Mode-sum regularization of the scalar self-force: Formulation in terms of a tetrad decomposition of the singular field

Roland Haas and Eric Poisson
Department of Physics, University of Guelph, Guelph, Ontario, Canada N1G 2W1
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We examine the motion in Schwarzschild spacetime of a point particle endowed with a scalar charge. The particle produces a retarded scalar field which interacts with the particle and influences its motion via the action of a self-force. We assume that the magnitude of the scalar charge is small, and that the deviations from geodesic motion produced by the self-force are small. This problem is analogous to that of an electric charge moving under the action of its electromagnetic self-force, and to that of a small mass moving under the action of its gravitational self-force. We exploit the spherical symmetry of the Schwarzschild spacetime and decompose the scalar field in spherical-harmonic modes. Although each mode is bounded at the position of the particle, a mode-sum evaluation of the self-force requires regularization because the sum does not converge: the retarded field is infinite at the position of the particle. The regularization procedure involves the computation of regularization parameters, which are obtained from a mode decomposition of the Detweiler-Whiting singular field; these are subtracted from the modes of the retarded field, and the result is a mode-sum that converges to the actual self-force. We present such a computation in this paper. While regularization parameters have been presented before in the literature, there are two main aspects of our work that are new. First, we define the regularization parameters as scalar quantities by referring them to a tetrad decomposition of the singular field. This is different from standard practice, which is to define regularization parameters as vectorial quantities. The advantage of dealing with tetrad components is that these, unlike vector components, are naturally decomposed in scalar spherical harmonics. Second, we calculate, for any bound orbit around a Schwarzschild black hole, four sets of regularization parameters (denoted schematically by $A$, $B$, $C$, and $D$) instead of the usual three ($A$, $B$, and $C$). While only the first three regularization parameters are needed to produce a convergent mode-sum, the inclusion of a fourth parameter has the practically important consequence of accelerating the convergence. The focus of this paper is entirely on the computation of regularization parameters for the scalar self-force. The techniques that we introduce in this work are not, however, restricted to this context. They will readily be exported to the electromagnetic and gravitational cases, but we leave this generalization for future work. As proof of principle that our methods are reliable, we calculate the self-force acting on a scalar charge in circular motion around a Schwarzschild black hole, and compare our answers with those recorded in the literature. We leave for future work the generalization of this calculation to generic orbits.

I. INTRODUCTION AND SUMMARY

A. Motivation

The capture of solar-mass compact objects by massive black holes residing in galactic centers has been identified as one of the most promising sources of gravitational waves for the Laser Interferometer Space Antenna [1]. The need for accurate templates for signal detection and source identification is currently motivating an intense effort from many workers to determine the motion of a relativistic two-body system with a small mass ratio. This is done in a treatment that goes beyond the test-mass description in which the small mass moves on a geodesic in the spacetime of the large black hole. An additional complication arises from the fact that the treatment cannot rely on a slow-motion or weak-field approximation. The accelerated motion of the small mass in the background spacetime of the large black hole is governed by the body’s gravitational self-force [2, 3], which encodes the influence of the body’s own gravitational field on its motion. To compute the gravitational self-force acting on a body moving on a bound orbit around a Schwarzschild or Kerr black hole is currently the focus of much work; for reviews, see Refs. [4, 5] and a special issue of Classical and Quantum Gravity [6] devoted to this topic.

The complexities associated with the computation and interpretation of the gravitational self-force have motivated the formulation of educational toy problems. These have the advantage of being much simpler to deal with, but they should nevertheless capture the essential physics of self-forced motions in curved spacetime. One such toy problem, which is based on real-world physics that is interesting in its own right, is the motion of an electrically charged particle in curved spacetime, subjected to a self-force produced by its own electromagnetic field. The foundations for this problem were laid more than 45 years ago by DeWitt and Brehme [7] (their work was later corrected by Hobbs [8]).

Another such problem is the motion of a particle endowed with a scalar charge; here the foundations were provided
more recently by Quinn [9]. In spite of its academic nature (there are no known macroscopic particles endowed with a scalar charge), the scalar self-force problem is a useful toy problem because of its relative technical simplicity, and because the motion of a particle under the influence of its scalar self-force is expected to resemble closely the motions obtained in the other contexts (electromagnetic and gravitational). The self-forced motion of a scalar charge in Schwarzschild spacetime is the focus of the work presented in this paper. We exploit this simple situation to introduce computational techniques that are required for the concrete evaluation of the self-force. These techniques, however, are not limited to the context of the scalar self-force, and they will readily be exported to the electromagnetic and gravitational cases. We leave this generalization for future work.

B. The problem

Consider a particle of mass \( m \) and scalar charge \( q \) moving on a world line \( \gamma \) of the Schwarzschild spacetime. The world line is described by the parametric relations \( z^\alpha(\tau) \) in which \( \tau \) is proper time. The particle produces a retarded scalar potential \( \Phi \) that satisfies the inhomogeneous wave equation [9]

\[
\Box \Phi(x) = -4\pi q \int_\gamma \delta_4(x,z) \, d\tau, \tag{1.1}
\]

in which \( \Box := g^{\alpha\beta} \nabla_\alpha \nabla_\beta \) is the covariant wave operator, and \( \delta_4(x,z) \) is a scalarized Dirac distribution with support on the world line. Except for the factor \(-4\pi\) inserted for convenience, the right-hand side of Eq. (1.1) is the particle’s scalar charge density.

The retarded potential \( \Phi(x) \) produces a field \( \Phi_\alpha(x) := \nabla_\alpha \Phi(x) \) that acts on the particle and influences its motion. As shown by Quinn [9], the particle’s acceleration is proportional to the components of \( \Phi_\alpha \) that are orthogonal to the particle’s velocity vector \( u^\alpha := dz^\alpha/d\tau \),

\[
ma^\alpha = q(g^{\alpha\beta} + u^\alpha u^\beta) \Phi_\beta(z), \tag{1.2}
\]

where \( a^\alpha := Du^\alpha/d\tau \) is the particle’s acceleration vector, the covariant derivative of the velocity vector along the world line. Quinn also showed that the longitudinal component of the retarded field is responsible for a change in the particle’s inertial mass,

\[
\frac{dm}{d\tau} = -qu^\alpha \Phi_\alpha(z). \tag{1.3}
\]

This effect was explored in cosmological situations in Refs. [10, 11].

Equations (1.2) and (1.3) have only formal validity because the singular field \( \Phi_\alpha(x) \) diverges as \( x \to z \): the field of a point charge is necessarily infinite at the position of the particle. Quinn [9] was able to regularize these equations so as to produce meaningful equations of motion for the charged particle. In breakthrough work that plays a central role in this paper, Detweiler and Whiting [12] showed that Quinn’s regularization procedure amounts to a decomposition of the retarded potential into uniquely defined singular and regular potentials,

\[
\Phi(x) = \Phi^S(x) + \Phi^R(x), \tag{1.4}
\]

with \( \Phi^S \) denoting the singular potential and \( \Phi^R \) the regular potential. As was shown by Detweiler and Whiting, the singular potential possesses the following properties: (i) it satisfies the same wave equation as the retarded potential, that is, it is a solution to Eq. (1.1); (ii) it displays the same singularity structure as the retarded potential near the particle’s world line; and (iii) it does not exert a force on the point charge. These properties make the decomposition of Eq. (1.4) unique. The regular potential, on the other hand, possesses the following properties: (i) it satisfies a homogeneous version of Eq. (1.1), with a zero right-hand side; (ii) it is smooth on and near the particle’s world line; and (iii) it alone determines the self-force acting on the particle. The conclusion, therefore, is that the actual equations of motion for the particle are Eqs. (1.2) and (1.3) with the regular field \( \Phi^R_\alpha := \nabla_\alpha \Phi^R \) substituted in place of the retarded field \( \Phi_\alpha \); this conclusion is in full agreement with Quinn’s earlier work. The Detweiler-Whiting decomposition also plays an essential role in the electromagnetic and gravitational self-force problems.

In Schwarzschild spacetime it is computationally advantageous to solve Eq. (1.1) after decomposing \( \Phi(x) \) in spherical harmonics \( Y_{lm}(\theta, \phi) \). Adopting the usual Schwarzschild coordinates \( [t, r, \theta, \phi] \), we would write

\[
\Phi(x) = \sum_{lm} \Phi_{lm}(t,r) Y_{lm}(\theta, \phi) \tag{1.5}
\]
and call $\Phi_{lm}(t,r)$ the spherical-harmonic modes of the retarded potential; the right-hand side involves a sum over all integers $l = 0, 1, 2, \ldots, \infty$ and a nested sum over all integers $m = -l, -l+1, \ldots, l-1, l$. After performing the decomposition of Eq. (1.5), Eq. (1.1) turns into a two-dimensional wave equation for each mode function $\Phi_{lm}(t,r)$; this equation is displayed in Eq. (1.12), below.

The spherical-harmonic modes of the retarded potential give rise to quantities $\Phi_{\alpha l}(x) := \nabla_\alpha \sum_m \Phi_{lm} Y_{lm}$, such that the retarded field can be expressed as the mode-sum

$$\Phi_{\alpha}(x) = \sum_l \Phi_{\alpha l}(x). \quad (1.6)$$

We call these quantities the multipole coefficients of the retarded field. Each $\Phi_{\alpha l}(x)$ is bounded in the limit $x \to z$, in spite of the fact that the retarded field is infinite on the particle’s world line. (The multipole coefficients are discontinuous at $x = z$.) The mode-sum of Eq. (1.6), of course, does not converge when the retarded field is evaluated on the world line.

The failure of Eq. (1.6) to converge when $x = z$ is exactly compensated for by the failure of

$$\Phi_{\alpha}^{S}(x) = \sum_l \Phi_{\alpha l}^{S}(x) \quad (1.7)$$

to converge, because (as was noted previously) the retarded and singular fields share the same singularity structure near the world line. Here, $\Phi_{\alpha l}^{S} := \nabla_\alpha \sum_m \Phi_{lm}^{S} Y_{lm}$ with $\Phi_{lm}^{S}$ denoting the spherical-harmonic modes of the singular potential. The regular field can thus be expressed as

$$\Phi_{\alpha}^{R}(z) = \lim_{x \to z} \left[ \Phi_{\alpha l}(x) - \Phi_{\alpha l}^{S}(x) \right], \quad (1.8)$$

in terms of a converging mode-sum. The limiting procedure involved in Eq. (1.8) is introduced to handle the (shared) discontinuity of the multipole coefficients $\Phi_{\alpha l}(x)$ and $\Phi_{\alpha l}^{S}(x)$ at $x = z$. After subtraction the multipole coefficients of the regular field are smooth at $x = z$, the mode-sum converges, and the limit can be taken without difficulty.

The prescription contained in Eq. (1.8) becomes a practical method to evaluate $\Phi_{\alpha}^{R}(z)$ — and therefore the self-force acting on the scalar charge — when one can actually compute the quantities $\Phi_{\alpha l}^{S}(x)$ for a field point $x$ close to the world line. As we shall review below, this computation is possible because the singular field $\Phi_{\alpha}^{S}(x)$ is known in a neighborhood of the world line; it can be expressed as a Laurent expansion in powers of the distance to the world line. The end result for the multipole coefficients takes the schematic form

$$\Phi_{\alpha l}(x) = q \left[ (l + \frac{1}{2}) A_{l} + B_{l} + \frac{C_{l}}{(l + \frac{1}{2})} + \cdots \right], \quad (1.9)$$

in which the quantities $A_{l}$, $B_{l}$, and $C_{l}$, known as regularizer parameters, are independent of $l$ but depend on the state of motion of the particle at $z$. Notice that the sum over the $A_{l}$ term would diverge quadratically, that the sum over the $B_{l}$ term would diverge linearly, and that the sum over the $C_{l}$ term would diverge logarithmically. The remaining terms, those designated with $(\cdots)$, lead to a converging sum that evaluates to zero by virtue of the Detweiler-Whiting axiom, according to which the singular field does not produce a force on the particle. This ensures that after removal of the $A_{l}$, $B_{l}$, and $C_{l}$ terms, the mode-sum of Eq. (1.8) will converge to the correct value for $\Phi_{\alpha l}^{R}(z)$.

Our central task in this paper is the computation of regularization parameters for a scalar charge moving on a bound orbit around a Schwarzschild black hole. We leave for future work the completion of a calculation of the actual self-force. This would involve, over and above the work presented here, a numerical determination of the retarded field produced by a scalar charge moving in Schwarzschild spacetime.

We consider the particle’s charge $q$ to be small, and we have in mind a perturbative implementation of Eqs. (1.2) and (1.3). In a zeroth-order approximation, the particle is taken to have a constant inertial mass and to move on a geodesic of the Schwarzschild spacetime. In a first-order approximation, the regular field $\Phi_{\alpha l}^{R}$ is computed for this geodesic motion and substituted on the right-hand sides of Eqs. (1.2) and (1.3). This iterative process could be continued, but we suppose that $q$ is sufficiently small that the process can be stopped after a single iteration. In this regime the self-force can be computed while assuming that the motion is geodesic. We adopt this small-charge approximation here, noting that it reflects the spirit of the small-mass-ratio approximation in the gravitational problem. The assumption leads to much simplification; the computation of regularization parameters for accelerated particles would be significantly more involved.
C. Past work

We are not the first researchers to define and compute regularization parameters for the mode-sum computation of self-forces in curved spacetime. In fact, the literature is vast and the field has already reached a fairly mature state. But as we argue below, we believe that this work (and our promise for extensions toward the electromagnetic and gravitational problems) is a significant addition.

The main ideas behind the mode-sum regularization of self-forces were first introduced by Barack and Ori in their pioneering work [13–15], which was later perfected and extended by an Israeli-Japanese consortium including Barack, Ori, Mino, Nakano, and Sasaki [16–19]. These authors computed regularization parameters for the mode-sum evaluation of the (scalar, electromagnetic, and gravitational) self-force acting on a particle moving on any bound geodesic of the Schwarzschild spacetime. Barack and Ori [20] were then able to extend these results to the Kerr spacetime. This early work on regularization parameters is nicely summarized in Ref. [21].

The (so-called early) work reviewed in the preceding paragraph was carried out before the discovery by Detweiler and Whiting of the retarded field’s decomposition into uniquely identified singular and regular pieces [12]. It relied on an alternative decomposition, in terms of “direct” and “tail” pieces, which did not come with the same degree of mathematical elegance. For example, unlike the regular field which is smooth everywhere in a neighborhood of the world line, the tail part of the retarded field is not differentiable on the world line. The Detweiler-Whiting decomposition provides a much sounder foundation for the definition and computation of regularization parameters. In follow-up works, Detweiler, Messaritaki, and Whiting [22] carried out such a computation for a scalar charge moving on a circular orbit around a Schwarzschild black hole, and Kim [23] extended these results to a generic orbit.

A number of works present explicit computations of the self-force (for various charges undergoing various motions in various spacetimes) by mode-sum techniques. Burko computed the self-force acting on an electric charge in circular motion in Minkowski spacetime [24]. He and his coworkers also considered scalar and electric charges kept stationary in a Schwarzschild spacetime [25], in a spacetime that contains a spherical matter shell (Burko, Liu, and Soen [26]), and in a Kerr spacetime (Burko and Liu [27]). In addition, Burko computed the scalar self-force acting on a particle in circular motion around a Schwarzschild black hole [28]. This calculation was since revisited by Detweiler, Messaritaki, and Whiting [22], as well as Diaz-Rivera, Messaritaki, and Whiting [29], who also considered the case of slightly eccentric motion. Barack and Burko dealt with a particle falling radially into a Schwarzschild black hole, and evaluated the scalar self-force acting on this particle [30]; Lousto [31], and Barack and Lousto [32], computed the gravitational self-force for radial infall.

D. This work

In this paper we present a new calculation of regularization parameters for the mode-sum evaluation of the scalar self-force acting on a particle moving on a bound geodesic of the Schwarzschild spacetime. Our calculation is new in two main respects.

First, we define the regularization parameters as scalar quantities by referring them to a tetrad decomposition of the singular field. In contrast, the original parameters of Eq. (1.9) are vectorial quantities that refer to $\Phi^S_\alpha$, the vectorial components of the singular field. The idea here is to introduce a basis of orthonormal vectors $e^{(\mu)}_{\alpha}(x)$ at every point of the Schwarzschild spacetime; the superscript $\alpha$ is the usual vectorial index, and the subscript $(\mu) = \{0, 1, 2, 3\}$ is a label that designates an individual member of the tetrad. The vectors satisfy $g_{\alpha\beta} e^{(\mu)}_{\alpha} e^{(\nu)} = \eta_{(\mu)(\nu)} = \text{diag}[-1, 1, 1, 1]$. The four quantities $\Phi^S_{(\mu)} := e^{(\mu)}_{(\mu)} \Phi^S$ are the frame components of the singular field, and these are scalar functions of the spacetime coordinates.

The advantage of introducing the tetrad and the associated decomposition of vector fields is that each frame component $\Phi^S_{(\mu)}(x)$ is a scalar function that can naturally be expanded in scalar spherical harmonics. This is quite unlike the vector $\Phi^S(x)$, which could be expanded elegantly in vectorial harmonics (a procedure that has not been adopted in the self-force literature) or inelegantly in scalar harmonics (as was done in all previous works on regularization parameters). By introducing the tetrad we are able, in this work, to provide an elegant definition for the regularization parameters. Our specific choice of tetrad will be specified below.

Second, we calculate an additional set of regularization parameters in order to accelerate the numerical convergence of the mode-sum. Together with our implementation of the tetrad decomposition, this amounts to replacing Eq. (1.8) by

$$\Phi^R_{(\mu)}(z) = \lim_{x \to z} \sum_l \left[ \Phi_{(\mu)l}(x) - \Phi^S_{(\mu)l}(x) \right], \quad (1.10)$$
and Eq. (1.9) by

$$\Phi_{(\mu)l}^S = q \left[ (l + \frac{1}{2})A_{(\mu)} + B_{(\mu)} + \frac{C_{(\mu)}}{(l + \frac{1}{2})} + \frac{D_{(\mu)}}{(l - \frac{1}{2})(l + \frac{3}{2})} + \cdots \right].$$

(1.11)

The regularization parameters $A_{(\mu)}$, $B_{(\mu)}$, and $C_{(\mu)}$ have already appeared (in vectorial form) in Eq. (1.9); the regularization parameters $D_{(\mu)}$ are new. We calculate all of these for a scalar charge moving on a bound geodesic of the Schwarzschild spacetime.

We note that the vectorial parameters $D_\alpha$ were computed by Detweiler, Messaritaki, and Whiting for the special case of circular motion in Schwarzschild spacetime [22], and that they were computed by Kim for generic orbits; Kim’s results are recorded in his PhD dissertation [33], but they have yet to appear in the peer-reviewed literature. We note also that because the vectors $e_\alpha^{(\mu)}$ contain an angular dependence, the operations of [multiplication by a tetrad vector] and [extraction of multipole coefficients] do not commute: $(e_\alpha^{(\mu)} \Phi_\alpha)_{l} \neq e_\alpha^{(\mu)} (\Phi_\alpha)_{l}$; as a consequence, our expressions for $A_{(\mu)}$, $B_{(\mu)}$, $C_{(\mu)}$, and $D_{(\mu)}$ cannot be compared directly to those for $A_\alpha$, $B_\alpha$, $C_\alpha$, and $D_\alpha$ that have appeared in the literature.

It is important to mention that the sum over $l$ of the $D_{(\mu)}$ term in Eq. (1.11) is actually zero, because $\sum_{l=0}^{\infty} (l - \frac{1}{2})(l + \frac{3}{2})^{-1} = 0$. And as we have seen, the same statement applies to the remaining terms in Eq. (1.11), those designated by (\cdots). These terms, therefore, do not contribute to the final value of $\Phi_{(\mu)l}(z)$ when the sum is evaluated in full, from $l = 0$ to $l = \infty$. Nevertheless, the $D_{(\mu)}$ term does play a useful role when the sum is truncated to some finite upper bound $l = l_{\text{max}}$; it produces a significant acceleration of the sum’s convergence. This property has practical importance, because to truncate the sum over $l$ is a computational necessity.

We compute the regularization parameters $A_{(\mu)}$, $B_{(\mu)}$, $C_{(\mu)}$, and $D_{(\mu)}$ by importing many techniques from the literature. In fact, an advantage of arriving late into this business is that we can pick and choose, from a variety of sources, the computational methods that are the most compelling. Thus, for example, in Sec. II we develop a covariant local expansion for $\Phi_\alpha^{(\mu)}(x)$ that is inspired by Mino, Nakano, and Sasaki [19], but which rests on the more secure Detweiler-Whiting decomposition of the retarded field into singular and regular pieces [12].

As another example, in Sec. V and the Appendix we employ the rotated angular coordinates $(\alpha, \beta)$ of Barack and Ori [17], but we do so without having to deal with vector components that are singular at the point of evaluation $(\alpha = 0)$. As a third example, in Sec. V we routinely make substitutions such as $\alpha \to \sqrt{2 - 2 \cos \alpha + O(\alpha^3)}$ to turn a function of the angles that is well-defined only in a neighborhood of $\alpha = 0$ to another function that is well-defined on the entire sphere. We stole this powerful idea from Detweiler, Messaritaki, and Whiting [22], but we implement it without making contact with the Thorne-Hartle-Zhang coordinates [34, 35]; these form an important part of their analysis, but they play no role here.

### E. Prescription

We summarize our main results in the form of a detailed prescription for the concrete evaluation of the scalar self-force. We recall that the force is acting on a particle moving on a generic orbit around a Schwarzschild black hole, and that it is evaluated within the small-charge approximation described near the end of Sec. I B.

**First step: Integrate the wave equation.** The first task that must be accomplished is to solve Eq. (1.1) for the retarded potential $\Phi(x)$. This is best accomplished by decomposing the potential in spherical harmonics, as in Eq. (1.5). Each mode $\Phi_{m}(t, r)$ of the retarded potential satisfies the reduced wave equation

$$\left\{ - \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r^2} - f \left[ \frac{l(l + 1)}{r^2} + \frac{2M}{r^3} \right] \right\} r \Phi_{lm} = - \frac{4\pi q f}{ru^3} Y_{lm}(\hat{\Phi}, 0) e^{-im\omega(t)} \delta(r - r(t)),

(1.12)$$

in which the right-hand side is the reduction of the scalar charge density $-4\pi q \int \delta_{s}(x, z) d\tau$. This equation must be solved while imposing ingoing-wave boundary conditions at the black-hole event horizon, and outgoing-wave boundary conditions at infinity. The modes are complex functions, related to each other by $\Phi_{l,m} = (-1)^m \Phi_{l,-m}$, in which an overbar indicates complex conjugation; this condition ensures that the scalar potential of Eq. (1.5) is real.

In Eq. (1.12), the reduced wave operator is written in terms of the tortoise coordinate $r^* = \int f^{-1} dr = r + 2M \ln(r/2M - 1)$, where $M$ is the black-hole mass and

$$f := 1 - \frac{2M}{r}.

(1.13)$$
The reduced charge density depends on the functions \( r(t) \) and \( \varphi(t) \), which give the coordinate positions of the world line. These are obtained by solving the geodesic equations,

\[
\dot{i} = \frac{E}{1 - 2M/r},
\]

\[
r^2 = E^2 - \left(1 - \frac{2M}{r}\right) \left(1 + \frac{L^2}{r^2}\right),
\]

\[
\dot{\varphi} = \frac{L}{r^2},
\]

in which an overdot indicates differentiation with respect to proper time \( \tau \). The constant \( E \) is the particle’s conserved energy per unit rest-mass, and the constant \( L \) is the conserved angular momentum per unit rest-mass. It is assumed that the particle is moving with \( \theta = \frac{\pi}{2} \) in the black hole’s equatorial plane, and the right-hand side of Eq. (1.12) also involves \( u^t = \dot{t} = E/f \), the time component of the particle’s velocity vector.

Equations (1.12) and (1.14)–(1.16) can be integrated with any reliable numerical method, and this procedure returns \( \Phi_{lm}(t, r) \) for selected values of \( l \) and \( m \). From this we extract the quantities

\[
\Phi_{lm}(t, r^+), \quad \frac{\partial}{\partial t} \Phi_{lm}(t, r^+), \quad \frac{\partial}{\partial r} \Phi_{lm}(t, r^+),
\]

which are evaluated at \( r = r^+ := r(t) + \Delta \), slightly away from the radial position of the particle at time \( t \). The symbol \( \Delta \) represents a small radial displacement, which can be either positive or negative. While the first two quantities listed above are actually continuous across \( r = r(t) \), the third quantity is discontinuous, and by evaluating it at \( r = r^+ \) we make its value unambiguous.

**Second step: Convert the modes.** The first step provides us with the spherical-harmonic modes of the retarded potential \( \Phi \). The computation of the self-force, however, as implemented in Eq. (1.10), requires the spherical-harmonic modes of \( \Phi_{(\mu)} := e_{\nu}^{(\mu)} \nabla_{\alpha} \Phi \), the frame components of the retarded field in the selected basis of orthonormal vectors. Our choice of tetrad is motivated in Sec. IV; in the usual ordering \([t, r, \theta, \phi] \) of the Schwarzschild coordinates we have

\[
e_{(0)}^\alpha = \left[ 1, \sqrt{f} \cos \theta \cos \phi, \frac{1}{r} \cos \theta \cos \phi, -\frac{\sin \phi}{r \sin \theta} \right],
\]

\[
e_{(1)}^\alpha = \left[ 0, \sqrt{f} \sin \theta \cos \phi, \frac{1}{r} \cos \theta \cos \phi, -\frac{\sin \phi}{r \sin \theta} \right],
\]

\[
e_{(2)}^\alpha = \left[ 0, \sqrt{f} \sin \theta \sin \phi, \frac{1}{r} \cos \theta \sin \phi, \frac{\cos \phi}{r \sin \theta} \right],
\]

\[
e_{(3)}^\alpha = \left[ 0, \sqrt{f} \cos \theta, \frac{1}{r} \sin \theta, 0 \right].
\]

In practice it is useful to introduce, as substitutes for \( e_{(1)}^\alpha \) and \( e_{(2)}^\alpha \), the complex combinations \( e_{(\pm)}^\alpha := e_{(1)}^\alpha \pm ie_{(2)}^\alpha \), or

\[
e_{(\pm)}^\alpha = \left[ 0, \sqrt{f} \sin \theta e^{\pm i \phi}, \frac{1}{r} \cos \theta e^{\pm i \phi}, \pm \frac{ie^{\pm i \phi}}{r \sin \theta} \right].
\]

As shown in Sec. IV, the spherical-harmonic modes \( \Phi_{(\mu)lm}(t, r) \) are given in terms of \( \Phi_{lm}(t, r) \) by

\[
\Phi_{(0)lm} = \frac{1}{\sqrt{f}} \frac{\partial}{\partial t} \Phi_{lm},
\]

\[
\Phi_{(+)}lm = -\sqrt{\frac{(l + m - 1)(l + m)}{(2l - 1)(2l + 1)}} \left(\sqrt{f} \frac{\partial}{\partial r} - \frac{l - 1}{r}\right) \Phi_{l-1,m+1}.
\]

\[
+\sqrt{\frac{(l - m + 1)(l - m + 2)}{(2l + 1)(2l + 3)}} \left(\sqrt{f} \frac{\partial}{\partial r} + \frac{l + 2}{r}\right) \Phi_{l+1,m-1},
\]

\[
-\sqrt{\frac{(l - m - 1)(l - m)}{(2l - 1)(2l + 1)}} \left(\sqrt{f} \frac{\partial}{\partial r} - \frac{l - 1}{r}\right) \Phi_{l-1,m-1}.
\]

\[

-\sqrt{\frac{(l + m + 1)(l + m + 2)}{(2l + 1)(2l + 3)}} \left(\sqrt{f} \frac{\partial}{\partial r} + \frac{l + 2}{r}\right) \Phi_{l+1,m+1},
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We have, in addition, after also involving Eq. (1.11).  

\[
\Phi_{(3)lm} = \sqrt{\frac{(l-m)(l+m)}{(2l-1)(2l+1)}} \left( \sqrt{\frac{r}{l}} \frac{\partial}{\partial r} - \frac{l-1}{r} \right) \Phi_{l-1,m} 
+ \sqrt{\frac{(l-m+1)(l+m+1)}{(2l+1)(2l+3)}} \left( \sqrt{\frac{r}{l}} \frac{\partial}{\partial r} + \frac{l+2}{r} \right) \Phi_{l+1,m}.
\] (1.26)

These can be evaluated at \( r = r^+ \) by substituting the values extracted in Eq. (1.17).

**Third step: Regularize the mode sum.** The spherical-harmonic modes \( \Phi_{(\mu)lm}(t, r) \) give rise to the multipole coefficients of the retarded field, which are defined by

\[
\Phi_{(\mu)lm}(t, r, \theta, \phi) := \sum_{m=-l}^{l} \Phi_{(\mu)lm}(t, r) Y_{lm}((\theta, \phi)).
\] (1.27)

The frame components of the retarded field are then given by

\[
\Phi_{(\mu)}(t, r, \theta, \phi) = \sum_{l} \Phi_{(\mu)l}(t, r, \theta, \phi).
\] (1.28)

This sum can be evaluated at \( r = r^+ := r(t) + \Delta, \theta = \frac{\pi}{2}, \) and \( \phi = \varphi(t) \). The sum would diverge in the limit \( \Delta \to 0 \), but it is to be regularized as in Eq. (1.10), which we rewrite as

\[
\Phi_{l}^R(t, r, \theta, \varphi) = \lim_{\Delta \to 0} \sum_{l} \left\{ \Phi_{(\mu)l}(t, r^+, \theta, \varphi) - q \left[ (l+\frac{1}{2})A_{(\mu)} + B_{(\mu)} + \frac{C_{(\mu)}}{(l+\frac{1}{2})^2} + \frac{D_{(\mu)}}{(l-\frac{1}{2})(l+\frac{1}{2})} + \cdots \right] \right\}
\] (1.29)

after also involving Eq. (1.11).

We compute the regularization parameters in Sec. V of this paper. We find

\[
A_{(0)} = \frac{i}{\sqrt{r^2 + L^2}} \text{sign}(\Delta),
\] (1.30)
\[
A_{(+)} = -e^{i\varphi} \frac{E}{\sqrt{r^2 + L^2}} \text{sign}(\Delta),
\] (1.31)
\[
A_{(3)} = 0,
\] (1.32)

where \( f := 1 - 2M/r \) and sign(\( \Delta \)) is equal to +1 if \( \Delta > 0 \) and to -1 if \( \Delta < 0 \). We have, in addition, \( A_{(-)} = \bar{A}_{(+)} \), \( A_{(1)} = \text{Re}[A_{(+)}] \), and \( A_{(2)} = \text{Im}[A_{(+)}] \).

We also find

\[
B_{(0)} = -\frac{E \dot{r}}{\sqrt{r^2 + L^2}^{3/2}} r^2 \delta + \frac{E \dot{r}}{2 \sqrt{r^2 + L^2}^{3/2}} \mathcal{K},
\] (1.33)
\[
B_{(+)} = e^{i\varphi} (B_{(+)}^c - iB_{(+)}^s),
\] (1.34)
\[
B_{(+)}^c = \left[ \frac{r^2}{\sqrt{r^2 + L^2}^{3/2}} + \frac{\sqrt{r^2 + L^2}}{2 \sqrt{r^2 + L^2}^{3/2}} \right] \delta,
\] (1.35)
\[
B_{(+)}^s = -\left[ \frac{(2 - \sqrt{r^2 + L^2}) \dot{r}}{2L \sqrt{r^2 + L^2}^{3/2}} \right] \delta + \frac{(2 - \sqrt{r^2 + L^2}) \ddot{r}}{2L \sqrt{r^2 + L^2}^{3/2}} \mathcal{K},
\] (1.36)
\[
B_{(3)} = 0.
\] (1.37)

We have, in addition, \( B_{(-)} = \bar{B}_{(+)} \), \( B_{(1)} = \text{Re}[B_{(+)}] = B_{(+)}^c \cos \varphi + B_{(+)}^s \sin \varphi \), and \( B_{(2)} = \text{Im}[B_{(+)}] = B_{(+)}^c \sin \varphi - B_{(+)}^s \cos \varphi \).

We have introduced the (rescaled) elliptic integrals

\[
\delta := \frac{2}{\pi} \int_{0}^{\pi/2} (1 - k \sin^2 \psi)^{1/2} d\psi = F(-\frac{1}{2}, \frac{1}{2}; 1; k)
\] (1.38)
and
\[ E := \frac{2}{\pi} \int_0^{\pi/2} (1 - k \sin^2 \psi)^{-1/2} d\psi = F\left(\frac{1}{2}, \frac{1}{2}; 1; k\right), \tag{1.39} \]
in which \( k := L^2/(r^2 + L^2) \). As indicated in Eqs. (1.38) and (1.39), the elliptic integrals can also be expressed in terms of hypergeometric functions.

We also find
\[ C_{(\mu)} = 0 \tag{1.40} \]
and
\[ D_{(0)} = -\left[ \frac{E r^3 (r^2 - L^2) r^3}{2 \sqrt{f(r^2 + L^2)^{3/2}}} + \frac{E(r^7 + 30 M r^6 - 7 L^2 r^5 + 114 M L^2 r^4 + 104 M L^4 r^2 + 36 M L^6 r)}{16 r^4 \sqrt{f(r^2 + L^2)^{5/2}}} \right] \mathcal{X}, \tag{1.41} \]
\[ D_{(+) - \mathcal{X}} \left[ \frac{E (5r^2 - 3L^2) r^3}{16 \sqrt{f(r^2 + L^2)^{7/2}}} + \frac{E(r^5 + 16 M r^4 - 3 L^2 r^3 + 42 M L^2 r^2 + 18 M L^4 r)}{16 r^2 \sqrt{f(r^2 + L^2)^{5/2}}} \right] \mathcal{X}, \tag{1.42} \]
\[ D_{(-)} = \frac{E (r^3 - 3L^2) r^3}{2 \sqrt{f(r^2 + L^2)^{3/2}}} + \frac{r^2}{4 (r^2 + L^2)^{3/2}} + \frac{3}{8 r^2 (r^2 + L^2)^{3/2}} \mathcal{X}, \tag{1.43} \]
\[ D_{(2)} = 0. \tag{1.44} \]

We have, in addition, \( D_{(-)} = \tilde{D}_{(+)}, D_{(1)} = \text{Re}[D_{(+) - \mathcal{X}}] = \tilde{D}_{(+) - \mathcal{X}} = \cos \varphi + \tilde{D}_{(+) - \mathcal{X}} \sin \varphi \), and \( D_{(2)} = \text{Im}[D_{(+) - \mathcal{X}}] = \tilde{D}_{(+) - \mathcal{X}} \sin \varphi - \tilde{D}_{(+) - \mathcal{X}} \cos \varphi \).

**Fourth step:** Construct the self-force. The mode-sum of Eq. (1.29) can now be evaluated. Thanks to the presence of the regularization parameters \( D_{(\mu)} \), the sum converges quickly to a precise estimate for the frame components \( \Phi_{(\mu)}^{R} \). The vector field \( \Phi^{R}_\alpha := \nabla_\alpha \Phi^{R} \) is related to these by
\[ \Phi^{R}_\alpha = -\Phi_{0}^{R} e_{(\alpha)} + \frac{1}{2} \Phi_{(+) - \mathcal{X}}^{R}(e_{(\alpha)} + \Phi_{(3)}^{R} e_{(\alpha)}, \Phi_{(3)}^{R} e_{(\alpha)}). \tag{1.46} \]
After involving Eqs. (1.18)–(1.22) and evaluating this at \( r = r(t), \theta = \frac{\pi}{2}, \) and \( \phi = \varphi(t) \), we obtain
\[ \Phi^{R}_t = \sqrt{f} \Phi^{R}_0, \tag{1.47} \]
\[ \Phi^{R}_\theta = \frac{1}{2 \sqrt{f}} \left( \Phi_{(+) - \mathcal{X}}^{R} e^{-i\varphi} + \Phi_{(3)}^{R} e^{i\varphi} \right), \tag{1.48} \]
\[ \Phi^{R}_\phi = -\frac{ir}{2} \left( \Phi_{(+) - \mathcal{X}}^{R} e^{-i\varphi} - \Phi_{(3)}^{R} e^{i\varphi} \right). \tag{1.49} \]
These, finally, can be substituted into Eqs. (1.2) and (1.3) for a concrete evaluation of the scalar self-force.
F. Case study: Particle on a circular orbit.

The prescription detailed in the preceding subsection will be fully implemented in a future publication. To convince ourselves that it actually works, we carried out the computations for the special case of a scalar charge moving on a circular orbit around a Schwarzschild black hole. Our results are not new: They reproduce some already obtained by Burko [28], Detweiler, Messaritaki, and Whiting [22], as well as Diaz-Rivera, Messaritaki, and Whiting [29]. Nevertheless, we present them here (without derivations) because they constitute a proof of principle that the prescription is valid.

We place a scalar charge on a circular, geodesic orbit at a radius $r_0 = 6M$ (this is the innermost stable circular orbit). We go through all the steps listed in Sec. I E and compute $\Phi^R$ as a function of multipole order $l$, in the interval $0 \leq l < 40$. The upper curve (in open triangles) represents the unregularized multipole coefficients of the retarded field; we plot this on a logarithmic scale as a function of $l$, in the interval $0 \leq l < 40$. This curve requires no regularization: the imaginary parts of $\Phi^R$ are approximately constant, so that its sum also diverges. The second curve (in open squares) is what is obtained after subtracting $(l+\frac{3}{2})A^\oplus$ from the first curve; we see here that the curve is approximately constant, so that its sum also diverges. The third curve (in open diamonds) is what is obtained after subtracting $C^\oplus(l+\frac{3}{2})/l$ from the second curve; this produces a function that decays as $1/l^2$, and this leads to a converging sum. The convergence is accelerated, however, with the fourth curve (in solid circles), which is obtained after subtracting $D^\oplus(l+\frac{3}{2})l(l+\frac{3}{2})$ from the third curve; this produces a function that decays as $1/l^4$.

Figure 1 constitutes a very robust test of our numerical and analytical computations: Any error would give rise to a gross violation of the properties listed above. For example, an error in the analytical form of $D^\oplus$ would produce a curve in full circles that would still decay as $1/l^2$ instead of the observed $1/l^4$. Similarly, a coding error would return a retarded field whose singularity structure would not be compatible with the regularization parameters, and this would again produce very visible effects in Fig. 1. We have tested this observation by deliberately inserting errors at various places in our numerical code. The results follow expectations and convince us that the prescription is valid. (We have not yet tested the $i \neq 0$ sector of the prescription.)

In Fig. 2 we plot $|\text{Im}\Phi^I|$, the absolute value of the imaginary part of $\Phi^I$, again we plot this on a logarithmic scale as a function of $l$, in the interval $0 \leq l < 40$. This curve requires no regularization: the imaginary parts of $A^\oplus$,
FIG. 2: Plot of $|\text{Im} \Phi_{(+)}|_l$ as a function of multipole order $l$.

$B_{(+)}$ and $D_{(+)}$ all vanish when $\dot{r} = 0$. (Recall that we evaluate the field at $t = 0$ and $\varphi := \Omega t = 0$.) Here we see that $|\text{Im} \Phi_{(+)}|_l$ decays exponentially as a function of $l$ (approximately as $e^{-l/2}$) until round-off errors start to dominate when $l$ is approximately equal to 20. This curve, of course, produces a rapidly converging sum.

Our final numerical results are

$$\frac{M^2}{q} \Phi^R_t \simeq 3.60907254 \times 10^{-4},$$

$$\frac{M^2}{q} \Phi^R_r \simeq 1.67730 \times 10^{-4},$$

$$\frac{M}{q} \Phi^R_\phi \simeq -5.30423170 \times 10^{-3},$$

with the number of significant digits reflecting our best estimation of the code’s numerical accuracy (the last digit is uncertain). The least accurate number is for $\Phi^R_r$, which is obtained after several rounds of regularization. Our numbers are consistent with results obtained by Diaz-Rivera, Messaritaki, and Whiting [29]: In their Table I they list $(M^2/q)\Phi^R_r = 1.6772834 \times 10^{-4}$ for $r_0 = 6M$.

G. Organization of this paper

The chain of calculations that lead to the prescription detailed in Sec. I E is a long one, and it occupies the remaining sections of the paper. Here is how the rest of the paper is organized.

We begin in Sec. II with the development of a covariant local expansion of the singular field $\Phi^S_\alpha(x)$ in the vicinity of the particle’s world line. The expansion is based on the assumption that the scalar charge follows a geodesic of a vacuum spacetime, but it is otherwise general; we do not yet, at this stage, assume that the metric is given by the Schwarzschild solution. The expansion must be carried out to a sufficient degree of accuracy to permit the determination of all four regularization parameters. Such an accurate expansion has never appeared in the literature, and we present it here for the first time.

In Sec. III we convert the covariant expansion into an explicit coordinate expansion that can be evaluated for any spacetime whose metric is expressed in any coordinate system. The methods by which we obtain the covariant and coordinate expansions rely heavily on the general theory of bitensors. These were introduced by Synge [36] and DeWitt and Brehme [7], and the theory is conveniently summarized in Poisson’s contribution to Living Reviews in Relativity [4]. This last paper is an essential resource for the calculations presented in Secs. II and III, and we will repeatedly refer to it as LRR.

In Sec. IV we motivate our choice of tetrad $e^\alpha_\mu$. We also work out the relationships listed in Eqs. (1.23)–(1.26) between $\Phi_{(\mu)lm}$, the spherical-harmonic modes of $\Phi_{(\mu)} := e^\alpha_\mu \nabla_\alpha \Phi$, and $\Phi_{lm}$, the modes of the scalar potential. As we explain in this section, the tetrad is selected so as to produce a simple relationship in which $\Phi_{(\mu)lm}$ is linked to the neighboring modes $\Phi_{l\pm 1,m}$ and $\Phi_{l, m\pm 1}$ only; other tetrads would lead to more complicated couplings and are best avoided.
In Sec. V we compute the regularization parameters $A_\mu$, $B_\mu$, $C_\mu = 0$, and $D_\mu$. We do this by expanding our local coordinate expansion for $\Phi^S_{(\mu)}(x)$ in Legendre polynomials and showing that the result takes the form of Eq. (1.11). To perform the Legendre decompositions we rely on techniques imported from Detweiler, Messaritaki, and Whiting [22]; these are summarized in the Appendix.

Many calculations presented in this paper are extremely tedious and could not have been carried out with pen and paper. We relied heavily on the symbolic manipulator GRTENSORII [37] working under MAPLE. Throughout the paper we use geometrized units in which $G = c = 1$, and we adhere to the sign conventions of Misner, Thorne, and Wheeler [38].

II. COVARIANT LOCAL EXPANSION OF THE SINGULAR FIELD

A. Singular field

The singular part $\Phi^S$ of the retarded potential $\Phi$ produced by a point scalar charge $q$ moving on an arbitrary world line of an arbitrary curved spacetime was first correctly identified by Detweiler and Whiting [12]. As they have shown, the singular potential possesses the following properties: (i) it satisfies the same wave equation as the retarded potential, Eq. (1.1); (ii) it displays the same singularity structure as the retarded potential near the particle’s world line; and (iii) it does not exert a force on the point charge. The scalar self-force acting on the charge therefore results from the sole action of the regular potential $\Phi^R = \Phi - \Phi^S$, which is smooth on the world line. The self-force is proportional to $\Phi^R := \nabla_\alpha \Phi^R$, and it can be calculated by first computing $\Phi^R := \nabla_\alpha \Phi$, then removing from this the singular part $\Phi^S := \nabla_\alpha \Phi^S$, and finally evaluating the result at the position of the particle.

Our task in this section is to develop a covariant expansion of the singular field $\Phi^S$ in powers of $\epsilon$, a book-keeping quantity that loosely represents the distance between the field point and the world line. (The distance to the world line will be defined precisely below.) As was justified near the end of Sec. I B, we shall restrict our attention to the case of a scalar charge $q$ that moves on a geodesic of a vacuum spacetime; the charge’s acceleration vector and the spacetime’s Ricci tensor will therefore be set equal to zero. The expansion begins with a term of order $\epsilon^{-2}$ and we shall keep it accurate through order $\epsilon^0$, neglecting terms of order $\epsilon^2$ and higher.

Our starting point is the expression displayed in Sec. 5.1.5 [Eq. (413)] of LRR [4],

$$\Phi^S_\alpha(x) = -\frac{q}{2r^2} U(x, x') \nabla_\alpha r - \frac{q}{2r_{\text{adv}}^2} U(x, x'') \nabla_\alpha r_{\text{adv}} + \frac{q}{2r} \left( \nabla_\alpha U(x, x') + u^\alpha \nabla_\alpha U(x, x') \nabla_\alpha u \right)$$

$$+ \frac{q}{2r_{\text{adv}}} \left( \nabla_\alpha U(x, x'') + u'^\alpha \nabla_\alpha U(x, x'') \nabla_\alpha v \right) + \frac{q}{2} \left( V(x, x') \nabla_\alpha u - V(x, x'') \nabla_\alpha v \right)$$

$$- \frac{q}{2} \int_x^v \nabla_\alpha V(x, z) \, d\tau. \tag{2.1}$$

We have introduced a large number of symbols. To begin, $x$ is the field point at which the singular field is evaluated, and the world line is described by parametric relations $z^\mu(\tau)$ involving the proper-time parameter $\tau$. The points $x'$ and $x''$ on the world line are known respectively as the retarded and advanced points associated with $x$; these are defined such that $x$ and $x'$ are linked by a unique future-directed null geodesic originating on the world line, while $x$ and $x''$ are linked by a past-directed null geodesic that also originates on the world line. We define the scalar field $u(x)$ as the value of the proper-time parameter at $x' \equiv z(\tau = u)$; this is known as the retarded time function of the field point $x$. Similarly, we define the advanced time function $v(x)$ as the proper time at $x'' \equiv z(\tau = v)$. With $\sigma(x, z)$ denoting Synge’s world function [36], equal to half the squared geodesic distance between the field point $x$ and the point $z$ on the world line, we have that $\sigma(x, x') = \sigma(x, x'') = 0$. (Our subsequent developments rely heavily on a working knowledge of the general theory of bitensors; this material is reviewed in Sec. 2 of LRR [4].)

The gradient of Synge’s world function is denoted $\sigma'_\alpha(x, z)$ if $\sigma(x, z)$ is differentiated with respect to its first argument, and $\sigma''_\alpha(x, z)$ if it is differentiated instead with respect to its second argument. The retarded distance $r(x)$ between $x$ and the world line refers to the retarded point $x'$ and is defined by

$$r := \sigma'_\alpha(x, x') u^\alpha; \tag{2.2}$$

where $u^\alpha$ is the particle’s velocity vector at $x'$; this is an affine-parameter distance along the null geodesic that links $x$ to $x'$. The advanced distance $r_{\text{adv}}(x)$ between $x$ and the world line refers instead to the advanced point $x''$ and is defined by

$$r_{\text{adv}} := -\sigma''_\alpha(x, x'') u^\alpha. \tag{2.3}$$
where \( u^{\alpha''} \) is the particle’s velocity vector at \( x'' \); this also is an affine-parameter distance. We note that the distance functions are both nonnegative. As a consequence of their defining relations (see Secs. 3.3.3 and 3.4.4 of LRR [4] for an extended discussion), we have the scaling relations \( r = O(\epsilon) \), \( r_{\text{adv}} = O(\epsilon) \), and \( v - u = O(\epsilon) \). It also follows from the defining relations that the gradients of \( u, v, r \), and \( r_{\text{adv}} \) that appear in Eq. (2.1) are given by

\[
\nabla_{\alpha} u = -\sigma_{\alpha}(x, x')/r, \quad (2.4)
\nabla_{\alpha} v = \sigma_{\alpha}(x, x'')/r_{\text{adv}}, \quad (2.5)
\nabla_{\alpha} r = \sigma_{\alpha'\beta} u'_{\beta} u_{\alpha''} \nabla_{\alpha} u + \sigma_{\alpha'\alpha} u_{\alpha'}, \quad (2.6)
\nabla_{\alpha} r_{\text{adv}} = -\sigma_{\alpha'\beta} v'_{\beta} u_{\alpha''} \nabla_{\alpha} v - \sigma_{\alpha'\alpha} u_{\alpha''}. \quad (2.7)
\]

Equations (2.4) and (2.5) are always valid and follow directly from the conditions \( \sigma(x, x') = \sigma(x, x'') = 0 \); Eqs. (2.6) and (2.7) are valid when the world line is a geodesic of the curved spacetime.

The biscalars \( U(x, z) \) and \( V(x, z) \) that appear in Eq. (2.1) are respectively the “direct” and “tail” parts of the retarded Green’s function \( G(x, z) \) associated with the scalar potential \( \Phi(x) \). (The general theory of scalar Green’s functions in curved spacetime is reviewed in Sec. 4.3 of LRR [4].) For our purposes in this section, the only relevant properties of these objects are the scaling relations

\[
U(x, x') = 1 + O(\epsilon^4), \quad (2.8)
U(x, x'') = 1 + O(\epsilon^4), \quad (2.9)
\n\nabla_{\alpha} U(x, x') = O(\epsilon^3), \quad (2.10)
\n\nabla_{\alpha'} U(x, x') = O(\epsilon^3), \quad (2.11)
\n\nabla_{\alpha} U(x, x'') = O(\epsilon^3), \quad (2.12)
\n\nabla_{\alpha'} U(x, x'') = O(\epsilon^3), \quad (2.13)
\nV(x, x') = O(\epsilon^2), \quad (2.14)
\nV(x, x'') = O(\epsilon^2), \quad (2.15)
\n\nabla_{\alpha} V(x, z) = O(\epsilon). \quad (2.16)
\]

These relations are valid in a Ricci-flat spacetime only.

Substituting Eqs. (2.8)–(2.16) into Eq. (2.1) produces the substantial simplification

\[
\Phi_{\alpha}^S(x) = -\frac{q}{2r^2} \nabla_{\alpha} r - \frac{q}{2r_{\text{adv}}^2} \nabla_{\alpha} r_{\text{adv}} + O(\epsilon^2). \quad (2.17)
\]

This will be turned into a more explicit expression in the course of the following subsections.

### B. Reference point on the world line

The expression of Eq. (2.17) refers to two separate points on the world line, the retarded point \( x' \equiv z(u) \) and the advanced point \( x'' \equiv z(v) \); each is linked to \( x \) by the null conditions \( \sigma(x, x') = \sigma(x, x'') = 0 \). We find it convenient to introduce a third point \( \bar{x} \equiv z(\bar{r}) \) on the world line, and to go through the lengthy procedure of re-expressing \( \nabla_{\alpha} \Phi_{\alpha}^S(x) \) solely in terms of tensorial quantities that are evaluated at \( \bar{x} \). An important aspect of this transcription is that we take the point \( \bar{x} \) to be completely arbitrary, except for the following restriction: We assume that \( \bar{x} \) is in a spacelike relation with \( x \), so that it lies after \( x' \) but before \( x'' \) on the world line; \( \bar{x} \) is otherwise arbitrary. The transcription produces two major advantages: First, it consolidates the dependence of the singular field on the world line to a single point instead of two; and second, it eliminates the dependence of \( \Phi_{\alpha}^S \) on \( x \) that is only implicitly contained in \( x'(x) \) and \( x''(x) \). Our resulting expression for the singular field, which appears in Eq. (2.62) below, contains a dependence on \( x \) that is fully explicit.

Having made arbitrary choices for the field point \( x \) and the reference point \( \bar{x} \) on the world line, we define the quantities

\[
\bar{r} := \sigma_{\alpha}(x, \bar{x}) u^{\alpha} \quad (2.18)
\]

and

\[
s^2 := (g^{\bar{\alpha} \bar{\beta}} + u_{\bar{\alpha}} u_{\bar{\beta}}) \sigma_{\alpha}(x, \bar{x}) \sigma_{\bar{\beta}}(x, \bar{x}). \quad (2.19)
\]
We note that $\tilde{r} = O(\epsilon)$ and that its definition is similar to that of $r$ and $r_{\text{adv}}$ provided in the preceding subsection. In fact, when $\tilde{x} \rightarrow x'$ we have that $\tilde{r} \rightarrow r$, while $\tilde{r} \rightarrow -r_{\text{adv}}$ when $\tilde{x} \rightarrow x''$; somewhere between $x'$ and $x''$ we have $\tilde{r}$ changing sign, from a positive value to a negative value. The quantity $s^2 = O(\epsilon^2)$ is the squared distance between $\tilde{x}$ and $x$ as measured by an observer at $\tilde{x}$ that is momentarily comoving with the charged particle. We note the useful identity $s^2 = 2\sigma(x, \tilde{x}) + \tilde{r}^2$, which shows that $s^2$ is necessarily positive when $x$ and $\tilde{x}$ are in a spacelike relation.

Our remaining task is to carry out the procedure described in the first paragraph of this subsection. This involves many steps, and lengthy calculations. The first step is to determine the positions of the retarded and advanced points relative to the reference point $\tilde{x}$; this we do in Sec. II C. The next steps involve computing the various pieces of the singular field of Eq. (2.17) and expressing them in terms of tensorial quantities defined at $\tilde{x}$. We begin in Sec. II D with a computation of $r$ and $r_{\text{adv}}$. We continue in Sec. II E with a computation of $\sigma_\alpha(x, x')$ and $\sigma_\alpha(x, x'')$, which appear in the expressions for $\nabla_\alpha u$ and $\nabla_\alpha v$ — see Eqs. (2.4) and (2.5). In Secs. II F and G we compute the quantities $\sigma_\alpha u^\alpha$, $\sigma_\alpha u^\alpha''$, $\sigma_\alpha'u^\alpha u^\beta$, and $\sigma_\alpha''u^\alpha u^\beta$, which are involved in the expressions for $\nabla_\alpha r$ and $\nabla_\alpha r_{\text{adv}}$ — see Eqs. (2.6) and (2.7). In Secs. II H and I we collect our results and compute $\nabla_\alpha u$, $\nabla_\alpha v$, $\nabla_\alpha r$, and $\nabla_\alpha r_{\text{adv}}$. And finally, in Sec. II J we produce our final expression for $\nabla_\alpha \Phi(x)$ — see Eq. (2.62) below. The reader who does not wish to go through these computations may simply jump to Sec. II J for the punch line.

### C. Retarded and advanced points

We wish to determine the positions of the retarded point $x' = z(u)$ and the advanced point $x'' = z(v)$ relative to the arbitrary reference point $\tilde{x} = z(\tilde{r})$ on the world line. We will achieve this by obtaining expressions for

\[
\Delta_+ := v - \tilde{r} > 0
\]

and

\[
\Delta_- := u - \tilde{r} < 0.
\]

These will be given in the form of expansions in powers of $\epsilon$. We will rely on Taylor-expansion techniques reviewed in Sec. 3.4 of LRR [4], as well as the standard bitensorial expansion (see, for example, Ref. [39])

\[
\sigma_{\tilde{\alpha}\tilde{\beta}} = g_{\tilde{\alpha}\tilde{\beta}} - \frac{1}{3} R_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta}} \sigma^\gamma \sigma^\delta + \frac{1}{12} R_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta};\tau} \sigma^\gamma \sigma^\delta \sigma^\tau + O(\epsilon^4),
\]

in which the metric, as well as the Riemann tensor and its covariant derivative, is evaluated at the reference point $\tilde{x}$.

We keep $x$ fixed and introduce the function

\[
\sigma(\tau) := \sigma(x, z(\tau))
\]

of the proper-time parameter $\tau$ on the world line. We note the special values $\sigma(u) = \sigma(v) = 0$, and that $\sigma(\tau)$ is positive in the interval $u < \tau < v$. We express $\sigma(\tau)$ as a Taylor expansion around the reference point $\tau = \tilde{r}$, and evaluate it at $\tau = \tilde{w}$, which stands collectively for either $u$ or $v$. With $\Delta := w - \tilde{r}$ (and therefore equal to either $\Delta_+$ or $\Delta_-$), the result is

\[
0 = \sigma + \dot{\sigma} \Delta + \frac{1}{2} \ddot{\sigma} \Delta^2 + \frac{1}{6} \sigma^{(3)} \Delta^3 + \frac{1}{24} \sigma^{(4)} \Delta^4 + \cdots,
\]

where $\dot{\sigma} := \sigma(\tilde{r})$ and all derivatives of $\sigma(\tau)$, which are indicated by overdots or a number within brackets, are evaluated at $\tau = \tilde{r}$. The computation of the derivatives is simplified by the fact that the motion is geodesic. From Eq. (2.23) we have $\dot{\sigma} = \sigma_\alpha u^\alpha$, $\ddot{\sigma} = \sigma_\alpha u^\alpha u^\beta$, $\sigma^{(3)} = \sigma_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}} u^{\tilde{\alpha}} u^{\tilde{\beta}} \tilde{\gamma}$, and $\sigma^{(4)} = \sigma_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta}} u^{\tilde{\alpha}} u^{\tilde{\beta}} u^{\tilde{\gamma}} u^{\tilde{\delta}}$. To evaluate the first derivative we simply involve Eq. (2.18) and get $\dot{\sigma} = \tilde{r}$. For the second derivative we involve Eq. (2.22) and obtain

\[
\ddot{\sigma} = -1 - \frac{1}{3} R_{u\sigma u \sigma} + \frac{1}{12} R_{\sigma u \sigma u \sigma} + O(\epsilon^4),
\]

where we have introduced the notation $R_{u\sigma u \sigma} := R_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta}} u^{\tilde{\alpha}} \sigma^{\tilde{\beta}} \sigma^{\tilde{\gamma}} \sigma^{\tilde{\delta}}$ and $R_{\sigma u \sigma u \sigma} := R_{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}\tilde{\delta} \lambda} u^{\tilde{\alpha}} \sigma^{\tilde{\beta}} u^{\tilde{\gamma}} \sigma^{\tilde{\delta}} \sigma^{\tilde{\lambda}}$; many variants of this notation will appear below. To evaluate the third derivative we begin by differentiating Eq. (2.22) to obtain an expansion for $\sigma^{(3)}$, which we then contract with the velocity vector. The result is

\[
\sigma^{(3)} = -\frac{1}{4} R_{u\sigma u \sigma u} + O(\epsilon^3).
\]
We proceed similarly for the fourth derivative and obtain

$$\sigma^{(4)} = O(\epsilon^2).$$  \hspace{1cm} (2.27)

Gathering the results, Eq. (2.24) becomes

$$0 = \bar{\sigma} + \bar{r}\Delta - \frac{1}{2} \left(1 + \frac{1}{3} R_{u\sigma u\sigma} - \frac{1}{12} R_{u\sigma u\sigma|\sigma}\right)\Delta^2 - \frac{1}{24} R_{u\sigma u\sigma|u}\Delta^3 + O(\epsilon^4).$$  \hspace{1cm} (2.28)

This equation must now be solved for \(\Delta\).

We assume that \(\Delta\) can be expressed as an expansion in powers of \(\epsilon\), in the form

$$\Delta = \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 + O(\epsilon^5)$$  \hspace{1cm} (2.29)

with \(\Delta_n = O(\epsilon^n)\). Substituting Eq. (2.29) into Eq. (2.28) and equating each coefficient of \(\epsilon^n\) to zero returns a hierarchy of equations to be solved. The first equation determines \(\Delta_1\):

$$\Delta_1^2 - 2\epsilon\Delta_1 - 2\bar{\sigma} = 0.$$  \hspace{1cm} (2.30)

In view of Eq. (2.19) we have that \(2\bar{\sigma} = s^2 - \bar{r}^2\), and the two solutions to Eq. (2.30) are \(\Delta_1^+ = \bar{r} + s\) and \(\Delta_1^- = \bar{r} - s\). The remaining equations produce \(\Delta_2 = 0\),

$$\Delta_3 = \frac{R_{u\sigma u\sigma}\Delta_1^2}{6(\bar{r} - \Delta_1)},$$  \hspace{1cm} (2.31)

and

$$\Delta_4 = \frac{(\Delta_1 R_{u\sigma u\sigma|u} - R_{u\sigma u\sigma|\sigma})\Delta_1^2}{24(\bar{r} - \Delta_1)}. $$  \hspace{1cm} (2.32)

The series expansion for \(\Delta\) is now determined to the required degree of accuracy.

Collecting our results, we conclude that \(\Delta_\pm\) is given by

$$\Delta_\pm = (\bar{r} \pm s) + \frac{(\bar{r} \pm s)^2}{6s} R_{u\sigma u\sigma} + \frac{(\bar{r} \pm s)^2}{24s} \left[ (\bar{r} \pm s) R_{u\sigma u\sigma|u} - R_{u\sigma u\sigma|\sigma} \right] + O(\epsilon^3).$$  \hspace{1cm} (2.33)

This determines the positions of the retarded point \(x' = z(u) = z(\bar{r} + \Delta_-)\) and the advanced point \(x'' = z(v) = z(\bar{r} + \Delta_+)\) relative to the reference point \(\bar{x} = z(\bar{r})\) on the world line. We note that in Eq. (2.33), the first term on the right-hand side is of order \(\epsilon\), the second term is of order \(\epsilon^3\), and the third term (involving the square brackets) is of order \(\epsilon^4\). We recall the notation introduced below Eq. (2.25): In Eq. (2.33),

$$R_{u\sigma u\sigma} := R_{\tilde{\alpha}\tilde{\mu}\tilde{\beta}\tilde{\nu}} u^\tilde{\alpha} \sigma^{\tilde{\beta} u} \sigma^{\tilde{\nu}}$$  \hspace{1cm} (2.34)

and

$$R_{u\sigma u\sigma|\sigma} := R_{\tilde{\alpha}\tilde{\mu}\tilde{\nu}\tilde{\lambda}} u^\tilde{\alpha} \sigma^{\tilde{\beta} u} \sigma^{\tilde{\nu} \tilde{\lambda}}.$$  \hspace{1cm} (2.35)

The notation is unambiguous and easily adaptable to other projections of the Riemann tensor; for example, \(R_{u\sigma u\sigma|u} := R_{\tilde{\alpha}\tilde{\mu}\tilde{\nu}\tilde{\gamma}} u^\tilde{\alpha} \sigma^{\tilde{\beta} u} \sigma^{\tilde{\nu} \tilde{\gamma}} u^\tilde{\gamma}\) also appears in Eq. (2.33).

D. Calculation of \(r\) and \(r_{adv}\)

We now wish to express \(r\) and \(r_{adv}\) in terms of tensorial quantities that are evaluated at \(\bar{x}\). Once more our strategy is to perform a Taylor expansion around \(\tau = \bar{\tau}\). From Eqs. (2.2) and (2.23) we obtain \(r = \bar{\sigma}(u)\), which may be expanded as

$$r = \bar{\sigma} + \bar{\sigma}\Delta_+ + \frac{1}{2} \sigma^{(3)}(3)\Delta_+^2 + \frac{1}{6} \sigma^{(4)}\Delta_+^3 + \cdots,$$  \hspace{1cm} (2.36)

where \(\Delta_\pm = u - \bar{\tau}\) and where all derivatives of \(\sigma(\tau)\) are evaluated at \(\tau = \bar{\tau}\). Similarly, Eqs. (2.3) and (2.23) give \(r_{adv} = -\bar{\sigma}(v)\) and

$$r_{adv} = -\bar{\sigma} - \bar{\sigma}\Delta_- - \frac{1}{2} \sigma^{(3)}(3)\Delta_-^2 - \frac{1}{6} \sigma^{(4)}\Delta_-^3 + \cdots,$$  \hspace{1cm} (2.37)
where $\Delta_+ = v - \bar{\tau}$. We recall that $\dot{\sigma} = \bar{\tau}$, and that expressions for the higher derivatives of $\sigma(\tau)$ were obtained in Eqs. (2.25)-(2.27). Furthermore, Eq. (2.33) gives $\Delta_+$ as an expansion in powers of $\epsilon$. Making these substitutions into Eqs. (2.36) and (2.37) produces, after some simplification,

$$r = s - \frac{\bar{\tau}^2 - s^2}{6s}R_{u\sigma u\sigma} - \frac{\bar{\tau} - s}{24s}[(\bar{\tau} - s)(\bar{\tau} + 2s)R_{u\sigma u\sigma} - (\bar{\tau} + s)R_{u\sigma u\sigma}] + O(\epsilon^5)$$

(2.38)

and

$$r_{\text{adv}} = s - \frac{\bar{\tau}^2 - s^2}{6s}R_{u\sigma u\sigma} - \frac{\bar{\tau} + s}{24s}[(\bar{\tau} + s)(\bar{\tau} - 2s)R_{u\sigma u\sigma} - (\bar{\tau} - s)R_{u\sigma u\sigma}] + O(\epsilon^5).$$

(2.39)

We note that in Eqs. (2.38) and (2.39), the first term on the right-hand side is of order $\epsilon$, the second term is of order $\epsilon^3$, and the third term (involving the square brackets) is of order $\epsilon^4$. Notice also that the difference between $r$ and $r_{\text{adv}}$ is of order $\epsilon^4$.

E. Calculation of $\sigma_\alpha(x, x')$ and $\sigma_\alpha(x, x'')$

We continue to keep $x$ fixed and introduce the vector-valued function

$$\sigma_\alpha(\tau) := \sigma_\alpha(x, z(\tau))$$

(2.40)

on the world line; the vectorial index refers to the fixed point $x$, and $\sigma_\alpha(\tau)$ is a set of four scalar functions of the argument $\tau$. In terms of this we have $\sigma_\alpha(x, x') = \sigma_\alpha(u)$ and $\sigma_\alpha(x, x'') = \sigma_\alpha(v)$, and we wish to express these in terms of tensorial quantities evaluated at $\bar{x}$. Letting $w$ stand for either $u$ or $v$, and re-introducing $\Delta := w - \bar{\tau}$, Taylor expansion gives

$$\sigma_\alpha(w) = \bar{\sigma}_\alpha + \delta \sigma_\alpha \Delta + \frac{1}{2} \bar{\sigma}_\alpha \Delta^2 + \frac{1}{6} \bar{\sigma}_\alpha^{(3)} \Delta^3 + \frac{1}{24} \bar{\sigma}_\alpha^{(4)} \Delta^4 + \cdots,$$

(2.41)

where $\bar{\sigma}_\alpha = \sigma_\alpha(\bar{\tau})$, and where all derivatives are evaluated at $\tau = \bar{\tau}$. We shall evaluate each term on the right-hand side of Eq. (2.41). We rely on results obtained in Sec. II C, as well as the standard bitensorial expansion (see, for example, Ref. [39])

$$\sigma_\alpha(x, \bar{x}) = g^{\bar{\beta}}_{\bar{\gamma}}\left[-g_{\alpha\beta} - \frac{1}{6} R_{\alpha\bar{\gamma}\bar{\beta}\bar{\delta}} \sigma^\bar{\gamma} \sigma^\bar{\delta} + \frac{1}{12} R_{\alpha\bar{\gamma}\bar{\beta}\bar{\delta}\bar{\epsilon}} \sigma^\bar{\gamma} \sigma^\bar{\delta} \sigma^\bar{\epsilon} + O(\epsilon^4) \right].$$

(2.42)

where $g^{\bar{\beta}}_{\bar{\gamma}}(x, \bar{x})$ is the parallel propagator, which takes a vector at $x$ and carries it to $\bar{x}$ by parallel transport.

We begin by recalling the identity $\sigma_\alpha(x, \bar{x}) = -g^{\alpha}_{\beta}(x, \bar{x})\bar{\sigma}_\beta(x, \bar{x})$, which follows from the geometrical interpretation of $\sigma_\alpha$ and $-\sigma_{\bar{\alpha}}$ as tangent vectors on the spacelike geodesic that links the points $x$ and $\bar{x}$. The identity allows us to write $\bar{\sigma}_\alpha$ as

$$\bar{\sigma}_\alpha = -g^{\alpha}_{\bar{\beta}}\bar{\sigma}_{\bar{\beta}}.$$  

(2.43)

The derivative of $\sigma_\alpha(\tau)$ is given by $\dot{\sigma}_\alpha = \sigma_{\alpha\beta}u^\beta$, and involving Eq. (2.42) produces

$$\dot{\sigma}_\alpha = g^{\alpha}_{\beta}\left[-u^\beta - \frac{1}{6} R_{\alpha\sigma u\sigma} + \frac{1}{12} R_{\alpha\sigma u\sigma}] + O(\epsilon^4) \right].$$

(2.44)

where we have introduced a notation similar to that of Eqs. (2.34) and (2.35): $R_{\alpha\sigma u\sigma} := R_{\alpha\bar{\mu}\bar{\beta}\bar{\lambda}}\sigma^\bar{\mu}u^\bar{\beta}u^\bar{\lambda}$ and $R_{\alpha\sigma u\sigma}] := R_{\bar{\alpha}\bar{\beta}\bar{\lambda}\bar{\mu}}\sigma^\bar{\mu}u^\bar{\beta}u^\bar{\lambda}$.

The second derivative of $\sigma_\alpha(\tau)$ is $\ddot{\sigma}_\alpha = \sigma_{\alpha\beta\gamma\delta}u^\beta u^\gamma \bar{\sigma}^\delta$, and an expression for $\sigma_{\alpha\beta\gamma\delta}$ can be obtained by differentiating Eq. (2.22) with respect to $x^\beta$. This expression is simplified with the help of Eq. (2.42), which can be truncated to $\ddot{\sigma}_\alpha = -g^{\alpha}_{\beta} + O(\epsilon^2)$ for the purposes of this computation. The end result, after contracting with $u^\alpha u^\beta$ and invoking the symmetries of the Riemann tensor, is

$$\ddot{\sigma}_\alpha = g^{\alpha}_{\beta}\left[ \frac{2}{3} R_{\alpha\beta u\sigma} - \frac{1}{12} (3R_{\alpha\sigma u\sigma}\sigma + R_{\alpha\sigma u\sigma}u) + O(\epsilon^4) \right].$$

(2.45)
Similar computations for the third and fourth derivatives produce

\[
\sigma^{(3)}_\alpha = g^\alpha_\alpha \left[ \frac{1}{2} R_{\tilde{\alpha} \tilde{\beta} \tilde{\gamma} \tilde{\delta}} u^{\tilde{\alpha}} R_{\tilde{\beta} \tilde{\gamma} \tilde{\delta}} u^{\tilde{\delta}} + O(\epsilon^2) \right]
\]

and

\[
\sigma^{(4)}_\alpha = O(\epsilon).
\]

Equations (2.43)–(2.47) can now be incorporated into Eq. (2.41), in which we also substitute Eq. (2.33). After simplification, the final results are

\[
\sigma_\alpha(x', x) = g^{\tilde{\alpha}}_\alpha \left\{ -\left[ \sigma_{\tilde{\alpha}} + (\bar{r} - s)u_{\tilde{\alpha}} \right] - \frac{1}{6}(\bar{r} - s)R_{\tilde{\alpha} \sigma \nu}u_{\sigma} \right. \\
\left. + \frac{1}{12}(\bar{r} - s)^2 R_{\tilde{\alpha} \sigma \nu \tau}u_{\sigma} - \frac{1}{6s} R_{\tilde{\alpha} \sigma \nu}u_{\sigma} - \frac{1}{3}(\bar{r} - s)^2 R_{\tilde{\alpha} \sigma \nu} \right\} + \frac{1}{12}(\bar{r} - s)^2 (3R_{\tilde{\alpha} \sigma \nu} + R_{\tilde{\alpha} \sigma \nu})u_{\sigma}
\]

and

\[
\sigma_\alpha(x'', x') = g^{\tilde{\alpha}}_\alpha \left\{ -\left[ \sigma_{\tilde{\alpha}} + (\bar{r} + s)u_{\tilde{\alpha}} \right] - \frac{1}{6}(\bar{r} + s)R_{\tilde{\alpha} \sigma \nu}u_{\sigma} - \frac{1}{6s} R_{\tilde{\alpha} \sigma \nu}u_{\sigma} - \frac{1}{3}(\bar{r} + s)^2 R_{\tilde{\alpha} \sigma \nu} \right\} + \frac{1}{12}(\bar{r} + s)^2 (3R_{\tilde{\alpha} \sigma \nu} + R_{\tilde{\alpha} \sigma \nu})u_{\sigma}
\]

In these equations, terms grouped within square brackets are of the same order of magnitude: The first group of terms is of order \(\epsilon\), the second group is of order \(\epsilon^3\), and the third group is of order \(\epsilon^4\). We recall the notation introduced below Eq. (2.44): In Eqs. (2.48) and (2.49), the various partial projections of the Riemann tensor are given by equations of the form

\[
R_{\tilde{\alpha} \sigma \nu} := R_{\tilde{\alpha} \tilde{\beta} \tilde{\gamma} \tilde{\delta}} u^{\tilde{\alpha}} R_{\tilde{\beta} \tilde{\gamma} \tilde{\delta}} u^{\tilde{\delta}}
\]

and

\[
R_{\tilde{\alpha} \sigma \nu \tau} := R_{\tilde{\alpha} \tilde{\beta} \tilde{\gamma} \tilde{\delta}} u^{\tilde{\alpha}} R_{\tilde{\beta} \tilde{\gamma} \tilde{\delta}} u^{\tilde{\delta}} u^{\tilde{\tau}}
\]

The full projections of the Riemann tensor have already been introduced in Eqs. (2.34) and (2.35).

F. Calculation of \(\sigma_{\alpha'\nu'}\) and \(\sigma_{\alpha''\nu''}\)

In terms of the vector-valued function \(\sigma_\alpha(\tau)\) introduced in Eq. (2.40), we have that \(\sigma_{\alpha'\nu'} = \dot{\sigma}_\alpha(u)\) and \(\sigma_{\alpha''\nu''} = \ddot{\sigma}_\alpha(v)\). With \(w\) standing for either \(u\) or \(v\), and with \(\Delta := w - \tilde{\tau}\), Taylor expansion gives

\[
\dot{\sigma}_\alpha(w) = \dot{\sigma}_\alpha + \ddot{\sigma}_\alpha \Delta + \frac{1}{2} \sigma^{(3)}_\alpha \Delta^2 + \frac{1}{6} \sigma^{(4)}_\alpha \Delta^3 + \cdots,
\]

in which all derivatives of \(\sigma_\alpha(\tau)\) are evaluated at \(\tau = \bar{r}\). These quantities are displayed in Eqs. (2.44)–(2.47), and Eq. (2.33) provides an expression for \(\Delta\). Substitution and simplification yields

\[
\sigma_{\alpha'\nu'} = g^{\tilde{\alpha}}_\alpha \left\{ -u_{\tilde{\alpha}} - \left[ \frac{1}{6} R_{\tilde{\alpha} \sigma \nu} - \frac{2}{3}(\bar{r} - s) R_{\tilde{\alpha} \sigma \nu} \right] + \frac{1}{12} R_{\tilde{\alpha} \sigma \nu \tau} \right\} + \frac{1}{12}(\bar{r} - s)^2 (3R_{\tilde{\alpha} \sigma \nu} + R_{\tilde{\alpha} \sigma \nu})u_{\sigma}
\]
Taylor expansion gives Eqs. (2.48) and (2.49). Combining these expansions produce expansions for the terms are of order \( \epsilon^0 \) and \( \epsilon^2 \), and the third group is of order \( \epsilon^3 \).

\[
\sigma_{\alpha\alpha'} u^\alpha'' = g_\alpha \left\{ -u_\alpha - \left[ \frac{1}{6} R_{\alpha\sigma\upsilon} - \frac{2}{3}(\bar{r} + s)R_{\bar{\alpha}u\sigma} \right] + \left[ \frac{1}{12} R_{\bar{\alpha}u\sigma} \right] \right. \\
- \frac{1}{12} (\bar{r} + s) \left( 3R_{\bar{\alpha}u\sigma|\sigma} + R_{\bar{\alpha}u\sigma|u} \right) + \frac{1}{4} (\bar{r} + s)^2 R_{\bar{\alpha}u\sigma|u} \Bigg\} + O(\epsilon^4). 
\]

(2.54)

In these equations, the first term on the right-hand side is of order \( \epsilon^0 \), the second group of terms is of order \( \epsilon^2 \), and the third group is of order \( \epsilon^3 \).

G. Calculation of \( \sigma_{\alpha\beta'} u^\alpha' u^\beta' \) and \( \sigma_{\alpha\beta'} u^\alpha'' u^\beta'' \)

Returning to the function \( \sigma(\tau) \) defined by Eq. (2.23), we have that \( \sigma_{\alpha\beta'} u^\alpha' u^\beta' = \tilde{\sigma}(u) \) and \( \sigma_{\alpha\beta'} u^\alpha'' u^\beta'' = \tilde{\sigma}(v) \). Taylor expansion gives

\[
\tilde{\sigma}(w) = \tilde{\sigma} + \sigma^{(3)} \Delta + \frac{1}{2} \sigma^{(4)} \Delta^2 + \ldots.
\]

(2.55)

Substitution of Eqs. (2.33) and (2.25)–(2.27) produces

\[
\sigma_{\alpha\beta'} u^\alpha' u^\beta' = -1 - \frac{1}{3} R_{u\sigma\upsilon} + \frac{1}{12} \left[ R_{u\sigma\upsilon} - 3(\bar{r} - s)R_{u\upsilon\sigma|u} \right] + O(\epsilon^4)
\]

(2.56)

and

\[
\sigma_{\alpha\beta'} u^\alpha'' u^\beta'' = -1 - \frac{1}{3} R_{u\sigma\upsilon} + \frac{1}{12} \left[ R_{u\sigma\upsilon} - 3(\bar{r} + s)R_{u\upsilon\sigma|u} \right] + O(\epsilon^4).
\]

(2.57)

In these equations, the first term on the right-hand side is of order \( \epsilon^0 \), the second term is of order \( \epsilon^2 \), and the bracketed terms are of order \( \epsilon^3 \).

H. Calculation of \( \nabla_\alpha u \) and \( \nabla_\alpha v \)

According to Eq. (2.4), the gradient of the retarded time function \( u(x) \) is \( \nabla_\alpha u = -\sigma_\alpha(x, x')/r \), and according to Eq. (2.5), the gradient of the advanced time function \( v(x) \) is \( \nabla_\alpha v = \sigma_\alpha(x, x'')/r_{\text{adv}} \). Expansions of \( r \) and \( r_{\text{adv}} \) in powers of \( \epsilon \) were obtained in Sec. II D and presented in Eqs. (2.38) and (2.39); these can easily be converted into expansions for \( 1/r \) and \( 1/r_{\text{adv}} \). Expansions for \( \sigma_\alpha(x, x') \) and \( \sigma_\alpha(x, x'') \) were developed in Sec. II E and presented in Eqs. (2.48) and (2.49). Combining these expansions produces

\[
\nabla_\alpha u = \frac{1}{s} g_\alpha \left\{ \sigma_\alpha + (\bar{r} - s)u_\alpha \right. \\
+ \frac{1}{6}(\bar{r} - s)R_{\bar{\alpha}u\sigma|\sigma} - \frac{1}{3}(\bar{r} - s)^2 R_{\bar{\alpha}u\sigma|u} + \frac{\bar{r}^2 - 4s}{6s^2} R_{u\sigma\upsilon} u_\alpha \\
+ \frac{(\bar{r} - s)^2(\bar{r} + 2s)}{6s^2} R_{u\sigma\upsilon} u_\alpha \left[ -\frac{1}{12}(\bar{r} - s)R_{u\sigma\upsilon|\sigma} + \frac{1}{8}(\bar{r} - s)^2 R_{u\upsilon\sigma|\sigma} + \frac{1}{24}(\bar{r} - s)^2 R_{u\upsilon\sigma|u} \right] \\
\left. - \frac{1}{12}(\bar{r} - s)^3 R_{u\upsilon\sigma|u} + \frac{\bar{r} - s}{24s^2} \left( (\bar{r} - s)^3 R_{u\upsilon\sigma|u} - (\bar{r} + s)R_{u\upsilon\sigma|\sigma} \right) u_\alpha \right\} + O(\epsilon^3) 
\]

(2.58)

and

\[
\nabla_\alpha v = -\frac{1}{s} g_\alpha \left\{ \sigma_\alpha + (\bar{r} + s)u_\alpha \right. \\
+ \frac{1}{6}(\bar{r} + s)R_{\bar{\alpha}u\sigma|\sigma} - \frac{1}{3}(\bar{r} + s)^2 R_{\bar{\alpha}u\sigma|u} + \frac{\bar{r}^2 - 4s}{6s^2} R_{u\sigma\upsilon} u_\alpha \\
+ \frac{(\bar{r} + s)^2(\bar{r} - 2s)}{6s^2} R_{u\sigma\upsilon} u_\alpha \left[ -\frac{1}{12}(\bar{r} + s)R_{u\sigma\upsilon|\sigma} + \frac{1}{8}(\bar{r} + s)^2 R_{u\upsilon\sigma|\sigma} + \frac{1}{24}(\bar{r} + s)^2 R_{u\upsilon\sigma|u} \right] \\
\left. - \frac{1}{12}(\bar{r} + s)^3 R_{u\upsilon\sigma|u} + \frac{\bar{r} + s}{24s^2} \left( (\bar{r} + s)^3 R_{u\upsilon\sigma|u} - (\bar{r} - s)R_{u\upsilon\sigma|\sigma} \right) u_\alpha \right\} + O(\epsilon^3) 
\]

(2.59)
Once more we have grouped terms of the same order of magnitude. The first group of (square-bracketed) terms within the curly brackets is of order $\epsilon$, the second group is of order $\epsilon^3$, and the third group is of order $\epsilon^4$. Noticing the factor $s^{-1}$ in front of the curly brackets, this means that $\nabla_\alpha u$ and $\nabla_\alpha v$ contain terms of order $\epsilon^0$, $\epsilon^2$, and $\epsilon^3$; the neglected terms are $O(\epsilon^4)$.

I. Calculation of $\nabla_\alpha r$ and $\nabla_\alpha r_{\text{adv}}$

The gradients of the retarded and advanced distance functions were defined in Eqs. (2.6) and (2.7); for example, $\nabla_\alpha r = \sigma_\alpha^y u^y u^\alpha \nabla_\alpha u + \sigma_{\alpha^y} u^\alpha$. Each piece of this expression was calculated separately in the preceding subsections: $\sigma_\alpha^y u^y u^\alpha$ was computed in Sec. II G, $\nabla_\alpha u$ was computed in Sec. II H, and $\sigma_{\alpha^y} u^\alpha$ was computed in Sec. II F. Combining all these results, we arrive at

$$\nabla_\alpha r = -\frac{1}{s} g_\alpha^\beta \left[ \left( \sigma_\alpha + \bar{\tau} u_\alpha \right) + \left[ \frac{1}{6} \bar{r} R_{\bar{\alpha} \sigma u} - \frac{1}{3} (\bar{r}^2 - s^2) R_{\bar{\alpha} u} + \frac{\bar{r}^2 + s^2}{6s^2} R_{u \sigma} \sigma_\alpha + \frac{\bar{r}(\bar{r}^2 - s^2)}{6s^2} R_{u \sigma} u_\alpha \right] \right]$$

and

$$\nabla_\alpha r_{\text{adv}} = -\frac{1}{s} g_\alpha^\beta \left[ \left( \sigma_\alpha + \bar{\tau} u_\alpha \right) + \left[ \frac{1}{6} \bar{r} R_{\bar{\alpha} \sigma u} - \frac{1}{3} (\bar{r}^2 - s^2) R_{\bar{\alpha} u} + \frac{\bar{r}^2 + s^2}{6s^2} R_{u \sigma} \sigma_\alpha + \frac{\bar{r}(\bar{r}^2 - s^2)}{6s^2} R_{u \sigma} u_\alpha \right] \right]$$

In these equations, the first group of (square-bracketed) terms within the curly brackets is of order $\epsilon$, the second group is of order $\epsilon^3$, and the third group is of order $\epsilon^4$. Noticing the common factor $s^{-1}$, this means that $\nabla_\alpha r$ and $\nabla_\alpha r_{\text{adv}}$ contain terms of order $\epsilon^0$, $\epsilon^2$, and $\epsilon^3$; the neglected terms are $O(\epsilon^4)$.

J. Final result: The singular field

Our final expression for the singular field is obtained by substituting the expansions of Eqs. (2.38), (2.39), (2.60), and (2.61) into Eq. (2.17). After a long computation and much simplification, we obtain

$$\Phi_\alpha^S(x) = \frac{q}{s^2} g_\alpha^\beta \left[ \left( \sigma_\alpha + \bar{\tau} u_\alpha \right) + \left[ \frac{1}{6} \bar{r} R_{\bar{\alpha} \sigma u} - \frac{1}{3} (\bar{r}^2 - s^2) R_{\bar{\alpha} u} + \frac{3\bar{r}^2 - s^2}{6s^2} R_{u \sigma} \sigma_\alpha + \frac{\bar{r}(\bar{r}^2 - s^2)}{2s^2} R_{u \sigma} u_\alpha \right] \right]$$

In this equation, terms grouped within square brackets are of the same order of magnitude. The first group of terms is of order $\epsilon$, the second group is of order $\epsilon^3$, and the third group is of order $\epsilon^4$. Noticing the common factor $s^{-3}$, this means that $\Phi_\alpha^S$ contains terms of order $\epsilon^{-2}$, $\epsilon^0$, and $\epsilon^1$; the neglected terms are $O(\epsilon^2)$. 
The dependence of $\Phi^S_\alpha := \nabla_\alpha \Phi^S$ on the field point $x$ is contained in the common factor $g^\alpha_\gamma(x, \bar{x})$, and also within the many occurrences of $\sigma_\alpha(x, \bar{x})$. This quantity appears explicitly in Eq. (2.62), and it is involved in the definitions of

$$\bar{r} := \sigma_\alpha u^\alpha$$

and

$$s^2 := (g^{\bar{\alpha}\bar{\beta}} + u^{\bar{\alpha}}u^{\bar{\beta}})\sigma_\alpha\sigma_\beta,$$

which were introduced in Eqs. (2.18) and (2.19), respectively. The gradient of the world function is also involved in the various projections of the Riemann tensor (and its covariant derivative) that appear in Eq. (2.62). We recall the notation introduced in the preceding subsections: A subscript in the various projections of the Riemann tensor (and its covariant derivative) that appear in Eq. (2.62). We recall which were introduced in Eqs. (2.18) and (2.19), respectively. The gradient of the world function is also involved.

The Riemann tensor, its derivatives, and the particle’s velocity vector $u^\alpha$ are all evaluated at the reference point $\bar{x}$ on the world line. We recall that $\bar{x}$ is chosen to be in a spacelike relation with $x$, but that it is otherwise arbitrary.

### III. COORDINATE EXPANSIONS OF BITENSORS

The calculations presented in Sec. II culminated into an expansion of the singular field $\Phi^S_\alpha(x)$ in powers of $\epsilon$, the distance between the field point $x$ and the reference point $\bar{x}$ on the world line. This expansion is fully covariant, and the dependence on $x$ is explicitly contained in $\sigma_\alpha(x, \bar{x})$ and $g^\alpha_\gamma(x, \bar{x})$. Our task in this section is to develop coordinate expansions for these two bitensors, in powers of

$$w^\alpha := x^\alpha - \bar{x}^\alpha,$$

the difference in the coordinate positions of the points $x$ and $\bar{x}$. These expansions, of course, will not be covariant; they will depend on the choice of coordinate system. When the coordinate expansions to be obtained here are substituted into Eq. (2.62) for the singular field, the result will be an explicit expression for the expanded $\Phi^S_\alpha(x)$ in the adopted system of coordinates.

#### A. Description of the geodesic linking $x$ to $\bar{x}$

We assume that there exists a unique geodesic segment that begins at $\bar{x}$ and ends at $x$. This geodesic segment is denoted $\beta$, and it is described, in the adopted coordinate system, by the parametric relations $p^\alpha(\lambda)$. We assume that the parameter $\lambda$ is an affine parameter, and that it is limited to the interval $0 \leq \lambda \leq 1$. We have that $p^\alpha(0) = \bar{x}^\alpha$ and $p^\alpha(1) = x^\alpha$, where $\bar{x}^\alpha$ are the coordinates assigned to $\bar{x}$, while $x^\alpha$ are the coordinates assigned to $x$.

The functions $p^\alpha(\lambda)$ may be expressed as Taylor expansions about $\lambda = 0$:

$$p^\alpha(\lambda) = p^\alpha(0) + \dot{p}^\alpha(0)\lambda + \frac{1}{2}\ddot{p}^\alpha(0)\lambda^2 + \frac{1}{6}\dddot{p}^\alpha(0)\lambda^3 + \frac{1}{24}\ddddot{p}^\alpha(0)\lambda^4 + \cdots,$$

in which overdots, or a number within brackets, indicate repeated differentiation with respect to $\lambda$. Equation (3.2) implies

$$\ddot{p}^\alpha(\lambda) = \ddot{p}^\alpha(0) + \dddot{p}^\alpha(0)\lambda + \frac{1}{2}\ddddot{p}^\alpha(0)\lambda^2 + \frac{1}{6}\dddeee\dddot{p}^\alpha(0)\lambda^3 + \cdots$$

and

$$\dddeee\dddot{p}^\alpha(\lambda) = \dddeee\dddot{p}^\alpha(0) + \dddeee\dddeee\dddot{p}^\alpha(0)\lambda + \frac{1}{2}\dddeee\dddeee\ddddot{p}^\alpha(0)\lambda^2 + \cdots.$$
These quantities are linked by the geodesic equation,

$$\ddot{\sigma}^\alpha(\lambda) + \Gamma^\alpha_{\beta\gamma}(\lambda)\dot{\sigma}^\beta(\lambda)\dot{\sigma}^\gamma(\lambda) = 0,$$

(3.5)
in which $\Gamma^\alpha_{\beta\gamma}(\lambda)$ is the Christoffel connection evaluated on $\beta$.

The range of the affine parameter $\lambda$ was chosen to ensure that the tangent vector $\dot{\sigma}^\alpha(\lambda)$ is intimately related to the gradient of Synge’s world function. In fact, as reviewed in Sec. 2.1.3 of LRR [4] — see in particular Eqs. (55) and (56) — we have that $\sigma^\alpha(x, \bar{x}) = g_{\alpha\beta}(x)\dot{\sigma}^\alpha(1)$ and

$$\sigma^\alpha(x, \bar{x}) = -g_{\alpha\beta}(\bar{x})\dot{\sigma}^\alpha(0).$$

(3.6)

B. Calculational strategy

Our first goal in this section is to obtain an expansion of $\sigma^\alpha(x, \bar{x})$ in powers of the coordinate difference of Eq. (3.1). This is given by $w^\alpha = p^\alpha(1) - p^\alpha(0)$, or

$$w^\alpha = \ddot{\sigma}^\alpha(0) + \frac{1}{2}\dddot{\sigma}^\alpha(0) + \frac{1}{6}\dddot{\sigma}^\alpha(3)(0) + \frac{1}{24}\dddot{\sigma}^\alpha(4)(0) + \cdots.$$  

(3.7)

We will achieve this in four steps. First (Sec. II C), we substitute Eqs. (3.3) and (3.4) into Eq. (3.5) and solve for $\ddot{\sigma}^\alpha(0)$, $p^\alpha(3)(0)$, and $p^\alpha(4)(0)$ in terms of $\dot{\sigma}^\alpha(0)$. Second (Sec. II D), we incorporate these results into Eq. (3.7) and obtain $w^\alpha$ as an expansion in powers of $\ddot{\sigma}^\alpha(0)$. Third (also Sec. II D), we invert this series to obtain $\dddot{\sigma}^\alpha(0)$ expanded in powers of $w^\alpha$. Finally (Sec. II E), we substitute the result into Eq. (3.6); our final expression for $\sigma^\alpha(x, \bar{x})$ is displayed in Eq. (3.19) below.

Our second goal in this section is to obtain an expression for $g^\alpha_{\alpha}(x, \bar{x})$ expanded in powers of $w^\alpha$. This calculation is carried out in Secs. II F and II G and it follows a very similar strategy; our final expression for the parallel propagator is displayed in Eq. (3.30) below.

The calculations that follow rely on the expansions displayed in Eqs. (3.2)–(3.4) and (3.7). We also will need the Taylor expansion of $\Gamma^\alpha_{\beta\gamma}(\lambda)$ about $\lambda = 0$. Starting with

$$\Gamma^\alpha_{\beta\gamma}(\lambda) = \Gamma^\alpha_{\beta\gamma}(0) + \Gamma^\alpha_{\beta\gamma,\mu}(0)[p^\mu(\lambda) - p^\mu(0)] + \frac{1}{2}\Gamma^\alpha_{\beta\gamma,\mu\nu}(0)[p^\mu(\lambda) - p^\mu(0)][p^\nu(\lambda) - p^\nu(0)] + \cdots$$

and involving Eq. (3.2), we arrive at

$$\Gamma^\alpha_{\beta\gamma}(\lambda) = \Gamma^\alpha_{\beta\gamma}(0) + \left[\Gamma^\alpha_{\beta\gamma,\mu}(0)\dot{p}^\mu(0)\right]\lambda + \frac{1}{2}\left[\Gamma^\alpha_{\beta\gamma,\mu\nu}(0)\dot{p}^\mu(0)\dot{p}^\nu(0) + \Gamma^\alpha_{\beta\gamma,\mu}(0)\ddot{p}^\mu(0)\right]\lambda^2 + \cdots,$$

(3.8)
in which the connection and its derivatives are evaluated at $\bar{x}$ on the right-hand side of the equation.

C. Calculation of $\ddot{\sigma}^\alpha(0)$, $p^\alpha(3)(0)$, and $p^\alpha(4)(0)$

We substitute Eqs. (3.3), (3.4), and (3.8) into Eq. (3.5) and collect terms that share the same power of $\lambda$. Setting the coefficient of the $\lambda^0$ term to zero yields the condition $0 = \ddot{\sigma}^\alpha(0) + \Gamma^\alpha_{\beta\gamma}(0)\dot{p}^\beta(0)\dot{p}^\gamma(0)$, or

$$\ddot{\sigma}^\alpha(0) = -\Gamma^\alpha_{\beta\gamma}\dot{p}^\beta(0)\dot{p}^\gamma(0).$$

(3.9)

It is understood that here, the connection is evaluated at the reference point $\bar{x}$. The same comment will apply below to all derivatives of the connection.

Setting the coefficient of the $\lambda^1$ term to zero yields

$$0 = p^\alpha(3)(0) + 2\Gamma^\alpha_{\beta\gamma}\ddot{p}^\beta(0)\dot{p}^\gamma(0) + \Gamma^\alpha_{\beta\gamma,\mu}\dddot{p}^\mu(0)\dot{p}^\gamma(0)\dot{p}^\mu(0),$$

and involving Eq. (3.9) gives

$$p^\alpha(3)(0) = -\Gamma^\alpha_{\beta\gamma\delta}\ddot{p}^\beta(0)\dot{p}^\gamma(0)\dot{p}^\delta(0)$$

(3.10)

with

$$\Gamma^\alpha_{\beta\gamma\delta} := -\Gamma^\alpha_{\beta\gamma,\delta} = -2\Gamma^\alpha_{\beta\delta,\gamma\mu}\Gamma^\mu_{\gamma\delta}.$$  

(3.11)
Setting the coefficient of the $\lambda^2$ term to zero yields
\[
0 = p^{(4)}(0) + 2\Gamma^\alpha_{\beta\gamma}(p^{(3)}(0) + \dot{p}^\beta(0)p^\gamma(0)) + \Gamma^\alpha_{\beta\gamma\mu}(4\dot{p}^\beta(0)p^\gamma(0)p^\mu(0) + \dot{p}^\beta(0)p^\gamma(0)p^\mu(0))
\]
and involving Eqs. (3.9)–(3.11) gives
\[
p^{(4)}(0) = -\Gamma^\alpha_{\beta\gamma\delta}\dot{p}^\beta(0)p^\gamma(0)p^\delta(0)\dot{p}^\epsilon(0)
\] (3.12)
with
\[
\Gamma^\alpha_{\beta\gamma\delta} := \Gamma^\alpha_{\beta\gamma\delta} - 4\Gamma^\alpha_{\beta\gamma\mu}\Gamma^\mu_{\delta\epsilon} - \Gamma^\alpha_{\beta\gamma\mu}\Gamma^\mu_{\delta\epsilon} - 2\Gamma^\alpha_{\beta\mu}\Gamma^\mu_{\gamma\delta\epsilon} + 4\Gamma^\alpha_{\beta\mu}\Gamma^\mu_{\gamma\delta\epsilon} + 2\Gamma^\alpha_{\gamma\delta\epsilon}\Gamma^\beta_{\mu\gamma}
\] (3.13)

It should be noted that $\Gamma^\alpha_{\beta\gamma\delta}$ and $\Gamma^\alpha_{\beta\gamma\delta\epsilon}$, as defined by Eqs. (3.10) and (3.12), are both fully symmetric in their lower indices. This symmetry has not, however, been implemented on the right-hand sides of Eqs. (3.11) and (3.13). Although this could easily be achieved, this operation is not necessary and we opt to leave these expressions as they are.

**D. Calculation of $\dot{p}^\alpha(0)$**

Combining Eqs. (3.7), (3.9), (3.10), and (3.12) gives
\[
u^\alpha = \dot{p}^\alpha(0) - \frac{1}{2}\Gamma^\alpha_{\beta\gamma}\dot{p}^\beta(0)p^\gamma(0) - \frac{1}{6}\Gamma^\alpha_{\beta\gamma\delta}\dot{p}^\beta(0)p^\gamma(0)p^\delta(0) - \frac{1}{24}\Gamma^\alpha_{\beta\gamma\delta\epsilon}\dot{p}^\beta(0)p^\gamma(0)p^\delta(0)\dot{p}^\epsilon(0) + \cdots
\] (3.14)

This is an expansion of $\nu^\alpha$ in powers of $\dot{p}^\alpha(0)$. The inverted series will take the form of
\[
\ddot{p}^\alpha(0) = \nu^\alpha + A^\alpha_{\beta\gamma}w^\beta w^\gamma + A^\alpha_{\beta\gamma\delta}w^\beta w^\gamma w^\delta + A^\alpha_{\beta\gamma\delta\epsilon}w^\beta w^\gamma w^\delta w^\epsilon + \cdots
\] (3.15)

and the coefficients $A^\alpha_{\beta\gamma}$, $A^\alpha_{\beta\gamma\delta}$, and $A^\alpha_{\beta\gamma\delta\epsilon}$ can be determined by inserting Eq. (3.15) into Eq. (3.14) and demanding that the substitution returns the identity $\ddot{p}^\alpha(0) = \ddot{p}^\alpha(0)$.

Elimination of the quadratic terms gives rise to the condition
\[
A^\alpha_{\beta\gamma} := -\frac{1}{2}\Gamma^\alpha_{\beta\gamma}
\] (3.16)

Elimination of the cubic terms produces $A^\alpha_{\beta\gamma\delta} = \frac{1}{3}\Gamma^\alpha_{\beta\gamma\delta} + A^\alpha_{\beta\mu}\Gamma^\mu_{\gamma\delta}$. This becomes
\[
A^\alpha_{\beta\gamma\delta} := \frac{1}{6}\left(\Gamma^\alpha_{\beta\gamma\delta} + \Gamma^\alpha_{\beta\mu}\Gamma^\mu_{\gamma\delta}\right)
\] (3.17)

after involving Eqs. (3.11) and (3.16). Elimination of the quartic terms produces
\[
A^\alpha_{\beta\gamma\delta\epsilon} = \frac{1}{24}\Gamma^\alpha_{\beta\gamma\delta\epsilon} + \frac{1}{3}A^\alpha_{\beta\mu}\Gamma^\mu_{\gamma\delta\epsilon} - \frac{1}{4}A^\alpha_{\mu\nu}\Gamma^\mu_{\beta\gamma}\Gamma^\nu_{\delta\epsilon} + \frac{1}{2}A^\alpha_{\beta\gamma\mu}\Gamma^\mu_{\delta\epsilon} + \frac{1}{2}A^\alpha_{\beta\mu}\Gamma^\mu_{\gamma\delta\epsilon} + \frac{1}{2}A^\alpha_{\gamma\delta\epsilon}\Gamma^\beta_{\mu\gamma}
\]

This becomes
\[
A^\alpha_{\beta\gamma\delta\epsilon} := -\frac{1}{24}\left(\Gamma^\alpha_{\beta\gamma\delta\epsilon} + \Gamma^\alpha_{\beta\gamma\mu}\Gamma^\mu_{\delta\epsilon} + 2\Gamma^\alpha_{\beta\mu}\Gamma^\mu_{\gamma\delta\epsilon} + \Gamma^\alpha_{\gamma\delta\epsilon}\Gamma^\beta_{\mu\gamma}\right)
\] (3.18)

after involving Eqs. (3.11), (3.13), (3.16), and (3.17).

Equation (3.15), with the coefficients of Eqs. (3.16)–(3.18), gives the expansion of $\dot{p}^\alpha(0)$ in powers of $\nu^\alpha = x^\alpha - \bar{x}^\alpha$. We recall that the coefficients involve the Christoffel connection and its partial derivatives evaluated at the reference point $\bar{x}$. We also remark that while $A^\alpha_{\beta\gamma\delta}$ and $A^\alpha_{\beta\gamma\delta\epsilon}$ have been defined in Eq. (3.15) as being fully symmetric in their lower indices, the right-hand sides of Eqs. (3.17) and (3.18) have been left in a non-symmetric form.

**E. Final result: $\sigma_\alpha$ expanded in powers of $\nu^\alpha$**

The gradient of Synge’s function is obtained by substituting Eq. (3.15) into Eq. (3.6). The result is
\[
-\sigma_\alpha(x, \bar{x}) = g_{\alpha\beta}w^\beta + A_{\alpha\beta\gamma}w^\beta w^\gamma + A_{\alpha\beta\gamma\delta}w^\beta w^\gamma w^\delta + A_{\alpha\beta\gamma\delta\epsilon}w^\beta w^\gamma w^\delta w^\epsilon + \cdots
\] (3.19)

where the coefficients $A_{\alpha\beta\gamma}$, $A_{\alpha\beta\gamma\delta}$, and $A_{\alpha\beta\gamma\delta\epsilon}$ are obtained from Eqs. (3.16)–(3.18) by lowering the first index with the spacetime metric (as if these quantities were tensors). We recall that the metric $g_{\alpha\beta}$, as well as the connection $\Gamma^\gamma_{\beta\gamma}$ and its partial derivatives, is evaluated at the reference point $\bar{x}$. Equation (3.19) is the required coordinate expansion of $\sigma_\alpha$ in powers of $\nu^\alpha = x^\alpha - \bar{x}^\alpha$, the difference in the coordinate positions of the points $x$ and $\bar{x}$. 
F. Parallel transport on the spacelike geodesic

We next turn to our second task, the development of a coordinate expansion for the parallel propagator \( g^a_\alpha (x, \bar{x}) \). We begin by introducing an arbitrary dual vector \( q_\alpha (\lambda) \) that we take to parallel transported on \( \beta \), the geodesic segment that links \( x \) to \( \bar{x} \). The definition of the parallel propagator implies that \( q_\alpha (1) \equiv q_\alpha (x) \) and \( q_\alpha (0) \equiv q_\alpha (\bar{x}) \) are related by

\[
q_\alpha (x) = g^a_\alpha (x, \bar{x}) q_\alpha (\bar{x}). \tag{3.20}
\]

We shall calculate the parallel propagator by expanding \( q_\alpha (\lambda) \) in a Taylor series about \( \lambda = 0 \), evaluating this at \( \lambda = 1 \), and comparing the result with Eq. (3.20).

The Taylor expansion is

\[
q_\alpha (\lambda) = q_\alpha (0) + \dot{q}_\alpha (0) \lambda + \frac{1}{2} \ddot{q}_\alpha (0) \lambda^2 + \frac{1}{6} q^{(3)}_\alpha (0) \lambda^3 + \cdots \tag{3.21}
\]

and it implies

\[
\dot{q}_\alpha (\lambda) = \dot{q}_\alpha (0) + \ddot{q}_\alpha (0) \lambda + \frac{1}{2} \dddot{q}_\alpha (0) \lambda^2 + \cdots. \tag{3.22}
\]

The dual vector is parallel transported on \( \beta \) if

\[
\dot{q}_\alpha (\lambda) - \Gamma^\mu_{\alpha \beta}(\lambda) q_\mu (\lambda) \dot{p}^\beta (\lambda) = 0. \tag{3.23}
\]

We shall work on this equation, using the expansions for \( \dot{p}^\beta (\lambda) \) and \( \Gamma^\mu_{\alpha \beta}(\lambda) \) that are displayed in Eqs. (3.3) and (3.8), respectively.

We make the substitutions in Eq. (3.23) and collect terms that share the same power of \( \lambda \). Setting the coefficient of the \( \lambda^0 \) term to zero yields the condition

\[
\dot{q}_\alpha (0) = \Gamma^\mu_{\alpha \beta} q_\mu (0) \dot{p}^\beta (0). \tag{3.24}
\]

We recall that the Christoffel symbols are evaluated at the reference point \( \bar{x} \); the same remark applies to their derivatives, which will appear in expressions below.

Setting the coefficient of the \( \lambda^1 \) term to zero yields

\[
\ddot{q}_\alpha (0) = \Gamma^\mu_{\alpha \beta} \left[ q_\mu (0) \ddot{p}^\beta (0) + \dot{q}_\mu (0) \dot{p}^\beta (0) \right] + \Gamma^\mu_{\alpha \beta, \nu} \ddot{p}^\nu (0) q_\mu (0) \dot{p}^\beta (0).
\]

This becomes

\[
\ddot{q}_\alpha (0) = Q^\mu_{\alpha \beta, \gamma} q_\mu (0) \ddot{p}^\beta (0) \dot{p}^\gamma (0)
\]

with

\[
Q^\mu_{\alpha \beta, \gamma} = \Gamma^\mu_{\alpha \beta, \gamma} - \Gamma^\mu_{\alpha \lambda} \Gamma^\nu_{\beta \gamma} + \Gamma^\mu_{\gamma \nu} \Gamma^\nu_{\alpha \beta}, \tag{3.26}
\]

after involving Eqs. (3.9) and (3.24).

Setting the coefficient of the \( \lambda^2 \) term to zero yields

\[
q^{(3)}_\alpha (0) = \Gamma^\mu_{\alpha \beta} \left[ q_\mu (0) \dddot{p}^\beta (0) + 2 \dot{q}_\mu (0) \ddot{p}^\beta (0) + 2 \dddot{q}_\mu (0) \dot{p}^\beta (0) \right] + 2 \Gamma^\mu_{\alpha \beta, \nu} \dddot{p}^\nu (0) \left[ q_\mu (0) \dot{p}^\beta (0) + \ddot{q}_\mu (0) \ddot{p}^\beta (0) \right]
\]

\[
+ \left[ \Gamma^\mu_{\alpha \beta, \nu} \dddot{p}^\nu (0) + \Gamma^\mu_{\alpha \beta, \nu \lambda} \ddot{p}^\nu (0) \dot{p}^\lambda (0) \right] q_\mu (0) \dot{p}^\beta (0).
\]

This becomes

\[
q^{(3)}_\alpha (0) = Q^\mu_{\alpha \beta, \gamma, \delta} q_\mu (0) \dddot{p}^\beta (0) \dot{p}^\gamma (0) \dot{p}^\delta (0) \tag{3.27}
\]

with

\[
Q^\mu_{\alpha \beta, \gamma, \delta} = \Gamma^\mu_{\alpha \beta, \gamma \delta} - \Gamma^\mu_{\alpha \lambda} \Gamma^\nu_{\beta \gamma \delta} + \Gamma^\nu_{\alpha \beta \gamma} \Gamma^\mu_{\nu \gamma \delta} - 2 \Gamma^\nu_{\beta \gamma} \Gamma^\mu_{\alpha \nu \delta} + 2 \Gamma^\mu_{\beta \nu} \Gamma^\nu_{\alpha \gamma \delta} - \Gamma^\nu_{\gamma \beta} \Gamma^\mu_{\alpha \delta \nu}
\]

\[
+ 2 \Gamma^\mu_{\alpha \nu} \Gamma^\nu_{\beta \lambda} \Gamma^\lambda_{\gamma \delta} - 2 \Gamma^\mu_{\beta \nu} \Gamma^\nu_{\alpha \lambda} \Gamma^\lambda_{\gamma \delta} - \Gamma^\mu_{\gamma \lambda} \Gamma^\nu_{\alpha \beta} \Gamma^\lambda_{\gamma \delta} + \Gamma^\mu_{\beta \nu} \Gamma^\nu_{\gamma \lambda} \Gamma^\lambda_{\alpha \delta}, \tag{3.28}
\]
after involving Eqs. (3.9)–(3.11), as well as Eqs. (3.24)–(3.26). Equations (3.21), (3.24), (3.25), and (3.27) combine to give

\[ q_\alpha(1) = q_\alpha(0) + \Gamma^\mu_{\alpha\beta} q_\mu(0) p^\beta(0) + \frac{1}{2} Q^\mu_{\alpha\beta\gamma} q_\mu(0) p^\beta(0) p^\gamma(0) + \frac{1}{6} Q^\mu_{\alpha\beta\gamma\delta} q_\mu(0) p^\beta(0) p^\gamma(0) p^\delta(0) + \cdots. \]

There is a common factor of \( q_\mu(0) \), and this equation has the same form as Eq. (3.20). The parallel propagator is therefore identified as

\[ g^\mu_\alpha(x, \bar{x}) = \delta^\mu_\alpha + \Gamma^\mu_{\alpha\beta} \bar{p}^\beta(0) + \frac{1}{2} Q^\mu_{\alpha\beta\gamma} \bar{p}^\beta(0) \bar{p}^\gamma(0) + \frac{1}{6} Q^\mu_{\alpha\beta\gamma\delta} \bar{p}^\beta(0) \bar{p}^\gamma(0) \bar{p}^\delta(0) + \cdots. \] (3.29)

The coefficients of this expansion in powers of \( \bar{p}^\alpha(0) \) are given by Eqs. (3.26) and (3.28).

**G. Final result:** \( g^\mu_\alpha \) expanded in powers of \( w^\alpha \)

Our final expression for the parallel propagator is obtained by substituting Eq. (3.15) into Eq. (3.29) and expanding the result in powers of \( w^\alpha \). After a lengthy calculation that involves Eqs. (3.16), (3.17), and (3.18), we obtain

\[ g^\mu_\alpha(x, \bar{x}) = \delta^\mu_\alpha + B^\mu_{\alpha\beta} w^\beta + B^\mu_{\alpha\beta\gamma} w^\beta w^\gamma + B^\mu_{\alpha\beta\gamma\delta} w^\beta w^\gamma w^\delta + \cdots. \] (3.30)

with

\[ B^\mu_{\alpha\beta} := \Gamma^\mu_{\alpha\beta}, \]

\[ B^\mu_{\alpha\beta\gamma} := \frac{1}{2} \left( \Gamma^\mu_{\alpha\beta\gamma} + \Gamma^\mu_{\beta\alpha} \Gamma^\nu_{\gamma\nu} \right), \]

\[ B^\mu_{\alpha\beta\gamma\delta} := \frac{1}{12} \left( 2 \Gamma^\mu_{\alpha\beta\gamma\delta} + 2 \Gamma^\nu_{\alpha\beta} \Gamma^\mu_{\nu\gamma\delta} - \Gamma^\mu_{\beta\gamma} \Gamma^\mu_{\alpha\nu\delta} + 4 \Gamma^\mu_{\beta\nu} \Gamma^\nu_{\alpha\gamma\delta} + \Gamma^\mu_{\beta\gamma} \Gamma^\mu_{\alpha\delta\nu} - \Gamma^\mu_{\beta\nu} \Gamma^\nu_{\alpha\lambda\gamma\delta} + \Gamma^\mu_{\nu\lambda} \Gamma^\nu_{\alpha\beta\gamma\delta} + 2 \Gamma^\mu_{\beta\nu} \Gamma^\nu_{\lambda\gamma} \Gamma^\lambda_{\alpha\delta} \right). \] (3.33)

The Christoffel connection and its partial derivatives are all evaluated at the reference point \( \bar{x} \).

**IV. SPHERICAL-HARMONIC DECOMPOSITION OF THE SCALAR FIELD**

Our task in this section is to identify a useful way of relating a spherical-harmonic decomposition of the field \( \Phi_\alpha := \nabla_\alpha \Phi \) to a spherical-harmonic decomposition of the potential \( \Phi \). As we shall see, the selected relation involves a decomposing of the vector \( \Phi_\alpha \) in terms of a tetrad of orthonormal vectors \( e^\alpha_{(\mu)} \). So instead of decomposing \( \Phi_\alpha \) in a set of vectorial harmonics, we shall decompose each frame component \( \Phi_\mu := \Phi_\alpha e^\alpha_{(\mu)} \) of the vector field — a scalar function of the spacetime coordinates — in scalar spherical harmonics. The decomposition of \( \Phi_\alpha \) in vectorial harmonics would make a viable alternative strategy, but one which would prove less convenient for the purposes of calculating regularization parameters — see Sec. V and the Appendix. Our scheme leaves open the choice of tetrad, which is a priori arbitrary; our particular choice is motivated by a desire to keep the spherical-harmonic decompositions of \( \Phi \) and \( \Phi_\mu \) as closely linked as possible.

**A. Spherical-harmonic decompositions**

Let \( \Phi(x^\alpha, \theta^A) \) be a scalar field on a spherically-symmetric spacetime. The spacetime manifold has the product structure \( \mathcal{M}^2 \times S^2 \), in which \( \mathcal{M}^2 \) is a two-dimensional submanifold that is orthogonal to the two-spheres \( S^2 \). We let \( x^\alpha \) stand for any coordinate system that charts an open domain of the submanifold \( \mathcal{M}^2 \); the lower-case Latin index \( a \) runs from 0 to 1. We let \( \theta^A \) be angular coordinates on the two-spheres; the upper-case Latin index \( A \) runs from 2 to 3. For a Schwarzschild spacetime charted with the usual coordinates \( [t, r, \theta, \phi] \), we have \( x^\alpha = [t, r] \) and \( \theta^A = [\theta, \phi] \). We shall leave the coordinates \( x^\alpha \) arbitrary for the time being, but we adopt the canonical angular coordinates \( \theta^A = [\theta, \phi] \).

We suppose that the scalar field is expressed as a decomposition in spherical-harmonic functions \( Y^{lm} \), so that

\[ \Phi(x^\alpha, \theta^A) = \sum_{lm} \Phi^{lm}(x^\alpha) Y^{lm}(\theta^A). \] (4.1)
The sum over the integer \( l \) extends from \( l = 0 \) to \( l = \infty \), while the sum over the integer \( m \) ranges from \( m = -l \) to \( m = l \). To keep \( \Phi \) real the mode functions \( \Phi^{lm} \) must satisfy \( \Phi^{l-m} = (-1)^m \bar{\Phi}^{lm} \), in which an overbar indicates complex conjugation. The gradient \( \nabla_a \Phi \) of the scalar field possesses the components

\[
\partial_a \Phi = \sum_{lm} \partial_a \Phi^{lm} Y^{lm} \tag{4.2}
\]

and

\[
\partial_A \Phi = \sum_{lm} \Phi^{lm} \partial_A Y^{lm}. \tag{4.3}
\]

These relations show that a natural basis of expansion for \( \nabla_a \Phi \) would involve the scalar harmonics \( Y^{lm} \) in the \( \mathcal{M}^2 \) sector, and the vectorial harmonics \( \partial_A Y^{lm} \) in the \( S^2 \) sector. Following this route, however, would introduce complications at a later stage (refer to the last paragraph of Sec. 2 in the Appendix), and we shall adopt an alternative strategy.

We introduce a tetrad of orthonormal vectors \( e^{\alpha}_{(\mu)} \) at every point in the spherically-symmetric spacetime. The superscript \( \alpha \) is the usual vectorial index, and the subscript \( (\mu) = \{(0),(1),(2),(3)\} \) is a label that designates an individual member of the tetrad. These vectors satisfy

\[
M_{\alpha \beta} \equiv \delta_{\alpha \beta} - e^{\alpha}_{(\mu)} e^{\beta}_{(\nu)} \delta_{(\mu)(\nu)} = 0 \tag{4.4}
\]

with \( \eta_{(\mu)(\nu)} = \text{diag}[-1,1,1,1] \). The four quantities

\[
\Phi_{(\mu)} := e^{\alpha}_{(\mu)} \nabla_\alpha \Phi \tag{4.5}
\]

are the \textit{frame components} of the vector \( \nabla_\alpha \Phi \); these are \textit{scalar functions} of the spacetime coordinates. The vector field can be reconstructed from its frame components by involving the dual tetrad \( e^{\alpha}_{(\mu)} \), which is defined by

\[
e^{\alpha}_{(\mu)} := \eta_{(\mu)(\nu)} g_{\alpha \beta} e^{\beta}_{(\nu)}, \tag{4.6}
\]

where \( \eta_{(\mu)(\nu)} = \text{diag}[-1,1,1,1] \) is the matrix inverse of \( \eta_{(\mu)(\nu)} \). It is easy to show that

\[
\nabla_\alpha \Phi = \Phi_{(\mu)} e^{(\mu)}_{\alpha}. \tag{4.7}
\]

Because each frame component \( \Phi_{(\mu)} \) is a scalar quantity, it is natural to decompose it in scalar harmonics. We therefore write

\[
\Phi_{(\mu)}(x^a, \theta^A) = \sum_{lm} \Phi^{lm}_{(\mu)}(x^a) Y^{lm}(\theta^A), \tag{4.8}
\]

and seek to determine the relation between \( \Phi^{lm}_{(\mu)} \), the modes of the frame components, and \( \Phi^{lm} \), the modes of the original scalar field.

The answer is provided by substituting Eq. (4.5) into the equations \( \Phi^{lm}_{(\mu)} = \int \Phi_{(\mu)} \tilde{Y}^{lm} \, d\Omega \), where \( d\Omega = \sin \theta \, d\theta d\phi \) is an element of solid angle. After involving Eqs. (4.2) and (4.3), we obtain

\[
\Phi^{lm}_{(\mu)} = \sum_{l'm'} \left[ C^{a}_{(\mu)}(l'm'|lm) \partial_a \Phi^{l'm'} + C_{(\mu)}(l'm'|lm) \Phi^{l'm'} \right], \tag{4.9}
\]

where the \textit{coupling coefficients} are given by

\[
C^{a}_{(\mu)}(l'm'|lm) := \int e^{a}_{(\mu)} Y^{l'm'} \tilde{Y}^{lm} \, d\Omega, \tag{4.10}
\]

and

\[
C_{(\mu)}(l'm'|lm) := \int e^{A}_{(\mu)} Y^{l'm'} \tilde{Y}^{lm} \, d\Omega. \tag{4.11}
\]

Here, \( e^{a}_{(\mu)} \) denotes the \((0,1)\) components of each basis vector, while \( e^{A}_{(\mu)} \) represents the angular components. The computation of the coupling coefficients requires the specification of the tetrad.

In the rest of this section we will make a specific choice of tetrad (Sec. II B), compute the coupling coefficients for this tetrad (Secs. II C and II D), and give an explicit form to Eq. (4.9). The tetrad is displayed in Eqs. (4.21)–(4.25) below, and the resulting relation between \( \Phi^{lm}_{(\mu)} \) and \( \Phi^{lm} \) was already displayed in Eqs. (1.23)–(1.26).
B. Choice of tetrad

The choice of tetrad is in principle free, but we wish to find a tetrad that leads to a simple structure for the coupling coefficients. Specializing to Schwarzschild spacetime and the usual coordinates \([t, r, \theta, \phi]\), a possible choice of tetrad would be the usual orthonormal frame

\[
e^\alpha_{(i)} = \begin{bmatrix} f^{-1/2}, 0, 0, 0 \end{bmatrix}, \quad (4.12)
\]

\[
e^\alpha_{(r)} = \begin{bmatrix} 0, f^{1/2}, 0, 0 \end{bmatrix}, \quad (4.13)
\]

\[
e^\alpha_{(\theta)} = \begin{bmatrix} 0, 0, r^{-1}, 0 \end{bmatrix}, \quad (4.14)
\]

\[
e^\alpha_{(\phi)} = \begin{bmatrix} 0, 0, 0, (r \sin \theta)^{-1} \end{bmatrix}, \quad (4.15)
\]

where

\[
f := 1 - \frac{2M}{r}. \quad (4.16)
\]

It is easy to show, however, that while this tetrad would lead to simple (diagonal) coupling coefficients \(C^a_{(\mu)}\), it would also lead to coefficients \(C_{(\mu)}\) that couple each \((lm)\) mode to an infinite number of \((l'm')\) modes. We shall not, therefore, make this choice of tetrad.

We shall instead introduce a “Cartesian frame” that is linked to the “spherical frame” of Eqs. (4.12)–(4.15) by the same relations that would hold in flat spacetime. Explicitly, our choice of tetrad is

\[
e^\alpha_{(0)} := e^\alpha_{(1)}, \quad (4.17)
\]

\[
e^\alpha_{(1)} := \sin \theta \cos \phi e^\alpha_{(r)} + \cos \theta \cos \phi e^\alpha_{(\theta)} - \sin \phi e^\alpha_{(\phi)}, \quad (4.18)
\]

\[
e^\alpha_{(2)} := \sin \theta \sin \phi e^\alpha_{(r)} + \cos \theta \sin \phi e^\alpha_{(\theta)} + \cos \phi e^\alpha_{(\phi)}, \quad (4.19)
\]

\[
e^\alpha_{(3)} := \cos \theta e^\alpha_{(r)} - \sin \theta e^\alpha_{(\phi)}. \quad (4.20)
\]

We may loosely think of \(e^\alpha_{(1)}\) as pointing in the “\(x\) direction,” of \(e^\alpha_{(2)}\) as pointing in the “\(y\) direction,” and of \(e^\alpha_{(3)}\) as pointing in the “\(z\) direction,” with \([x, y, z]\) representing a quasi-Cartesian frame related in the usual way to the quasi-spherical coordinates \([r, \theta, \phi]\). Independently of this heuristic interpretation, we note that the transformation of Eqs. (4.17)–(4.20) defines a legitimate set of orthonormal vectors. And as we shall see, this tetrad has the desirable property of leading to a simple structure for the coupling coefficients.

Combining Eqs. (4.17)–(4.20) with Eqs. (4.12)–(4.15) produces the following explicit expressions for the basis vectors:

\[
e^\alpha_{(0)} = \begin{bmatrix} \frac{1}{\sqrt{f}}, 0, 0, 0 \end{bmatrix}, \quad (4.21)
\]

\[
e^\alpha_{(1)} = \begin{bmatrix} 0, \sqrt{f} \sin \theta \cos \phi, \frac{1}{r} \cos \theta \cos \phi, -\frac{\sin \phi}{r \sin \theta} \end{bmatrix}, \quad (4.22)
\]

\[
e^\alpha_{(2)} = \begin{bmatrix} 0, \sqrt{f} \sin \theta \sin \phi, \frac{1}{r} \cos \theta \sin \phi, \frac{\cos \phi}{r \sin \theta} \end{bmatrix}, \quad (4.23)
\]

\[
e^\alpha_{(3)} = \begin{bmatrix} 0, \sqrt{f} \cos \theta, -\frac{1}{r} \sin \theta, 0 \end{bmatrix}. \quad (4.24)
\]

It is useful to introduce, as substitutes for \(e^\alpha_{(1)}\) and \(e^\alpha_{(2)}\), the complex combinations

\[
e^\alpha_{(\pm)} := e^\alpha_{(1)} \pm ie^\alpha_{(2)} = \begin{bmatrix} 0, \sqrt{f} \sin \theta e^{\pm i\phi}, \frac{1}{r} \cos \theta e^{\pm i\phi}, \pm \frac{ie^{\pm i\phi}}{r \sin \theta} \end{bmatrix}. \quad (4.25)
\]

In terms of the complex vectors we have \(e^\alpha_{(1)} = [e^\alpha_{(+)} + e^\alpha_{(-)}]/2 = \text{Re}[e^\alpha_{(+)}/(2i)]\) and \(e^\alpha_{(2)} = [e^\alpha_{(+)}/(2i)] - [e^\alpha_{(-)}]/(2i) = \text{Im}[e^\alpha_{(+)}/(2i)]\). In the sequel we will work primarily in terms of the complex tetrad \(e^\alpha_{(0)}, e^\alpha_{(+)}, e^\alpha_{(-)},\) and \(e^\alpha_{(3)}.\)
C. Calculation of \(C^a_{(\mu)}(l'm'|lm)\)

We may now substitute the tetrad of Eqs. (4.21)–(4.25) into Eq. (4.10) and calculate \(C^a_{(\mu)}(l'm'|lm)\), the first set of coupling coefficients. Our computations will rely on the standard identities (see, for example, Sec. 12.9 of Ref. [40])

\[
\cos \theta Y^{l,m} = \sqrt{\frac{(l - m + 1)(l + m + 1)}{(2l + 1)(2l + 3)}} Y^{l+1,m} + \sqrt{\frac{(l - m)(l + m)}{(2l - 1)(2l + 1)}} Y^{l-1,m},
\]

(4.26)

\[
\sin \theta e^{i\phi} Y^{l,m} = -\sqrt{\frac{(l + m + 1)(l + m + 2)}{(2l + 1)(2l + 3)}} Y^{l+1,m+1} + \sqrt{\frac{(l - m)(l - m - 1)}{(2l - 1)(2l + 1)}} Y^{l-1,m+1},
\]

(4.27)

\[
\sin \theta e^{-i\phi} Y^{l,m} = \sqrt{\frac{(l - m + 1)(l + m + 2)}{(2l + 1)(2l + 3)}} Y^{l+1,m-1} - \sqrt{\frac{(l + m)(l + m - 1)}{(2l - 1)(2l + 1)}} Y^{l-1,m-1},
\]

(4.28)

involving spherical-harmonic functions.

Substituting Eq. (4.21) into Eq. (4.10) and involving the orthonormality relations of the spherical harmonics reveals that the only nonvanishing component of \(C^a_{(0)}(l'm'|lm)\) is

\[
C^a_{(0)} = \frac{1}{\sqrt{f}} \delta_{ll'} \delta_{mm'}. \tag{4.29}
\]

Substituting Eq. (4.25) into Eq. (4.10) and involving Eq. (4.27) shows that

\[
C^r_{(+) \mp} = -\sqrt{\frac{(l + m - 1)(l + m)}{(2l - 1)(2l + 1)}} \sqrt{f} \delta_{l,l'} \delta_{m,m'} + \sqrt{\frac{(l - m + 1)(l - m + 2)}{(2l + 1)(2l + 3)}} \sqrt{f} \delta_{l,l'} \delta_{m,m'} \tag{4.30}
\]

is the only nonvanishing component of \(C^a_{(+) \mp}(l'm'|lm)\). Similarly,

\[
C^r_{(-)} = \sqrt{\frac{(l - m - 1)(l + m)}{(2l - 1)(2l + 1)}} \sqrt{f} \delta_{l,l'} \delta_{m,m'} - \sqrt{\frac{(l + m + 1)(l + m + 2)}{(2l + 1)(2l + 3)}} \sqrt{f} \delta_{l,l'} \delta_{m,m'} \tag{4.31}
\]

is the only nonvanishing component of \(C^a_{(-)}(l'm'|lm)\). Finally, substituting Eq. (4.24) into Eq. (4.10) and involving Eq. (4.26) reveals that

\[
C^r_{(3)} = \sqrt{\frac{(l - m)(l + m)}{(2l - 1)(2l + 1)}} \sqrt{f} \delta_{l,l'} \delta_{m,m'} + \sqrt{\frac{(l - m + 1)(l + m + 1)}{(2l + 1)(2l + 3)}} \sqrt{f} \delta_{l,l'} \delta_{m,m'} \tag{4.32}
\]

is the only nonvanishing component of \(C^a_{(3)}(l'm'|lm)\).

D. Calculation of \(C_{(\mu)}(l'm'|lm)\)

We now substitute the tetrad of Eqs. (4.21)–(4.25) into Eq. (4.11) and calculate \(C_{(\mu)}(l'm'|lm)\), the second set of coupling coefficients. These computations also will rely on the standard identities listed in Eqs. (4.26)–(4.28), as well as (see, for example, Sec. 12.7 of Ref. [40])

\[
e^{i\phi} (\partial_\theta + i \cot \theta \partial_\varphi) Y^{l,m} = \sqrt{(l - m)(l + m + 1)} Y^{l,m+1} \tag{4.33}
\]

and

\[
e^{-i\phi} (\partial_\theta - i \cot \theta \partial_\varphi) Y^{l,m} = -\sqrt{(l + m)(l - m + 1)} Y^{l,m-1}. \tag{4.34}
\]

Substituting Eq. (4.21) into Eq. (4.11) immediately gives

\[
C_{(0)} = 0, \tag{4.35}
\]
Substituting Eq. (4.25) into Eq. (4.11) produces an integral for $C_{(+)}$ that involves the factor
\[ \cos \theta e^{i\phi} \partial_y Y^{l'm'} + \frac{ie^{i\phi}}{\sin \theta} \partial_y Y^{l'm'}. \]
The first term can be expressed as
\[ e^{i\phi} \partial_y (\cos \theta Y^{l'm'}) + \sin \theta e^{i\phi} Y^{l'm'}. \]
The second term, on the other hand, can be expressed as
\[ ie^{i\phi} \cot \theta \partial_y (\cos \theta Y^{l'm'}) + i \sin \theta e^{i\phi} Y^{l'm'}. \]
The sum becomes
\[ e^{i\phi} (\partial_y + i \cot \theta \partial_y) (\cos \theta Y^{l'm'}) - (m' - 1) \sin \theta e^{i\phi} Y^{l'm'}, \]
and this is in such a form that Eqs. (4.26), (4.27), and (4.33) can now be involved. Multiplying by $\bar{Y}^{lm}$ and evaluating the integrals returns
\[ C_{(+)} = \sqrt{\frac{(l + m - 1)(l + m)}{(2l - 1)(2l + 1)}} \frac{l - 1}{r} \delta_{l',l-1} \delta_{m',m-1} + \sqrt{\frac{(l - m + 1)(l - m + 2)}{(2l + 1)(2l + 3)}} \frac{l + 2}{r} \delta_{l',l+1} \delta_{m',m-1}. \] (4.36)
We similarly obtain
\[ C_{(-)} = -\sqrt{\frac{(l - m - 1)(l - m)}{(2l - 1)(2l + 1)}} \frac{l - 1}{r} \delta_{l',l-1} \delta_{m',m+1} - \sqrt{\frac{(l + m + 1)(l + m + 2)}{(2l + 1)(2l + 3)}} \frac{l + 2}{r} \delta_{l',l+1} \delta_{m',m+1}. \] (4.37)

Substituting Eq. (4.24) into Eq. (4.11) produces an integral that involves the factor
\[ \sin \theta \partial_y Y^{l'm'} = \partial_y (\sin \theta Y^{l'm'}) - \cos \theta Y^{l'm'}. \]
This can be expressed as
\[ e^{-i\phi} (\partial_y - i \cot \theta \partial_y) (\sin \theta e^{i\phi} Y^{l'm'}) + ie^{-i\phi} \cot \theta \partial_y (\sin \theta e^{i\phi} Y^{l'm'}) - \cos \theta Y^{l'm'}, \]
or as
\[ e^{-i\phi} (\partial_y - i \cot \theta \partial_y) (\sin \theta e^{i\phi} Y^{l'm'}) - (m' + 2) \cos \theta Y^{l'm'}, \]
which is now in a useful form. After involving Eqs. (4.26), (4.27), and (4.34), then multiplying by $\bar{Y}^{lm}$, and finally evaluating the integrals, we arrive at
\[ C_{(3)} = -\sqrt{\frac{(l - m)(l + m)}{(2l - 1)(2l + 1)}} \frac{l - 1}{r} \delta_{l',l-1} \delta_{m'm} + \sqrt{\frac{(l - m + 1)(l + m + 1)}{(2l + 1)(2l + 3)}} \frac{l + 2}{r} \delta_{l',l+1} \delta_{m'm}. \] (4.38)

E. Final result: $\Phi^{lm}_{(\mu)}$ in terms of $\Phi^{lm}$

We substitute Eqs. (4.29)–(4.32), (4.35)–(4.38) into Eq. (4.9), which gives $\Phi^{lm}_{(\mu)}$, the spherical-harmonic modes of the frame components $\Phi_{(\mu)}$, in terms of $\Phi^{lm}$, the modes of the original scalar field $\Phi$. After evaluating the sums over $l'$ and $m'$, we find that the relationship is given by Eqs. (1.23)–(1.26), which are displayed back in Sec. I E. Inspection of these equations reveals that the relationship is simple: The structure of the coupling coefficients is such that $\Phi^{lm}_{(\mu)}$ is linked to the neighboring modes $\Phi^{l\pm 1,m}$ and $\Phi^{l, m\pm 1}$ only. This simplicity is a benefit of the choice of tetrad made in Sec. IV B; as was discussed in that subsection, other choices would produce more complicated relationships.
V. REGULARIZATION PARAMETERS

The self-force acting on a scalar charge $q$ moving on a geodesic of the Schwarzschild spacetime is proportional to $\Phi_\alpha := \nabla_\alpha \Phi^R$, where $\Phi^R$ is the regular potential that remains after the singular potential $\Phi^S$ is subtracted from the retarded potential $\Phi$. A local covariant expansion for $\Phi^R := \nabla_\alpha \Phi^S$ was worked out in Sec. II, and with the help of the results presented in Sec. III, this can be turned into an explicit expansion in Schwarzschild coordinates. In Sec. IV we introduced a tetrad of orthonormal vectors $e_\mu^\alpha$ to resolve the vector fields $\Phi_\alpha$, $\Phi_\mu^R$, and $\Phi_\mu^S$ in terms of their frame components $\Phi_{(\mu)} := \Phi_\alpha e_\mu^\alpha$, $\Phi_{(\mu)}^R := \Phi_\mu^R e_\mu^\alpha$, and $\Phi_{(\mu)}^S := \Phi_\mu^S e_\mu^\alpha$, respectively. We have

$$\Phi_{(\mu)}^R := \Phi_{(\mu)} - \Phi_{(\mu)}^S. \quad (5.1)$$

Also in Sec. IV we showed how the spherical-harmonic modes of the frame components $\Phi_{(\mu)}$ can be related to those of the scalar potential $\Phi$. Our task in this section is to compute the spherical-harmonic modes of the singular field $\Phi_{(\mu)}^S$ and to extract from them the quantities known as regularization parameters. We will rely on the results obtained in Sec. II, III, and IV, as well as multipole-decomposition techniques reviewed in the Appendix. Most of the computations that are described below were carried out with the symbolic manipulation software MAPLE, with the help of the tensor package GRTENSORII [37]. We will describe how these calculations were performed, but space considerations compel us to leave most details hidden.

A. Definition of the regularization parameters

The scalar charge $q$ moves on an arbitrary geodesic of the Schwarzschild spacetime. We place this geodesic in the equatorial plane, and we assign to the particle the coordinates $t = t(\tau)$, $r = r(\tau)$, $\theta = \frac{\pi}{2}$, and $\phi = \phi(\tau)$, in which $\tau$ is proper time on the geodesic. These functions are determined by integrating the geodesic equations, which take the form $\ddot{t} = E/f$, $\ddot{r} = E^2 - f(1 + L^2/r^2)$, and $\ddot{\phi} = L/r^2$, in which an overdot indicates differentiation with respect to $\tau$. The constant $E$ is the particle’s conserved energy per unit rest-mass, and the constant $L$ is the conserved angular momentum per unit rest-mass. We also have introduced the metric function $f := 1 - 2M/r$ and its value $f_0 := f(r_0)$. At some instant $\tau = \tau_0$, the particle is found at the point $\vec{x} = [t_0, r_0, \frac{\pi}{2}, \phi_0]$ on its world line. We let $f_0 := 1 - 2M/r_0$, and we wish to evaluate the scalar self-force at that instant.

We are interested in the value of $\Phi_{(\mu)}$ at the point $\vec{x}$. We calculate this by first decomposing the retarded field $\Phi_{(\mu)}$ into spherical-harmonic modes, then subtracting the modes associated with the singular field $\Phi_{(\mu)}^S$, and finally summing over all modes. This mode-sum converges because the regular field is smooth on the world line; the mode-sums associated with the retarded and singular fields do not converge on their own, because both fields diverge on the world line. In spite of this singular behavior, each spherical-harmonic mode $\Phi_{(\mu)lm}$ and $\Phi_{(\mu)lm}^S$ is bounded as $x \to \vec{x}$; the singularity merely gives rise to a jump discontinuity of each mode at $\vec{x}$. The value of each mode is therefore ambiguous at $\vec{x}$, but this ambiguity is of no consequence because the discontinuity disappears when the mode of the singular field is subtracted from the mode of the retarded field. The modes of the regular field are continuous (and differentiable) at $\vec{x}$ because $\Phi_{(\mu)}^R$ is smooth on the world line. In practice the discontinuity can be handled by evaluating each mode of the retarded and singular fields slightly away from $\vec{x}$, performing the subtraction, and then taking the limit $x \to \vec{x}$.

A practical implementation of this prescription is contained in

$$\Phi_{(\mu)}^R(t_0, r_0, \vec{\pi}, \phi_0) = \lim_{\Delta \to 0} \sum_{l=1}^{\infty} \left[ \Phi_{(\mu)l} - \Phi_{(\mu)l}^S \right], \quad (5.2)$$

where

$$\Phi_{(\mu)l} := \sum_{m=-l}^{l} \Phi_{(\mu)lm}(t_0, r_0 + \Delta)Y^{lm}(\vec{\pi}, \phi_0') \quad (5.3)$$

are the multipole coefficients of the retarded field, while

$$\Phi_{(\mu)l}^S := \sum_{m=-l}^{l} \Phi_{(\mu)lm}(t_0, r_0 + \Delta)Y^{lm}(\vec{\pi}, \phi_0') \quad (5.4)$$

are the multipole coefficients of the singular field. The quantities $\Phi_{(\mu)lm}$ and $\Phi_{(\mu)lm}^S$ are the spherical-harmonic modes of the retarded and singular fields, respectively. (Details regarding the spherical-harmonic decomposition of a scalar
function on $S^2$ are provided in the Appendix.) In Eqs. (5.2)–(5.4) we choose the displaced point to be at the same
time coordinate $t_0$ as $\bar{x}$, but at a displaced radius $r_0 + \Delta$ and a displaced azimuthal angle
\[
\phi'_0 := \phi_0 - c\Delta;
\] (5.5)
the constant $c$ will be selected for convenience below, in Sec. V D — see Eq. (5.20). (The idea of introducing a
displacement in the $\phi$ direction goes back to Mino, Nakano, and Sasaki [19].)

Equations (5.2)–(5.4) are at the core of the procedure to calculate the self-force. We imagine that the modes $\Phi_{(\mu)lm}$
of the retarded field can be computed with a convenient numerical method. From these we obtain the multipole
coefficients $\Phi_{(\mu)l}$, from which we subtract $\Phi^S_{(\mu)l}$, the multipole coefficients of the singular field. These can be computed
analytically, which is our task in this section. As we shall see, they have the form
\[
\Phi_{(\mu)l} = q \left[ (l + \frac{1}{2}) A_{(\mu)} + B_{(\mu)} + \frac{C_{(\mu)}}{(l + \frac{1}{2})} + \frac{D_{(\mu)}}{(l - \frac{1}{2})(l + \frac{3}{2})} + \cdots \right],
\] (5.6)
in which the coefficients $A_{(\mu)}$, $B_{(\mu)}$, $C_{(\mu)}$, and $D_{(\mu)}$, known as regularization parameters, are independent of $l$ but
depend on the state of motion of the particle at $\bar{x}$. The subtraction produces $\Phi_{(\mu)l}$, which we sum over all values of $l$
to get $\Phi_{(\mu)}^R$ and eventually the self-force.

**B. Rotation of the angular coordinates**

Efficient techniques to compute multipole coefficients were devised by Detweiler, Messaritaki, and Whiting [22];
these are reviewed in the Appendix. They rely on a rotation of the angular coordinates that maps the special point
$(\theta = \frac{\pi}{2}, \phi = \phi'_0)$ to the North pole of the new angular coordinates. (This idea goes back to Barack and Ori [17, 18].)
This rotation is described by
\[
\begin{align*}
\sin \theta \cos(\phi - \phi'_0) &= \cos \alpha, \\
\sin \theta \sin(\phi - \phi'_0) &= \sin \alpha \cos \beta, \\
\cos \theta &= \sin \alpha \sin \beta,
\end{align*}
\] (5.7–5.9)
which is a slightly modified version of the rotation described by Eqs. (A.5)–(A.7). The new angles are $\alpha$ and $\beta$, and
it is easy to see that the rotation does indeed map the point $(\theta = \frac{\pi}{2}, \phi = \phi'_0)$ to $(\alpha = 0, \beta = ?)$, with $\beta$ undetermined.

As reviewed in the Appendix, each multipole coefficient $\Phi_{(\mu)l}^S$ is computed by first expressing the singular field $\Phi_{(\mu)}^S$
as a function of the angles $\alpha$ and $\beta$, then extracting the Legendre projection
\[
\Phi_{(\mu)l}^S(\beta) := \frac{1}{2} \int_{-1}^{1} \Phi_{(\mu)}^S(t_0, r_0 + \Delta, \alpha, \beta) P_l(\cos \alpha) d \cos \alpha,
\] (5.10)
and finally averaging over the angles $\beta$,
\[
\Phi_{(\mu)l}^S = \frac{1}{2\pi} \int_{0}^{2\pi} \Phi_{(\mu)l}^S(\beta) d \beta.
\] (5.11)
The decomposition of $\Phi_{(\mu)l}(t_0, r_0 + \Delta, \alpha, \beta)$ in Legendre polynomials relies on the techniques reviewed in Sec. 4 of the
Appendix. The averaging over all angles $\beta$ relies on the techniques reviewed in Sec. 5 of the Appendix. A summary
of the key results is provided in Sec. 6 of the Appendix.

**C. Calculation of $\Phi_{(\mu)(t_0, r_0 + \Delta, \alpha, \beta)$**

The starting point of these computations is the calculation of the singular field $\Phi_{(\mu)}^S$ at a position $x$, to which we
assign the (unrotated) coordinates $(t_0, r_0 + \Delta, \theta, \phi)$. This point is slightly displaced from $\bar{x}$, to which we have assigned
the coordinates $(t_0, r_0, \frac{\pi}{2}, \phi_0)$. The displacement vector is
\[
w^\alpha := x^\alpha - \bar{x}^\alpha = [0, \Delta, \theta - \frac{\pi}{2}, \phi - \phi_0],
\] (5.12)
and an expression for $\Phi_{(\mu)l}(x)$ expanded in powers of $w^\alpha$ can be obtained by combining the results displayed in Sec. II
J [Eq. (2.62) and following], Sec. III E [Eq. (3.19)], Sec. III G [Eq. (3.30)], and Sec. IV B [Eqs. (4.21)–(4.25)]. The
computation of the singular field is extremely tedious, and was handled by the symbolic manipulator GRTENSORII [37] operating under MAPLE.

The singular field is now expressed in terms of $w^\theta$ and $w^\phi$ (in addition to $\Delta$), but these components of the displacement vector are functions of $\alpha$ and $\beta$ that can be determined from Eqs. (5.7)–(5.9). We have

$$w^\theta = -\arcsin(\sin \alpha \sin \beta)$$  \hspace{1cm} (5.13)

and

$$w^\phi = \arcsin\left(\frac{\sin \alpha \cos \beta}{\sqrt{1 - \sin^2 \alpha \sin^2 \beta}}\right) - c\Delta,$$  \hspace{1cm} (5.14)

where the (as yet unassigned) constant $c$ was introduced back in Eq. (5.5). These quantities are small wherever $\alpha$ and $\Delta$ are small, and the expansion of $\Phi^S_{(\mu)}$ in powers of $w^\alpha$ is valid in a small neighborhood of the North pole ($\alpha = 0, \beta = ?$).

Defining

$$Q := \sqrt{1 - \cos \alpha},$$  \hspace{1cm} (5.15)

we observe that $w^\theta$ and $w^\phi$ can each be expressed as an expansion in powers of $Q$:

$$w^\theta = -\sqrt{2}Q \sin \beta - \frac{\sqrt{2}}{12}Q^3(1 - 4 \cos^2 \beta) \sin \beta + O(Q^5),$$  \hspace{1cm} (5.16)

$$w^\phi = -c\Delta + \sqrt{2}Q \cos \beta + \frac{\sqrt{2}}{12}Q^3(9 - 8 \cos^2 \beta) \cos \beta + O(Q^5).$$  \hspace{1cm} (5.17)

We note that $Q$ is formally of the same order of magnitude as $\alpha$. Making these substitutions within the singular field returns a double expansion in powers of $\Delta$ and $Q$, which are formally considered to be of the same order of magnitude. This new representation of $\Phi^S_{(\mu)}$ involves the rotated angles $\alpha$ and $\beta$, which is required for its substitution into Eqs. (5.10) and (5.11). Moreover, the singular field is expressed in terms of $\sin \beta, \cos \beta, \text{ and } Q$, functions that are globally well-defined on the sphere. This property is critical for the successful decomposition of $\Phi^S_{(\mu)}$ in Legendre polynomials, and the subsequent averaging over $\beta$. We stole this powerful idea from Detweiler, Massaritaki, and Whiting [22].

It should be acknowledged that the global extension of the singular field beyond the neighborhood of the North Pole is not unique. This ambiguity, however, is of no consequence, because a different extension $\Phi^S_{(\mu)}$ that continues to respect the local expansion through order $\epsilon^1$ will be such that $\Phi^S_{(\mu)} - \Phi^S_{(\mu)} = O(\epsilon^2)$. Because the difference vanishes at the position of the particle, the value of the self-force, and the value of the four regularization parameters, will not be affected.

D. Squared-distance function

An important piece of $\Phi^S_{(\mu)}$, as can be seen from Eq. (2.62), is $s^2$, the squared distance between the points $x$ and $\bar{x}$. This is defined by Eq. (2.19), and its leading term in an expansion in powers of $w^\alpha$ is obtained by substituting $\sigma^\alpha (x, \bar{x}) = -g_{\alpha\beta}w^\beta$, a truncated version of Eq. (3.19), into Eq. (2.19). The result is

$$\rho^2 := (g_{\alpha\beta} + u_\alpha u_\beta)w^\alpha w^\beta,$$  \hspace{1cm} (5.18)

in which the metric and the velocity vector are evaluated at $\bar{x} = [t_0, r_0, \frac{\pi}{2}, \phi_0]$. With $w^\alpha := [E/f_0, \dot{r}_0, 0, L/r_0^2]$ we obtain

$$\rho^2 = \frac{r_0(r_0 - 2M + r_0^2)}{(r_0 - 2M)^2} \Delta^2 + r_0^2w^\theta^2 + (r_0^2 + L^2)(w^\phi)^2 + \frac{2r_0L\dot{r}_0}{r_0 - 2M} \Delta w^\phi.$$  \hspace{1cm} (5.19)

We re-express this result in terms of $w^\phi' := \phi - \phi' = w^\phi + c\Delta$ and choose $c$ in order to eliminate the term proportional to $\Delta w^\phi$. With [19]

$$c := \frac{r_0L\dot{r}_0}{(r_0 - 2M)(r_0^2 + L^2)}$$  \hspace{1cm} (5.20)
we find that Eq. (5.19) becomes
\[
\dot{\rho}^2 = \frac{r_0^4 E^2}{(r_0 - 2M)^2 (r_0^2 + L^2)} \Delta^2 + r_0^2 (w^\theta)^2 + (r_0^2 + L^2) (w^\phi)^2,
\]
(5.21)
where we have also used the geodesic equation to eliminate \( r_0^2 \), the square of the radial velocity at \( \tilde{x} \), in favor of \( E^2 \).

We define a “squared-distance function” \( \rho^2 \) by making the substitutions \( w^\theta = -\sqrt{2Q} \sin \beta \) and \( w^\phi = \sqrt{2Q} \cos \beta \) into Eq. (5.21); these are truncated versions of Eq. (5.16) and (5.17), respectively. We find that this is given by
\[
\rho^2 := \frac{r_0^4 E^2}{(r_0 - 2M)^2 (r_0^2 + L^2)} \Delta^2 + 2(r_0^2 + L^2) \chi Q^2,
\]
(5.22)
in which
\[
\chi := 1 - k \sin^2 \beta, \quad k := \frac{L^2}{r_0^2 + L^2}
\]
(5.23)
contains the dependence of \( \rho^2 \) on \( \beta \). Recalling the definition of \( Q \) from Eq. (5.15), the squared-distance function can also be written as
\[
\rho^2 = 2(r_0^2 + L^2) \chi (\delta^2 + 1 - \cos \alpha),
\]
(5.24)
with
\[
\delta^2 := \frac{E^2 r_0^4}{2(r_0^2 + L^2)^2 (r_0 - 2M)^2} \frac{\Delta^2}{\chi}.
\]
(5.25)

E. Calculation of \( \Phi^S_{(\mu)} \) — continued

At the end of Sec. V C we had computed the singular field \( \Phi^S_{(\mu)}(t_0, r_0 + \Delta, \alpha, \beta) \) and expressed it in terms of \( \sin \beta \), \( \cos \beta \), \( Q := \sqrt{1 - \cos \alpha} \), and \( \Delta \); the expression also involves world-line quantities such as \( r_0 \), \( E \), \( L \), and \( \tilde{r}_0 \) which describe the state of motion of the particle at \( \tilde{x} \). We interrupted this computation in Sec. V D to introduce the “squared-distance function” \( \rho^2 \), which is the leading term in an expansion of \( s^2 := (g^\delta)^{\beta\gamma} + u^{\beta} u^{\gamma} \sigma_{\alpha} \sigma_{\beta} \) in powers of \( Q \) and \( \Delta \); this is itself a function of \( \sin \beta \), \( Q \) and \( \Delta \), as well as world-line quantities.

The squared-distance function is introduced to eliminate the dependence of \( \Phi^S_{(\mu)} \) on all even powers of \( Q \), and to reduce all odd powers of \( Q \) to something linear in \( Q \). The idea is to solve Eq. (5.22) for \( Q^2 \) and to substitute the resulting expression \( Q^2(\rho^2, \Delta^2) \) into our current representation of the singular field. We thus systematically replace each factor \( Q^{2n} \) in \( \Phi^S_{(\mu)} \) by \( [Q^2(\rho^2, \Delta^2)]^{n} \), and each factor \( Q^{2n+1} \) by \( [Q^2(\rho^2, \Delta^2)]^{n} Q \). This yields a representation of the singular field which separates into a first set of terms that is independent of \( Q \) and a second set of terms that is proportional to \( Q \); each set contains a dependence on \( \sin \beta \), \( \cos \beta \), \( \rho \), and \( \Delta \), as well as on the world-line quantities.

Our final expression for \( \Phi^S_{(\mu)}(t_0, r_0 + \Delta, \alpha, \beta) \) is too long to be displayed here. In fact, it is too long to be displayed anywhere, and we have taken measures to keep it hidden within the depths of our MAPLE worksheets. Its basic structure is as follows. The singular field admits an expansion in powers of \( \epsilon \) of the form
\[
\Phi^S_{(\mu)} = \Phi^S_{(\mu),-2} + \Phi^S_{(\mu),-1} + \Phi^S_{(\mu),0} + \Phi^S_{(\mu),+1} + O(\epsilon^2),
\]
(5.26)
in which \( \Phi^S_{(\mu),-2} \) is of order \( \epsilon^{-2} \), \( \Phi^S_{(\mu),-1} \) of order \( \epsilon^{-1} \), and so on. (Recall from Sec. II that \( \epsilon \) loosely measures the distance between \( x \) and \( \tilde{x} \); we have that \( \rho \) and \( \Delta \) are both of order \( \epsilon \).) The terms that appear on the right-hand side of Eq. (5.26) possess the schematic form
\[
\Phi^S_{(\mu),-2} = M_{(\mu),-2}(\Delta/\rho^3) + O(Q \cos \beta / \rho^3),
\]
(5.27)
\[
\Phi^S_{(\mu),-1} = M_{(\mu),-1}(1/\rho) + O(Q \cos \beta \Delta / \rho^4) + O(\Delta^2 / \rho^5) + O(Q \cos \beta \Delta^2 / \rho^6) + O(\Delta^4 / \rho^6),
\]
(5.28)
\[
\Phi^S_{(\mu),0} = O(Q \cos \beta / \rho) + O(\Delta / \rho) + O(Q \cos \beta \Delta / \rho^3) + O(\Delta^2 / \rho^3) + O(Q \cos \beta \Delta^2 / \rho^5) + O(\Delta^5 / \rho^5) + O(Q \cos \beta \Delta^3 / \rho^7) + O(\Delta^7 / \rho^7),
\]
(5.29)
\[
\Phi^S_{(\mu),+1} = M_{(\mu),+1}(\rho) + O(Q \cos \beta / \rho) + O(\Delta^2 / \rho) + O(Q \cos \beta \Delta / \rho^3) + O(\Delta^4 / \rho^3)
\]
\[
+ O(Q \cos \beta \Delta^2 / \rho^5) + O(\Delta^6 / \rho^5) + O(Q \cos \beta \Delta^3 / \rho^7) + O(\Delta^8 / \rho^7)
\]
\[
+ O(Q \cos \beta \Delta^5 / \rho^7) + O(\Delta^{10} / \rho^9).
\]
(5.30)
These equations display the dependence of each term on \( Q, \cos \beta, \Delta, \) and \( \rho; \) the remaining dependence on \( \beta \) is contained entirely in the function \( \chi := 1 - k \sin^2 \beta \) defined by Eq. (5.23). This dependence, as well as the dependence on the world-line quantities, is left implicit in Eqs. (5.27)–(5.30). The terms in these equations that involve the coefficients \( M(0), -2, M(0), -1, \) and \( M(0), +1 \) are important: As we shall see, only these terms actually contribute to the regularization parameters. The coefficients depend on \( \chi \) and the world-line quantities. The remaining terms in Eqs. (5.27)–(5.30), those represented by the various symbols \( O(\ ) \), are unimportant: They do not contribute to the regularization parameters.

F. Final results: Regularization parameters

The singular field of Eqs. (5.26)–(5.30) can now be substituted into Eqs. (5.10) and (5.11) to compute the multipole coefficients \( \Phi^S_{(\mu)} \). The techniques reviewed in the Appendix provide us with efficient calculational rules. Equation (A.34), for example, implies that all terms involving \( \cos \beta \) in Eqs. (5.27)–(5.30) average to zero and do not contribute to the multipole coefficients. Equations (A.38)–(A.42), on the other hand, imply that all remaining \( O(\ ) \) terms in Eqs. (5.27)–(5.30) vanish in the limit \( \Delta \to 0 \); recall that this limiting procedure was introduced back in Eq. (5.2).

The only surviving contributions come from the terms involving the coefficients \( M(0), -2, M(0), -1, \) and \( M(0), +1 \). These are handled with the help of Eqs. (A.35)–(A.37), and this shows that \( \Phi^S_{(\mu)} \) does indeed take the form of Eq. (5.6). We remark that the neglected \( O(\varepsilon^2) \) terms in Eq. (5.26) are those which would produce the neglected \( \cdots \) terms in Eq. (5.6); the \( \cdots \) terms sum to zero because the \( O(\varepsilon^2) \) terms in the singular field vanish at the position of the particle.

The regularization parameters \( A(\mu) \) are produced by the term involving the coefficient \( M(0), -2 \) in Eq. (5.27). Our results are listed back in Sec. I E, in Eqs. (1.30)–(1.32). Notice that in these equations, we changed our notation from \( r_0 \) to \( r(t) \), and from \( \phi_0 \) to \( \phi(t) \). The regularization parameters \( B(\mu) \) are produced by the term involving the coefficient \( M(0), -1 \) in Eq. (5.28). Our results are listed back in Sec. I E, in Eqs. (1.33)–(1.37). The regularization parameters \( C(\mu) \) would normally have originated from Eq. (5.29). Because, however, there are no surviving contributions from \( \Phi^S_{(\mu),0} \), we conclude that \( C(\mu) = 0 \). The regularization parameters \( D(\mu) \) are produced by the term involving the coefficient \( M(0), +1 \) in Eq. (5.30). Our results are listed back in Sec. I E, in Eqs. (1.41)–(1.45).

The regularization parameters of Eqs. (1.30)–(1.45) are expressed in terms of the (rescaled) elliptic functions \( \delta := \frac{2}{\pi} \int_0^{\pi/2} (1 - k \sin^2 \psi)^{1/2} d\psi = F(-\frac{1}{2}, \frac{1}{2}; 1; k) \) and \( \lambda := \frac{2}{\pi} \int_0^{\pi/2} (1 - k \sin^2 \psi)^{-1/2} d\psi = F(\frac{1}{2}, \frac{1}{2}; 1; k) \), where \( k := L^2/(r^2_0 + L^2) \) was introduced in Eq. (5.23). As indicated, the elliptic integrals can also be expressed in terms of hypergeometric functions. These appear naturally in the course of averaging over the angles \( \beta \) — see Eq. (A.28). In fact, to obtain our final expressions for the regularization parameters we have rationalized their dependence on the hypergeometric functions by using the recurrence relation of Eq. (A.31).

As a final remark we note that the regularization parameters depend on \( \Delta \) in two distinct ways. First, Eqs. (1.30)–(1.32) show that the parameters \( A(\mu) \) are proportional to \( \text{sign}(\Delta) \) and therefore discontinuous across \( \Delta = 0 \); this behavior accounts for the discontinuity across \( r = r_0 \) of each mode \( \Phi_{(\mu)lm} \) of the retarded field. Second, the regularization parameters contain an implicit dependence on \( \Delta \) that is not shown in Eqs. (1.30)–(1.45). Indeed, the right-hand side of each equation should include terms that depend on \( \Delta \) but vanish in the limit \( \Delta \to 0 \); they are not displayed precisely because they do not survive the limiting procedure described in Eq. (5.2).

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APPENDIX: MULTIPOLe COEFFICIENTS

In this Appendix we collect results from the literature that are required for the computation of the regularization parameters in Sec. V. Mostly we rely on the techniques developed in the paper by Detweiler, Messaritaki, and Whiting [22].
1. Decomposition of a scalar function in spherical harmonics

Let \(f(\theta, \phi)\) be a scalar field on \(S^2\). We decompose it in spherical harmonics as

\[
f(\theta, \phi) = \sum_{lm} f_{lm} Y_{lm}(\theta, \phi),
\]

(A.1)

The sum over the integer \(l\) extends from \(l = 0\) to \(l = \infty\), while the sum over the integer \(m\) ranges from \(m = -l\) to \(m = l\). The field’s spherical-harmonic modes are given by

\[
f_{lm} = \int f(\theta, \phi) \bar{Y}_{lm}(\theta, \phi) \, d\Omega,
\]

(A.2)

where an overbar indicates complex conjugation, and \(d\Omega = \sin \theta \, d\theta \, d\phi\) is an element of solid angle.

We shall be interested in the value of \(f\) at the special point \((\theta = \frac{\pi}{2}, \phi = 0)\). This we express as

\[
f\left(\frac{\pi}{2}, 0\right) = \sum_l f_l,
\]

(A.3)

with

\[
f_l = \sum_{m=-l}^l f_{lm} Y_{lm}\left(\frac{\pi}{2}, 0\right).
\]

(A.4)

The quantities \(f_l\) associated with \(f(\theta, \phi)\) play a fundamental role below; we shall refer to them as the multipole coefficients of the function \(f\).

2. Rotation of the angular coordinates

A convenient way to calculate the multipole coefficients \(f_l\) is to perform a rotation of the angular coordinates that maps the special point \((\frac{\pi}{2}, 0)\) to the North pole of the new coordinate system \([17, 18]\). The rotation from the old angles \((\theta, \phi)\) to the new angles \((\alpha, \beta)\) is defined by the equations

\[
\begin{align*}
\sin \theta \cos \phi &= \cos \alpha, \\
\sin \theta \sin \phi &= \sin \alpha \cos \beta, \\
\cos \theta &= \sin \alpha \sin \beta.
\end{align*}
\]

(A.5) (A.6) (A.7)

It is easy to see that this does indeed map the point \((\theta = \frac{\pi}{2}, \phi = 0)\) to the point \((\alpha = 0, \beta = ?)\), with \(\beta\) undetermined. The special point is a singular point of the new coordinate system, and as we shall see, this property is a source of simplification in the computation of \(f_l\).

It is well-known that a rotation of the angular coordinates changes \(Y_{lm}(\theta, \phi)\) into a mixture of functions \(Y_{lm'}(\alpha, \beta)\) with \(m'\) ranging from \(-l\) to \(l\); the rotation mixes \(m\) but leaves \(l\) invariant. Because the quantity on the left-hand-side of Eq. (A.3) is a scalar, whose value is unchanged by the rotation, we may conclude that the rotation leaves the multipole coefficients \(f_l\) invariant. We shall make use of this simple fact to find an efficient way to compute these quantities.

The function \(f(\alpha, \beta)\) can be expanded in spherical harmonics \(Y_{lm}(\alpha, \beta)\) as in Eq. (A.1), and such a decomposition implies that \(f(0, ?) = \sum_{lm} f_{lm} Y_{lm}(\alpha = 0, \beta = ?) = \sum_l \sqrt{(2l+1)/(4\pi)} f_{l0}\), after evaluating the spherical harmonics. Comparing this with Eq. (A.3) yields

\[
f_l = \sqrt{\frac{2l+1}{4\pi}} f_{l0}[\alpha, \beta \text{ decomposition}].
\]

(A.8)

As indicated in Eq. (A.8), the multipole coefficient \(f_l\) is proportional to the axisymmetric mode \(f_{l0}\) of the function \(f(\alpha, \beta)\). This is given by

\[
f_{l0} = \sqrt{\frac{2l+1}{4\pi}} \int f(\alpha, \beta) P_l(\cos \alpha) \, d\cos \alpha \, d\beta,
\]

(A.9)
where \( P_l(\cos \alpha) \) is a Legendre polynomial. Equation (A.8) therefore becomes
\[
f_l = \frac{2l + 1}{4\pi} \int f(\alpha, \beta) P_l(\cos \alpha) \, d\cos \alpha \, d\beta. \tag{A.10}
\]
If we let
\[
f_l(\beta) := \frac{1}{2} (2l + 1) \int_{-1}^{1} f(\alpha, \beta) P_l(\cos \alpha) \, d\cos \alpha,
\tag{A.11}
\]
then Eq. (A.10) can be expressed as
\[
f_l = \langle f_l(\beta) \rangle, \tag{A.12}
\]
where
\[
\langle f_l(\beta) \rangle = \frac{1}{2\pi} \int_{0}^{2\pi} f_l(\beta) \, d\beta \tag{A.13}
\]
is the average of \( f_l(\beta) \) over all angles \( \beta \). Equations (A.11) and (A.12) summarize the method by which the multipole coefficients of the function \( f(\theta, \phi) \) are computed.

Equations (A.11)–(A.13) can be given an alternative interpretation that turns out to be useful for our purposes. Suppose that we are presented with a scalar function \( f(\alpha, \beta) \) and that we wish to represent its dependence on \( \alpha \) in terms of an expansion in Legendre polynomials. We would write
\[
f(\alpha, \beta) = \sum_l f_l(\beta) P_l(\cos \alpha), \tag{A.14}
\]
and we would use the orthogonality properties of the Legendre polynomials to express \( f_l(\beta) \) as in Eq. (A.11). The average of \( f_l(\beta) \) over the angles \( \beta \), as defined by Eq. (A.13), would then give us the multipole coefficients \( f_l \). It is this interpretation that will be emphasized in the rest of this Appendix.

We pause here to remark that the calculational method described above to compute the multipole coefficients \( f_l \) takes advantage of the fact that the rotation of the coordinate system maps the special point \( (\theta = \pi/2, \phi = 0) \) to the singular point \( (\alpha = 0, \beta = ?) \). It is the singular nature of the new coordinates at the North pole which produces the equality of Eq. (A.8). This method works well because \( f(\theta, \phi) \) is a scalar function of the angles: the rotation does not change the numerical value of the function. The method would not work as well for vectorial or tensorial functions of the angles, because the map to a singular point would alter the value of the functions by a singular factor. While this difficulty can be averted [17, 18], this introduces complications that can be avoided by choosing to deal with scalar functions only. It is this observation that has motivated us to work in terms of the scalars \( \Phi(\mu) := \Phi_{\alpha_e^\mu} \) instead of the vector \( \Phi_{\alpha} := \nabla_{\alpha} \Phi \).

### 3. Distance function \( \rho(\alpha, \beta) \)

The multipole coefficients that are required in Sec. V are those associated with negative and positive powers of the “distance function” \( \rho(\alpha, \beta) \), defined by Eq. (5.24),
\[
\rho^2 := 2(r_0^2 + L^2) \chi(\delta^2 + 1 - \cos \alpha). \tag{A.15}
\]
The symbols that appear in Eq. (A.15) are all introduced in Sec. V D. We have \( r_0 \) denoting the current radial position of the charged particle, and \( L \) is the particle’s (conserved) angular momentum per unit mass. In addition,
\[
\delta^2 := \frac{E^2 r_0^4}{2(r_0^2 + L^2)^2(r_0^2 - 2M)^2} \Delta^2 \chi, \tag{A.16}
\]
in which \( E \) denotes the particle’s (conserved) energy per unit mass, and
\[
\Delta := r - r_0, \tag{A.17}
\]
is the radial component of the displacement vector \( w^\alpha = x^\alpha - \bar{x}^\alpha \). And finally,
\[
\chi := 1 - k \sin^2 \beta, \quad k := \frac{L^2}{r_0^2 + L^2} \tag{A.18}
\]
contains the dependence of \( \rho \) on the angle \( \beta \).
4. Decomposition of $(\delta^2 + 1 - \cos \alpha)^{-n-1/2}$ in Legendre polynomials

The calculation of the regularization parameters described in Sec. V requires multipole coefficients for various powers of $\rho$, in a context in which $\Delta$, and therefore $\delta$, is small. The relevant powers of $\rho$ are $\rho^{-2n-1}$ with $n = -1, 0, 1, 2$, and so on. As was explained in the paragraph surrounding Eq. (A.14), the starting point of a computation of multipole coefficients is the decomposition of those selected powers of $\rho$ in Legendre polynomials. The most important piece of $\rho$ for these decompositions is the factor $(\delta^2 + 1 - \cos \alpha)^{1/2}$, and we therefore need expansions of the corresponding powers of this quantity in Legendre polynomials.

Quoting from Detweiler, Messaritaki, and Whiting [22], we have

\[ (\delta^2 + 1 - \cos \alpha)^{-n-1/2} = \sum_l A_l^n(\delta)P_l(\cos \alpha), \]  

(A.19)

in terms of coefficients $A_l^n(\delta)$ that can be expanded in powers of $\delta$. They obey the recurrence relation

\[ A_l^{n+1} = -\frac{1}{(2n+1)\delta} \frac{d}{d\delta} A_l^n, \]  

(A.20)

and special values are given by

\[ A_l^0 = \sqrt{2} - (2l + 1)\delta + \frac{(2l + 1)^2}{2\sqrt{2}} \delta^2 - \frac{1}{3} (l+1)(2l+1)\delta^3 + O(\delta^4), \]  

(A.21)

\[ A_l^1 = \frac{2l + 1}{\delta} - \frac{(2l + 1)^2}{\sqrt{2}} + l(l+1)(2l+1)\delta + O(\delta^2), \]  

(A.22)

\[ A_l^2 = \frac{2l + 1}{3\delta^3} - \frac{l(l+1)(2l+1)}{3\delta} + O(\delta^0), \]  

(A.23)

\[ A_l^3 = \frac{2l + 1}{5\delta^5} - \frac{l(l+1)(2l+1)}{15\delta^3} + O(\delta^{-1}), \]  

(A.24)

\[ A_l^4 = \frac{2l + 1}{7\delta^7} - \frac{l(l+1)(2l+1)}{35\delta^5} + O(\delta^{-3}). \]  

(A.25)

By induction from Eq. (A.20) we infer that

\[ A_l^n = \frac{2l + 1}{(2n-1)\delta^{2n-1}} \left[ 1 + O(\delta^2) \right], \quad n \geq 2. \]  

(A.26)

We shall also need a decomposition for $(\delta^2 + 1 - \cos \alpha)^{1/2}$. This is given by Eq. (A.19) with $n = -1$, and in this case the expansion coefficients are [22]

\[ A_l^{-1} = -\frac{2\sqrt{2}}{(2l - 1)(2l + 3)} + O(\delta^2). \]  

(A.27)

The results displayed in this subsection can be used in concert with Eq. (A.15) to calculate the Legendre coefficients \([\rho^{-2n-1}]_n(\beta)\) associated with the functions $\rho^{-2n-1}(\alpha, \beta)$. The dependence on $\beta$ is contained in the factors of $\chi$ that are hidden in the definitions of $\delta$ and $\rho$; refer back to Eqs. (A.16) and (A.18).

5. Averaging over $\beta$

The next step in the calculation of the multipole coefficients $(\rho^{-2n-1})_l$ is to carry out the averaging over the angles $\beta$, as defined by Eq. (A.13). Because the dependence on $\beta$ is contained in $\chi$, what we need are expressions for $\langle \chi^{-p} \rangle$, where $p$ is some (positive or negative) number.

Again quoting from Detweiler, Messaritaki, and Whiting [22], we have

\[ \langle \chi^{-p} \rangle = F_p := F(p, \frac{1}{2}; 1; k), \]  

(A.28)

where $F(a, b; c; x)$ is the hypergeometric function, and where $k$ is the constant defined by Eq. (A.18). Special cases of Eq. (A.28) are

\[ \langle \chi^{-1} \rangle = \frac{1}{\sqrt{1-k}} = \frac{\sqrt{r_0^2 + L^2}}{r_0} \]  

(A.29)
and

\[ \langle \chi \rangle = 1 - \frac{1}{2} k. \]  

(A.30)

The hypergeometric functions \( F_p \) are linked by a recurrence relation displayed in Eq. (15.2.10) of Ref. [41]. When specialized to our specific situation, this equation reads

\[ F_{p+1} = \frac{p-1}{p(k-1)} F_{p-1} + \frac{1-2p + (p - \frac{1}{2}) k}{p(k-1)} F_p. \]  

(A.31)

Using this identity we find that the \( F_p \)'s that appear in our expressions for \( (\rho^{-2n-1})_l \) below — see Eqs. (A.36) and (A.37) — can all be expressed in terms of \( F_{1/2} \) and \( F_{-1/2} \). These, in turn, can be expressed in terms of complete elliptic integrals: According to Eqs. (17.3.1), (17.3.9), and (17.3.10) of Ref. [41], we have

\[ F_{1/2} = K(k) := \frac{2}{\pi} \int_0^{\pi/2} (1 - k \sin^2 \psi)^{-1/2} d\psi \]  

(A.32)

and

\[ F_{-1/2} = E(k) := \frac{2}{\pi} \int_0^{\pi/2} (1 - k \sin^2 \psi)^{1/2} d\psi. \]  

(A.33)

Our results in Sec. V E are expressed in terms of the elliptic integrals.

We conclude this subsection with the remark that any function \( g \) of \( \chi \) is necessarily periodic with period \( \pi \) in the interval \( 0 < \beta < 2\pi \); the function’s behavior in the interval \( 0 < \beta < \pi \) is replicated in the interval \( \pi < \beta < 2\pi \). Furthermore, any such function is necessarily symmetric (even) about \( \beta = \pi/2 \) in the first interval, and about \( \beta = 3\pi/2 \) in the second interval. These properties are sufficient to infer that

\[ \langle g(\chi) \sin \beta \rangle = \langle g(\chi) \cos \beta \rangle = \langle g(\chi) \sin \beta \cos \beta \rangle = 0 \]  

(A.34)

for any function \( g(\chi) \).

6. Final results: Multipole coefficients \( (\rho^{-2n-1})_l \)

The multipole coefficients of the functions \( \rho^{-2n-1}(\alpha, \beta) \) are denoted \( (\rho^{-2n-1})_l \). They are calculated by implementing the procedure described around Eq. (A.14). Most of the required computations were already performed in Secs. 4 and 5 of this Appendix; here we simply collect the results and put it all together.

By combining Eqs. (A.15), (A.16), (A.19), (A.21)–(A.27), as well as Eqs. (A.28) and (A.29), we arrive at the following listing of multipole coefficients:

\[ \Delta (\rho^{-3})_l = (l + \frac{1}{2}) \frac{r_0 - 2M}{E r_0} \text{sign}(\Delta) + O(\Delta), \]  

(A.35)

\[ (\chi^{-p}\rho^{-1})_l = \frac{F_{p+1/2}}{\sqrt{r_0^2 + L^2}} + O(\Delta), \]  

(A.36)

\[ (\chi^{-p}\rho^{-2})_l = \frac{\sqrt{r_0^2 + L^2} F_{p-1/2}}{(l - \frac{1}{2})(l + \frac{3}{2})} + O(\Delta^2). \]  

(A.37)

We also record the scaling relations

\[ \Delta (\chi^{-p}\rho^{-1})_l = O(\Delta), \]  

(A.38)

\[ \Delta^2 (\chi^{-p}\rho^{-2})_l = O(\Delta), \]  

(A.39)

\[ \Delta^4 (\chi^{-p}\rho^{-5})_l = O(\Delta), \]  

(A.40)

\[ \Delta^7 (\chi^{-p}\rho^{-7})_l = O(\Delta^2), \]  

(A.41)

\[ \Delta^{10} (\chi^{-p}\rho^{-9})_l = O(\Delta^3), \]  

(A.42)

which follow from the same set of equations.

