Neutrino Mass Squared Differences in the
Exact Solution of a 3-3-1 Gauge Model without
Exotic Electric Charges

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Abstract

The mass splittings for the Majorana neutrinos in the exact solution of a particular 3-3-1 gauge model are computed here in detail. Since both $\sin^2 \theta_{13} \approx 0$ and the mass splittings ratio $r_\Delta \approx 0.033$ are taken into account, the analytical calculations seem to predict an inverted mass hierarchy and a mixing matrix with a texture based on a very close approximation to the bi-maximal mixing. The resulting formulas for the mass squared differences can naturally accommodate the available data if the unique free parameter ($a$) gets very small values ($\sim 10^{-15}$).

Consequently, the smallness of the parameter requires (according to our method) a large breaking scale $< \phi > \sim 10^6 - 10^7$ TeV in the model. Hence, the results concerning the neutrino mass splittings may lead to a more precise tuning in the exact solution of the 3-3-1 model of interest, being able - at the same time - to recover all the Standard Model phenomenology and predict the mass spectrum of the new gauge bosons $Z', X, Y$ in accordance with the actual data. The minimal absolute mass in the neutrino sector is also obtained - $m_0 \approx 0.0035$ eV - in the case of our suitable approximation for the bi-maximal mixing.

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Key words: neutrino mass splittings, 3-3-1 models, exact solution

1 Introduction

The neutrino mass issue is one of the most effervescent still open questions in the nowadays physics. Since the successful Standard Model (SM) deals only with massless neutrinos that pair charged leptons in three left-handed flavor generations of the gauge group $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$, it appears obviously that any theoretical mechanism designed to generate neutrino masses must invoke some new physics beyond the SM and even a radically new framework. One such a natural extension of the SM emerged in the literature with the papers of Pisano, Pleitez and Frampton [1] - based on the new gauge group $SU(3)_c \otimes SU(3)_L \otimes U(1)_Y$ - that opened a plethora of theoretical ways [2] to generate neutrino masses. These mathematical strategies have been developed
ever since in order to accommodate the available data regarding the solar and atmospheric neutrino oscillation experiments, but a final answer to this challenge has not been given yet.

The author proposed an original mechanism in order to provide neutrino masses using the exact solution of a particular class of 3-3-1 models (namely, model D in [5]). The neutrinos are considered as Majorana fields and their mass matrix arises naturally from the Lagrangian of the model when particular tensor products among Higgs triplets are involved. This strategy works and relates the neutrino masses to the charged lepton ones, since the SSB takes place in the manner proposed by the general method of solving electro-weak gauge models with high symmetries [6]. It relies on a special parametrization of the Higgs sector that finally provides a one-parameter mass scale. The one-parameter solution of the 3-3-1 model under consideration here has the advantage that it recovers all the SM phenomenology, it can predict the mass spectrum for the new bosons in the model and it needs no massive Majorana seesaw partners for physical neutrinos (even no seesaw mechanism at all). All these results are achieved just by tuning the unique remaining free parameter of the model. Besides, a large overall breaking scale seems to be required due to the smallness of the free parameter $a$. Calculation of the concrete expressions for the mass splittings within the one-parameter exact solution of the model is accomplished here by taking into account the condition $\sin^2 \theta_{13} \approx 0$ which seems to be most likely, since the actual experimental investigations are not sensitive to any CP-phase violation in the lepton sector.

The paper is organized as follows. In Sec. 2 the neutrino mass matrix is obtained within the framework of the exact solution of the 3-3-1 model of interest. Then, the mass matrix is analytically diagonalized by solving an appropriate set of equations (Sec. 3). Consequently, the mass squared differences are computed and some phenomenological implications are discussed in Sec.4, where a certain texture to the mixing matrix is proposed. Our conclusions are sketched in the last section of the paper (Sec. 5).

## 2 Neutrino Mass Matrix in the 3-3-1 Model without Exotic Electric Charges

In the following we look for the neutrino masses, recalling the main results of the exact solution of a 3-3-1 model without exotic electric charges (model D in [5]) and assuming the neutrino mixing in the manner presented in the excellent reviews of Bilenky [7]. All the standard notations of the field are considered and used below.

The neutrino mass splittings can be addressed only after defining the physical neutrino mass eigenstates. A unitary mixing matrix $U$ (with $U^+ U = 1$) is necessary for this purpose. It links the gauge-flavor basis to the physical one of the massive neutrinos:

$$\nu_{\alpha L}(x) = \sum_{i=1}^{3} U_{\alpha i} \nu_{i L}(x)$$

where $\alpha = e, \mu, \nu$ (denoting gauge flavor-eigenstates) and $i = 1, 2, 3$ (denoting massive physical eigenstates). We consider hereafter the physical neutrinos as Majorana
fields, i.e. \( \nu_{1L}^i(x) = \nu_{1L}(x) \). The neutrino mass term in the Yukawa sector yields then:

\[
-\mathcal{L}_Y = \frac{1}{2} \nu_L M \nu^*_L + H.c
\]

(2)

with \( \nu_L = | \nu_e \nu_\mu \nu_\tau \rangle^T_L \) where the superscripts \( T \) and \( L \) mean "transposed" and "charge conjugation" respectively, while subscript \( L \) means "left-handed" component.

The proposed symmetric matrix \( S \) for the neutrinos involved in the model:

\[
\eta
\]

parameter matrix \( G \)

obviously determined by the parameter choice (namely, the bijective mapping \( \eta, \rho, \chi \) → (1, 2, 3)) that establish a certain VEV alignment. Therefore, in order to accommodate the data concerning the sum of the neutrino masses, one can accept only.

\[
\sum_m m_i = 4 < \phi(\eta) > < \phi(\chi) > < \phi(\rho) > < \phi >
\]

(5)

where the left hand term of Eq. (5) is phenomenologically restricted by the actual data.
the Case I (see for details Sec. 4.3 in [4]) for which the parameter \( a \) (that determines the VEV alignment) has to be in the range \( a < 0.118 \cdot 10^{-3} \) [4]. It follows naturally that \( \langle \phi_1 \rangle, \langle \phi_3 \rangle \ll \langle \phi_2 \rangle \). Under these circumstances, the mass matrix becomes:

\[
M = \begin{pmatrix} m(c) & D & E \\ D & m(\mu) & F \\ E & F & m(\tau) \end{pmatrix} \left( \frac{2a}{1 - a} \sqrt{1 - 2 \sin^2 \theta_W} \right) \frac{\cos^2 \theta_W}{\sqrt{1 - a}}
\]

where - rigorously speaking - the new \( D, E, F \) in Eq. (6) differ from those in Eq. (4) by a factor \( \langle \phi' \rangle > \). However this is not a reason to change the notations since they are amounts to be eliminated afterwards in our calculations.

### 3 Calculating Neutrino Masses

At the present level of knowledge, the absolute masses of the physical neutrinos are experimentally irrelevant. Instead, the mass squared differences - defined as \( \Delta m_{ij}^2 = m_j^2 - m_i^2 \) - can offer significant data by observing the neutrino oscillations phenomenon. Their right order of magnitude is \( 7.1 \times 10^{-5} \text{ eV}^2 \leq \Delta m_{12}^2 \leq 8.9 \times 10^{-5} \text{ eV}^2 \) (from solar and KamLAND data [3]) and \( 1.4 \times 10^{-3} \text{ eV}^2 \leq \Delta m_{13}^2 \leq 3.3 \times 10^{-3} \text{ eV}^2 \) (from Super Kamiokande atmospheric data [4]). A systematic approach of the way to deal with them can be found in [10] and references therein. These data are compatible with either a normal mass hierarchy \( \Delta m_{23}^2 > 0 \) \( (m_1 < m_2 < m_3) \) or an inverted one \( \Delta m_{23}^2 < 0 \) \( (m_3 \ll m_1 \approx m_2) \).

For our purpose, the coupling constants in Eq. (4) are regarded as unknown variables and try to eliminate them by diagonalization the matrix \( M \). This is equivalent (in the Majorana case) to a set of 6 linear equations with 9 variables which can supply the following generic solution for the physical neutrino masses:

\[
m_i = f_i [m(e), m(\mu), m(\tau)] \left( \frac{2a}{1 - a} \right) \frac{\sqrt{1 - 2 \sin^2 \theta_W}}{\cos^2 \theta_W} \]  

with \( i = 1, 2, 3 \).

The concrete forms of \( f_i \)'s remain to be determined by solving the following set of equations:

\[
\begin{align*}
\begin{array}{l}
m_1 = c^2_2 m(e) + c^2_1 c^2_2 m(\mu) + s^2_1 s^2_2 m(\tau) - 2 c_1 c_2 s_1 s_2 D + 2 s_1 s_2 c_2 E - 2 c_1 s_1 s_2 F \\
0 = c_2 s_2 m(e) - c^2_1 c_2 s_2 m(\mu) - s^2_1 s_2 c_2 m(\tau) - (1 - 2 s^2_2) s_1 E + 2 s_1 s_2 c_1 c_2 F \\
0 = -c_1 s_1 s_2 m(\mu) + c_1 s_1 s_2 m(\tau) + c_2 s_1 D + c_1 c_2 E - (1 - 2 s^2_2) s_2 F \\
m_2 = s^2_2 m(e) + c^2_1 c^2_2 m(\mu) + s^2_1 s^2_2 m(\tau) + 2 c_1 c_2 s_1 D - 2 s_1 s_2 c_2 E - 2 c_1 s_1 c_2 F \\
0 = s_1 c_1 c_2 m(\mu) - s_1 c_1 c_2 m(\tau) + s_1 s_2 D + c_1 s_2 E + (1 - 2 s^2_2) c_2 F \\
m_3 = s^2_1 m(\mu) + c^2_1 m(\tau) + 2 c_1 s_1 F \\
\end{array}
\end{align*}
\]

Since actual data are not sensitive to any CP-phase violation in the lepton sector we have taken into account from the very beginning \( \sin^2 \theta_{13} \approx 0 \) - as it can be easily observed by inspecting the shape of Eq. (8) - but the proposed values for the other two
mixing angles will be embedded only in the resulting formulas for the neutrino masses (9). Thus, one obtains after a few manipulations the following analytical equations:

\[
m_1 = \frac{m(\tau) \sin^2 \theta_{12} \sin^2 \theta_{23} - m(\mu) \sin^2 \theta_{12} (1 + \sin^2 \theta_{23})}{(1 - 2 \sin^2 \theta_{23}) (1 - 2 \sin^2 \theta_{12})} + \frac{m(e) (1 - \sin^2 \theta_{12})}{(1 - 2 \sin^2 \theta_{12})},
\]

\[
m_2 = \frac{m(\mu)(1 - \sin^2 \theta_{12} - \sin^2 \theta_{12} + 3 \sin^2 \theta_{12} \sin^2 \theta_{23}) - m(\tau) \sin^2 \theta_{23} (1 - \sin^2 \theta_{12})}{(1 - 2 \sin^2 \theta_{23}) (1 - 2 \sin^2 \theta_{12})} + \frac{m(e) \sin^2 \theta_{12}}{(1 - 2 \sin^2 \theta_{12})},
\]

\[
m_3 = \frac{m(\tau) (1 - \sin^2 \theta_{23}) - m(\mu) \sin^2 \theta_{23}}{1 - 2 \sin^2 \theta_{23}}.
\]

(9)

A rapid investigation of the resulting equations can be made even at this stage.

• Note that some of the masses could come out negative (for certain combinations of angles), but this is not an impediment since for any fermion field a \( \gamma_5 \psi \) transformation can be performed at any time, without altering the physical content of the theory. As a result of this manipulation the mass sign changes.

• Let’s observe that the analytical equations for the masses (9) strictly impose \( \sin^2 \theta_{12} \neq 0.5 \) and \( \sin^2 \theta_{12} \neq 0.5 \), yet this does not forbid any closer approximation to the bi-maximal neutrino mixing.

• These equations also do not contradict the trace condition which requires indeed a finite neutrino mass sum independently of the values of the mixing angles. As a matter of fact, if one sums the three masses in (9), then the troublesome denominators disappear and the required (by Eq. (6)) value is recovered.

• The particular shape of the analytical neutrino masses is due to both the choice of the \( \theta_{13} = 0 \) and the diagonal entries in the mixing matrix that are proportional to the charged lepton masses. Any other choice definitely leads to a different set of equations to be solved and, thus, to a different form of the solution.

• The mass splittings ratio is independent of the free parameter of the model, therefore its right value can be determined only by choosing appropriate values for the mixing angles.

• Assuming the structure of the mass spectrum in the charged lepton sector \([9]\), one can neglect in Eqs. (9) all the terms except for those multiplying \( m(\tau) \). That is, our analysis will rely on the following suitable approximation:

\[
m_1 \simeq \frac{m(\tau) \sin^2 \theta_{12} \sin^2 \theta_{23}}{(1 - 2 \sin^2 \theta_{23}) (1 - 2 \sin^2 \theta_{12})}.
\]
\[ |m_2| \simeq \frac{m(\tau) (1 - \sin^2 \theta_{12}) \sin^2 \theta_{23}}{(1 - 2 \sin^2 \theta_{23}) (1 - 2 \sin^2 \theta_{12})} \tag{10} \]
\[ m_3 \simeq \frac{m(\tau) (1 - 2 \sin^2 \theta_{12}) (1 - \sin^2 \theta_{23})}{(1 - 2 \sin^2 \theta_{23}) (1 - 2 \sin^2 \theta_{12})} \]

4 Phenomenological Implications

One of the striking features of the above presented solution is that it requires in the most likely case (see below) an inverted hierarchy \( m_3 \ll m_1 \simeq m_2 \), since the data \[10\] suggest the following limits for the solar mixing angle: \((1/3) \leq \sin^2 \theta_{12} \leq (1/2)\). Hence, \( m_1 \leq |m_2| \) holds in (10) as long as \( \sin^2 \theta_{12} \leq 1/2 \). If the maximal atmospheric angle is embedded, then \( m_3 < m_1 \) is accomplished in (10) only if \( \sin^2 \theta_{12} > 1/3 \) which is quite a plausible prediction.

4.1 Case 1 (\( \sin^2 \theta_{23} \simeq 0.5 \))

We have already established that the most likely setting with maximal atmospheric angle occurs within the inverted hierarchy. In the following we exploit the mass splittings ratio \( r_\Delta = \Delta m^2_{12}/\Delta m^2_{23} \) that has to be fulfilled \( (r_\Delta \simeq 0.033) \) by a certain value of the solar mixing angle. A suitable approximation will be performed in the denominators - which are identical for all the three terms in Eqs. (10) - only as the last step of the calculation, while for the moment we focus on the numerators which determine the splitting ratio. They are:

\[ m_1 \sim m(\tau) \sin^2 \theta_{12} \]
\[ |m_2| \sim m(\tau) (1 - \sin^2 \theta_{12}) \tag{11} \]
\[ m_3 \sim m(\tau) (1 - 2 \sin^2 \theta_{12}) \]

Under these circumstances - i.e. inverted hierarchy - the mass ratio will have the form:

\[ r_\Delta = \frac{(1 - 2 \sin^2 \theta_{12})}{(2 - 3 \sin^2 \theta_{12}) \sin^2 \theta_{12}} \tag{12} \]

Since one takes as a good approximation in the numerator of Eq. (12) \( \sin^2 \theta_{12} \simeq 0.495 \), a reasonable value for \( r_\Delta \) can result very close to the phenomenological value 0.033. (Actually, the value for \( \sin^2 \theta_{12} \) was inferred by solving the resulting equation (12) with the solar mixing angle as variable; a unique solution out of the two is suitable, while the other one is rejected not belonging to \([-1, 1]\))
Once the solar mixing angle has been established, the mass squared differences can be computed. The solar neutrino mass splitting is:

\[ \Delta m^2_{12} = \frac{m^2(\tau)}{(1 - 2 \sin^2 \theta_{12}) (1 - 2 \sin^2 \theta_{23})} \left( \frac{a^2}{1 - a} \right) \frac{1 - 2 \sin^2 \theta_W}{16 \cos^4 \theta_W} \sim 10^{25} a^2 \text{eV}^2 \]

(13)

According to the data supplied by [9] for the Weinberg angle and tau lepton and assuming the approximation \( \sin^2 \theta_{23} = 0.4995 \), the resulting value for parameter \( a \) has to be \( \sim 2 \times 10^{-15} \) in order to fit the experimental condition \( \Delta m^2_{12} \simeq 8 \times 10^{-5} \text{eV}^2 \). Consequently, this value of the parameter fits the atmospheric neutrino mass splitting \( \Delta m^2_{23} \simeq 2.4 \times 10^{-3} \text{eV}^2 \) as well. Such a small value for the free parameter determines a large VEV \(< \phi > \sim 10^6 - 10^7 \text{TeV} \).

Thus, the bi-maximal condition seems to successfully accomodate our method. Moreover, it can predict even the minimal absolute mass in the neutrino sector. It is now expressed by:

\[ m_3 \simeq \frac{m(\tau)}{(1 - 2 \sin^2 \theta_{23})} \left( \frac{a}{\sqrt{1 - a}} \right) \sqrt{1 - 2 \sin^2 \theta_W} \cos^2 \theta_W \]

(14)

and has the approximate value \( m_3 \simeq 0.0035 \text{eV} \).

### 4.2 Case 2 (\( \sin^2 \theta_{12} \simeq 0.3 \))

Let’s have a look at this particular case. The question is whether the lower limit \( \sin^2 \theta_{12} \simeq 0.3 \) can supply a suitable setting in our method. A rapid estimation in this case confirms also an inverted hierarchy where the numerators of the individual neutrino masses can be sufficiently well approximated as:

\[ m_1 \sim 0.3m(\tau) \sin^2 \theta_{23} \]

\[ |m_2| \sim 0.7m(\tau) \sin^2 \theta_{23} \]

(15)

\[ m_3 \sim 0.4m(\tau) (1 - \sin^2 \theta_{23}) \]

A normal hierarchy would require - when comparing \( m_3 > m_2 \) - an unacceptable upper limit for the atmospheric angle \( \sin^2 \theta_{23} < 4/11 \sim 0.36 \). Thus, \( \sin^2 \theta_{23} > 4/7 \sim 0.57 \) which ensures \( m_3 < m_1 \) seems to be the unique acceptable possibility. Unfortunately, this leads to a similar equation to (12):

\[ r_{\Delta} = \frac{4 \sin^2 \theta_{23}}{3.3 \sin^4 \theta_{23} + 3.2 \sin^2 \theta_{23} - 1.6} \]

(16)

that gives unacceptable solutions for \( \sin^2 \theta_{23} \) (both resulting values not belonging to \([-1, 1]\)) when \( r_{\Delta} = 0.033 \) is embedded in (16). Therefore, this case is ruled out.
5 Concluding Remarks

In conclusion, we consider that we have convincingly proved that our method of exactly solving gauge models with high symmetries [4, 6] when applied to the particular 3-3-1 model without exotic electric charges can provide a realistic solution [4]. That is, it can recover all the SM phenomenology and predict a plausible mass spectrum for the new bosons \( Z', X, Y \) (their masses are \( \geq 1 \, \text{TeV} \), in good accordance to [9]). Furthermore, if a special Yukawa mechanism based on tensor products among Higgs triplets is implemented into the model, then neutrino masses can be generated in a natural way at the tree level without any approximation [4]. The resulting values can accommodate the available data concerning the mass splitting [8, 9, 10] just by tuning a unique free parameter \( (\alpha) \) as we have just shown in the previous section, even though this requires a large breaking scale for one of the three VEVs \( \phi \sim 10^6 - 10^7 \text{TeV} \) (the overall scale). If one wants to separate the VEV question from the neutrino mass issue (and hence, getting lighter non-SM bosons), one has to add a new small parameter in a proper way in the parameter matrix of the Higgs sector. This mathematical artifice [11] will generate a canonical seesaw mechanism that can be naturally implemented in the model even at a low breaking scale \( \sim 1 \text{TeV} \) (the procedure to be presented elsewhere).

Concerning the texture of the mixing matrix obtained above (Sec. 4.1), note that this kind of settings were widely discussed in the literature [12, 13, 14]. It implies - along with the inverted hierarchy - the approximate symmetry \( L_e - L_\mu - L_\tau \) [13]. Although some predictions of this scenario (one of them: the maximal solar mixing requirement, in the leading order of perturbations) are not in vogue, analyzing the topic is worthwhile (see [14] and references therein) since the issue of the neutrino matrix texture awaits more accurate evidence in the forthcoming experiments.

Our method definitely favors the bi-maximal mixing and inverted hierarchy for physical neutrinos.

References


