Energy and enstrophy dissipation in steady state 2-d turbulence

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Implications of our results are discussed.

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I. INTRODUCTION

The study of two dimensional turbulence was originally justified as a simplified version of the more complex three dimensional turbulence, but it has come to be regarded as an interesting research field in its own right with deep connections to geophysical and astrophysical problems such as as strongly rotating stratified flows. A large number of experimental methods have been devised to constrain flows in two dimensions (e.g. soap films) allowing some theories theories to be tested in the lab. Direct numerical simulations are far easier than the three dimensional case, and this has enabled researchers to investigate two dimensional turbulence computationally at much higher Reynolds numbers. As a result, there are more simulation data to test theories of two dimensional turbulence. Nevertheless many fundamental questions remain open; see [1] for a recent review.

The inviscid conservation of enstrophy as well as energy in two dimensions results in two cascading quadratic invariants that make the phenomenology of two dimensional turbulence somewhat more complex than three dimensional turbulence and not derivable from simple dimensional arguments. Theoretical studies of turbulence usually employ a statistical description and often involve assumptions about homogeneity isotropy and the nature of the interactions. Based on physical arguments, Kraichnan [2], Leith [10] and Batchelor [11] conjectured that there is a dual cascade in two dimensional turbulence: energy flows to the larger scales while enstrophy moves to the small scales (when the system is driven at some intermediate scale). Kraichnan–Leith-Batchelor (KLB) theory assumes isotropy and homogeneity in the limit of infinite domain size, and in the zero viscosity limit predicts a $k^{-5/3}$ energy spectrum for the large scales and a $k^{-3}$ energy spectrum for the small scales. The assumptions of the KLB theory, as well as the power index and the universality of the two energy spectra has been questioned in the literature [12, 13, 14, 15, 16, 17, 18].

In this paper we derive some simple rigorous bounds for the long time averaged bulk energy and enstrophy dissipation rates for two dimensional statistically stationary flows sustained by a variety of driving forces. The study of physically relevant rigorous bounds on the energy dissipation rate, i.e., the power consumption of turbulence, for a class of boundary-driven flows can be traced back to the seminal work of Howard [19], and in recent years an alternative approach [20] has renewed interest in those kinds of problems, providing direct connections to experiments in some cases. Bounds for the energy dissipation of steady body forced flows—more convenient for theoretical and numerical investigations—in three dimensions have been derived by Foias [21] and others [22, 23, 24, 25]. Not unexpectedly, Foias et al. [26] also derived a bound for the enstrophy dissipation rate in the statistically stationary states of two-dimensional turbulence driven by a restricted class of forces. Bounds for the energy and enstrophy dissipation in two dimensional flow driven by a monochromatic forcing were derived in [27] and [10]. The case of temporally white noise forcing was studied by Eyink [28]. More recently Tran el at [29, 30] derived bounds of the enstrophy dissipation for freely decaying two dimensional turbulence in terms of the initial ideal invariants of the flow. (See also [31] for the treatment of the same problem in terms of the inviscid Euler Equations). Finally, we mention that bounds on the dimension of attractor of the 2D Navier-Stokes have been derived in [32, 33, 34, 35, 36] and more recently by [37].

The results for the energy and enstrophy dissipation of forced flows derived in this paper apply to a more general type of forcing than the single scale forcing that was considered in [10, 27]. We also consider forces that are
smoothly varying in time, unlike the white noise forcing investigated in [28], and we are particularly interested in the behavior of the long times averaged dissipation rates in the vanishing viscosity limit. Because the viscosity is a dimensional quantity we must specify what we mean by “small” viscosity. To be precise, we measure the magnitude of the viscosity in terms of the Reynolds number in the statistically steady state,

\[ Re = \frac{U}{k_f \nu} \]  

(1)

where \( U \) is the root mean square velocity and \( k_f \) is a natural wavenumber (inverse length scale) in the driving force. The dissipation rates are also dimensional quantities, so we measure them in terms of the inviscid scales determined by \( U \) and \( k_f \). That is, we estimate

\[ \beta = \frac{\epsilon}{k_f U^3} \quad \text{and} \quad \gamma = \frac{\chi}{k_f^2 U^3} \]  

(2)

in terms of \( Re \) and focus on the \( Re \to \infty \) limit holding other parameters (such as the large scale eddy turnover time in the most general case) fixed. For a broad class of external driving we find that

\[ \beta \lesssim Re^{-1/2} \quad \text{and} \quad \gamma \lesssim Re^0, \]  

(3)

consistent with an enstrophy cascade of sorts.

However, for special cases of forcing such as “ultra narrow band” monochromatic (i.e., involving on a single length scale, albeit with a broad range of time dependence) forcing, or for a fixed energy flux forcing popular for direct numerical simulations, a stronger bound holds:

\[ \beta \lesssim Re^{-1} \quad \text{and} \quad \gamma \lesssim Re^{-1}. \]  

(4)

This kind of \( Re^{-1} \) scaling suggests “laminar” flows where the energy is concentrated at or above the smallest length scale of the forcing. This sort of scaling has been shown before in the literature [16, 27] for monochromatic forcing and for white noise in time forcing [28].

In every case the bounds derived here are strictly less than those available—or expected—for three dimensional turbulence. The upper bounds [33] on the energy and enstrophy dissipation for two dimensional flows derived here are in a sense a consequence of combining the approaches in [27] and [16, 27] applied to a class of forcing functions concentrated in a finite range of length scales. Even though some steps in our analysis have been taken before, in order to make this paper self-contained the complete (but nevertheless short) proofs will be presented here.

The rest of this paper is organized as follows. In the next section II we introduce the problem and basic definitions, and perform the analysis leading to \[ \beta \] for the simplest case of time independent body forces. Section III generalizes the analysis to include a broad class of time-dependent forces. In section IV we briefly review the results for time-dependent but single-length scale forcing and “fixed-flux” forces in order to establish the stronger results in [34]. The concluding section V contains a brief discussion of the results and their implications.

II. TIME-INDEPENDENT FORCING

Consider a two dimensional periodic domain \([0, L]^2\), i.e., \( \mathbb{T}_L^2 \), filled with an incompressible fluid of unit density evolving according to the Navier-Stokes equation:

\[ \partial_t u + u \cdot \nabla u = -\nabla p + \nu \nabla^2 u + f \]  

(5)

where \( u = i u_x(x, y, t) + j u_y(x, y, t) \) is the incompressible (divergence-free) velocity field, \( p(x, y, t) \) is the pressure, \( \nu \) is the viscosity, and \( f(x, y) = \hat{i} f_x(x, y) + \hat{j} f_y(x, y) \) is a smooth, mean zero, divergence-free body force with characteristic length scale \( \sim k_f \) (defined precisely below). The scalar vorticity \( \omega = \partial_x u_y - \partial_y u_x \) satisfies

\[ \partial_t \omega + u \cdot \nabla \omega = \nu \nabla^2 \omega + \phi \]  

(6)

where \( \phi = \hat{k} \cdot \nabla \times f = \partial_x f_y - \partial_y f_x \).

The Reynolds number is defined in [16] where

\[ U \equiv \langle |u|^2 \rangle^{1/2} \]  

(7)

is the root-mean-square velocity with \( \langle \cdot \rangle \) representing the space-time average

\[ \langle g \rangle = \lim_{T \to \infty} \frac{1}{T} \int_0^T \left( \frac{1}{L^2} \int_{\mathbb{T}_L^2} g(x, y, t) \, d^2 x \right) \, dt. \]  

(8)

(The limit in the time average is assumed to exist for all the quantities of interest.) The forcing length scale associated with the wavenumber \( k_f \) is defined by

\[ k_f^2 \equiv \frac{\| \nabla^2 f \|}{\| f \|} \]  

(9)

where \( \| \cdot \| \) is the \( L_2 \) norm on \( \mathbb{T}_L^2 \). It is apparent that we are restricting ourselves to sufficiently smooth forcing functions.

The time and space averaged energy dissipation rate is

\[ \epsilon \equiv \nu \langle |\nabla u|^2 \rangle = \nu \langle \omega^2 \rangle, \]  

(10)

the second expression resulting from integrations by parts utilizing incompressibility. The bulk averaged enstrophy dissipation rate is

\[ \chi \equiv \nu \langle |\nabla \omega|^2 \rangle = \nu \langle |\nabla^2 u|^2 \rangle. \]  

(11)

We think of \( \beta = \epsilon / k_f U^3 \) and \( \gamma = \chi / k_f^2 U^3 \) as functions of \( Re \) and the functional form or “shape” of the forcing, but not explicitly on its amplitude

\[ F = \frac{\| f \|}{L} \]  

(12)
except indirectly through its influence on $U$.

We are considering the Reynolds number to be the “control parameter” even though it is defined in terms of the emergent quantity $U$. Strictly speaking the flow is determined by the structure and amplitude of the body force (and the initial data) so the Grashof number such as $Gr = F/k_0^2
u^2$ should naturally be used as the relevant dimensionless control parameter indicating the intensity of the driving and the resulting flow. Indeed, while we can always realize any given value of $Gr$, it is not at all evident that every particular value of $Re$ can be achieved. Nevertheless, in order to express the results in terms of quantities familiar to the theory of homogeneous isotropic turbulence we will (without loss of rigor) express the bounds in terms of $Re$.

Poincare’s inequality applied to (11) and (13) immediately yield the lower estimates

$$\epsilon \geq \frac{4\pi^2}{L^2} U^2 \quad \text{and} \quad \chi \geq \frac{16\pi^4}{L^4} U^2 - 1$$

so that

$$\beta \geq 4\pi^2 \alpha^2 Re^{-1} \quad \text{and} \quad \gamma \geq 16\pi^4 \alpha^4 Re^{-1}$$

where $\alpha = (k_f L)^{-1} \leq (2\pi)^{-1}$ is the ratio of the forcing to domain length scales. If $\beta$ and $\gamma$ scale both as $\sim Re^{-1}$ then we say that the flow exhibits laminar behavior because the energy is then necessarily concentrated at relatively long length scales determined by the prefactor, rather than over a broad range of scales that increases as $Re \rightarrow \infty$.

On the other hand if $\beta \sim Re^0$, the scaling expected in three dimensional turbulence, the flow exhibits finite (residual) dissipation in the limit of zero viscosity indicating the presence of an active and effective energy cascade to small scales. It was recently shown $22, 24$ that $\beta \leq \epsilon Re^0$ for the vanishing viscosity limit of three dimensional versions of the systems under consideration here and in section III, where the coefficient $\epsilon$ is uniform in $\nu$, $L$, and $F$. There is, however, no known a priori enstrophy dissipation rate bound for the three dimensional turbulence; this is related to the outstanding open question of the regularity of solution for the three dimensional Navier-Stokes equations. As the results of this paper suggest quantitatively, the dissipation rates of two dimensional turbulence falls somewhere between laminar scalings and the rates for three dimensional turbulence.

To prove the two dimensional bounds we first take the inner product of the vorticity equation (11) with $\omega$ and average to obtain the enstrophy production-dissipation balance

$$\chi = \langle \omega \phi \rangle \quad (15)$$

where the time derivative term drops out when we take the long time average. Integrating by parts to move the $\mathbf{k} \cdot \nabla \chi$ from $\omega$ onto $\phi$ and the Cauchy-Schwarz inequality, we easily obtain

$$\chi \leq k_f^2 U F. \quad (16)$$

For the second step, consider a smooth incompressible vector field $\mathbf{v}(x, y)$. Take the inner product of $\mathbf{v}$ with the Navier-Stokes equation, integrate by parts and average to obtain

$$\frac{1}{L^2} \int_{T_L^2} \mathbf{v} \cdot \mathbf{f} d^2x = -\langle \mathbf{u} \cdot (\nabla \mathbf{v}) \cdot \mathbf{u} + \nu \mathbf{u} \cdot \nabla^2 \mathbf{v} \rangle. \quad (17)$$

Using the Cauchy-Schwarz and Hölder’s inequality (as in $22$) we deduce

$$F \times \frac{1}{L^2 \parallel \mathbf{f} \parallel} \int_{T_L^2} \mathbf{v} \cdot \mathbf{f} d^2x \leq U^2 \parallel \nabla \mathbf{v} \parallel_\infty + \nu \frac{U}{L^2} \parallel \nabla^2 \mathbf{v} \parallel \quad (18)$$

where $\parallel \cdot \parallel_\infty$ is the $L_\infty$ norm on $T_L^2$. In order for the inequality to be non-trivial we need to restrict $\mathbf{v}$ so that $\int_{T_L^2} \mathbf{v} \cdot \mathbf{f} d^2x > 0$. This is easy to arrange. For example the choice $\mathbf{v} = \mathbf{f}/F$ will satisfy this condition if $\mathbf{f}$ is sufficiently smooth that the right hand side of (13) is finite. If it is not so smooth, then for instance we can take $\mathbf{v} \sim K \mathbf{f}$ where $K(x, y, x', y')$ is a (positive) smoothing kernel. In any case we can choose $\mathbf{v}$ appropriately and use (13) to eliminate $F$ in (11) so that:

$$\chi \leq U^3 k_f^3 \left( C_1 + \frac{C_2}{Re} \right) \Rightarrow \gamma \leq \left( C_1 + \frac{C_2}{Re} \right) \quad (19)$$

where the dimensionless coefficients $C_1$ and $C_2$ are independent of $k_f$ and $L$, depending only on the functional “shape” of $\mathbf{v}$ (and thus also on the shape of $\mathbf{f}$) but not on its amplitude $F$ or the viscosity $\nu$. Explicitly they are

$$C_1 = \frac{\parallel \nabla \mathbf{v} \parallel_\infty}{\langle \mathbf{v} \cdot \mathbf{f}/F \rangle} \quad \text{and} \quad C_2 = \frac{\langle |\nabla^2 \mathbf{v}| \rangle^{1/2}}{|\mathbf{v} \cdot \mathbf{f}/F|} \quad (20)$$

where $\nabla_l$ has the gradient with respect to the non-dimensional coordinate $k_f x$. An upper bound for the enstrophy dissipation rate like that in (13) was first obtained in (21). Note that for strictly band-limited forces, i.e., if the Fourier transform of the force is supported on wavenumbers with $|k| \leq (k_{min}, k_{max})$ with $0 < k_{min} < k_{max} < \infty$, then the coefficients $C_1$ and $C_2$ are bounded by pure numbers depending only on $k_{max}/k_{min}$.

For the final step of the proof we use integrations by parts and the Cauchy-Schwarz inequality to see that

$$\langle \omega^2 \rangle = \langle (\mathbf{u} \cdot \nabla \times (k \omega))^2 \rangle \leq \langle |\mathbf{u}|^2 \rangle \langle |\nabla \omega|^2 \rangle . \quad (21)$$

Combining (21) with (15) we deduce

$$\langle \omega^2 \rangle \leq \frac{k_f U^5}{\nu} \left( C_1 + \frac{C_2}{Re} \right), \quad (22)$$

and in terms of the energy dissipation rate this is the announced result

$$\beta \leq Re^{-1/2} \left( C_1 + \frac{C_2}{Re} \right)^{1/2} \quad (23)$$
III. TIME-DEPENDENT FORCES

Now consider the Navier-Stokes equation where the time dependent body force \( f(x, y, t) \) is smooth and incompressible with characteristic length scale \( \sim k_f^{-1} \) given by

\[
k_f^2 = \frac{\langle |\nabla^2 f|^2 \rangle}{\langle |f|^2 \rangle},
\]

and time scale \( \Omega_f^{-1} \) defined by

\[
\Omega_f^2 = \frac{\langle |\partial_t f|^2 \rangle}{\langle |f|^2 \rangle}.
\]

We define

\[
\tau = \frac{\Omega_f}{k_f U},
\]

the ratio of the “eddy turnover” time \( (k_f U)^{-1} \) to the forcing time scale \( \Omega_f^{-1} \). In this time-dependent setting the amplitude \( F \) of the force is

\[
F = \langle |f|^2 \rangle^{1/2}.
\]

As before, the space and time average of \( \omega \) times the vorticity equation yields the enstrophy balance equation, and integration by parts and the Cauchy-Schwarz inequality implies

\[
\chi \leq k_f^2 U F.
\]

For the second step here we introduce a smooth incompressible vector field \( \mathbf{v}(x, y, t) \) and take space and time average of the inner product of with the Navier-Stokes equation to obtain

\[
\langle \mathbf{v} \cdot f \rangle = - \langle \mathbf{u} \cdot \partial_t \mathbf{v} + \mathbf{u} \cdot (\nabla \mathbf{v}) \cdot \mathbf{u} + \nu \mathbf{u} \cdot \nabla^2 \mathbf{v} \rangle.
\]

Cauchy-Schwarz and H"older’s inequalities then imply

\[
F \langle \mathbf{v} \cdot f \rangle \leq U \langle |\partial_t \mathbf{v}|^2 \rangle^{1/2} + U^2 \sup_t \| \nabla \mathbf{v} \|_\infty + \nu U \langle \nabla^2 \mathbf{v} \rangle^{1/2}.
\]

Now we need to be able to choose \( \mathbf{v} \) satisfying \( \langle \mathbf{v} \cdot f \rangle > 0 \) such that all the coefficients on the right hand side are all finite. Our ability to do this depends on details of \( f(x, y, t) \).

For example if \( f \) is sufficiently smooth in space and appropriately uniformly bounded in time then we can choose \( \mathbf{v} \sim f \). We could also choose \( \mathbf{v} \) as an appropriately filtered version of \( f \) to cover more general cases. For the purposes of this study and to display the results in the clearest (if not the sharpest or most general form) we will simply presume that \( f \) is sufficiently regular that we can take \( \mathbf{v} = f \). In that case becomes

\[
F \leq \Omega_f U + U^2 \sup_t \| \nabla f \|_\infty + \nu k_f^2 U.
\]

Then using this to eliminate \( F \) from we have

\[
\chi \leq k_f^2 U^3 \left( \tau + C_3 + \frac{1}{Re} \right) \Rightarrow \gamma \leq \left( \tau + C_3 + \frac{1}{Re} \right)^{1/2},
\]

where the coefficient \( C_3 \) is

\[
C_3 = \frac{\sup_t \| \nabla f \|_\infty}{F}.
\]

with \( \nabla_i \) denoting the gradient with respect to the non-dimensional coordinate \( k_f x \). The dimensionless number \( C_3 \) is independent of the scales of \( F \), \( k_f \), \( L \), etc., depending only on the “shape” of \( f \). For example if \( f \) is quasi-periodic with \( N \) frequencies and involves only wavenumbers \( k \) with \( 0 < k_{min} < |k| < k_{max} < \infty \), then \( C_3 \) is bounded by \( \sqrt{N} \) times a function of \( k_{max}/k_{min} \).

The final step again uses the inequality

\[
e^2 = \nu^2 \langle \omega^2 \rangle = \nu^2 \langle \mathbf{u} \cdot \nabla \times (\hat{k} \omega) \rangle^2 \leq \nu U^2 \chi
\]

and it then follows immediately from that

\[
\beta \leq Re^{-1/2} \left( \tau + C_3 + \frac{1}{Re} \right)^{1/2}.
\]

Note in this case \( \tau \) depends on \( U \) and features of the forcing through \( k_f \) and \( \Omega_f \), but not on \( \nu \).

IV. MONOCHROMATIC AND CONSTANT FLUX FORCES

An even sharper scaling bound on the energy and enstrophy dissipation rates can be derived when the driving is monochromatic in space, whether it is steady or time dependent. Suppose the body force involves only a single length scale, i.e.,

\[
-\nabla^2 f = k_f^2 f.
\]

This does not preclude complex time-dependence for \( f(x, y, t) \), just that it involves only spatial modes with wavenumbers \( k \) with \( |k| = k_f \). Then the enstrophy production-dissipation balance implies

\[
\chi = \langle \omega \phi \rangle = \langle \mathbf{u} \cdot (-\nabla^2 f) \rangle = k_f^2 \langle \mathbf{u} \cdot f \rangle = k_f^2 \epsilon.
\]

Combining this with , we observe that

\[
\epsilon \leq \nu k_f^2 U^2 \chi
\]

so that

\[
\epsilon \leq \nu k_f^2 U^2 \quad \text{and} \quad \chi \leq \nu k_f^2 U^2
\]

implying that both \( \beta \) and \( \gamma \) are bounded by \( Re^{-1} \). Note that this kind of monochromatic forcing is a special case that has been shown in the literature for some cases to lead to a laminar flow that never loses stability.
An other type of forcing that results in this scaling is
\[ f(x, y, t) = \epsilon \frac{\mathcal{P} \mathbf{u}}{L^{-2}||\mathcal{P} \mathbf{u}||^2} \tag{40} \]
where \( \mathcal{P} \) is the projector onto spatial modes of wavenumber \( \mathbf{k} \) with \( |k| \in [k_{\text{min}}, k_{\text{max}}] \), and the coefficient \( \epsilon \) is now the control parameter. This type of forcing is often applied in numerical simulations of homogeneous isotropic turbulence. With this forcing in the Navier-Stokes equations constitutes an autonomous dynamical system with kinetic energy injected at a constant rate \( \epsilon \) at wavenumbers with \( |k| \in [k_{\text{min}}, k_{\text{max}}] \). The rms speed \( U \) (i.e., the Reynolds number) and the enstrophy dissipation rate \( \chi \) are then emergent quantities, determined by the imposed energy flux \( \epsilon \). The mean power balance for solutions is still
\[ \nu(\langle \nabla \mathbf{u}^2 \rangle) = \nu(\langle \omega^2 \rangle) = \epsilon, \tag{41} \]
and the enstrophy production-dissipation balance reads
\[ \chi = \nu(\langle \nabla \omega^2 \rangle) = \epsilon \left( \frac{\langle |\nabla \mathcal{P} \mathbf{u}|^2 \rangle}{||\mathcal{P} \mathbf{u}||^2} \right). \tag{42} \]
Because the forcing only involves wavenumbers in \( [k_{\text{min}}, k_{\text{max}}] \) with positive energy injection at each wavenumber, at each instant of time
\[ k_{\text{min}}^2 \nu ||\mathcal{P} \mathbf{u}||^2 \leq ||\nabla \mathcal{P} \mathbf{u}||^2 \leq k_{\text{max}} \nu ||\mathcal{P} \mathbf{u}||^2. \tag{43} \]
Then \( \epsilon \) implies that
\[ k_{\text{min}}^2 \epsilon \leq \chi \leq k_{\text{max}} \epsilon. \tag{44} \]
Using this with \( \epsilon \) we see that
\[ \epsilon^2 \leq \nu U^2 \chi \leq \nu U^2 k_{\text{max}}^2 \epsilon, \tag{45} \]
and we conclude
\[ \epsilon \leq \nu k_{\text{max}}^2 U^2 \quad \text{and} \quad \chi \leq \nu k_{\text{max}}^4 U^2. \tag{46} \]
Hence also in this case both \( \beta \) and \( \gamma \) are bounded \( \sim Re^{-1} \).

Note that in both these derivations a condition like \( \chi \) or the stronger condition \( \epsilon^4 \) was used. It is an open question whether such a condition holds for more general and more “realistic” forcing functions. The results \( \epsilon \) and \( \epsilon^4 \) give restrictions on the energy and enstrophy dissipation rate for a broader class of driving, but it is natural to wonder how broad of a class of forcing functions would actually result in the \( Re^{-1} \) scalings in the vanishing viscosity limit.

V. DISCUSSION

These quantitative bounds show that for two dimensional turbulence sustained by forces as described in the previous sections, there is no residual dissipation in the vanishing viscosity limit defined by \( Re \to \infty \) at fixed \( U, L, k_f \) and \( \Omega_I \). To be precise, \( \epsilon \) vanishes at least as fast as \( Re^{-1/2} \) in this limit. This confirms that there is no forward energy cascade in the steady state in the inviscid limit. On the other hand the residual enstrophy dissipation allowed by \( \epsilon^4 \) in this limit does not rule out a forward enstrophy cascade. This combination, \( \epsilon \to 0 \) with \( \chi = O(1) \) in the inviscid limit, is consistent with the dual-cascade picture of two-dimensional turbulence developed by Kraichnan, Leith, and Batchelor. Note however that the absence of forward energy cascade is not necessarily true for any finite value of \( Re \). The \( \beta \to Re^{-1/2} \) scaling allowed by the bound is less severe than what a laminar flow (or a flow with only inverse cascade of energy) would predict, and as a result \( \epsilon^4 \) does not exclude the presence of some direct sub-dominant cascade of energy when the Reynolds number is finite as suggested by \( \epsilon \).

On the other hand the direct cascade of enstrophy is necessarily absent for some forcing functions (see \( \epsilon^4 \) and \( \epsilon \)). When the forcing acts at a single scale or constant power is injected in a finite band of wavenumbers, both \( \epsilon \) and \( \chi \) vanish \( \sim Re^{-1} \). This suggests an essentially laminar behavior for these flows: if the energy spectrum follows a power law \( E(k) \sim k^{-\alpha} \) for large wavenumbers then the exponent must be \( \alpha \leq -5 \) for \( \chi \) to vanish in the vanishing viscosity limit. These results have been interpreted as absence of enstrophy cascade in finite domains. However, both of these results rely on the condition \( \epsilon^4 \) which is not guaranteed for a general forcing functions. Whether \( \epsilon^4 \) might hold for more general forcing functions is an open question; the results \( \epsilon \) and \( \epsilon^4 \) give the restrictions on the energy and enstrophy dissipation rate for a general forcing. Note that the \( \beta \sim Re^{-1/2} \) does not impose significant restriction on the energy spectrum given the bound on \( \chi \). These considerations suggest that it is possible therefore that in two dimensional turbulence the steady state energy spectrum depends on the type of forcing used, even within the class of relatively narrow-band driving. High resolution numerical simulations with forcing that does not necessarily satisfy \( \epsilon^4 \) would be useful at this point to resolve this issue.

We conclude by noting that an interesting question that follows from these results is that of the \( Re^{-1} \)-scaling of the energy dissipation in systems that almost have two dimensional behavior like strongly rotating, strongly stratified or conducting fluids in the presence of a strong magnetic field. For example, is there a critical value of the rotation such that the scaling of the energy dissipation rate with the Reynolds number transitions from \( \epsilon \sim Re^0 \) to \( \epsilon \sim Re^{-1/2} ? \) These questions remain for future studies.

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