Exploring twisted mass Lattice QCD with the Clover term

D. Bećirević\textsuperscript{a}, Ph. Boucaud\textsuperscript{a}, V. Lubicz\textsuperscript{b,c},
G. Martinelli\textsuperscript{d}, F. Mescia\textsuperscript{e}, S. Simula\textsuperscript{c}, C. Tarantino\textsuperscript{f}

\textsuperscript{a}Laboratoire de Physique Théorique (Bât. 210), Université de Paris XI, Centre d’Orsay, 91405 Orsay-Cedex, France
\textsuperscript{b}Dipartimento di Fisica, Università di Roma Tre, Via della Vasca Navale 84, I-00146 Rome, Italy
\textsuperscript{c}INFN, Sezione di Roma III, Via della Vasca Navale 84, I-00146 Roma, Italy
\textsuperscript{d}Dipartimento di Fisica, Università di Roma “La Sapienza”, and INFN, Sezione di Roma, P.le A. Moro 2, I-00185 Rome, Italy
\textsuperscript{e}INFN, Laboratori Nazionali di Frascati, Via E. Fermi 40, I-00044 Frascati, Italy
\textsuperscript{f}Physik Depart., Technische Universität München, D-85748 Garching, Germany

Abstract

It has been shown that in the twisted mass formulation of Lattice QCD at maximal twist large cutoff effects are generated when the quark mass becomes of $O(a\Lambda_{QCD}^2)$. In general, these effects can be suppressed in two ways: either by choosing the critical quark mass in an “optimal way”, or by adding the Clover term to the twisted action. We investigate the second option by performing a quenched lattice QCD simulation with twisted Clover fermions and pion masses as low as 280 MeV. We show that the Clover term is indeed efficient in reducing the large cutoff effects. In particular, the so-called bending phenomenon observed in the determination of the pion decay constant is cured in this way. In addition, by using maximally twisted Clover fermions, we provide a non-perturbative determination of the vector current renormalization constant $Z_V$ as well as of the non-perturbatively renormalized light quark masses. Finally, we calculate the connected contribution to the charged-neutral pseudoscalar meson mass splitting, finding that the introduction of the Clover term in the twisted action is also beneficial, in the quenched approximation, in reducing cutoff effects related to the isospin symmetry breaking at finite lattice spacing.
1 Introduction

The twisted mass formulation of Lattice QCD (tmLQCD) [1, 2] offers a number of interesting advantages. The absence of real unphysical zero modes, which affect the standard Wilson discretization of the fermionic action, prevents the appearance in this framework of the exceptional configurations in quenched and partially-quenched numerical simulations. This allows to investigate QCD dynamics with much lighter valence quarks. In addition, twisting the Wilson term in the action (in the physical basis) considerably simplifies the renormalization pattern of several local operators. Notable examples in this respect are the determination of the pion decay constant, which can be performed without introducing any renormalization factor [1], and of the kaon $B_K$ parameter, which does not require subtraction of wrong chirality operators [1, 3].

Furthermore, tmLQCD with a maximal twist angle ($\alpha = \pi/2$), denoted as maximally twisted mass LQCD (Mtm-LQCD), ensures automatic (or almost automatic) $O(a)$-improvement for the physically interesting quantities computed on the lattice [6].

The drawback, however, is that with tmLQCD both parity and flavor symmetries are explicitly broken at finite lattice spacing. The breaking of parity generates a mixing with wrong parity states which may affect, in some relevant cases, the large time behavior of lattice correlation functions. The breaking of flavor symmetry is responsible for generating a mass splitting between otherwise degenerate isospin hadronic partners. Although the breaking effects are parametrically of $O(a^2)$ at maximal twist, this splitting has been found to be numerically significant in the case of the neutral and charged pseudoscalar mesons [7].

Another important issue of practical interest for Mtm-LQCD is the presence of large cutoff effects which are generated when the quark mass $m_q$ becomes of $O(a\Lambda_{QCD}^2)$ or smaller. This phenomenon has been the subject of recent numerical and theoretical investigations. From the point of view of the Symanzik expansion, these large cutoff effects have been studied in Ref. [8]. In correlation functions they are related to multiple insertions of the dimension-five operator $L_5$ entering the Symanzik effective Lagrangian for Mtm-LQCD close to the continuum limit. Since, at maximal twist, $L_5$ has the quantum numbers of the neutral pion, its insertions can create from the vacuum Goldstone bosons with zero momentum, thus generating discretization effects which are proportional, at the leading order, to $(a/M_\pi^2)^2 \sim (a/m_q)^2$. In this situation, the value of the quark mass is limited to the region $m_q \gtrsim a\Lambda_{QCD}^2$. Being proportional to $(a/m_q)^2$, such lattice artifacts have been also called “infra-red divergent” (IRD) discretization effects. Note, however, that they actually do not represent any infra-red divergence, since the chiral limit

---

1For an alternative approach which avoids this subtraction even when working with ordinary Wilson fermions see Refs. [4, 5].
can be taken only after the continuum extrapolation has been performed.

A further step forward in understanding the IRD cutoff effects has been made in Ref. [9], where it is shown that their contribution to the Symanzik expansion can be explicitly resummed using twisted-mass chiral perturbation theory (tmχPT) [10]. For a simple quantity like the pion decay constant $f_\pi$, the leading order result of tmχPT reads

$$f_\pi = f \frac{\mu}{\sqrt{\mu^2 + \delta m^2}} \simeq f \left[ 1 - \frac{1}{2} \left( \frac{\delta m}{\mu} \right)^2 + \ldots \right],$$

where $f$ is the pion decay constant in the chiral limit and $\mu$ and $\delta m$ represent the (renormalized) twisted and non-twisted quark masses respectively. Since at maximal twist $\delta m$ is a quantity of $O(a)$ introduced by a discretization error in the determination of the critical mass, Eq. (1) shows that the deviation of $f_\pi$ from its value in the chiral limit has the form of an IRD cutoff effect. The leading correction is proportional to $(\delta m/\mu)^2 \sim (a/M_\pi^2)^2$ and its contribution becomes sizable when the twisted mass $\mu$ is small and comparable to $\delta m$, i.e., $\mu \sim O(a\Lambda^2_{QCD})$.

The requirement $\mu \gtrsim a\Lambda^2_{QCD}$ is in practice rather restrictive. For a typical inverse lattice spacing $a^{-1} \simeq 2$ GeV, it implies pion masses $M_\pi \gtrsim 500$ MeV. In order to allow for simulations with smaller values of the quark masses (down to $\mu \gtrsim a^2\Lambda^3_{QCD}$), and keep discretization effects under control, two strategies have been proposed [8]. Either one uses an “optimal” determination of the critical mass $m_{cr}$ or the (twisted) Clover term is introduced in the quark action. In the latter case, the coefficient $c_{SW}$ has to be fixed to the same value that guarantees on-shell $O(a)$-improvement with ordinary Wilson fermions.

The idea behind the first strategy, namely the optimal determination of the critical mass is to tune the $O(a)$ contribution to $m_{cr}$ in such a way that the dimension-five operator in the Symanzik effective Lagrangian has vanishing matrix element between the vacuum and the single pion state, i.e. $\langle 0 | L_5 | \pi^0 \rangle = 0$ [8]. Since the IRD cutoff effects have been shown to be proportional at the leading order to the previous matrix element, this procedure guarantees their suppression and allows much smaller values of the quark masses to be safely reached in the simulation. In practice, this procedure corresponds to fix the value of the critical mass by requiring the vanishing of the PCAC quark mass in the twisted basis, as it was anticipated in Refs. [11, 12]. In the following, we will denote as $m_{cr}^{opt}$ this “optimal” determination of the critical mass.

The numerical studies reported in Refs. [13, 14] show that such a strategy is indeed efficient. In particular, in agreement with theoretical expectations, it helps in suppressing the pronounced bending behaviour of the pion decay constant, i.e., the appearance of large lattice artifacts in this quantity when the quark mass becomes of $O(a\Lambda^2_{QCD})$ or smaller. Note, however, that one has to make a preliminary
numerical study, with twisted fermions, to determine the optimal value of the critical mass. Such a determination may be computationally expensive, particularly in the case of unquenched simulations.

In this paper we explore the second strategy of Ref. [8], namely the simulation of Mtm-LQCD with the inclusion of the Clover term in the action. In this case, in the Symanzik expansion of the lattice effective Lagrangian at $\mathcal{O}(a)$ only a term proportional to the square of the twisted quark mass survives [15], that is $\mathcal{L}_5 = \mathcal{O}(a\mu^2) = \mathcal{O}(aM_\pi^4)$. Thus, the introduction of the Clover term is expected to be efficient in cancelling the leading order IRD cutoff effects as much as the tuning of the critical mass in the twisted Wilson case.

From the previous discussion it should be clear that the Mtm-Clover approach does not require any specific prescription for the determination of the critical mass. We show here that this is indeed the case. We have performed two numerical simulations with Mtm-Clover fermions and with a critical quark mass determined in the non-twisted theory either by requiring the vanishing of the pion mass in the chiral limit (denoted in the following as $m_{\text{cr}}^{\text{pion}}$) or by imposing the vanishing of the PCAC mass ($m_{\text{cr}}^{\text{PCAC}}$). In both cases, we do not observe any bending phenomenon in the pion decay constant, even at values of quark masses smaller than $\mathcal{O}(a\Lambda_{QCD}^2)$. We also compare our results for the pion mass and decay constant with those obtained by the $\chi$LF Collaboration [13] using Mtm-Wilson fermions with the optimal choice of the critical mass, finding a very good agreement. The appealing feature of the Mtm-Clover approach is that the determination of the critical mass does not require any specific tuning and can be obtained from a single simulation performed with non-twisted Clover fermions.

Besides investigating the effectiveness of Mtm-Clover fermions in suppressing the IRD cutoff effects, we also present a non-perturbative determination of the local vector current renormalization constant $Z_V$ which renormalizes, in Mtm-LQCD, the local axial-vector current (in the physical basis). This constant is expected to be identical, up to $\mathcal{O}(a^2)$ effects, to the value determined by using non-twisted Clover fermions. Indeed, we find a very good agreement between our result and the non-twisted determination of $Z_V$ performed in Ref. [16] with the non-perturbative RI-MOM method.

In addition, we present a calculation of the strange and of the average up/down quark masses using the value of the quark mass renormalization constant determined non-perturbatively in Ref. [16]. With respect to previous quenched determinations of light quark masses, here we benefit from the use of Mtm-LQCD with the Clover term to probe ever lower values of the quark masses, and thereby reduce the errors due to the chiral extrapolation when determining the light up/down quark masses on the lattice.

Finally, we investigate the isospin symmetry breaking effects, induced at finite
lattice spacing by the tmLQCD action, by studying the contribution of connected diagrams to the charged-neutral pion mass splitting. This connected contribution can be interpreted as the charged-neutral mass splitting between pseudoscalar mesons composed by quarks and antiquarks belonging to different, though degenerate, isospin doublets. By comparing the results obtained with Mtm-Wilson and Mtm-Clover fermions, we find that the introduction of the Clover term is beneficial also in reducing these cutoff effects, at least for the pure connected contribution considered in this study and in the quenched approximation.

It is also worth noting that the same $O(a^2)$ operator that provides the leading contribution to the pion mass splitting in Mtm-LQCD also affects the determination of both the charged and the neutral pion masses when working with ordinary Wilson/Clover fermions. In this respect, therefore, the use of Mtm-LQCD does not introduce any additional uncertainty.

The plan of the paper is as follows. In Section 2 we give the details of the lattice simulation and present our results for the pseudoscalar meson mass and decay constant; we show the effectiveness of the introduction of the Clover term in curing the bending phenomenon. We present the determination of the renormalization constant $Z_V$ in Section 3.1 and the calculation of the light quark masses in Section 3.2. In Section 3.3 we evaluate the contribution of connected diagrams to the charged-neutral pion mass splitting and compare the results with those obtained by using Mtm-Wilson fermions with the optimal choice of the critical mass. Our conclusions are summarized in Section 4.

2 Pseudoscalar meson masses and decay constants with the Mtm-Clover action

The numerical results presented in this study have been obtained from a set of 300 gauge configurations generated, in the quenched approximation, with the standard Wilson gauge action at $\beta = 6.0$, which corresponds to an inverse lattice spacing $a^{-1} \simeq 2.0$ GeV. The lattice volume is $16^3 \times 32$ so that the physical length of the lattice is about 1.6 fm in the spatial directions. Using this set of gauge configurations, quark propagators have been computed by implementing three different choices for the quark action:

a) A simulation with Mtm-Clover fermions and the critical mass determined from the PCAC relation in the non-twisted theory, $m_{cr} = m_{cr}^{PCAC}$. The value of the critical hopping parameter, $\kappa_{cr}^{PCAC} = 0.135217$, corresponding to $am_{cr}^{PCAC} \simeq -0.3022$, has been obtained by using the results of a previous simulation done by our collaboration with standard Clover fermions at the same value of the gauge coupling on the volume $24^3 \times 64$ [16].
b) A simulation with Mtm-Clover fermions and the critical mass determined from the vanishing of the pion mass in the non-twisted theory, \( m_{\text{cr}} = m_{\text{cr}}^{\text{pion}} \). The value of the critical hopping parameter, \( \kappa_{\text{cr}}^{\text{pion}} = 0.135293 \), corresponds to \( am_{\text{cr}}^{\text{pion}} \approx -0.3043 \) and has been determined from the same set of non-twisted Clover data used to determine the value of \( m_{\text{cr}}^{\text{PCAC}} \) in simulation a).

c) A simulation with Mtm-Wilson fermions with the critical mass determined in the optimal way, that is \( m_{\text{cr}} = m_{\text{cr}}^{\text{opt}} \). We used the value \( \kappa_{\text{cr}}^{\text{opt}} = 0.157409 \) obtained by the \( \chi \)LF Collaboration from a simulation at \( \beta = 6.0 \) on the volume \( 16^{3} \times 32 \) [13].

In simulations a) and b) with Mtm-Clover fermions the coefficient of the Clover term has been fixed to the value determined non-perturbatively in Ref. [17], namely \( c_{\text{SW}} = 1.769 \).

In all three sets of simulations we have computed quark propagators with the same 9 values of the bare twisted quark mass \( a\mu \) (cf. Table 1), which are equal to those used in Ref. [13]. The range of chosen \( a\mu \) values covers pion masses between approximately 280 MeV and 1.1 GeV. In order to invert the Dirac operator for such a broad set of quark masses we employed the multiple mass solver algorithm [18].

To examine the impact of cutoff effects when working with maximally twisted fermions at small values of the quark mass, we investigate whether or not the inclusion of the Clover term is efficient in curing the bending phenomenon observed in the determination of the pion decay constant with Mtm-Wilson fermions, when the critical mass is determined in a non-optimal way.

Pseudoscalar meson masses and decay constants have been evaluated by studying the large time behavior of the following two-point correlation functions:

\[
C_{PP}(t) \equiv \sum_{\vec{x}} \langle P_{5}(t, \vec{x}) P_{5}^{\dagger}(0) \rangle , \quad C_{AP}(t) \equiv \sum_{\vec{x}} \langle A_{0}(t, \vec{x}) P_{5}^{\dagger}(0) \rangle ,
\]

where \( P_{5} = \bar{u}\gamma_{5}d \) and \( A_{\mu} = \bar{u}\gamma_{\mu}\gamma_{5}d \). Throughout this paper we choose to work in the physical basis and use mass-degenerate \( u \) and \( d \) quarks.

Using the completeness relation and considering large enough time separations, one gets

\[
C_{PP}(t) \quad t \gg 1 \quad \frac{Z_{PS}}{M_{P}} \frac{e^{-M_{P}T/2}}{\cosh \left( M_{P}(T/2 - t) \right)} ,
\]

\[
C_{AP}(t) \quad t \gg 1 \quad \frac{f_{P}M_{P}}{Z_{V}} \frac{\sqrt{Z_{PS}}}{M_{P}} \frac{e^{-M_{P}T/2}}{\sinh \left( M_{P}(T/2 - t) \right)} ,
\]

where \( M_{P} \) is the mass of the lightest pseudoscalar meson state, \( \sqrt{Z_{PS}} = |\langle 0 | P_{5}(0) | \pi \rangle| \) and \( f_{P}M_{P} = Z_{V} \langle 0 | A_{0}(0) | \pi \rangle \). \( Z_{V} \) is the renormalization constant of the local axial-vector current with Mtm-LQCD, which corresponds to the renormalization
constant of the local vector current with the non-twisted quark action [6]. The pseudoscalar decay constant can therefore be extracted by studying the ratio $C_{AP}(t)/C_{PP}(t)$ at large time distances,

$$f_P = Z_V \sqrt{Z_{PS}} \frac{C_{AP}(t)}{C_{PP}(t)} \coth (M_P(T/2 - t)) \right|_{t \gg 1}. \tag{4}$$

An alternative determination of the decay constant [1], which does not require the introduction of any renormalization constant, can be obtained by exploiting the consequences of the axial-vector Ward identity (AWI) for Mtm-LQCD. Up to $O(a^2)$ correction, the relevant identity reads

$$Z_V \sum_{\vec{x}} \langle \partial_0 A_0(t, \vec{x}) P_5^\dagger(0) \rangle = 2\mu \sum_{\vec{x}} \langle P_5(t, \vec{x}) P_5^\dagger(0) \rangle , \tag{5}$$

where $\mu$ is the bare twisted mass. Combining the above relation with Eqs. (3) one gets

$$f_P = 2\mu \sqrt{Z_{PS} \frac{M_P^2}{}} , \tag{6}$$

where both $Z_{PS}$ and $M_P$ can be determined from the correlation function $C_{PP}(t)$ of Eq. (3). We have evaluated the pseudoscalar meson decay constant $f_P$ using both the “direct” method (4) and the “indirect” one (6), obtaining always a very good agreement within the statistical errors.\footnote{This agreement is further improved if the factor $M_P^2$ in the denominator of Eq. (6) is replaced with $M_P \sinh(\alpha M_P)/\alpha$, where the sinh is originated from the discretized symmetric lattice version of the derivative in Eq. (5). Our results from Eq. (6) have been obtained with this replacement always implemented.} For the direct method, we used the non-perturbative RI-MOM determination $Z^{(Clover)}_V = 0.772$ [16] and the one-loop boosted perturbative estimate $Z^{(Wilson)}_V = 0.63$ [19, 20] in the Mtm-Clover and Wilson case respectively. In what follows, we only quote the results for $f_P$ obtained from the indirect method (6).

Our result for the pseudoscalar meson masses and decay constants obtained from simulations a), b) and c) are collected in Table 1. These results are also shown in Figs. 1 and 2, as a function of the bare quark mass $a\mu$. The Mtm-Wilson results from our simulation c) agree well with those presented in Ref. [13]. For comparison, we also show in these plots the results obtained in [13] by employing Mtm-Wilson fermions with the non-optimal choice of the critical mass, as well as the results obtained with non-twisted Clover fermions (at $\beta = 6.0$) using data produced by our collaboration for the studies described in Refs. [16, 21]. In the non-twisted case, the quark mass $a\mu$ in Figs. 1 and 2 represents the bare AWI quark mass $am^{\text{AWI}}$ [21] multiplied by the axial-vector renormalization constant.
Fig. 1: Values of the pion mass squared obtained with different $O(a)$-improved actions, as specified in the legend and in the text. The solid and dashed lines represent quadratic fits in $a\mu$.

Fig. 2: Same as in Fig. 1, but for the pseudoscalar decay constant. The lines represent quadratic fit in $a\mu$ except for the results obtained with Mtm-Wilson fermions and the $m^{\text{pion}}_{\text{cr}}$ determination of the critical mass. In this case, the fit also takes into account the tm$\chi$PT prediction expressed by Eq. (1).
Table 1: Values of the bare twisted quark mass $a\mu$, of the pseudoscalar meson mass $aM_P$ and of the pseudoscalar meson decay constant $af_P$ from the three sets of simulations performed in this study. The quoted errors are purely statistical.

<table>
<thead>
<tr>
<th></th>
<th>a) Mtm-Clover, $m_{\text{cr}}^\text{PCAC}$</th>
<th>b) Mtm-Clover, $m_{\text{cr}}^\text{pion}$</th>
<th>c) Mtm-Wilson, $m_{\text{cr}}^\text{opt}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a\mu$</td>
<td>0.0038</td>
<td>0.0057</td>
<td>0.0076</td>
</tr>
<tr>
<td></td>
<td>0.1359 (66)</td>
<td>0.1607 (52)</td>
<td>0.1811 (46)</td>
</tr>
<tr>
<td></td>
<td>0.0690 (30)</td>
<td>0.0710 (22)</td>
<td>0.0727 (18)</td>
</tr>
<tr>
<td></td>
<td>0.1308 (70)</td>
<td>0.1571 (51)</td>
<td>0.1787 (47)</td>
</tr>
<tr>
<td></td>
<td>0.0739 (34)</td>
<td>0.0741 (19)</td>
<td>0.0748 (19)</td>
</tr>
<tr>
<td>$aM_P$</td>
<td>0.0038</td>
<td>0.0057</td>
<td>0.0076</td>
</tr>
<tr>
<td></td>
<td>0.1359 (66)</td>
<td>0.1607 (52)</td>
<td>0.1811 (46)</td>
</tr>
<tr>
<td></td>
<td>0.0690 (30)</td>
<td>0.0710 (22)</td>
<td>0.0727 (18)</td>
</tr>
<tr>
<td></td>
<td>0.1308 (70)</td>
<td>0.1571 (51)</td>
<td>0.1787 (47)</td>
</tr>
<tr>
<td></td>
<td>0.0739 (34)</td>
<td>0.0741 (19)</td>
<td>0.0748 (19)</td>
</tr>
<tr>
<td>$af_P$</td>
<td>0.0038</td>
<td>0.0057</td>
<td>0.0076</td>
</tr>
<tr>
<td></td>
<td>0.1359 (66)</td>
<td>0.1607 (52)</td>
<td>0.1811 (46)</td>
</tr>
<tr>
<td></td>
<td>0.0690 (30)</td>
<td>0.0710 (22)</td>
<td>0.0727 (18)</td>
</tr>
<tr>
<td></td>
<td>0.1308 (70)</td>
<td>0.1571 (51)</td>
<td>0.1787 (47)</td>
</tr>
<tr>
<td></td>
<td>0.0739 (34)</td>
<td>0.0741 (19)</td>
<td>0.0748 (19)</td>
</tr>
</tbody>
</table>

$Z_A$. Indeed, this is the quantity that renormalizes with the same renormalization constant $Z_\mu = 1/Z_P$ of the bare twisted mass $a\mu$ in the twisted formulation of LQCD.\(^3\)

From the plots in Figs. 1 and 2 we observe that:

- There is a very good agreement among the results for the pseudoscalar meson masses calculated using various quark actions, all of which are $O(a)$-improved. This may signal the smallness of discretization effects on this quantity at the value of the lattice spacing considered in the present study. Further investigations at different lattice spacings could better clarify this point.

- IRD cutoff effects are clearly visible in the pseudoscalar meson decay constant at low quark masses ($a\mu \lesssim 0.02 \sim a^2\Lambda_{\text{QCD}}^2$) when computed by using Mtm-Wilson fermions with the $m_{\text{cr}}^\text{pion}$ determination of the critical mass. This is the bending phenomenon observed in Ref. [13]. As shown in Fig. 2, these effects are strongly reduced either by choosing the optimal value of $m_{\text{cr}}$, or by introducing the Clover term in the quark action. The reduction of IRD effects is almost the same within the two approaches. This is the main result of the present study.

\(^3\)It should be also mentioned that the bare twisted quark mass $a\mu$ with Mtm-Clover fermions renormalizes differently from its counterpart with Mtm-Wilson fermions. However, the ratio of the corresponding renormalization constants, $Z_P^\text{Clover} / Z_P^\text{Wilson}$, is very close to unity (it is 0.97 when using the one-loop boosted perturbative expressions).
• As shown in Fig. 1, IRD cutoff effects are negligible in the case of the pseudoscalar meson mass. This is consistent with the observation that IRD cutoff effects in the determination of this mass are softened by an additional factor of $M_P^2$ [8].

• The results for the pseudoscalar meson masses obtained with the Mtm-Clover action and the $m_{cr}^{\text{pion}}$ determination of the critical mass are not shown in Fig. 1, because they are practically indistinguishable from those obtained with $m_{cr}^{\text{PCAC}}$ (cf. Table 1). In the case of the decay constant, instead, the two Mtm-Clover determinations with different choices of $m_{cr}$ show a small spread at very low quark masses. This may signal the contribution of higher-order IRD cutoff effects that we cannot cure.

For illustrative purposes we quote the results for the pion decay constants obtained by quadratically extrapolating the lattice data to the physical pion mass. We obtain $f_\pi = \{138 (8), 144 (9), 138 (8)\}$ MeV in simulations a), b) and c) respectively, to be compared with the experimental value $f_\pi = 132$ MeV.

3 More numerical results

3.1 The renormalization constant $Z_V$

A non-perturbative determination of the renormalization constant $Z_V$ can be obtained using the two-point AWI of Eq. (5) as

$$Z_V^{-1} = \frac{(C_{AP}(t + 1) - C_{AP}(t - 1))/2}{2 \mu C_{PP}(t)},$$

where in the numerator we used the symmetric ($O(a)$)-improved version of the lattice time derivative.

We have computed $Z_V$ from Eq. (7) using both the Wilson and the Clover action at maximal twist. In the Clover case, the results obtained from simulations a) and b), which only differ by the choice of the critical mass, are in good agreement within the statistical errors. For this reason, only data from simulation a) and c) will be discussed below. Our results for $Z_V$ are shown in Fig. 3 as a function of the bare twisted quark mass.

The dependence of $Z_V$ on the quark mass is induced by discretization effects, which are predicted to be of $O(a^2)$ at maximal twist. Thus one can expect both

---

4In this paper we follow the standard practice to denote with $Z_V$ the renormalization constant of the axial-vector current in Mtm-LQCD. The reason is that the axial-vector current corresponds to the vector current in the twisted basis, which also implies that $Z_V$ coincides with the vector current renormalization constant of the standard non-twisted theory.
a quadratic dependence on the quark mass, due to contributions of $\mathcal{O}(a^2\mu^2)$, and a linear dependence induced by terms of $\mathcal{O}(a^2\Lambda_{QCD}\mu)$.

From the results shown in Fig. 3, it can be clearly seen that terms of $\mathcal{O}(a^2\Lambda_{QCD}\mu)$ are dominant in the Mtm-Clover case, while the quadratic contributions of $\mathcal{O}(a^2\mu^2)$ are larger in the Wilson case, except that at very small values of the quark mass.

In both the Wilson and Clover cases, we can correct for mass-dependent discretization effects by extrapolating the results for $Z_V$ to the chiral limit. In this way, we obtained the estimates

$$Z_V^{(\text{Wilson})} = 0.643(1), \quad Z_V^{(\text{Clover})} = 0.768(1).$$

Our result for $Z_V^{(\text{Wilson})}$ agrees with the one reported in Ref. [22], namely $Z_V^{(\text{Wilson})} = 0.6424(4)$.

The determinations in Eq. (8) can be also compared to the estimates of $Z_V$ obtained in the corresponding non-twisted theories (for which $\mu = 0$), since renormalization constants are independent of the quark mass. For the standard Clover action, non-perturbative determinations of $Z_V$ have been obtained using several

---

Footnote:\footnote{We thank R. Frezzotti and G. Rossi for having drawn our attention to this point.}
approaches. In Ref. [16], by using the RI-MOM method the estimate $Z_{V}^{\text{Clover}} = 0.772(2)(2)$ has been obtained, in good agreement with our result in Eq. (8) (this comparison is also illustrated in Fig. 3). Similar estimates have been obtained with the other approaches [23–25]. The determinations of $Z_{V}$ with standard Wilson fermions are affected, instead, by much larger uncertainties, since the action is not improved at $\mathcal{O}(a)$. Typically, the non-perturbative estimates lie in the range $Z_{V}^{\text{Wilson}} \simeq 0.6 \div 0.8$ [25–27]. At variance with these estimates, however, we remind that our determination in Eq. (8) is improved at $\mathcal{O}(a)$. Another interesting comparison is provided by the estimates $Z_{V}^{(\text{Wilson})} \simeq 0.63$ and $Z_{V}^{(\text{Clover})} \simeq 0.81$ obtained using one-loop tadpole-improved boosted perturbation theory [19, 20].

3.2 Light quark masses

In the quenched approximation, one of the main advantages of tmLQCD is the possibility to perform simulations at small values of the quark masses, without encountering the problem of exceptional configurations. This can be very beneficial for the extrapolations towards the chiral limit. A notable example, in this respect, is the calculation of the light quark masses and, in particular, of the average up/down quark mass.

Moreover, the determination of quark masses with Mtm-LQCD is simpler than with the standard Wilson and/or Clover actions. To emphasize this point we remind the reader that the renormalized quark mass in Mtm-LQCD is obtained from the twisted mass parameter $\mu$ as $\hat{m}_{q} = Z_{\mu} \mu$, with $Z_{\mu} = 1/Z_{P}$. Thus $\mu$ plays a role similar to the bare $m_{\text{AWI}}$ in the standard, non-twisted, case. But contrary to $m_{\text{AWI}}$, which is obtained from the ratio of the correlation functions and thus suffers from both statistical and systematic errors, the bare twisted mass $\mu$ is known without uncertainties. In addition to the automatic $\mathcal{O}(a)$-improvement of the results obtained by using either the Mtm-Wilson or the Mtm-Clover action, this makes the calculation of the light quark mass particularly attractive.

To exemplify the benefits of using the Mtm-LQCD we now apply the same strategy explained in detail in Ref. [21], and determine the values of the strange and the average up/down quark masses. Concerning the renormalization constant, in the Mtm-Clover case we use the non-perturbative estimate $Z_{P}^{\text{RI/MOM}}(2 \text{ GeV}) = 0.525$ [16], which corresponds to $Z_{P}^{\text{MTS}}(2 \text{ GeV}) = 0.624$. In the Mtm-Wilson case, instead, we will use the one-loop boosted perturbative result, namely $Z_{P}^{\text{MTS}}(2 \text{ GeV}) = \ldots$

\[ \ln(P) = -\frac{1}{3} \hat{g}^{2} (3.40/a) \left[ 1 - 1.1905 \frac{\hat{g}^{2} (3.40/a)}{4\pi} \right] \]  \hspace{1cm} (9)
The physical values of strange and of the average up/down quark masses have been extracted by fitting the results for the pseudoscalar meson masses collected in Table 1 as a function of the twisted quark mass. We find that these results are consistent with a quadratic dependence of the form

\[(aM_P)^2 = P_1 + P_2(a\mu) + P_3(a\mu)^2.\]  

(10)

The coefficient \(P_1\) parameterizes small, though non-vanishing, \(O(a^2)\)-discretization effects: we obtain \(P_1 = \{6(2), 5(2), 3(2)\} \cdot 10^{-3}\) in simulations a), b) and c) respectively. In the range of our simulated quark masses we do not observe any deviation from the simple quadratic dependence of \(M_P^2\) on the quark masses which could signal the contribution of non analytical chiral logs. In Fig. 4 we show that the Mtm-LQCD indeed allows a much better control over the extrapolation to the physical pion mass than it was the case with the standard non-twisted Wilson/Clover action.

Our final results for the average up/down and strange quark masses are collected in Table 2. The determinations obtained with the various Mtm-LQCD actions are consistent with each other and in agreement with most of the existing quenched estimates available in literature (for a recent review see e.g. Ref. [28]).

### 3.3 Isospin breaking effects

The twisted mass formulation of LQCD breaks both parity and flavour symmetries at finite lattice spacing. At maximal twist, these breakings are expected to be of \(O(a^2)\).

A clear manifestation of isospin breaking effects is a difference between the neutral and the charged pion mass. In the Mtm-Wilson case, this splitting has been investigated in Ref. [7]. The main observations are: i) isospin breaking effects are numerically large, even for reasonably fine lattice spacings; quantitatively, it is found \(r_0^2(M_{\pi^0}^2 - M_{\pi^+}^2) = c(a/r_0)^2\) with \(c \approx 10\). ii) The contribution of disconnected diagrams helps in significantly reducing this splitting; without adding

<table>
<thead>
<tr>
<th>Mtm-LQCD</th>
<th>(m_\ell) (MeV)</th>
<th>(m_s) (MeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mtm-Clover, (m_{cr}^{\text{PCAC}})</td>
<td>4.3 (2)</td>
<td>105 (5)</td>
</tr>
<tr>
<td>Mtm-Clover, (m_{cr}^{\text{pion}})</td>
<td>4.2 (2)</td>
<td>103 (5)</td>
</tr>
<tr>
<td>Mtm-Wilson, (m_{cr}^{\text{opt}})</td>
<td>4.2 (2)</td>
<td>103 (6)</td>
</tr>
</tbody>
</table>

Table 2: Values of the average up/down \((m_\ell)\) and of the strange \((m_s)\) quark masses renormalized in the \(\overline{MS}\) scheme at the scale of 2 GeV as obtained using the various Mtm-LQCD actions at \(\beta = 6.0\).
Fig. 4: Pseudoscalar meson masses squared, in physical units, as a function of the quark mass $\overline{m}_q$ renormalized in the $\overline{MS}$ scheme at the scale $\mu = 2$ GeV. The dashed line represents the result of the fit to Eq. (10). In the $M_P^2$ values shown in the plot the pure lattice artifact contribution represented by the coefficient $P_1$ in Eq. (10) has been subtracted. Solid lines indicate the physical values of the pion and kaon masses. The results obtained with the non-twisted Clover action in Ref. [21] are also shown for comparison.

the disconnected piece, the value of $c$ is about twice bigger. iii) The optimal choice of the critical mass, which minimizes parity breaking effects, does not help in reducing discretization effects related to isospin breaking.

We cannot make the study along the lines presented in Ref. [7], since we explore the benefits of using the Mtm-Clover action at single lattice spacing. Furthermore, we neglect the contribution of disconnected diagrams, the calculation of which is computationally expensive and statistically noisy. One observes that the pure connected contribution provides by itself the neutral-charged mass splitting between pseudoscalar mesons composed by different, though mass-degenerate, isospin doublets, namely $P^i = \overline{q}_i \frac{\gamma_5}{2} q_j$ with $q \neq q'$. In the absence of isospin breaking effects the neutral and charged mesons would be degenerate, and thus their splitting still represents a measure of isospin breaking discretization effect of $O(a^2)$.

In Fig. 5 we compare the results for the neutral-charged mass splitting obtained by using the Mtm-Wilson and the Mtm-Clover actions, with the optimal and the
Fig. 5: Relative mass splitting between charged and neutral pseudoscalar mesons as obtained by using the Mtm-Wilson and the Mtm-Clover actions. The results are presented as a function of the bare quark mass. For both actions, the calculation of the neutral meson mass only includes the contribution of connected diagrams (see text for explanations).

PCAC determination of the critical mass respectively. In the Mtm-Clover case, the results obtained with the two different choices for the critical mass ($m_{\text{PCAC}}$ and $m_{\text{pion}}$) do not show significant differences. From Fig. 5, it appears that in the Mtm-Clover case the isospin breaking effects are significantly reduced with respect to the situation in which the Mtm-Wilson action is used, at least for the pure connected contribution considered in this study and in the quenched approximation.

4 Conclusions

In this paper we have presented an exploratory numerical study of Mtm-LQCD with the Clover term in the action. We have performed quenched simulations with both Wilson and Clover twisted fermions at $\beta = 6.0$, with pseudoscalar meson masses ranging from about 1.1 GeV down to approximately 280 MeV. We have investigated, in particular, the role of the Clover term in reducing the large (IRD) cutoff effects which affect, in general, simulations with Mtm-LQCD when the quark
mass becomes $O(a\Lambda_{QCD}^2)$. As anticipated in Ref. [8], we find that adding the Clover term to the twisted mass quark action is very effective in reducing these effects, as much as using Mtm-Wilson fermions with the optimal choice of the critical mass.

By using Mtm-Clover fermions, we have also performed other interesting numerical studies. We have computed the renormalization constant $Z_V$, finding good agreement with the estimates obtained with the standard Clover action. We have presented a calculation of the strange and the average up/down quark masses, which uses non-perturbative renormalization and much lighter valence quark masses that could not have been simulated with standard Wilson/Clover fermions, due to the problem of exceptional configurations. Finally, we have calculated the contribution of connected diagrams to the charged-neutral pion mass splitting, finding that the use of the Mtm-Clover action is also beneficial in reducing cutoff effects related to isospin breaking, at least for this pure connected contribution in the quenched approximation. We believe that all these results encourage to pursue this project in exploring the unquenched QCD dynamics at small quark masses.

**Acknowledgments**

We warmly thank R. Frezzotti, M. Papinutto, C. Pena and G. Rossi for useful discussions on the subject of this paper. We also thank the $\chi$LF Collaboration for having provided us with their numerical results obtained in the simulation with twisted Wilson fermions at maximal twist.

**References**


