Estimates for the two-dimensional Navier-Stokes equations in terms of the Reynolds number

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Abstract

The tradition in Navier-Stokes analysis of finding estimates in terms of the Grashof number $Gr$, whose character depends on the ratio of the forcing to the viscosity $\nu$, means that it is difficult to make comparisons with other results expressed in terms of Reynolds number $Re$, whose character depends on the fluid response to the forcing. The first task of this paper is to apply the approach of Doering and Foias [23] to the two-dimensional Navier-Stokes equations on a periodic domain $[0, L]^2$ by estimating quantities of physical relevance, particularly long-time averages $\langle \cdot \rangle$, in terms of the Reynolds number $Re = U\ell/\nu$, where $U^2 = L^{-2} \langle \|u\|_2^2 \rangle$ and $\ell$ is the forcing scale. In particular, the Constantin-Foias-Temam upper bound [1] on the attractor dimension converts to $a_\ell^2 Re (1 + \ln Re)^{1/3}$, while the estimate for the inverse Kraichnan length is $(a_\ell^2 Re)^{1/2}$, where $a_\ell$ is the aspect ratio of the forcing. Other inverse length scales, based on time averages, and associated with higher derivatives, are estimated in a similar manner. The second task is to address the issue of intermittency: it is shown how the time axis is broken up into very short intervals on which various quantities have lower bounds, larger than long time-averages, which are themselves interspersed by longer, more quiescent, intervals of time.

1 Introduction

1.1 General introduction

In the last two decades the notion of global attractors in parabolic partial differential equations has become a well-established concept [1, 2, 3, 4]. The general nature of the dynamics on the attractor $\mathcal{A}$, in a time averaged sense, can roughly be captured by identifying sharp estimates
of the Lyapunov (or fractal or Hausdorff) dimension of $A$, or the number of determining modes \[5\], with the number of degrees of freedom. Introduced by Landau \[6\], this latter idea says that in a dynamical system of spatial dimension $d$ of scale $L$, the number of degrees of freedom $N$ is roughly defined to be that number of smallest eddies or features of scale $\lambda$ and volume $\lambda^d$ that fit into the system volume $L^d$

$$N \sim \left( \frac{L}{\lambda} \right)^d.$$  \hspace{1cm} (1.1)

This is the origin of the much-quoted $N \sim \Re^{9/4}$ result associated with the three-dimensional Navier-Stokes equations which rests on taking $\lambda \sim \lambda_k \sim L\Re^{-3/4}$, where $\lambda_k$ is the Kolmogorov length scale. In the absence of a proof of existence and uniqueness of solutions of the three-dimensional Navier-Stokes equations, at best this is no more than a rule of thumb result. It rests on a more solid and rigorous foundation, however, for the closely related three-dimensional LANS-$\alpha$ equations for which Foias, Holm and Titi \[7\] have proved existence and uniqueness of solutions. Following on from this, Gibbon and Holm \[8\] have demonstrated that the dimension of the global attractor for this system has an upper bound proportional to $\Re^{9/4}$. An important milestone has been passed recently in another closely related problem with the establishment by Cao and Titi \[9\] of an existence and uniqueness proof for Richardson’s three-dimensional primitive equations for the atmosphere.

For the Navier-Stokes equations the idea sits more naturally in studies in the two-dimensional context. The existence and uniqueness of solutions has been a closed problem for many decades and the nature of the global attractor has been well-established [1-5, 10-14]. While the two- and three-dimensional equations have the same velocity formulation, in reality, the former have a tenuous connection with the latter because of the absence of the drastic property of vortex stretching. As a result, the presence of vortex stretching in three dimensions, and perhaps other more subtle properties, have set up seemingly unsurmountable hurdles even on periodic boundary conditions. For problems on non-periodic boundaries, such as lid-driven flow, solving the two-dimensional Navier-Stokes equations is a technically more demanding problem – see some references in [10] [15] [16].

The sharp estimate found by Constantin, Foias & Temam \[11\] for the Lyapunov dimension of
the global attractor $\mathcal{A}$ expressed in terms of the Grashof number $Gr$

$$d_L(\mathcal{A}) \leq c_1 Gr^{2/3} (1 + \ln Gr)^{1/3},$$

(1.2)

has been one of the most significant results in two-dimensional Navier-Stokes analysis on a
periodic domain $\Omega = [0, L]^2_{\text{per}}$. The traditional length scale in the two-dimensional Navier-
Stokes equations is the Kraichnan length, $\eta_k$, which plays an equivalent role in two dimensions
to that of the Kolmogorov length, $\lambda_k$, which is more important in three dimensions. In two
dimensions, $\eta_k$ and $\lambda_k$ are defined respectively in terms of the enstrophy and energy dissipation
rates $\epsilon_{\text{ens}}$ and $\epsilon$

$$\epsilon_{\text{ens}} = \nu L^{-2} \left\langle \int_{\Omega} | \nabla \omega |^2 dV \right\rangle, \quad \epsilon = \nu L^{-2} \left\langle \int_{\Omega} | \omega |^2 dV \right\rangle,$$

(1.3)

where the pair of brackets $\langle \cdot \rangle$ denote a long-time average defined as $[2,3,10-13].$

$$\langle g(\cdot) \rangle = \lim_{t \to \infty} \limsup_{g(0)} \frac{1}{t} \int_0^t g(\tau) d\tau.$$  

(1.4)

The inverse Kraichnan length $\eta_k^{-1}$ and the inverse Kolmogorov length $\lambda_k^{-1}$ are defined in terms
of $\epsilon_{\text{ens}}$ and $\epsilon$ as

$$\eta_k^{-1} = \left( \frac{\epsilon_{\text{ens}}}{\nu^3} \right)^{1/6}, \quad \lambda_k^{-1} = \left( \frac{\epsilon}{\nu^3} \right)^{1/4}.$$  

(1.5)

It has been shown by Constantin, Foias and Temam [11] that instead of using an estimate for $\epsilon_{\text{ens}}$ in terms of $Gr$, the upper bound for $d_L$ can be re-expressed in terms of $L\eta_k^{-1}$ (see other
literature on this topic [17,18,19])

$$d_L \leq c_2 \left( L\eta_k^{-1} \right)^2 \left\{ 1 + \ln \left( L\eta_k^{-1} \right) \right\}^{1/3}.$$  

(1.6)

If $d_L$ is identified with the number of degrees of freedom $\mathcal{N}$, this result is consistent with the idea
expressed in (1.1) that in a two-dimensional domain, the average length scale of the smallest
vortical feature $\lambda$ can be identified with the Kraichnan length $\eta_k$, to within log-corrections. The
result in (1.2) has also been improved by Foias, Jolly, Manley and Rosa [20, 21] to an estimate
proportional to $Gr^{1/2}$ (to within logarithmic corrections) provided Kraichnan’s theory of fully
developed turbulence is implemented [22].
While these results display a pleasing convergence between rigorous estimates and scaling methods in the two-dimensional case, the tradition in Navier-Stokes analysis of finding estimates in terms of the Grashof number $Gr$, whose character depends on the ratio of the forcing to the viscosity $\nu$, means that it is difficult to compare with the results of scaling theories whose results are expressed in terms of Reynolds number. One of the tasks of this paper is to estimate quantities of physical relevance, particularly long-time averages, in terms of the Reynolds number, whose character depends on the fluid response to the forcing, and which is intrinsically a property of Navier-Stokes solutions. Doering and Fojas \[23\] have addressed this problem and have shown that in the limit $Gr \to \infty$, solutions of the $d$-dimensional Navier-Stokes equations must satisfy

$$Gr \leq c (Re^2 + Re), \quad (1.7)$$

while the energy dissipation rate $\epsilon$ has a lower bound proportional to $Gr$. The problem, however, is not as simple as replacing standard estimates in terms of $Gr$ by $Re^2$ from (1.7). Estimates such as that for $d_L$ in (1.2) and the inverse Kraichnan and Kolmogorov lengths defined in (1.5), depend upon long time-averages of the enstrophy and energy dissipation rates defined in (1.3). Other estimates of inverse length scales (to be discussed in \[1.2\]) also depend upon long time-averages. When estimated in terms of $Re$ all these turn out to be better than straight substitution using (1.7). These results are summarized in \[1.2\] and worked out in detail in \[1.2\].

The second topic to be addressed in this paper is that of intermittency. Originally this important effect was considered to be a high Reynolds number phenomenon associated with three-dimensional Navier-Stokes flows. First discovered by Batchelor and Townsend \[24\], it manifests itself in violent fluctuations of very short duration in the energy dissipation rate $\epsilon$. These violent fluctuations away from the average are interspersed by quieter, longer periods in the dynamics. This is a well established, experimentally observable phenomenon \[25, 26, 27\]; its appearance in systems other than the Navier-Stokes equations has been discussed in an early and easily accessible paper by Frisch & Morf \[28\]. One symptom of its occurrence is the deviation of the ‘flatness’ of a velocity signal (the ratio of the 4th order moment to the square of the 2nd

\footnote{This result is not advertised in \[23\] but follows immediately from their equation (48).}
order moment) from the value of 3 that holds for Gaussian statistics.

Recent analysis discussing intermittency in three-dimensional Navier-Stokes flows shows that while it may be connected with loss of regularity, the two are subtly different issues [29]. This is reinforced by the fact that although solutions of the two-dimensional Navier-Stokes equations remain regular for arbitrarily long times, nevertheless many of its solutions at high \(Re\) are known to be intermittent \([30,31,32,33,34]\). While three-dimensional analysis of the problem is based on the assumption that a solution exists \([29,35]\), so that the higher norms can be differentiated, no such assumption is necessary in the two-dimensional case where existence and uniqueness are guaranteed. The result in both dimensions is such that the time-axis is broken up into good and bad intervals: on the latter there exist large lower bounds on certain quantities, necessarily resulting in their extreme narrowness and thus manifesting themselves as spikes in the data. This is summarized in \(\S 1.2\) and worked out in detail in \(\S 4\).

### 1.2 Summary and interpretation of results

For simplicity the forcing \(f(x)\) in the two-dimensional Navier-Stokes equations (\(\text{div} \ u = 0\))

\[
\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = \nu \Delta \mathbf{u} - \nabla p + f(x)
\]

(1.8)
is taken to be divergence-free and smooth of narrow-band type, with a characteristic single length-scale \(\ell\) such that \([23,29,35]\)

\[
\|\nabla^n f\|_2 \approx \ell^{-n} \|f\|_2.
\]

(1.9)

Moreover, the aspect ratio of the forcing length scale to the box scale is defined as

\[
a_\ell = L/\ell.
\]

(1.10)

With \(f_{\text{rms}} = L^{-d/2}\|f\|_2\), the usual definition of the Grashof number \(Gr\) appearing in (1.7) in \(d\)-dimensions is

\[
Gr = \frac{\ell^3 f_{\text{rms}}}{\nu^2}.
\]

(1.11)

The Reynolds number \(Re\) in (1.7) is defined as

\[
Re = \frac{U \ell}{\nu}, \quad U^2 = L^{-d} \langle \|\mathbf{u}\|_2^2 \rangle,
\]

(1.12)
where \( \langle \cdot \rangle \) is the long-time average defined in (1.4). One of the main results of this paper is the following theorem whose proof is given in §2.1. All generic constants are designated as \( c \).

**Theorem 1.1.** Let \( u(x, t) \) be a solution of the two-dimensional Navier-Stokes equations (1.8) on a periodic domain \([0, L]^2\), and subject to smooth, divergence-free, narrow-band forcing \( f(x) \). Then estimates in terms of the Reynolds number \( Re \) and the aspect ratio \( \alpha \ell \) for the inverse Kraichnan length \( \eta_k^{-1} \), the attractor dimension \( d_L \), and the inverse Kolmogorov length \( \lambda_k^{-1} \) are given by

\[
L \eta_k^{-1} \leq c (\alpha^2 \ell Re)^{1/2},
\]

\[
d_L \leq c a^2 \ell Re [1 + \ln Re]^{1/3},
\]

\[
L \lambda_k^{-1} \leq c a \ell Re^{5/8}.
\]

In the short proof of this theorem in §2.1 the estimate for \( d_L \) in (1.14) is not re-worked from first principles but is derived from a combination of (1.13) and (1.14). The result in (1.15) comes from a \( Re^{5/2} \) bound on \( \langle H_1 \rangle \) and has also recently been found by Alexakis and Doering [36]. It implies that

\[
\frac{L}{U^3} \leq c a \ell Re^{-1/2},
\]

whereas in three-dimensions the right hand side is \( O(1) \). The estimate in (1.14) is also consistent with the result of Foias, Jolly, Manley and Rosa [20] when their \( Gr^{1/2} \) estimate is converted to one proportional to \( Re \). Their estimate, however, was based on the implementation of certain features of the Kraichnan model [22], while (1.14) is true for all solutions and requires no assumption of fully developed turbulence.

The estimates for \( \eta_k^{-1} \) and \( d_L \) are consistent with the long-standing belief that \( Re^{1/2} \times Re^{1/2} \) grid points are needed to numerically resolve a flow; indeed, when the aspect ratio is taken into account, Theorem 1.1 is consistent with \( a \ell Re^{1/2} \times a \ell Re^{1/2} \). However, both these estimates are dependent upon only the time average of low moments of the velocity field. For non-Gaussian flows, low-order moments are not sufficient to uniquely determine the statistics of a flow. Thus it
is necessary to find ways of estimating small length scales associated with higher-order moments. In §2.2 we follow the way of defining inverse length scales associated with derivatives higher than two, introduced elsewhere [18], by combining the forcing with higher derivatives of the velocity field such that

\[
F_n = \int_{\Omega} (|\nabla^n u|^2 + \tau^2|\nabla^n f|^2) \, dV, \tag{1.17}
\]

where \( \tau = \ell^2 \nu^{-1} (G r (1 + \ln G r))^{-\frac{3}{4}} \) is a characteristic time: this choice of \( \tau \) is discussed in Appendix A. The gradient symbol \( \nabla^n \) within (1.17) refers to all derivatives of every component of \( u \) of order \( n \) in \( L^2(\Omega) \). The \( F_n \) are used to define a set of time-dependent inverse length scales

\[
\kappa_{n,r}(t) = \left( \frac{F_n}{F_r} \right)^{\frac{1}{2(n-r)}}. \tag{1.18}
\]

Actually, \( \kappa_{n,0}^{2n} \) behaves as the \( 2n \)-th-moment of the energy spectrum as shown by

\[
\kappa_{n,0}^{2n} = \frac{\int_{2\pi/L}^{\infty} k^{2n} (|\hat{u}|^2 + \tau^2|\hat{f}|^2) \, dV_k}{\int_{2\pi/L}^{\infty} (|\hat{u}|^2 + \tau^2|\hat{f}|^2) \, dV_k}. \tag{1.19}
\]

More relevant to the two-dimensional case, \( \kappa_{n,1}^{2(n-1)} \) behaves as the \( 2(n-1) \)-th-moment of the enstrophy spectrum. Using Landau’s argument the dimension of the global attractor \( d_L(A) \) was identified with the number of degrees of freedom \( N \). In [19] a definition was introduced to represent the number of degrees of freedom associated with all higher derivatives of the velocity field represented by \( \kappa_{n,r} \), which is itself an inverse length. This naturally leads to the definition of the infinite set

\[
N_{n,r} = L^2 \langle \kappa_{n,r}^2 \rangle. \tag{1.20}
\]

Using the definition of the quantities \( \Lambda_{n,0} \) and \( \Lambda_{n,1} \) (\( n \geq 2 \))

\[
\Lambda_{n,0} = \frac{3n - 2}{2n}, \quad \Lambda_{n,1} = \frac{3n - 4}{2(n - 1)}, \tag{1.21}
\]

the second main result of the paper is a theorem whose proof is given in §2.2:

**Theorem 1.2.** Let \( \kappa_{n,r} \) be the moments of a two-dimensional Navier-Stokes velocity field defined in (1.18). Then in a two-dimensional periodic box of side \( L \) the numbers of degrees of freedom \( N_{n,1} \) and \( N_{n,0} \) defined in (1.20) are estimated as \( n \geq 2 \)

\[
N_{n,1} \leq c_{n,1} (a^2 \ell^2 Re)^{\Lambda_{n,1}} (1 + \ln Re)^{1/2}, \tag{1.22}
\]

where \( \ell = \tau^2 \nu^{-1} (G r (1 + \ln G r))^{-\frac{3}{4}} \)
\[ \mathcal{N}_{n,0} \leq c_{n,0}(a_1^2 Re)^{\Lambda_{n,0}} (1 + \ln Re)^{1/2} \]  

(1.23)

where \( \Lambda_{n,0} \) and \( \Lambda_{n,1} \) are defined in (1.21).

Note that \( \Lambda_{2,0} = \Lambda_{2,1} = 1 \). Thus the estimate for the first in each sequence, \( \mathcal{N}_{2,1} \) and \( \mathcal{N}_{1,0} \), are of the same order as the estimate for \( d_L \), namely \( a_1^2 Re (1 + \ln Re)^{1/3} \) except in the exponent of the logarithm. The exponents in (1.22) and (1.23) provide an estimate of the extra resolution that is needed to take account of energy at sub-Kraichnan scales. Notice that in the limit \( n \to \infty \) both exponents converge to 3/2.

The intermittency results of §4 show that there can exist small intervals of time where there are large lower bounds on \( \kappa_{n,1}^2 \) that are much larger than the upper bound on the long-time average for \( \langle \kappa_{n,1}^2 \rangle \). Translated into pictorial terms, Figure 1 in §4 is consistent with the existence of spiky data whose duration must be very short. Estimates are found for the width of these spikes which turn out to be in terms of a negative exponent of \( Re \).

2 Time average estimates in terms of \( Re \)

2.1 Proof of Theorem 1.1

The first step in the proof of Theorem 1.1, which has been expressed in §1.2, is to find an upper bound on \( \langle H_2 \rangle \) in terms of \( Re \). Consider the equation for the two-dimensional Navier-Stokes vorticity \( \omega = \omega \hat{k} \)

\[ \frac{\partial \omega}{\partial t} + u \cdot \nabla \omega = \nu \Delta \omega + \text{curl } f, \]  

(2.1)

and let \( H_n \) be defined by \( (n \geq 0) \)

\[ H_n = \int_{\Omega} |\nabla^n u|^2 dV. \]  

(2.2)

For a periodic, divergence-free velocity field \( u \)

\[ H_1 = \int_{\Omega} |\nabla u|^2 dV = \int_{\Omega} |\omega|^2 dV. \]  

(2.3)
Then the evolution equation for $H_1$ is

\[
\frac{d}{dt} H_1 = -\nu H_2 + \int_\Omega \omega \cdot \text{curl} f \, dV \quad (2.4)
\]

\[
\leq -\nu H_2 + \|u\|_2 \|\nabla^2 f\|_2 \quad (2.5)
\]

\[
\leq -\nu H_2 + \ell^{-2} \|u\|_2 \|f\|_2, \quad (2.6)
\]

where the forcing term has been integrated by parts in (2.4) and the narrow-band property has been used to move from (2.5) to (2.6). Using the definitions of $Re$, $Gr$, and $a_\ell$ in (1.12), (1.11) and (1.10), the long-time average of $H_2$ is estimated as

\[
\langle H_2 \rangle \leq L^2 \ell^{-6} \nu^2 Re Gr \quad (2.7)
\]

\[
\leq c a_\ell^2 \ell^{-4} \nu^2 Re^3 + O(Re^2). \quad (2.8)
\]

This holds the key to the three results in Theorem 1.1.

The inverse Kraichnan length $\eta_k^{-1} = \epsilon_{\text{ens}}/\nu^3$ with $\epsilon_{\text{ens}} = \nu L^{-2} \langle H_2 \rangle$, can now be estimated by noting that

\[
L^6 \epsilon_{\text{ens}} \leq c a_\ell^6 \nu^3 Re^3 \quad (2.9)
\]

and so

\[
L \eta_k^{-1} \leq c (a_\ell^2 Re)^{1/2}, \quad (2.10)
\]

which is (1.13) of Theorem (1.1). The estimate for $d_L$ in (1.14) then follows immediately from the relation between the estimate for $d_L$ in (1.6) and (2.10).

Finally, we turn to proving the estimate for $\langle H_1 \rangle$ in (1.15) which turns around the use of the simple inequality $H_1^2 \leq H_2 H_0$. The next step is to use the fact that

\[
\langle H_1 \rangle \leq \langle H_2 \rangle^{1/2} \langle H_0 \rangle^{1/2} \quad (2.11)
\]

\[
= \nu a_\ell Re \langle H_2 \rangle^{1/2}. \quad (2.12)
\]

Using the upper bound in (2.7) gives

\[
\langle H_1 \rangle \leq c \nu^2 a_\ell^2 \ell^2 Re^{5/2}, \quad (2.13)
\]

which then gives (1.15) in Theorem (1.1). In fact, (2.13) is an improvement in the bound for $\langle H_1 \rangle$ from $Re^3$ to $Re^{5/2}$. This result has also been found recently by Alexakis and Doering [35].
2.2 Proof of Theorem 1.2

Having introduced the notation for $H_n$ in (2.2), similar quantities are used that contain the forcing $[35, 29]$, namely

$$F_n = \int_\Omega \left( |\nabla^nu|^2 + \tau^2 |\nabla^nf|^2 \right) dV, \quad (2.14)$$

defined first in (1.17), and the moments $\kappa_{n,r}$ defined in (1.18)

$$\kappa_{n,r}(t) := \left( \frac{F_n}{F_r} \right)^{\frac{1}{n-r}}. \quad (2.15)$$

The parameter $\tau$ in (2.14) is a time scale and needs to be chosen appropriately. The idea is that it should be chosen in such a way that the forcing does not dominate the behavior of the moments of the velocity field. Defining $\omega_0 = \ell^{-2}\nu$, it is shown in Appendix A that this end is achieved if $\tau^{-1}$ is chosen as

$$\tau^{-1} = \omega_0[Gr(1 + \ln Gr)]^{1/2} \leq c\omega_0Re(1 + \ln Re)^{1/2}. \quad (2.16)$$

As a preliminary to the proof of Theorem 1.2, we state the ladder theorem proved in [35, 29].

**Theorem 2.1.** The $F_n$ satisfy the differential inequalities

$$\frac{1}{2} \dot{F}_0 \leq -\nu F_1 + c \tau^{-1} F_0, \quad (2.18)$$

$$\frac{1}{2} \dot{F}_1 \leq -\nu F_2 + c \tau^{-1} F_1, \quad (2.19)$$

and, for $n \geq 2$, either

$$\frac{1}{2} \dot{F}_n \leq -\nu F_{n+1} + c_{n,1} \left( \|\nabla u\|_\infty + \tau^{-1} \right) F_n, \quad (2.20)$$

or

$$\frac{1}{2} \dot{F}_n \leq -\frac{1}{2} \nu F_{n+1} + c_{n,2} \left( \nu^{-1} \|u\|_\infty^2 + \tau^{-1} \right) F_n. \quad (2.21)$$

The $L^\infty$-inequalities in Theorem 2.1, particularly $\|\nabla u\|_\infty$ in (2.20), can be handled using a modified form of the $L^\infty$-inequality of Brezis and Gallouet that has already been proved in [18]:
Lemma 2.1. In terms of the $F_n$ of (2.14) and $\kappa_{3,2}$ of (2.15), a modified form of the two-dimensional $L^\infty$-inequality of Brezis and Gallouet is
\[
\|\nabla u\|_\infty \leq c F_2^{1/2} [1 + \ln(L\kappa_{3,2})]^{1/2}.
\] (2.22)

This lemma directly leads to an estimate for $\langle \kappa_{n,r}^2 \rangle$ for $r \geq 2$.

Lemma 2.2. For $n > r \geq 2$, to leading order in $Re$,
\[
L^2 \langle \kappa_{n,r}^2 \rangle \leq c (a_r^2 Re)^{3/2} (1 + \ln Re)^{1/2}.
\] (2.23)

Proof: By dividing (2.20) by $F_n$ and time averaging, we have
\[
\nu \langle \kappa_{n+1,n}^2 \rangle \leq c_{n,1} (\|\nabla u\|_\infty) + c \omega_0 Re (1 + \ln Re)^{1/2}.
\] (2.24)

However, because $\kappa_{n,r} \leq \kappa_{n+1,n}$ for $r < n$, for every $2 \leq r < n$, in combination with Lemma 2.1, we have
\[
\nu \langle \kappa_{n,r}^2 \rangle \leq c \left( F_2^{1/2} [1 + \ln(L\kappa_{3,2})]^{1/2} \right) + c \omega_0 Re (1 + \ln Re)^{1/2}.
\] (2.25)

The logarithm is a concave function and $\kappa_{3,2} \leq \kappa_{n,r}$ so Jensen’s inequality gives
\[
L^2 \langle \kappa_{n,r}^2 \rangle \leq L^2 \nu^{-1} c \left( F_2 \right)^{1/2} \left( [1 + \ln\{L^2 \langle \kappa_{n,r}^2 \rangle\}]^{1/2} \right) + c a_r^2 Re (1 + \ln Re)^{1/2}.
\] (2.26)

The estimate for $\langle F_2 \rangle$ can be found from $\langle H_2 \rangle$ in (2.4); the extra term $\tau^2 \|\nabla^2 f\|^2_2$ is no more than $O(Re^2)$. Standard properties of the logarithm turn inequality (2.26) into (2.23). ■

Lemma 2.2 gives estimates for $\langle \kappa_{n,r}^2 \rangle$ for $r \geq 2$. These are used in the following theorem to give better estimates for the cases $r = 0$ and $r = 1$. Prior to this, it is necessary to state the results that immediately derive from (2.18) and (2.19) by respectively dividing through by $F_0$ and $F_1$ before time averaging
\[
N_{1,0} \equiv L^2 \langle \kappa_{1,0}^2 \rangle \leq c a_0^2 Re (1 + \ln Re)^{1/2}, \quad N_{2,1} \equiv L^2 \langle \kappa_{2,1}^2 \rangle \leq c a_1^2 Re (1 + \ln Re)^{1/2}.
\] (2.27)

With the estimates in (2.27), we are now ready to complete the proof of Theorem 1.2.

Proof of Theorem 1.2: Let us return to (2.28) in Lemma 2.2 and use the fact that
\[
\langle \kappa_{n,1}^2 \rangle = \left( \frac{F_n}{F_2} \right)^{1/2} \left( \frac{F_0}{F_1} \right)^{1/2} = \langle \kappa_{n,2}^{m-2} \kappa_{2,1}^{2} \rangle,
\] (2.28)
and thus
\[ \langle \kappa_{n,1}^2 \rangle \leq \langle \kappa_{n,2}^2 \rangle \frac{\kappa_{2,1}^2}{n-2} \langle \kappa_{2,1}^2 \rangle \frac{1}{n-1}. \] (2.29)

Using (2.23) in Lemma 2.2 together with (2.27), for \( n \geq 2, \)
\[ N_{n,1} = L^2 \langle \kappa_{n,1}^2 \rangle \leq c_{n,1} \left( a_\ell^2 R e \right) \frac{2n-4}{(n-1)} \left[ 1 + \ln Re \right]^{1/2}, \] (2.30)
which coincides with \( a_\ell^2 Re (1 + \ln Re)^{1/2} \) at \( n = 2 \) but converges to \( Re^{3/2} (1 + \ln Re)^{1/2} \) as \( n \to \infty. \)
The exponent \( \Lambda_{n,1} \) is defined in (1.21).

Likewise, in the same manner as (2.28) we have
\[ \langle \kappa_{n,0}^2 \rangle \leq \langle \kappa_{n,1}^2 \rangle \frac{n-1}{n} \langle \kappa_{1,0}^2 \rangle \frac{1}{n}. \] (2.31)
Thus we find that for \( n \geq 2 \)
\[ N_{n,0} = L^2 \langle \kappa_{n,0}^2 \rangle \leq c_{n,0} \left( a_\ell^2 R e \right) \frac{4n-2}{2n} \left[ 1 + \ln Re \right]^{1/2}. \] (2.32)
The exponent \( \Lambda_{n,0} \) is defined in (1.21).

3 Point-wise Estimates

Let us consider the differential inequalities for \( H_0 \) and \( H_1: \)
\[ \frac{1}{2} \dot{H}_0 \leq -\nu H_1 + \| f \|_2 H_0^{1/2}, \] (3.1)
\[ \frac{1}{2} \dot{H}_1 \leq -\nu H_2 + \ell^{-2} \| f \|_2 H_0^{1/2}, \] (3.2)
having used the narrow-band property on (3.2). Upon combining Poincaré’s inequality with
Lemmas 13.1 and 13.2 in Appendix B we obtain
\[ \lim_{t \to \infty} H_0 \leq ca_\ell^2 \nu^2 Gr^2 \leq ca_\ell^2 \nu^2 Re^4, \] (3.3)
and
\[ \lim_{t \to \infty} H_1 \leq c \ell^{-2} a_\ell^2 \nu^2 Gr^2 \leq c \ell^{-2} a_\ell^2 \nu^2 Re^4. \] (3.4)
The additive forcing terms in \( F_1 \) and \( F_0 \) are of a lower order in \( Re \) so we end up with
\[ \lim_{t \to \infty} F_0 \leq ca_\ell^6 \nu^2 Re^4 + O(Re^2), \] (3.5)
\[ \lim_{t \to \infty} F_1 \leq c \ell^{-2} a_\ell^6 \nu^2 \text{Re}^4 + O(\text{Re}^2). \]  

(3.6)

The estimate for \( F_1 \) enables us to obtain point-wise estimates on \( F_n, n \geq 2 \) [18 sec. 7.2]. In fact we have the following lemma.

**Lemma 3.1.** As \( Gr \to \infty \)

\[ \lim_{t \to \infty} F_n \leq c_n \nu^2 \ell^{-2n} a_\ell^{6n} \text{Re}^{4n}. \]  

(3.7)

**Proof:** Applying a Gagliardo–Nirenberg inequality in two-dimensions to \( \nabla u \) we obtain

\[ \| \nabla u \|_\infty \leq c \| \nabla^n u \|_2^a \| \nabla u \|_2^{1-a} \leq c F_n^{\frac{a}{2}} F_1^{\frac{1-a}{2}}, \]  

(3.8)

with \( a = \frac{1}{n-1} \). Using this in (2.20) gives

\[ \frac{1}{2} \dot{F}_n \leq -\nu F_{n+1} + c_n F_n^{1+\frac{a}{2}} F_1^{\frac{1-a}{2}} + c \omega_0 \text{Re}(1 + \ln \text{Re})^{1/2} F_n. \]  

(3.9)

Moreover the following inequality can easily be proved using Fourier transforms

\[ F_{N+p}^{p+q} \leq F_N^q F_{N-p}^p F_{N+q}, \]  

(3.10)

from which, with \( N = n, p = n-1, q = 1 \), it can be deduced that

\[ -F_{n+1} \leq -\frac{F_{n-1}^n}{F_1^{n-1}}. \]  

(3.11)

We now use (3.11) in (3.9) to obtain

\[ \frac{1}{2} \dot{F}_n \leq -\nu \frac{F_{n-1}^n}{F_1^{n-1}} + c_n F_n^{1+\frac{a}{2}} F_1^{\frac{1-a}{2}} + c \omega_0 \text{Re}(1 + \ln \text{Re})^{1/2} F_n, \]  

(3.12)

with \( a = \frac{1}{n-1} \). We use now estimate (3.6) in (3.12) with the further use of Lemma B.2 to obtain

\[ \lim_{t \to \infty} F_n \leq c_n \nu^2 \ell^{-2n} a_\ell^{6n} Gr^{2n}, \]  

(3.13)

which leads to the result.

The above Lemma enables us to obtain an estimate on the wave-numbers \( \kappa_{n,r} \).
Lemma 3.2. For \( n > r \geq 0 \), as \( Gr \to \infty \)

\[
\lim_{t \to \infty} (L \kappa_{n,r}) \leq c_n a_{\epsilon}^{\frac{4n-r-1}{n-r}} Re^{\frac{2n-1}{n-r}} (1 + \ln Re)^{\frac{1}{2(n-r)}}.
\] (3.14)

Proof: Essentially one uses the upper bound on \( F_n \) and the lower bound on \( F_r \) which can be calculated from the forcing part in terms of \( Gr \), leading to the result (see also [18, Ch. 7]). ■

4 Intermittency: good and bad intervals

The issue of intermittency in solutions of the two-dimensional Navier-Stokes equations is now addressed. While the \( F_n \) and \( \kappa_{n,r} \) are bounded from above for all time, nevertheless it is possible that their behaviour could be spiky in an erratic manner. To show how this might come about, consider the definition of \( \kappa_{n,r} \) in [1,18] from which we find

\[
F_{n+1} = \kappa_{n,r}^2 \left( \frac{\kappa_{n+1,r}}{\kappa_{n,r}} \right)^{2(n+1-r)} F_n.
\] (4.1)

Now consider inequality (3.9) re-written as

\[
\frac{1}{2} \frac{\dot{F}_n}{F_n} \leq -\nu \kappa_{n,1}^2 \left( \frac{\kappa_{n+1,1}}{\kappa_{n,1}} \right)^{2n} + c_n \left( \frac{\kappa_{n+1,1}}{\kappa_{n,1}} \right)^{n} \kappa_{n,1} F_1^{1/2} + c \omega_0 Re (1 + \ln Re)^{1/2}.
\] (4.2)

where we have used (4.1) and the fact that \( \kappa_{n,1} \leq \kappa_{n+1,1} \) in the middle term. Using Young’s inequality on this same term we end up with

\[
\frac{1}{2} \frac{\dot{F}_n}{F_n} \leq -\nu \kappa_{n,1}^2 \left( \frac{\kappa_{n+1,1}}{\kappa_{n,1}} \right)^{2n} + c_n \nu^{-1} F_1 + c \omega_0 Re (1 + \ln Re)^{1/2}.
\] (4.3)

The main question is whether, for Navier-Stokes solutions, the lower bound on

\[
\frac{\kappa_{n+1,1}}{\kappa_{n,1}} \geq 1
\] (4.4)

can be raised from unity. A variation on the interval theorem proved in [29] is used.

Theorem 4.1. For any value of the parameter \( \mu \in (0,1) \), the ratio \( \kappa_{n+1,1}/\kappa_{n,1} \) obeys the long-time averaged inequality (\( n \geq 2 \))

\[
\left\langle \left[ c_n \left( \frac{\kappa_{n+1,1}}{\kappa_{n,1}} \right)^{2} \right]^{1/\mu - 1} - \left[ \frac{(L^2 \kappa_{n,1}^2)^{\mu}}{(a_{\epsilon}^2 Re)^{\Lambda_{n,1}} (1 + \ln Re)^{1/2}} \right]^{1/\mu - 1} \right\rangle \geq 0,
\] (4.5)
where the $c_n$ are the same as those in Theorem 1.2. Hence there exists at least one interval of time, designated as a ‘good interval’, on which the inequality

$$c_n \left( \frac{\kappa_{n+1,1}}{\kappa_{n,1}} \right)^2 \geq \frac{(L^2 \kappa_{n,1}^2)^\mu}{(a_1^2 \text{Re})^{\lambda_{n,1}} (1 + \ln \text{Re})^{1/2}}$$

(4.6)

holds. Those other parts of the time-axis on which the reverse inequality

$$c_n \left( \frac{\kappa_{n+1,1}}{\kappa_{n,1}} \right)^2 < \frac{(L^2 \kappa_{n,1}^2)^\mu}{(a_1^2 \text{Re})^{\lambda_{n,1}} (1 + \ln \text{Re})^{1/2}}$$

(4.7)

holds are designated as ‘bad intervals’.

**Remark:** In principle, the whole time-axis could be a good interval, where the positive time average in (4.5) ensures that the complete time-axis cannot be ‘bad’. This paper is based on the worst-case supposition that bad intervals exist, that they could be multiple in number, and that the good and the bad are interspersed. The precise distribution and occurrence of the good/bad intervals and how they depend on $n$ remains an open question. The contrast between the two-dimensional and three-dimensional Navier-Stokes equations is prominent; while no singularities can occur in the $\kappa_{n,1}$ in the two-dimensional case, in three dimensions it is within these bad intervals that they can potentially occur.

**Proof:** Take two parameters $0 < \mu < 1$ and $0 < \alpha < 1$ such that $\mu + \alpha = 1$. The inverses $\mu^{-1}$ and $\alpha^{-1}$ will be used as exponents in the Hölder inequality on the far right hand side of

$$\langle \kappa_{n,1}^{2\alpha} \rangle \leq \langle \kappa_{n+1,1}^{2\alpha} \rangle = \left( \frac{\kappa_{n+1,1}}{\kappa_{n,1}} \right)^{2\alpha} \langle \kappa_{n,1}^{2\alpha} \rangle \leq \left( \frac{\kappa_{n+1,1}}{\kappa_{n,1}} \right)^{2\alpha/\mu} \langle \kappa_{n,1}^{2\alpha/\mu} \rangle \langle \kappa_{n,1}^{\alpha \mu} \rangle,$$

(4.8)

thereby giving

$$\langle \left( \frac{\kappa_{n+1,1}}{\kappa_{n,1}} \right)^{2\alpha/\mu} \rangle \geq \left( \frac{\langle \kappa_{n,1}^{2\alpha} \rangle}{\langle \kappa_{n,1}^{\alpha \mu} \rangle} \right)^{1/\mu} = \langle \kappa_{n,1}^{2\alpha} \rangle \left( \frac{\langle \kappa_{n,1}^{2\alpha} \rangle}{\langle \kappa_{n,1}^{\alpha \mu} \rangle} \right)^{\alpha/\mu}.$$  

(4.9)

Two-dimensional Navier-Stokes information can be injected into these formal manipulations: the upper bound on $\langle \kappa_{n,1}^{2\alpha} \rangle$ from Theorem 1.2 and the lower bound $L \kappa_{n,1} \geq 1$ are used in the ratio on the far right hand side of 4.9 to give 4.15, with the same $c_n$ as in Theorem 1.2. ■

Now consider what must happen on bad intervals. It is always true that $\kappa_{n+1,1}/\kappa_{n,1} \geq 1$, so 4.7 implies that on these intervals there is a lower bound

$$L^2 \kappa_{n,1}^2 > c_n (a_1^2 \text{Re})^{\lambda_{n,1}/\mu} (1 + \ln \text{Re})^{1/2 \mu}.$$  

(4.10)
This lower bound cannot be greater than the upper point-wise bound in (3.14), which means that $\mu$ is restricted by

$$\frac{\Lambda_{n,1}}{\mu} < 2 \left( \frac{2n - 1}{n - 1} \right).$$

(4.11)

Moreover, the factor of $1/\mu$ in the exponent makes the lower bound in (4.10) much larger than the upper bound on the average $\langle \kappa_{n,1}^2 \rangle$ given in Theorem (1.2). These intervals must therefore be very short. To estimate how large they can be requires an integration of (4.3) over short times $\Delta t = t - t_0$ which, in turn, requires the time-integral of $H_1$ for short times $\Delta t$. We use the notation $\int_{\Delta t} = \int_{t_0}^t$, with the definition $\omega_0 = \nu \ell^{-2}$.

**Lemma 4.1.** To leading order in $Re$

$$\int_{\Delta t} F_1 \, dt \leq \nu a_t^4 \left[ c_1 a_t^2 + c_2 \omega_0 \Delta t \right] Re^4. \tag{4.12}$$

**Proof:** Integrating (3.1) over a short time $\Delta t$ gives

$$\nu \int_{\Delta t} H_1 \, dt \leq \frac{1}{2} H_0(t_0) + \Delta t \left[ \ell^{-2} \nu^3 a_t^4 Gr^2 \right]$$

$$\leq c_1 a_t^6 \nu^2 Re^4 + \Delta t \left[ c_2 \ell^{-2} \nu^3 a_t^4 Re^4 \right], \tag{4.13}$$

having used (3.3) for the $\frac{1}{2} H_0(t_0)$-term. The forcing term in $F_1$ is only $O(Re^2)$. ■

![Figure 1: A cartoon, not to scale, of good/bad intervals for some value of $n \geq 3$.](image-url)
Now we wish to estimate $\omega_0 \Delta t$ in terms of $Re$. Integrating (4.3), using (4.13) and the lower bound (4.10) and multiplying by $\ell^2$, we have

$$\frac{1}{2} \ell^2 [\ln F_n(t) - \ln F_n(t_0)] + \frac{1}{2} c_n \nu a_t^{-2} (a_t^2 Re)^{\Lambda_n,1/\mu} (1 + \ln Re)^{1/2\mu} \Delta t \leq \ell^2 a_t^4 [c_1 a_t^2 + c_2 \omega_0 \Delta t] Re^4 + c \ell^2 \omega_0 \Delta t Re (1 + \ln Re)^{1/2}. \quad (4.14)$$

As $Gr \to \infty$, the dominant terms are

$$\omega_0 \Delta t \left\{ a_t^{-2} (a_t^2 Re)^{\Lambda_n,1/\mu} (1 + \ln Re)^{1/2\mu} - a_t^6 Re^4 \right\} \leq c_1 a_t^6 Re^4. \quad (4.15)$$

Choosing $\mu$ in the range, to leading order we have

$$\mu < \frac{1}{4} \Lambda_{n,1}, \quad (4.16)$$

then $\Delta t$ must satisfy

$$\omega_0 \Delta t \leq c (a_t^2 Re)^{4-\Lambda_{n,1}/\mu}. \quad (4.17)$$

Because the exponent in (4.17) is necessarily negative these intervals are very small and decreasing with increasing $Re$. Combining (4.17) with (4.16) we have

$$\frac{(n-1)}{2(2n-1)} \Lambda_{n,1} < \mu < \frac{1}{4} \Lambda_{n,1}, \quad (4.18)$$

which actually holds for every $n \geq 1$. Figure 1 is a cartoon-like figure displaying the lower bound on the bad intervals of width $(\Delta t)_b$ and also the maximum of $\kappa_{n,1}$ allowed by (3.14) in Lemma 3.2. The full dynamics of two-dimensional Navier-Stokes is actually determined by the intersection of all cartoons for every $n \geq 3$ on the grounds that the position and occurrence of the bad intervals varies with $n$. Thus we are interested in the limit $n \to \infty$ which determines that the range of $\mu$ is squeezed between

$$\frac{3}{8} \left( 1 - \frac{5}{6n} \right) < \mu < \frac{3}{8} \left( 1 - \frac{1}{3n} \right). \quad (4.19)$$

Thus, in the limit, $\mu$ takes a value just under $3/8$. We conclude that the interval theorem (Theorem 4.1) reproduces the effects of intermittency in a two-dimensional flow by manifesting very large lower bounds within bad intervals and suppressing spiky behaviour within the good
intervals which must be quiescent for long intervals, otherwise the long-time average would be violated.

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### A Forcing & the fluid response

For technical reasons, we must address the possibility that in their evolution the quantities $H_n$ might take small values. Thus we need to circumvent problems that may arise when dividing by these (squared) semi-norms. We follow Doering and Gibbon who introduced the modified quantities

$$F_n = H_n + \tau^2 \|\nabla^n f\|_2^2,$$

where the “time-scale” $\tau$ is to be chosen for our convenience. So long as $\tau \neq 0$, the $F_n$ are bounded away from zero by the explicit value $\tau^2 L_3^3 \ell^{-2n} f_{rms}^2$. Moreover, we may choose $\tau$ to depend on the parameters of the problem such that $\langle F_n \rangle \sim \langle H_n \rangle$ as $Gr \to \infty$. To see how to achieve this, let us define

$$\tau = \ell^2 \nu^{-1} [Gr(1 + \ln Gr)]^{-1/2}. \quad (A.2)$$

Then the additional term in (A.1) is

$$\tau^2 \|\nabla^n f\|_2^2 = L_3^3 \nu^{-2} \ell^{4-2n} f_{rms}^2 [Gr(1 + \ln Gr)]^{-1}$$

$$= \nu^2 \ell^{-\left(2n+2\right)} L_3^3 Gr(1 + \ln Gr)^{-1}. \quad (A.3)$$

Now Doering & Foias proved that in $d$-dimensions, the energy dissipation rate $\epsilon$ has a lower bound of the form

$$\epsilon \geq c \nu^3 \ell^{-3} L^{-1} Gr. \quad (A.4)$$
Using this on the far right hand side of (A.3) we arrive at

\[ \tau^2 \| \nabla^n f \|^2 \leq c_6 \epsilon \ell^{-(2n-1)} L^4 \nu^{-1} (1 + \ln Gr)^{-1} \]

\[ = c_6 \left( \frac{L}{\ell} \right)^{(2n-1)} L^{-2(n-1)} \langle H_1 \rangle (1 + \ln Gr)^{-1}. \]  \tag{A.5}

Using Poincaré’s inequality in the form \( H_1 \leq (2\pi L)^{2(n-1)} H_n \), as \( Gr \to \infty \) we have

\[ \frac{\tau^2 \| \nabla^n f \|^2}{\langle H_n \rangle} \leq c_6 \epsilon^{2(n-1)} (1 + \ln Gr)^{-1}. \]  \tag{A.6}

Hence, the additional forcing term in (A.1) becomes negligible with respect to \( \langle H_n \rangle \) as \( Gr \to \infty \), so the forcing does not dominate the response.

B Comparison theorems for ODEs

We present a comparison theorem for ODE which is useful for obtaining various estimates. We start with the following classical result.

**Lemma B.1.** Let \( f : [0, T] \times \mathbb{R} \to \mathbb{R} \) be a continuous function which is locally Lipschitz uniformly in \( t \): for all intervals \( [a, b] \subset \mathbb{R} \) there exists a constant such that \( |f(s, x) - f(s, y)| \leq C|x - y| \) for all \( x, y \in [a, b] \) and all \( s \in [0, T] \). Furthermore, let \( x \in AC([0, T], \mathbb{R}) \) be such that

\[ \dot{x}(t) \leq f(t, x(t)) \]

for all \( t \in [0, T] \) and let \( y(t) \) be the solution of \( \dot{y}(t) = f(t, y(t)) \) on \( [0, T] \). Assume further that \( x(0) \leq y(0) \). Then, \( x(t) \leq y(t) \) for all \( t \in [0, T] \).

We can use this Lemma to prove the following useful result.

**Lemma B.2.** Let \( x : [0, T] \to [0, \infty) \) be an absolutely continuous function with \( x(0) > 0 \) which satisfies

\[ \dot{x} \leq \Delta_0 x + F x^{n_1} - E x^{n_2}, \]  \tag{B.1}

where \( \Delta_0, F, E > 0 \) and \( 1 < n_1 < n_2 \). Then

\[ \limsup_{t \to \infty} x(t) \leq (4\Delta_0 E^{-1})^{\frac{1}{n_2-1}} + (2FE^{-1})^{\frac{1}{n_2-n_1}}. \]  \tag{B.2}
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