - \omega$ mixing where the mass differences are large.

It is not always a good approximation to suppose that K consists of poles alone, even if we are interested only in an energy region close to the poles. If the pole residues are very small, as they are in the scattering description of the K_L - K_S system, then the non-pole background may actually dominate the scattering. In the K_L - K_S example, the background is overwhelming; it is due to strong and electromagnetic interactions, while the pole terms arise by virtue of weak interactions. It is important, then, to extend our formalism so as to allow background, while maintaining unitarity. If a completely general background to the poles of K is allowed, we cannot write the T matrix in a form sufficiently explicit to be useful in phenomenology. If the background as it appears in the S matrix is constant, however, we can derive a useful formula. We write

$$S = S_B + S_P = S_B + i \sum_i \frac{\Gamma_i}{E_i - w} \rho^{1/2}(E_i) (\psi \hat{\psi}^\dagger)_i \rho^{1/2}(E_i). \quad (2.63)$$

It is easy to show that unitarity of S implies unitarity of S_B , when S_B is constant. Consequently, $S_B^{\frac{1}{2}}$ and $S_B^{-\frac{1}{2}}$ exist and are unitary, and we can write

$$S = S_B^{1/2} (1 + S_B^{-1/2} S_P S_B^{-1/2}) S_B^{1/2} = S_B^{1/2} \bar{S} S_B^{1/2}. \quad (2.64)$$

The matrix $\bar{S}$ is unitary, and it consists of just two poles. We may introduce a Hermitian $\bar{K}$ defined as follows:

$$\bar{S} = (1 + i\bar{K})(1 - i\bar{K})^{-1}. \quad (2.65)$$

Now we can apply all of the above calculations to $\bar{K}$, which consists of just two poles with Hermitian, factorized residues. Of course, one puts $\rho = 1$ in our previous equations, since ρ does not appear in the relations of the barred matrices:

$$\bar{S} = 1 + 2i\bar{T} = 1 + 2i\bar{K}(1 - i\bar{K})^{-1}. \quad (2.66)$$

3. THE K MATRIX DESCRIPTION OF MIXING

In the previous section we analyzed the fully "mixed" situation. Now we wish to "unmix" the states; i.e., we wish to relate the physical scattering matrix to an idealized scattering matrix in which there are two separate states which do not communicate at all. Attention is focused on the inverse of K , which will be denoted as follows:

$$H(\omega) = K^{-1}(\omega). \quad (3.1)$$

This notation improves the appearance of our equations, and emphasizes the fact that K^{-1} plays a role in our perturbation theory similar to the Hamiltonian in Rayleigh-Schrödinger perturbation theory.

We suppose that there are two neighbouring poles of the T matrix, which are associated with two vanishing eigenvalues of H :

$$H(\omega^{(1)})\bar{\Phi}^{(1)} = 0, \quad H(\omega^{(2)})\bar{\Phi}^{(2)} = 0. \quad (3.2)$$

We assume that there is a degenerate limit of the theory in which H becomes a direct sum:

$$H(\omega) \rightarrow H_0(\omega) = \begin{bmatrix} H_{01}(\omega) & 0 \\ 0 & H_{02}(\omega) \end{bmatrix}. \quad (3.3)$$

The unmixed states are defined as follows:

$$H_0(\omega_{01})\bar{\Phi}_{01} = 0, \quad H_0(\omega_{02})\bar{\Phi}_{02} = 0, \quad (3.4)$$

$$\bar{\Phi}_{01} = \begin{bmatrix} \phi_1 \\ 0 \end{bmatrix}, \quad \bar{\Phi}_{02} = \begin{bmatrix} 0 \\ \phi_2 \end{bmatrix}. \quad (3.5)$$

More generally we are interested in the case where

$$H_0(\omega) = \begin{bmatrix} H_{01}(\omega) & 0 & 0 \\ 0 & H_{02}(\omega) & 0 \\ 0 & 0 & H_{03}(\omega) \end{bmatrix}, \quad (3.6)$$

but in which H_{03} does not have vanishing eigenvalues for ω near ω_{01} , ω_{02} . This is the situation for the K_L - K_S problem, discussed in a later paper. The following formalism applies to either (3.3) or (3.6).

It is convenient to introduce a parameter λ to describe the passage from the physical situation to the degenerate limit. We define $H(\omega, \lambda)$ as a Hermitian matrix for $0 \leq \lambda \leq 1$ in such a way that

$$H(\omega, 1) = H(\omega), \quad H(\omega, 0) = H_0(\omega). \quad (3.7)$$

We shall require some differentiability with respect to λ . This is not too restrictive, since it is always possible to introduce λ so that H is differentiable to all orders. For instance, we could take

$$H(\omega, \lambda) = H_0(\omega) + \lambda [H(\omega) - H_0(\omega)]. \quad (3.8)$$

We begin with the simplest case where the unperturbed states are exactly degenerate in mass: $\omega_{01} = \omega_{02} = \omega_0$. We wish to solve the equation

$$H(\omega(\lambda), \lambda) \Phi(\lambda) = 0. \quad (3.9)$$

To that end, we assume that the perturbed energy and eigenvectors may be expanded as follows:

$$\begin{aligned} w &= w_0 + \lambda w_1 + O(\lambda^2), \\ \Phi &= \Phi_0 + \lambda \Phi_1 + O(\lambda^2). \end{aligned}$$

(3.10)

We also assume that $w(\lambda)$ is differentiable, and that

$$\frac{dw}{d\lambda} = w_1 + O(\lambda).$$

(3.11)

It is not necessary to suppose that there are complete power series for w and Φ . Equations (3.10) may be understood in the sense of finite Taylor expansions with remainder. As in ordinary degenerate perturbation theory, the zeroth order eigenvector is a linear combination of the two degenerate unperturbed states:

$$\Phi_0 = \sum_i c_i \Phi_{0i}.$$

(3.12)

Equation (3.9) takes the form

$$\left[H(w_0, 0) + \lambda [H_w(w_0, 0)w_1 + H_\lambda(w_0, 0)] + O(\lambda^2) \right] \cdot \left[\sum c_i \Phi_{0i} + \lambda \Phi_1 + O(\lambda^2) \right] = 0,$$

(3.13)

where $H_w = (\partial/\partial w)H$, etc. We now form the Hermitian scalar product of (3.13) with Φ_{01} , and then with Φ_{02} , and retain only the first order terms. This gives us two equations to determine the "right linear combinations" of unperturbed vectors:

$$\begin{aligned} \Delta_{11} c_1 + \Delta_{12} c_2 &= w_1 \beta_1 c_1, \\ \Delta_{21} c_1 + \Delta_{22} c_2 &= w_1 \beta_2 c_2, \end{aligned}$$

(3.14)

$$\Delta_{ij} = (\Phi_{oi}, H_\lambda(w_o, o) \Phi_{oj}) , \quad (3.15)$$

$$\beta_i = 2/\gamma_i = -(\Phi_{oi}, H_w(w_o, o) \Phi_{oi}) . \quad (3.16)$$

Note that β_i is just the reciprocal of the K matrix residue factor $\gamma_i/2$ for the unperturbed system [cf., Eq. (2.13)]. Since the unperturbed states are decoupled, $1/\beta_i$ is the unperturbed reduced half width.

Since the two unperturbed reduced widths are not equal in general, the equations (3.14) do not have the form of a standard eigenvalue problem. We put them into standard form by multiplying with $\beta^{-1/2}$, where β is the diagonal matrix $[\beta_i \delta_{ij}]$. Thus,

$$\Delta M \underline{v} = w_i \underline{v} , \quad (3.17)$$

$$\Delta M = \begin{bmatrix} \Delta_{11}/\beta_1 & \Delta_{12}/(\beta_1 \beta_2)^{1/2} \\ \Delta_{21}/(\beta_2 \beta_1)^{1/2} & \Delta_{22}/\beta_2 \end{bmatrix} , \quad \underline{v} = \begin{bmatrix} c_1 \beta_1^{1/2} \\ c_2 \beta_2^{1/2} \end{bmatrix} . \quad (3.18)$$

From the secular equation of (3.17) we get the perturbed energies to first order:

$$\begin{aligned} w^{(1,2)} &= w_o + w_i^{(1,2)} = \\ &= w_o + \frac{1}{2} \left[\Delta M_{11} + \Delta M_{22} \pm [(\Delta M_{11} - \Delta M_{22})^2 + 4 \Delta M_{12}^2]^{1/2} \right] \end{aligned} \quad (3.19)$$

$$= \frac{1}{2} \left[M_{11} + M_{22} \pm [(M_{11} - M_{22})^2 + 4 M_{12}^2]^{1/2} \right] , \quad (3.20)$$

where

$$M = \omega_0 \cdot I + \Delta M . \quad (3.21)$$

It is always possible to choose the phases of the vectors Φ_{0i} so that $\Delta M_{12} = \Delta M_{21}^*$ is real, whether or not the theory is invariant under time reversal. We have made that choice in (3.19), and retain it in the following.

Since M is Hermitian, the two solutions $\underline{v}^{(1)}, \underline{v}^{(2)}$ of (3.17) will be orthogonal, provided the degeneracy is removed in first order. If the degeneracy is not removed, we shall choose two orthogonal vectors from an infinite set. The coefficient vectors $\underline{c}^{(i)} = \beta^{-\frac{1}{2}} \underline{v}^{(i)}$ are orthogonal if and only if $\beta_1 = \beta_2$. The zeroth order eigenvectors,

$$\Phi_o^{(i)} = \sum_j c_j^{(i)} \Phi_{oj} , \quad (3.22)$$

are orthogonal if and only if the unperturbed reduced widths are equal. This fact is important for a proper understanding of mixing. A lack of orthogonality means that the mixing angle determined by diagonalizing the mass matrix does not fix coupling constant ratios. An additional parameter, β_1/β_2 , is needed to determine the latter.

The vectors $\underline{c}^{(i)}$ may be written as follows:

$$\begin{aligned} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}^{(1)} &= a_1 \begin{bmatrix} 1 \\ \frac{(\beta_1/\beta_2)^{1/2} \Delta M_{12}}{(\Delta M_{11} - \Delta M_{22}) d} \end{bmatrix} , \\ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}^{(2)} &= a_2 \begin{bmatrix} \frac{(\beta_2/\beta_1)^{1/2} \Delta M_{12}}{(\Delta M_{22} - \Delta M_{11}) d} \\ 1 \end{bmatrix} , \end{aligned} \quad (3.23)$$

$$2d = 1 + \left[1 + \left(\frac{2\Delta M_{12}}{\Delta M_{11} - \Delta M_{22}} \right)^2 \right]^{1/2}. \quad (3.24)$$

The square root in (3.24) is positive, and the normalization constants a_i are left arbitrary for the moment.

If the theory is invariant under time reversal, then all of the above eigenvectors and matrices may be chosen to be real. It then becomes useful to define two real mixing angles θ_1, θ_2 :

$$\begin{aligned} \bar{\Phi}_0^{(1)} &= \cos \theta_1 \bar{\Phi}_{01} - \sin \theta_1 \bar{\Phi}_{02}, \\ \bar{\Phi}_0^{(2)} &= \sin \theta_2 \bar{\Phi}_{01} + \cos \theta_2 \bar{\Phi}_{02}. \end{aligned} \quad (3.25)$$

We have

$$(\bar{\Phi}_0^{(1)}, \bar{\Phi}_0^{(2)}) = \sin(\theta_2 - \theta_1). \quad (3.26)$$

Neither θ_1 nor θ_2 is equal, in general, to the angle θ of the orthogonal matrix U which diagonalizes the mass matrix M :

$$U = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}. \quad (3.27)$$

One has $\theta_1 = \theta_2 = \theta$ when $\beta_1 = \beta_2$.

In certain problems, one gets a first rough sketch of mixing from just the zeroth-order eigenvectors (3.22) and the first-order energies (3.20). In other problems, (for instance, that of neutral K mesons), it is essential to have also the first-order eigenvectors, since the zeroth-order vectors have non-zero components only for closed channels. To obtain $\bar{\Phi}_1$, we begin with the first-order part of (3.13); namely,

$$H(w_0, 0)\Phi_1 = -[H_w(w_0, 0)w_1 + H_\lambda(w_0, 0)] \sum_i c_i \Phi_{0i} . \quad (3.28)$$

Let $\{\Phi_{0i}\}$ be a complete orthonormal set of eigenvectors of $H(w_0, 0)$, where k_i^{-1} is the eigenvalue of Φ_{0i} . Since Φ_{01} and Φ_{02} have vanishing eigenvalues, we have

$$H(w_0, 0) = \sum_{i \neq 1, 2} k_i^{-1} (\Phi_0 \Phi_0^\dagger)_i . \quad (3.29)$$

Since the c_i satisfy (3.14), the right side of (3.28) is orthogonal to Φ_{01} and Φ_{02} . This means that the following operator has the same effect as the unit operator when applied to the right side of (3.28):

$$\sum_{i \neq 1, 2} (\Phi_0 \Phi_0^\dagger)_i . \quad (3.30)$$

A solution of (3.28) may, therefore, be constructed in the form

$$\Phi_1 = - \sum_{i \neq 1, 2} k_i (\Phi_0 \Phi_0^\dagger)_i [H_w(w_0, 0)w_1 + H_\lambda(w_0, 0)] \sum_{j=1}^2 c_j \Phi_{0j} . \quad (3.31)$$

The solution of (3.28) given in (3.31) is not unique. One may add to (3.31) any linear combination of Φ_{01} and Φ_{02} , and still have a solution. One must, in fact, make use of this freedom if the perturbation series is carried on to the second order. A term must be added to (3.31) in order to satisfy the second order equation. There is, of course, a similar situation in ordinary degenerate perturbation theory²⁸⁾. Our second order equation reads as follows

$$H_0 \Phi_2 = - (H_w w_1 + H_\lambda) \Phi_1 - \frac{1}{2} (H_{ww} w_1^2 + 2 H_{w\lambda} w_1 + H_{ww} w_2 + H_{\lambda\lambda}) \Phi_0 , \quad (3.32)$$

where all matrices are evaluated at $(w, \lambda) = (w_0, 0)$.

A necessary condition for (3.32) to have a solution is that the right-hand side be orthogonal to every solution of the homogeneous equation $H_0 \Phi_2 = 0$.

To meet that condition, we substitute

$$\Phi_1 = \sum_{i=1}^2 d_i \Phi_0^{(i)} + \hat{\Phi}_1,$$

where the vector (3.31) is now denoted by $\hat{\Phi}_1$, and also put $w_1 = w_1^{(1)}$ and $\Phi_0 = \Phi_0^{(1)}$ in (3.32). Since

$$\begin{aligned} (\Phi_0^{(j)}, [H_w w_1^{(1)} + H_\lambda] \Phi_0^{(1)}) &= 0, \\ (\Phi_0^{(j)}, [H_w w_1^{(1)} + H_\lambda] \Phi_0^{(2)}) &= (w_1^{(1)} - w_1^{(2)}) (\Phi_0^{(j)}, H_w \Phi_0^{(2)}), \end{aligned} \quad (3.33)$$

formation of the scalar product of (3.32) with $\Phi_0^{(i)}$ yields

$$\begin{aligned} \delta_{i2} (w_1^{(1)} - w_1^{(2)}) (\Phi_0^{(2)}, H_w [d_2 \Phi_0^{(2)} + \hat{\Phi}_1]) + \\ + \frac{1}{2} (\Phi_0^{(i)}, [H_{ww} w_1^{(1)2} + 2 H_{w\lambda} w_1^{(1)} + H_{ww} w_2^{(1)} + H_{\lambda\lambda}] \Phi_0^{(1)}) &= 0. \end{aligned} \quad (3.34)$$

By the choice $i=1$, Eq. (3.34) determines the second order energy $w_2^{(1)}$, and then with $i=2$ the coefficient d_2 is fixed. The coefficient d_1 is not determined by (3.29). It is possible to show that the correct choice is $d_1=0$, since any other choice gives the same eigenvector (up to a constant multiplier) if the perturbation series is carried to all orders.

We now turn to the case where the unperturbed states are nearly, but not exactly, degenerate. We attempt the following expansions:

$$w^{(i)} = w_{0i} + \lambda w_1^{(i)} + O(\lambda^2), \quad (3.35)$$

$$\Phi^{(i)} = \sum c_j^{(i)} \Phi_{oj} + \lambda \bar{\Phi}_i^{(i)} + O(\lambda^2). \quad (3.36)$$

Proceeding as in (3.13) ff., we obtain the following equations for $i = 1$:

$$\Delta_{11}^{(1)} c_1^{(1)} + \Delta_{12}^{(1)} c_2^{(1)} = w_1^{(1)} \beta_1^{(1)} c_1^{(1)}, \quad (3.37)$$

$$\Delta_{21}^{(1)} c_1^{(1)} + \Delta_{22}^{(1)} c_2^{(1)} = (w_1^{(1)} \beta_2^{(1)} - D_2 \lambda^{-1}) c_2^{(1)}, \quad (3.38)$$

$$\Delta_{ij}^{(k)} = (\Phi_{oi}, H_\lambda(w_{ok}, 0) \Phi_{oj}), \quad (3.39)$$

$$D_i = (\bar{\Phi}_{oi}, H(w_{oj}, 0) \Phi_{oi}), \quad i \neq j, \quad (3.40)$$

$$\beta_i^{(k)} = -(\bar{\Phi}_{oi}, H_w(w_{ok}, 0) \Phi_{oi}). \quad (3.41)$$

Because of the term D_2 , which vanishes if $w_{01} = w_{02}$, the zeroth and first order equations cannot be satisfied separately. The zeroth order equation is $D_2 c_2^{(1)} = 0$. If this were satisfied by taking $c_2^{(1)} = 0$, we would then be back to non-degenerate perturbation theory. Instead, we may combine the zeroth and first order equations, as in (3.37) and (3.38). Solution of these equations gives $w_1^{(1)}$ and $c_1^{(1)}$ as functions of λ . That is to say, the equations do not actually contain only the zeroth and first powers of λ , but rather represent a certain summation of all powers. As far as magnitudes of the terms retained are concerned, the equations

should be about the same at $\lambda = 1$ as Eqs. (3.14) provided that $|w_{01} - w_{02}|$ is less than or comparable to the first order energy shift $w_1^{(1)}$. This is seen clearly if we note the expansion of D_2 to lowest order in $w_{01} - w_{02}$:

$$\begin{aligned} D_2 &\cong (\Phi_{02}, [H(w_{02}, 0) + H_w(w_{02}, 0)(w_{01} - w_{02})] \Phi_{02}) \\ &= -\beta_2^{(2)} (w_{01} - w_{02}). \end{aligned} \quad (3.42)$$

Thus, if

$$|w_{01} - w_{02}| \leq w_1^{(1)}, \quad (3.43)$$

then at $\lambda = 1$ the term $-D_2/\lambda$ does not change the rough order of magnitude of the coefficient of $c_2^{(1)}$ on the right side of (3.38). One can still use this method when (3.43) does not hold, however.

An argument in favour of the above prescription is that it has the correct limit as λ tends to zero. One of the two roots of the secular equation of (3.37)-(3.38) is the physical one which is to be identified with $w_1^{(1)}$, while the other is spurious and tends to infinity as λ goes to zero. Thus,

$$\begin{aligned} w_1^{(1)} &= \\ &\frac{1}{2} \left[\Delta M_{11}^{(1)} + \Delta M_{22}^{(1)} + (\Delta M_{11}^{(1)} - \Delta M_{22}^{(1)}) \left[1 + \frac{4 |\Delta M_{12}^{(1)}|^2}{(\Delta M_{11}^{(1)} - \Delta M_{22}^{(1)})^2} \right]^{1/2} \right], \end{aligned} \quad (3.44)$$

where

$$\Delta M_{11}^{(1)} = \Delta_{11}^{(1)} / \beta_1^{(1)}, \quad \Delta M_{22}^{(1)} = (\Delta_{22}^{(1)} + D_2 \lambda^{-1}) / \beta_2^{(1)}, \quad (3.45)$$

$$\Delta M_{12}^{(1)} = \Delta M_{21}^{(1)*} = \Delta_{12}^{(1)} / (\beta_1^{(1)} \beta_2^{(1)})^{1/2}. \quad (3.46)$$

For small λ one has

$$w_1^{(1)} = \Delta M_{11}^{(1)} - \lambda \beta_2^{(1)} |\Delta M_{12}^{(1)}|^2 / D_2 + O(\lambda^2), \quad (3.47)$$

which tends to the correct result $\Delta M_{11}^{(1)}$ of non-degenerate perturbation theory in the limit $\lambda = 0$. If the unperturbed levels are close, then D_2 is small, and there is a rapid departure from the result of non-degenerate theory as λ increases. The coefficient $c_2^{(1)}$ vanishes as λ , while $c_1^{(1)}$ tends to a non-zero constant:

$$\frac{c_2^{(1)}}{c_1^{(1)}} = \frac{w_1^{(1)} - \Delta M_{11}^{(1)}}{(\beta_2/\beta_1)^{(1)} \frac{1}{2} \Delta M_{12}^{(1)}} = -\lambda \Delta_{21}^{(1)} / D_2 + O(\lambda^2) \quad (3.48)$$

$$\cong \frac{\lambda \Delta_{21}^{(1)}}{\beta_2^{(2)} (w_{01} - w_{02})} + O(\lambda^2). \quad (3.49)$$

Note that large D_2 is equivalent to small λ . That is, when the zeroth order part of our mixed zeroth-first-order equations (3.37), (3.38) is relatively large, we get the simplification expressed in (3.47) and (3.48).

To find the first correction $\Phi_1^{(1)}$ to the eigenvector, we must solve the equation

$$H(w_{0i}, 0) \Phi_1^{(1)} = -[H(w_{0i}, 0) + H_w(w_{0i}, 0) w_1^{(1)} + H_\lambda(w_{0i}, 0)] \Phi_0^{(1)}. \quad (3.50)$$

Let $\{\Phi_{0j}^{(i)}\}$ be a complete orthonormal set of eigenvectors of $H(w_{0i}, 0)$, where $\Phi_{0i}^{(i)} = \Phi_{0i}$ is the eigenvector with eigenvalue zero. By the argument which led to Eq. (3.37), we see that one solution of (3.50) is

$$\begin{aligned} \Phi_1^{(1)} = & - \sum_{j \neq i} k_j^{(1)} (\Phi_0 \Phi_0^\dagger)_j^{(1)} [H(w_{0i}, 0) + H_w(w_{0i}, 0) w_1^{(1)} \\ & + H_\lambda(w_{0i}, 0)] \Phi_0^{(1)} \end{aligned} \quad (3.51)$$

In contrast to the exactly degenerate case, it is not necessary to add a solution of the homogeneous equation in order to satisfy the next equation in the hierarchy. The second order equation is

$$\begin{aligned} H(w_{0i}, 0) \Phi_2^{(i)} = & - [H_w(w_{0i}, 0) w_1^{(i)} + H_\lambda(w_{0i}, 0)] \Phi_1^{(i)} - \\ & - \frac{1}{2} [H_{ww}(w_{0i}, 0) w_1^{(i)2} + 2H_{w\lambda}(w_{0i}, 0) w_1^{(i)} + H_{ww}(w_{0i}, 0) w_2^{(i)} + \\ & + H_{\lambda\lambda}(w_{0i}, 0)] \Phi_0^{(i)}. \end{aligned} \quad (3.53)$$

The necessary condition that Φ_{0i} be orthogonal to the right side of (3.53) merely determines the second-order energy shift $w_2^{(i)}$, as in non-degenerate perturbation theory. The present scheme has a formal resemblance to non-degenerate theory, although it contains the physics of the degenerate case.

In the preceding discussion there are twice as many parameters as in the case of exact degeneracy. There are two different mass matrices, $\Delta M^{(1)}$ and $\Delta M^{(2)}$, and two ratios $(\beta_1/\beta_2)^{(1)}$, $(\beta_1/\beta_2)^{(2)}$. In particular applications, it may happen that the full set of eight parameters is effectively reduced to a smaller set. Very often we should be able to make the approximation

$$\left(\frac{\beta_1}{\beta_2}\right)^{(1)} \approx \left(\frac{\beta_1}{\beta_2}\right)^{(2)}, \quad (3.54)$$

which says that the matrix elements of $H(w, 0)$ have constant slope between w_{01} and w_{02} . We cannot claim that $\Delta M_{ii}^{(1)} \approx \Delta M_{ii}^{(2)}$ if the D_i are large, but when the D_i are sufficiently large, then the coefficient ratios $(c_1/c_2)^{(i)}$, which are usually the main quantities of interest, take the simplified form (3.49). Combining this with (3.54), we see that the coefficient ratios are determined by the three parameters $\Delta M_{21}^{(1)}$, $\Delta M_{21}^{(2)}$ and β_1/β_2 , (in addition to $w_{01}-w_{02}$, which is usually known). As a rough approximation, poorer than (3.54), we can also say that $\Delta M_{21}^{(1)} \approx \Delta M_{21}^{(2)}$. Of course, if

we put $\lambda = 1$ and let w_{01} and w_{02} become equal, then the formalism goes over into the exactly degenerate one. The criterion for the two formalisms to be nearly the same is that $|w_{01} - w_{02}| \ll w_1^{(i)}$.

When the D_i are large [i.e., when $|w_{01} - w_{02}| \ll w_1^{(i)}$] we can reproduce (nearly!) the results of the above method by means of non-degenerate perturbation theory. We make the following expansions for state 1:

$$w^{(1)} = w_{01} + \lambda w_1^{(1)} + O(\lambda^2), \quad (3.55)$$

$$\Phi^{(1)} = \Phi_{01} + \lambda \bar{\Phi}_1^{(1)} + O(\lambda^2), \quad (3.56)$$

$$H(w^{(1)}(\lambda), \lambda) = H(w_{01}, 0) + \lambda [H_w(w_{01}, 0) w_1^{(1)} + H_\lambda(w_{01}, 0)] + O(\lambda^2). \quad (3.57)$$

We obtain

$$w_1^{(1)} = \Delta M_{11}^{(1)} = - \frac{(\Phi_{01}, H_\lambda(w_{01}, 0) \Phi_{01})}{(\Phi_{01}, H_w(w_{01}, 0) \Phi_{01})}, \quad (3.58)$$

and

$$\bar{\Phi}_1^{(1)} = - \sum_{j \neq 1}^{\infty} \kappa_j^{(1)} (\Phi_0 \Phi_0^+)_j^{(1)} [H_w(w_{01}, 0) w_1^{(1)} + H_\lambda(w_{01}, 0)] \Phi_{01}. \quad (3.59)$$

If $w_{01} - w_{02}$ is sufficiently small [irrespective of its magnitude in comparison with $w_1^{(i)}$] then $\Phi_{02}^{(1)}$ will be quite close to Φ_{02} . Suppose, in fact, that $\Phi_{02}^{(1)} = \Phi_{02}$ to a sufficient approximation. Then we can work out the eigenvalue $k_2^{(1)}$ as follows, assuming as in (3.54) that $H(w, 0)$ has constant slope between w_{01} and w_{02} :

$$H(\omega_{01}, 0) \Phi_{02} = [H(\omega_{02}, 0) + H_W(\omega_{02}, 0)(\omega_{01} - \omega_{02})] \Phi_{02} \quad (3.60)$$

$$= (\omega_{01} - \omega_{02}) H_W(\omega_{02}, 0) \Phi_{02} = (1/k_2^{(1)}) \Phi_{02}. \quad (3.61)$$

Taking the scalar product with Φ_{02} , we have

$$k_2^{(1)} = \frac{1}{\beta_2(\omega_{02} - \omega_{01})}. \quad (3.62)$$

Now, since $H_W(\omega_{01}, 0)$ does not connect Φ_{01} and Φ_{02} , the contribution of Φ_{02} to the sum (3.59) is simply

$$\frac{\Delta_{21}^{(1)}}{\beta_2(\omega_{01} - \omega_{02})} \Phi_{02} = \left(\frac{\beta_1}{\beta_2}\right)^{1/2} \frac{\Delta M_{21}^{(1)}}{\omega_{02} - \omega_{01}} \Phi_{02}. \quad (3.63)$$

Comparing with (3.49), we see that we have the same coefficient of Φ_{02} that was obtained in the zeroth order state before.

4. RHO-OMEGA MIXING

A good example in which to illustrate our formalism is the electromagnetic mixing of ρ_0 and ω mesons.

The mixing is relatively weak, and thereby susceptible to our perturbation treatment. The S matrix description applies most directly to the colliding beam reaction $e_+e_- \rightarrow \pi_+\pi_-$ (29). With further assumptions of peripheral production, etc., our description may be applied to other reactions in which $\rho - \omega$ interference has been reported (18); for instance, $\pi_+p \rightarrow \pi_+\pi_-\Delta_{++}$. For the present we shall deal only with $e_+e_- \rightarrow \pi_+\pi_-$, since our purpose is more to illustrate the scheme than to do comprehensive phenomenology. We hope to publish detailed fits to data elsewhere.

We assume that the T matrix for $e_+e_- \rightarrow \pi_+\pi_-$ is entirely dominated by the ρ and ω poles. The excuse for neglecting background is that the Orsay data as reported in Ref. 29) are fitted well enough by the poles alone. In principle, the pure pole model might be improved slightly by using the full formula (2.35), which includes the correct threshold factors $\rho(w)$. That procedure is somewhat involved, however, and will not be carried out here. Instead, we shall compromise by putting $\rho(w)$ equal to a constant everywhere, except in the ρ meson width (30). It is then convenient to rid the equations of $\rho(w)$ by working with the matrices $\hat{T} = \rho^{\frac{1}{2}} T \rho^{\frac{1}{2}}$, $\hat{K} = \rho^{\frac{1}{2}} K \rho^{\frac{1}{2}}$, which are henceforth denoted in this section by T and K. Thus, $T = K(1 - iK)^{-1}$, the quantities ρ_{12} and χ_{12} of Section 3 are both equal to the scalar product $\chi = (\Phi_1, \Phi_2)$, and ρ_i is replaced by 1. At the end Γ_ρ is replaced by $\Gamma_\rho(w)$ in the ρ pole denominator only, as in Ref. 29).

To apply the formalism of the preceding section, we must first choose the unperturbed H matrix. It will not do to let H_0 be the limit of H as the electric charge vanishes. In that limit the scattering in electromagnetic channels such as $\pi_0\gamma$, e_+e_- , $\mu_+\mu_-$, vanishes completely, and H_0 would contain blocks of zeros.

Our perturbation theory would then be inconsistent if carried beyond the zeroth order. The left and right sides of the following first-order equation could not agree:

$$H_0 \Phi_1 + H_w w_1 \Phi_0 = -H_\lambda \Phi_0. \quad (4.1)$$

The components of the left side corresponding to electromagnetic channels would be zero, while the corresponding components on the right would not vanish. In general, consistency of the perturbation method requires that H_0 have no null-space in the space of all eigenvectors of H , except at the isolated energies where $H_0 \Phi_0 = 0$.

We can avoid this difficulty, and at the same time avoid the theoretically doubtful idea of turning off the electromagnetic interaction. Consider the submatrix T_M of the physical T matrix, which has $J^P = 1^-$ and which refers only to the major channels 2π ($C = -1, Q = 0$), 3π ($C = -1, Q = 0$), and $\pi_0 \gamma$. We write

$$T_M = \begin{bmatrix} T_1 & T_{12} \\ T_{21} & T_2 \end{bmatrix}, \quad (4.2)$$

where T_1 refers to 2π only, and T_2 refers to 3π and $\pi_0 \gamma$. Since the minor channels such as $\pi_+ \pi_- \pi_+ \pi_-$, $e_+ e_-$, $\mu_+ \mu_-$, etc., must contribute very little to unitarity sums, and since $\rho_0 \rightarrow \pi_0 \gamma$ is not seen experimentally, it follows that each of the matrices T_M , T_1 and T_2 will obey the unitarity condition almost exactly. It is useful, therefore, to define Hermitian matrices K_i associated with the T_i :

$$T_i = K_i (1 - iK_i)^{-1}, \quad i = 1, 2, M. \quad (4.3)$$

Now H_0 in (3.3) will be defined as follows:

$$H_0(\omega) = \begin{bmatrix} K_1^{-1}(\omega) & 0 \\ 0 & K_2^{-1}(\omega) \end{bmatrix}. \quad (4.4)$$

The perturbed H will be

$$H(\omega) = K_M^{-1}(\omega), \quad (4.5)$$

and we may define

$$H(\omega, \lambda) = H_0(\omega) + \lambda [H(\omega) - H_0(\omega)]. \quad (4.6)$$

The perturbation theory will serve to construct the matrix T_M as the sum of two poles. Finally, to obtain the full T matrix, we augment the residue vectors of T_M with components for the minor channels. The augmented T_M will be almost exactly unitary, so that we need not bother with the exact unitarity constraints on minor channels.

The unmixed ρ and ω are defined as null vectors of H_0 :

$$H_0(\omega_{01})\bar{\Phi}_{01} = 0, \quad H_0(\omega_{02})\bar{\Phi}_{02} = 0, \quad (4.7)$$

$$\bar{\Phi}_{01} = \begin{bmatrix} \phi_1 \\ 0 \end{bmatrix}, \quad \bar{\Phi}_{02} = \begin{bmatrix} 0 \\ \phi_2 \end{bmatrix}, \quad (\bar{\Phi}_{0i}, \bar{\Phi}_{0i}) = 1. \quad (4.8)$$

With this definition the unmixed states are rather close to the physical states. They include some, but not all, radiative corrections. To an accuracy of a few per cent, we should have

$$\begin{aligned} m_\rho &\cong \omega_{01} ; \quad \Gamma_\rho/2 \cong g_{01} = -(\bar{\Phi}_{01}, H_\omega(\omega_{01}, 0)\bar{\Phi}_{01})^{-1}, \\ m_\omega &\cong \omega_{02} ; \quad \Gamma_\omega/2 \cong g_{02} = -(\bar{\Phi}_{02}, H_\omega(\omega_{02}, 0)\bar{\Phi}_{02})^{-1}. \end{aligned} \quad (4.9)$$

It will not be necessary to use the approximations (4.9) as equalities; they are merely for orientation.

Since the unperturbed states are not exactly degenerate in mass, we must decide whether to use the non-degenerate perturbation theory of Eq. (3.55) ff., or the "almost degenerate" method of Eq. (3.32) ff. We choose the former, since the electromagnetic transition mass ΔM_{12} turns out to be somewhat smaller than $w_{02} - w_{01}$. We could as well use the almost degenerate formalism, and obtain the same results in a less transparent way. We must first examine the perturbed eigenvectors of H :

$$\Phi^{(i)} = \Phi_{0i} + \Phi_1^{(i)}. \quad (4.10)$$

We have to know only the (2π) part of $\Phi_1^{(2)}$ and the $(3\pi, \pi_0 \gamma)$ part of $\Phi_1^{(1)}$. The former contributes directly to the partial width $\Gamma_{\omega \rightarrow 2\pi}$, while the latter enters into the overlap χ . The overlap, which is required for constructing the T matrix, is

$$\begin{aligned} \chi &= (\Phi^{(1)}, \Phi^{(2)}) \cong (\Phi_{01}, \Phi_1^{(2)}) + (\Phi_1^{(1)}, \Phi_{02}) \\ &= (\Phi_{01}, \Phi_{1(2\pi)}) + (\Phi_{02}, \Phi_{1(3\pi, \pi_0 \gamma)}). \end{aligned} \quad (4.11)$$

According to (3.59), we have the following convenient formulas:

$$\Phi_{1(2\pi)}^{(2)} = -K_1(w_{02}) H_\lambda(w_{02}, 0) \Phi_{02}, \quad (4.12)$$

$$\Phi_{1(3\pi, \pi_0 \gamma)}^{(1)} = -K_2(w_{01}) H_\lambda(w_{01}, 0) \Phi_{01}. \quad (4.13)$$

The matrices K_1 and K_2 are essentially the physical K matrices for (2π) and $(3\pi, \pi_0 \gamma)$ scattering, respectively. We suppose that the π - π scattering in (4.12), a single channel process at the

energy $w_{02} \approx m_\omega$, is given by the unperturbed ρ pole alone. This follows more or less directly from our assumption of pole dominance in $e_+e_- \rightarrow \pi_+\pi_-$ ³¹⁾. Equation (4.12) then reads

$$\begin{aligned} \Phi_{1(2\pi)}^{(2)} &= -\Phi_{01} \frac{g_{01}}{w_{01} - w_{02}} (\Phi_{01}, H_\lambda(w_{02}, 0) \Phi_{02}) \\ &= \Phi_{01} \frac{g_{01}}{w_{02} - w_{01}} \Delta_{12}^{(2)}. \end{aligned} \quad (4.14)$$

The vector Φ_{01} has 1 for its first component, and zeros for its other components. For the evaluation of (4.13) note that we have not assumed ω dominance of $(3\pi, \pi_0\gamma)$, even though it might be reasonable to do so. One contribution in (4.13) comes from the tail of the unperturbed ω pole:

$$-\Phi_{02} \frac{g_{02}}{w_{02} - w_{01}} \Delta_{12}^{(1)}. \quad (4.15)$$

There may be other significant contributions to (4.13), but we can argue that they are nearly orthogonal to Φ_{02} , and therefore do not influence the overlap χ . At $w = w_{02}$, Φ_{02} is one of the eigenvectors of K_2 , and is orthogonal to all of the other eigenvectors which make up the non-resonant background. This background should be slowly varying, and should remain orthogonal to Φ_{02} in the small energy interval $w_{01} < w < w_{02}$. Then from (4.11), (4.14) and (4.15), we obtain a simple formula for the overlap:

$$\chi = \frac{g_{01} \Delta_{12}^{(2)}}{w_{02} - w_{01}} \left[1 - \frac{g_{02}}{g_{01}} \frac{\Delta_{12}^{(1)}}{\Delta_{12}^{(2)}} \right]. \quad (4.16)$$

It is difficult to say how much change will occur in Δ_{12} between w_{01} and w_{02} , but a change of more than 30% or so would be surprising, since this matrix element involves no resonance poles. We suppose, then, that

$$.07 \leq \frac{g_{02}}{g_{01}} \frac{\Delta_{12}^{(1)}}{\Delta_{12}^{(2)}} \leq .13 ,$$

(4.17)

since $g_{02}/g_{01} \approx 1/10$, by (4.9). This yields

$$\frac{g_{01} \Delta_{12}^{(2)}}{w_{02} - w_{01}} = \chi \eta , \quad \eta = 1.1 \pm .05 .$$

(4.18)

Except for the small uncertainty in η , and the uncertainty in the physical masses and widths of ρ and ω , the only freedom in the theory is to adjust the real constant χ .

It turns out that the small overlap approximation, as developed following Eq. (2.53), is not valid in the present problem. It is necessary to apply the exact equations (2.37), (2.47), (2.48), (2.49) and (2.50). The physical pole positions E_i are given by (2.37), where w_i and g_i appear in the perturbed reaction matrix for major channels, i.e.,

$$K_M = \sum_{i=1}^2 \frac{g_i}{w_i - w} (\Phi \Phi^\dagger)^{(i)} .$$

(4.19)

The E_i will be taken as input from experiment. Then if χ is given, we can determine the w_i and g_i from (2.37). That is,

$$W_i = w_i - i g_i = \frac{1}{2} (E_1 + E_2 \pm [(E_1 - E_2)^2 + 4 g_1 g_2 \chi^2]^{1/2}) .$$

(4.20)

To compute W_i , we first evaluate the right side of (4.20) with some $\chi = \chi_0$, and with $g_i = \Gamma_i/2$. Since the correct value of g_i is $-\text{Im } W_i$ rather than $\Gamma_i/2$, this evaluation actually gives the W_i for the value of χ which satisfies the equation

$$\operatorname{Im} W_1 \operatorname{Im} W_2 \chi^2 = (\Gamma_1 \Gamma_2 / 4) \chi_0^2. \quad (4.21)$$

The values of the W_i for various χ and the E_i from Ref. 29) are shown in Table 1. The biggest difference between the W_i and the E_i comes in the ω width: g_2 and $\Gamma_2/2$ can differ appreciably.

Now that g_i and w_i are known as functions of χ , we can easily work out T_M as a function of χ . The residue vectors of T_M [cf. (2.52)] are

$$\psi^{(i)} = \sum_j b_j^{(i)} \Phi^{(j)}, \quad (4.22)$$

where by (2.48) we have

$$\underline{b}^{(1)} = a_1 \begin{bmatrix} 1 \\ B_1 \end{bmatrix}; \quad \underline{b}^{(2)} = a_2 \begin{bmatrix} B_2 \\ 1 \end{bmatrix}; \quad (4.23)$$

$$B_1 = \frac{g_2 (W_1 - E_1 + i g_1 \chi^2)}{g_1 \chi (w_2 - E_1)}; \quad B_2 = \frac{g_1 (W_2 - E_2 + i g_2 \chi^2)}{g_2 \chi (w_1 - E_2)}. \quad (4.24)$$

The complex normalization constants a_i are determined, up to a sign, by Eqs. (2.49) and (2.50). These equations yield

$$a_i^2 = \frac{\mu_i}{(\Gamma_i/2)(E_j - E_i)} \frac{1}{1 + 2\chi B_i + B_i^2},$$

$$\mu_i = g_1 (W_2 - E_i) + g_2 (W_1 - E_i) + 2i g_1 g_2 \chi^2. \quad (4.25)$$

Finally, we augment the vectors $\psi^{(i)}$ with additional components $\xi^{(i)}$ for the minor channels not included in T_M :

$$\Psi^{(i)} = \begin{bmatrix} \psi^{(i)} \\ \xi^{(i)} \end{bmatrix}. \quad (4.26)$$

The complete T matrix is

$$T = \sum_i \frac{\Gamma_i/2}{E_i - \omega} (\Psi \Psi^T)^{(i)}. \quad (4.27)$$

For comparison with experiment we are interested in the ratio of ρ and ω pole residues in the amplitude for $e_+e_- \rightarrow \pi_+\pi_-$. According to (4.14), (4.18) and (4.22), the ω residue is

$$(\Gamma_\omega/2) (b_1^{(2)} + \gamma \chi b_2^{(2)}) \xi_{e_+e_-}^{(2)}. \quad (4.28)$$

Here we have neglected terms which are higher order in electromagnetism; i.e., we have retained just the coefficient of $(\Phi_{01})_{2\pi} = 1$. The ρ residue is

$$(\Gamma_\rho/2) b_1^{(1)} \xi_{e_+e_-}^{(1)}, \quad (4.29)$$

where higher order effects are again ignored.

The partial width for $\omega \rightarrow 2\pi$ is given by

$$(\Gamma_{\omega \rightarrow 2\pi} / \Gamma_\rho)^{1/2} = (\Gamma_\omega / \Gamma_\rho)^{1/2} |b_1^{(2)} + \gamma \chi b_2^{(2)}| \quad (4.30)$$

From Ref. 29) we have the experimental parameters

$$(\Gamma_{\omega \rightarrow 2\pi} / \Gamma_\rho)^{1/2} = .06 \pm .02, \quad (4.31)$$

$$m_\rho = 773.5 \pm 5.4 \text{ MeV}, \quad \Gamma_\rho = 110.7 \pm 5.3 \text{ MeV}. \quad (4.32)$$

The following ω parameters were taken as input to the fits of Ref. 29):

$$m_\omega = 783 \text{ MeV}, \quad \Gamma_\omega = 12.2 \text{ MeV}. \quad (4.33)$$

If we compute (4.30) as a function of χ , using the central values of (4.32) with (4.33), we find that (4.31) implies

$$.45 < \chi < .67. \quad (4.34)$$

For the central value 0.06 of (4.31), we have $\chi = 0.58$. The corresponding magnitudes of the dimensionless parameters of (2.53) are $|\Delta| = 0.085$, $|\delta_1| = 0.79$, $|\delta_2| = 0.14$. These values are much too large to allow expansion of square roots as in (2.54) ff. We found it most convenient to evaluate the W_i and the $b_j^{(i)}$ on a digital computer, from the exact formulas.

It is usually assumed that the amplitudes $\xi_{e^+e^-}^{(i)}$ (essentially the photon-vector-meson couplings) are nearly real. Reality would follow from time reversal invariance if the ρ and ω were stable³²⁾. In that case we get the relative phase of the ρ and ω residues as

$$\begin{aligned} \Theta \pmod{\pi} &= \arg \left[(b_1^{(2)} + \gamma \chi b_2^{(2)}) / b_1^{(1)} \right] \\ &\cong \arg \left[b_1^{(2)} + \gamma \chi b_2^{(2)} \right]. \end{aligned} \quad (4.35)$$

The phase of $b_1^{(1)}$ is negligible, being always less than 1.5° . With the central values from (4.32), the uncertainty in η assumed in (4.18) allows the following range of θ , where θ increases with η :

$$83^\circ < \theta < 117^\circ \quad (\text{mod } \pi). \quad (4.36)$$

For a given η , θ is almost independent of χ in the interval $0 < \chi < 0.7$. For sufficiently small χ , one can see analytically why θ is independent of χ , and why a θ near 90° is preferred. In the small overlap approximation,

$$b_1^{(2)} + \eta \chi b_2^{(2)} = \chi \left[\frac{i g_1}{W_1 - W_2} + \eta \right]. \quad (4.37)$$

The real part of the first term in (4.37) is big compared to the imaginary part, but is cancelled almost exactly by the real second term. This cancellation of relatively large real parts persists as χ is increased, even though the $b_i^{(j)}$ acquire a significant non-linear dependence on χ .

The value of θ depends fairly strongly on the values of m_ρ and Γ_ρ . According to the tabulation of Ref. 33)

$$m_\rho = 765 \pm 10 \text{ MeV}, \quad \Gamma_\rho = 125 \pm 20 \text{ MeV}. \quad (4.38)$$

If we allow the uncertainties expressed in (4.38) together with the uncertainties in η , then we have a slightly bigger range of θ :

$$80^\circ < \theta < 124^\circ \quad (\text{mod } \pi). \quad (4.39)$$

A survey of the dependence on m_ρ and Γ_ρ is given in Table 2, which shows that for the smaller values of m_ρ , θ is large and relatively independent of η .

The above results agree substantially with the results of Gourdin, Stodolsky and Renard ¹⁰⁾, although we admit a larger uncertainty in θ . These authors obtain $\theta = 101^\circ$, and then add in an extra 10° as their estimate of the phase of $\xi_{e^+e^-}^{(2)}/\xi_{e^+e^-}^{(1)}$. Horn ¹²⁾ has made an analysis similar to that of Gourdin et al., but has been a bit more cautious about concluding that θ should be close to 90° . Our discussion takes such a different form that it is difficult to make a step-by-step comparison with Refs. 10) and 12).

The phase of $\xi_{e^+e^-}^{(2)}/\xi_{e^+e^-}^{(1)}$ was estimated as 11° in Ref. 10) by assuming that ρ and ω correspond to some discrete eigenstates, $\rho^{(0)}$ and $\omega^{(0)}$, of an unperturbed Hamiltonian. The SU(6) ratio of 3 between the $\gamma-\rho^{(0)}$ and $\gamma-\omega^{(0)}$ couplings was also assumed. We would like to avoid the former assumption, but we are then left without a phase estimate.

The Orsay experiment yielded the following relative phase of ρ and ω residues:

$$\theta = 164^\circ \pm 28^\circ.$$

(4.40)

[This is "Fit 2", which goes with the mass and width values of (4.32).] If one adds 10° or so for the phase of $\xi_{e^+e^-}^{(2)}/\xi_{e^+e^-}^{(1)}$ to (4.36), then there is no really clear disagreement between (4.36) and (4.40). It takes some imagination to see $\rho-\omega$ interference at all in the Orsay data, so the result (4.40) could be subject to a large modification on refinement of the experiment. Similarly, we should not take the result (4.34) for χ too seriously. As noted above, our calculation of θ is independent of χ .

Finally, we estimate the "transition mass" ΔM_{12} of the K matrix formalism. Define

$$\Delta M_{12} = (g_{01} g_{02})^{1/2} \Delta_{12}^{(2)}.$$

(4.41)

Then, by (4.18), (4.9) and (4.34),

$$\Delta M_{12} = (g_{02}/g_{01})^{1/2} \chi \eta (\omega_{02} - \omega_{01}), \quad (4.42)$$

$$1.6 < |\Delta M_{12}| < 2.3 \quad (\text{MeV}), \quad (4.43)$$

if $\omega_{02} - \omega_{01} \cong 10 \text{ MeV}$.

The real part of the mass matrix element $m_{\rho\omega}$, which is not to be confused with ΔM_{12} , has roughly the same order of magnitude as ΔM_{12} . Gourdin, Stodolsky and Renard deduced $|\text{Re } m_{\rho\omega}| \cong 7 \text{ MeV}$ from the Orsay results. Theoretical estimates of this quantity, to be found in Refs. 12), 17), 34) and 35), range from 2 to 5 MeV.

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χ	W_1	W_2
0.0995	773.0-55.43i	783.0-6.166i
0.196	772.9-55.23i	783.0-6.364i
0.309	772.8-54.80i	783.1-6.804i
0.392	772.7-54.29i	783.3-7.314i
0.501	772.5-53.24i	783.5-8.362i
0.592	772.1-51.18i	783.9-9.784i
0.700	771.2-48.86i	784.8-12.74i

Table 1: Values of the $W_i = w_i - ig_i$ for various choices of the overlap χ . The W_i were obtained from Eq. (4.20), with the following E_i :

$$E_1 = 773 - 55.5i$$

$$E_2 = 783 - 6.1i$$

$\rho \backslash m$	+	0	-
+	80	97	109
0	80	99	112
-	80	102	116

$$\eta = 1.05$$

$\rho \backslash m$	+	0	-
+	97	108	115
0	94	107	117
-	91	108	120

$$\eta = 1.1$$

$\rho \backslash m$	+	0	-
+	113	116	120
0	106	114	121
-	102	114	124

$$\eta = 1.15$$

Table 2: The phase θ of Eq. (4.35) in degrees, for various values of m_ρ , Γ_ρ and η . The values of m_ρ and Γ_ρ are $(m_+, m_0, m_-) = (775, 765, 755)$ MeV, $(\Gamma_+, \Gamma_0, \Gamma_-) = (145, 125, 105)$ MeV.

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- 32) We can take the photon γ and the stable states $\rho^{(0)}$, $\omega^{(0)}$ to be invariant under the time reversal operator Θ . Then $\chi_{e_+e_-}^{(\rho)}$ is a real constant times

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