THERMODYNAMIC INSTABILITY OF A SYSTEM OF GRAVITATING FERMIONSP. Hertel^{*)}

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ABSTRACT

We discuss the temperature-dependent Thomas-Fermi equation for a system of many fermions with gravitational interaction. According to a previous investigation this equation becomes exact in a suitable thermodynamic limit. For fixed number of particles and large enough confining volume the total energy, as a function of temperature, jumps by a finite amount at a certain transition temperature. This corresponds to a transition between a nearly homogeneous phase of medium mass density and a phase with a high density core surrounded by an atmosphere of low density. In the microcanonical ensemble the thermodynamic instability is due to negative heat capacity.

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1. Introduction

Gravitational forces are of long range and always attractive. Although negligible if only few elementary particles are involved they will compete with electromagnetic and even strong interactions if aggregates of $\sim 10^{57}$ nucleons, such as stars, are under discussion. During the first $\sim 10^{10}$ years of a star's life thermonuclear reactions prevent it from radiating its gravitational energy and thereby collapsing. Later the Fermi pressure can guarantee a stable state. If the mass, and consequently the tendency to collapse, is small enough the pressure of the not too relativistic electrons is sufficient (white dwarfs). If the electrons become highly relativistic their pressure is insufficient, they will eventually be squeezed into the protons. Then it is the pressure of the neutrons which may halt the collapse (neutron stars). If the mass is too large even that does not work and the star will disappear in the black hole dug by its own gravitation.

We shall demonstrate in this paper that under certain conditions the balance between gravitational pull and fermion pressure is indeed thermodynamically unstable. More precisely, we study the equilibrium thermodynamics of a system of N identical fermions with gravitational interaction. We shall treat the fermions as well as gravitation non-relativistically. It is found that such a system exhibits a phase transition: if the remaining parameters are suitably chosen and kept fixed, the total energy as a function of temperature has a discontinuity.

A rather simple argument shows that such a behaviour is to be expected: Assume that N neutrons are homogeneously distributed within a spherical

volume $V = 4\pi R^3/3$. The potential energy is ⁽¹⁾

$$E_p = -3N^2/5R . \quad (1.1)$$

As kinetic energy we choose the expression for noninteracting fermions:

$$E_K = 2V \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{2} \frac{1}{1 + e^{\frac{1}{2}\beta p^2 + \alpha}} \quad (1.2)$$

where α has to be determined from

$$N = 2V \int \frac{d^3p}{(2\pi)^3} \frac{1}{1 + e^{\frac{1}{2}\beta p^2 + \alpha}} . \quad (1.3)$$

The virial theorem (see chapter 2) tells us that the pressure P is given by

$$3PV = 2E_K + E_p \quad (1.4)$$

which has a positive and a negative contribution. In fig. 1 the kinetic energy is plotted as a function of R^{-1} for various temperatures β^{-1} .

The straight line is $-\frac{1}{2}E_p$ so that the difference between the curves and the straight line is $3PV/2$. We see that for temperatures below a certain critical value the above simple reasoning would lead to a negative pressure in a region $R_1 < R < R_2$. This will be avoided by a strongly inhomogeneous mass distribution in the critical region. Thus, if R is very large, the effect of gravity is small and we find a nearly homogeneous distribution of matter. Keeping the temperature fixed and compressing the system will result in a more and more inhomogeneous mass distribution. At smaller R we shall find an inner core of high and rather homogeneous density and an atmosphere of low density which contributes little to E_K and E_p .

The essential finding of this paper is: the transition between the large and small R cases is discontinuous. At very small R the zero point motion dominates again and the distribution of particles within the volume be-

comes homogeneous.

Our results are, of course, limited to nonrelativistic kinematics where a complete collapse for masses beyond Chandrasekhar's limit does not occur.

In chapter 2 we state the result of a previous investigation where the asymptotic equality of the free energy and the effective field approximation thereof was proven. Although this required considerable analytical effort the result is rather plausible in view of the long range of the gravitational potential ⁽⁶⁾. The temperature-dependent Thomas-Fermi equation results and a few consequences are discussed. These include the virial theorem as well as the equation for hydrostatic equilibrium. We discuss the computer solution of the Thomas-Fermi equation in chapter 3. It is explained how the phase transition comes about. In the appendix we supplement our previous investigation: it is shown that approximating the r^{-1} - potential and solving the Thomas Fermi equation are interchangeable operations.

2. The Thomas-Fermi equation

We consider a system of N fermions interacting with their gravitational forces only and being enclosed within a spherical volume $V = 4\pi R^3/3$.

The free energy of this system is defined by

$$e^{-\beta F(N, \beta, V)} = \text{Trace}_{\mathcal{H}(N, V)} e^{-\beta H(N)}. \quad (2.1)$$

Here $\mathcal{H}(N, V)$ is the Hilbert space of square-integrable complex wave functions $\Psi(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_N)$ which are antisymmetric in all N arguments and vanish if one $\vec{x}_j$ is outside V . $H(N)$ is the Hamiltonian of the system⁽¹⁾:

$$H(N) = \frac{1}{2} \sum_{i=1}^N \left\{ \vec{p}_i^2 - \sum_{\substack{j=1 \\ j \neq i}}^N |\vec{x}_i - \vec{x}_j|^{-1} \right\}. \quad (2.2)$$

In (2) and in the appendix to this paper we have rigorously proved the following statement:

$$\lim_{\lambda \rightarrow \infty} \lambda^{-7/3} F(\lambda N, \lambda^{-4/3} \beta, \lambda^{-1} V) = F_U(N, \beta, V) \quad (2.3)$$

where $U(r)$ is that potential which corresponds to the smallest stationary point of (3)

$$U \rightarrow F_U(N, \beta, V) = - \frac{N \alpha_U}{\beta} - \frac{1}{2} \int_0^R dr 4\pi r^2 \rho_U(r) U(r) - \frac{2}{\beta} \int_0^R dr 4\pi r^2 \int_0^\infty \frac{d\varepsilon \sqrt{\varepsilon}}{\sqrt{2} \pi^2} \log(1 + e^{-\beta(\varepsilon + U(r)) - \alpha_U}). \quad (2.4)$$

Here ρ_U is the mass distribution which generates U :

$$U(r) = - \int_{|\vec{x}'| \leq R} d^3 x' |\vec{x} - \vec{x}'|^{-1} \rho(r') \quad (r = |\vec{x}|, r' = |\vec{x}'|) \\ = - \int_0^R dr' 4\pi r'^2 \rho(r') \left\{ \frac{\theta(r - r')}{r} + \frac{\theta(r' - r)}{r'} \right\} \quad (2.5)$$

and α_U the (unique) solution of

$$N = 2 \int_0^R dr 4\pi r^2 \int_0^\infty \frac{d\varepsilon \sqrt{\varepsilon}}{\sqrt{2} \pi^2} \frac{1}{1 + e^{\beta(\varepsilon + u(r)) + \alpha_U}}. \quad (2.6)$$

We have to take spherically symmetric mass distributions and potentials since we start with spherically symmetric boundary conditions from the outset. The particular limit in equation (2.3) means that we consider a sequence of systems with an increasing number of particles, at increasing temperature, becoming more and more compressed. Thereby the free energy increases with N like $N^{7/3}$. This behaviour is to be expected since, according to an investigation by Lévy-Leblond ⁽⁴⁾, the ground state energy of the Hamiltonian (2.2) for large N also decreases like $-N^{7/3}$. ⁽⁵⁾ Although the free energy per particle has no limit ($N \rightarrow \infty$) the entropy per particle has since $\beta \sim N^{-4/3}$. Moreover, the thermodynamic limit as defined by equation (2.3) coincides with the usual limit if no interaction is taken into account, i.e. for an ideal gas (IG), since

$$\lambda^{-7/3} F_{IG}(\lambda N, \lambda^{-4/3} \beta, \lambda V) = \lambda^{-1} F_{IG}(\lambda N, \beta, \lambda V). \quad (2.7)$$

This statement is a simple consequence of the fact that $x \rightarrow \lambda^{2/3} x$, $p \rightarrow \lambda^{-2/3} p$ is a canonical transformation and that the Hamiltonian is quadratic in the momenta.

If N is very large (e.g. $\sim 10^{57}$ for a neutron star), $N^{4/3} \beta / m^{-5} \hbar^2 x^{-2}$ as well as $N^{1/3} R / m^{-3} \hbar^2 x^{-1}$ of order unity the limit in (2.3) is practically reached for $\lambda \approx 1$ so that then the right-hand side of (2.3) is the desired free energy.

Let us now derive the condition for the potential so that F_U is stationary.

F_U is differentiable with respect to U , we have

$$\delta F_U(N, \beta, V) = -\frac{N\delta\alpha}{\beta} - \int_0^R dr 4\pi r^2 \rho(r) \delta U(r) \quad (2.8)$$

$$+ \frac{2}{\beta} \int_0^R dr 4\pi r^2 \int_0^\infty \frac{d\varepsilon \sqrt{\varepsilon}}{\sqrt{2}\pi^2} \frac{\beta \delta U(r) + \delta\alpha}{1 + e^{\beta(\varepsilon + U(r)) + \alpha_U}}.$$

In virtue of equation (2.6) the contributions from $\delta\alpha$ cancel. Therefore,

$U \rightarrow F_U$ is stationary for a potential that satisfies

$$U(r) = \int_0^R dr' 4\pi r'^2 \left\{ \frac{\theta(r'-r)}{r'} + \frac{\theta(r-r')}{r} \right\} 2 \int_0^\infty \frac{d\varepsilon \sqrt{\varepsilon}}{\sqrt{2}\pi^2} \frac{1}{1 + e^{\beta(\varepsilon + U(r')) + \alpha_U}} \quad (2.9)$$

where α_U is defined by (2.6).

Equation (2.9) is the temperature dependent Thomas-Fermi equation: the potential is generated by a Fermi distribution of particles in which the local energy is the sum of kinetic energy ε and potential energy $U(r)$.

This allows to determine $U(r)$ and $\rho(r)$ selfconsistently.

If, for given N, β , and R one finds more than one solution of the Thomas-Fermi equation (2.9) the analysis in ⁽²⁾ shows that one has to choose that for which F_U is smallest. This situation actually occurs for a suitable choice of N, β , and R and gives rise to a phase transition.

However, before discussing in chapter 3 the numerical solution of the selfconsistency equation (2.9) let us derive a few formulae which are a direct consequence of the fact that F_U is stationary at U .

The chemical potential of our system is

$$\mu = \frac{\partial}{\partial N} F_U(N, \beta, V) = -\frac{\alpha_U}{\beta} \quad (2.10)$$

For calculating the derivative of the free energy with respect to N

(or β , or V) an implicate dependence via U of it plays no rôle because

the potential U corresponds to a stationary point.

The total energy is

$$E = \frac{\partial}{\partial \beta} \beta F_U(N, \beta, V) \\ = \int_0^R dr 4\pi r^2 2 \int_0^\infty \frac{d\varepsilon \sqrt{\varepsilon}}{\sqrt{2} \pi^2} \frac{\varepsilon + \frac{1}{2} U(r)}{1 + e^{\beta(\varepsilon + U(r)) + \alpha_U}} \quad (2.11)$$

i.e. the sum of kinetic energy

$$E_K = \int_0^R dr 4\pi r^2 2 \int_0^\infty \frac{d\varepsilon \sqrt{\varepsilon}}{\sqrt{2} \pi^2} \frac{\varepsilon}{1 + e^{\beta(\varepsilon + U(r)) + \alpha_U}} \quad (2.12)$$

and potential energy

$$E_P = \frac{1}{2} \int_0^R dr 4\pi r^2 \rho(r) U(r) \quad , \quad (2.13)$$

as it should be.

The thermodynamically defined pressure P of the system,

$$P = - \frac{\partial}{\partial V} F_U(N, \beta, V)$$

is most conveniently calculated in the following way:

One can easily verify the law of corresponding states:

$$F_U(N, \beta, V) = \lambda^{-7/3} F_{U_\lambda}(\lambda N, \lambda^{-4/3} \beta, \lambda^{-1} V) \quad (2.14)$$

where $\lambda^{-4/3} U_\lambda(\lambda^{-1/3} r) = U(r)$.

Differentiating (2.14) twice with respect to λ and taking $\lambda = 1$ gives

$$0 = -\frac{7}{3} F_U(N, \beta, V) + \mu N - \frac{4}{3} (E - F_U) + PV \quad . \quad (2.15)$$

Since we can rewrite (2.4) by partial integration,

$$F_U = -N \alpha_U \beta^{-1} - \frac{2}{3} E_K - E_P \quad , \quad (2.16)$$

we obtain

$$3PV = 2E_K + E_P \quad (2.17)$$

which is the virial theorem for gravitational or coulomb interaction. Of course this result can also be derived for finite N from (2.1) by studying a suitable scale transformation.

With

$$\varepsilon_K(r) = 2 \int_0^\infty \frac{d\varepsilon \sqrt{\varepsilon}}{\sqrt{2} \pi^2} \frac{\varepsilon}{1 + e^{\beta(\varepsilon + u(r)) + \alpha_u}} \quad (2.18)$$

as density of kinetic energy one derives

$$\frac{d}{dr} \left(\frac{2}{3} \varepsilon_K(r) \right) = - \varrho(r) \frac{du(r)}{dr} = - \frac{\varrho(r) M(r)}{r^2}, \quad (2.19)$$

where

$$M(r) = \int_0^r dr' 4\pi r'^2 \varrho(r') \quad (2.20)$$

is the mass contained within a sphere of radius r.

(2.19) is the equation for hydrostatic equilibrium since $2 \varepsilon_K(r)/3$

is the local pressure. If $\varrho(r)$ is given at a certain point one can

determine $U(r) + \alpha_u$ from it and then calculate the local pressure.

Hence the equation of state by which the equation for hydrostatic equi-

librium has to be supplemented is given by the well-known dependence of

the pressure of noninteracting fermions on β and the density. Both

equations together are clearly equivalent to the Thomas-Fermi equation (6).

By integrating (2.19):

$$\begin{aligned} \frac{4\pi}{3} \int_0^R dr r^3 \frac{d}{dr} \frac{2}{3} \varepsilon_K(r) &= V \frac{2}{3} \varepsilon_K(R) - \frac{2}{3} E_K \\ &= - \frac{1}{3} \int_0^R dr 4\pi r^2 \varrho(r) \int_0^R dr' 4\pi r'^2 \varrho(r') \theta(r-r') \\ &= \frac{1}{3} \left\{ - \frac{1}{2} \int_0^R dr 4\pi r^2 \varrho(r) \int_0^R dr' 4\pi r'^2 \varrho(r') \left\{ \frac{\theta(r-r')}{r} + \frac{\theta(r'-r)}{r'} \right\} \right\} = \frac{1}{3} E_P \end{aligned} \quad (2.21)$$

and comparing with (2.17) one sees that the thermodynamically defined pressure of the system is equal to the hydrostatic pressure at the surface,

$$P = \frac{2}{3} \epsilon_K(R) \quad , \quad (2.22)$$

which is a satisfactory result.

3. Numerical solution of the Thomas-Fermi equation

Let us define

$$\xi^{-1} W(\xi) = \beta U(\beta^{1/4} \xi) + \alpha u. \quad (3.1)$$

From (A.14) and (2.9) we see that $U(r)$ is bounded and consequently twice differentiable so that the Thomas-Fermi equation is equivalent to

$$W''(\xi) = \frac{\sqrt{32}}{\pi} \xi \int_0^\infty d\eta \sqrt{\eta} \frac{1}{1 + e^{\eta + W(\xi)/\xi}} \quad (3.2)$$

and
$$\lim_{\xi \rightarrow 0} \xi^{-1} W(\xi) = \text{finite} \quad (3.3)$$

which is one initial condition.

Although the right-hand side of (3.2), considered as a mapping

$$\xi, W \longrightarrow \frac{\sqrt{32}}{\pi} \xi \int_0^\infty d\eta \sqrt{\eta} \frac{1}{1 + e^{\eta + W/\xi}},$$

has a singularity for $\xi = 0$ existence and uniqueness of a local solution satisfying (3.3) and $W'(0)$ fixed can be established by the usual methods.

Since $W'' \geq 0$ we have $W(\xi) \geq W'(0) \cdot \xi$ and from this the estimate

$$W''(\xi) \leq \frac{\sqrt{32}}{\pi} \xi \int_0^\infty d\eta \sqrt{\eta} (1 + e^{\eta + W'(0)})^{-1} \leq c \xi.$$

Thus we find
$$W'(0) \xi \leq W(\xi) \leq W'(0) \xi + \frac{c}{6} \xi^3$$

and no problem arises in continuing the solution from a neighbourhood of

$\xi = 0$ to the point ξ_1 determined from the normalization condition

$$N = \left(\frac{\xi_1}{R} \right)^3 \int_0^{\xi_1} d\xi \xi^2 \frac{\sqrt{32}}{\pi} \int_0^\infty d\eta \sqrt{\eta} \frac{1}{1 + e^{\eta + W(\xi)/\xi}}. \quad (3.4)$$

Since the right-hand side of (3.4) increases indefinitely with ξ_1 there is, for given $W'(0)$, exactly one solution ξ_1 and consequently exactly one $\beta = (R/\xi_1)^4$ to which the solution $W(\xi)$ belongs.

However, it turns out that for $RN^{1/3} \geq 12.8 \text{ m}^{-3} \hbar^2 \kappa^{-1}$ the following phenomenon is encountered: β is not a monotonic function of $W'(0)$. Three different values $W'(0)$ with three different potentials and mass distributions correspond to the same set of parameters (N, β, V) . According to what was stated in chapter 2. only that potential $U(r)$ with the smallest $F_U(N, \beta, V)$ corresponds to the thermodynamic equilibrium state, the remaining solutions are to be discarded.

In fig. 2 the whole curve $W'(0) \rightarrow \beta$, $W'(0) \rightarrow -\beta F$ is plotted in parametric form for $N = 10^{57}$ neutrons, $R = 30$ and 100 km . The thin part of the $R = 100 \text{ km}$ curve corresponds to solutions of the Thomas-Fermi equation which are no equilibrium states because they do not correspond to (global) minima. Since the derivative of βF with respect to β is E we see that the total energy has a jump at the temperature where two of the three competing solutions give rise to the same minimal free energy. In this way we obtain the phase transition we spoke of in the introduction. For its occurrence it is necessary and sufficient that $RN^{1/3}$ be larger than a critical value.

Fig. 3 is a plot of the total energy as a function of β . It should be noted that the jump of the energy, for a system of 10^{57} neutrons (the sun consists of roughly as many nucleons), is comparable with the amount of energy a star like the sun can produce by fusion throughout its life. The free energy is a continuous function of β . Consequently the transition temperature is such that the two shaded areas in the E vs. β plot are of equal size. This is an analogy of Maxwell's straight line

construction in the theory of the van der Waals gas.

In fig. 4 the entropy, as defined by $S = \beta (E - F)$, is plotted vs. E for 10^{57} neutrons and $R = 30, 100$ km. The $R = 100$ km curve has a piece of convex curvature which would mean that the heat capacity of the system is negative there. It agrees with previous studies ⁽⁷⁾ of long range attractive forces that the phase transition (as calculated in the canonical ensemble) is linked with negative heat capacity (as calculated in the microcanonical ensemble ⁽⁸⁾). In the canonical ensemble positivity of heat capacity is insured by bridging the region of negative heat capacity which consists of the (canonically) nonequilibrium states.

Fig. 5 contains mass distributions for various temperatures. It is only for zero temperature that mass density and pressure vanish identically from a certain radius ($r = 15.9$ km) on. One can clearly see how, by passing the transition temperature ($\beta = 0.165 \text{ MEV}^{-1}$), the mass is reshuffled so that a large amount of gravitational energy is liberated. For temperatures below the transition temperature the central density is more than 10^5 times larger than the density at the surface.

In fig. 6 the degree of degeneracy is plotted:

$$\xi(r) = \frac{3}{2} \frac{S(r)}{\epsilon_K(r)} \quad (3.6)$$

measures the deviation from total Fermi degeneracy ($\xi = 0$) or to which degree matter obeys Boltzmann statistics ($\xi = 1$).

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Appendix

In ref. 2 we had to approximate the original Newton potential

$$v_N(\vec{x}, \vec{x}') = |\vec{x} - \vec{x}'|^{-1} \quad (\text{A.1})$$

by a continuous potential

$$v_\mu(\vec{x}, \vec{x}') = \frac{1 - e^{-\mu |\vec{x} - \vec{x}'|}}{|\vec{x} - \vec{x}'|} \quad (\text{A.2})$$

$v_\mu : S_1 \times S_1 \rightarrow \mathbb{R}$ can be considered as the kernel of a positive integral operator and be diagonalized:

$$v_{\mu s}(\vec{x}, \vec{x}') = \sum_{a=1}^s v_a \varphi_a(\vec{x}) \varphi_a(\vec{x}') \quad (\text{A.3})$$

converges uniformly in $S_1 \times S_1$ towards v_μ if $s \rightarrow \infty$.⁽⁹⁾ We shall always choose s such that $v_{\mu s}$ is invariant under rotations, such as is v_μ and v_N .

The relation between the mass distribution ρ and the potential U is now

$$U(x) = - \int_{S_1} d^3x' v_{\mu s}(\vec{x}, \vec{x}') \rho(x') \quad (\text{A.4})$$

instead of (2.5). It is inessential - see equation (2.4) - that this relation is not one-to-one. Only rotationally invariant mass distributions and potentials need be and are considered here.

The result of our previous investigation can be formulated as follows:

(i) For $\mu > 0$ and $s \in \mathbb{N}$ the problem of finding the lowest stationary point of $U \rightarrow F_U$ (as defined by (2.4), (A.4) and (2.6)) has a solution which we shall call $U_{\mu s}$.

(ii)
$$\lim_{\lambda \rightarrow \infty} \lambda^{-7/3} F(\lambda N, \lambda^{-4/3} \beta, \lambda^{-1} V) = \lim_{\mu \rightarrow \infty} \lim_{s \rightarrow \infty} F_{U_{\mu s}}$$

where $F(N, \beta, V)$ is defined by (2.1).

We shall now show that the main statement of chapter 2 follows from the foregoing.

The condition for $U \rightarrow F_U$ being stationary is (1.23) of ref. 2, i.e.

$$U_{\mu s}(x) = - \int_{S_1} d^3x' v_{\mu s}(\vec{x}, \vec{x}') 2 \int_0^\infty \frac{d\varepsilon \sqrt{\varepsilon}}{\sqrt{2} \pi^2} \frac{1}{1 + e^{\beta(\varepsilon + U_{\mu s}(x')) + \alpha_{\mu s}}} . \quad (A.5)$$

A mass distribution which generates this potential according to (A.4) is

$$\rho_{\mu s}(x) = 2 \int_0^\infty \frac{d\varepsilon \sqrt{\varepsilon}}{\sqrt{2} \pi^2} \frac{1}{1 + e^{\beta(\varepsilon + U_{\mu s}(x)) + \alpha_{\mu s}}} . \quad (A.6)$$

We have

$$\rho_{\mu s}(x) \geq 0 \quad \text{and} \quad \int_{S_1} d^3x \rho_{\mu s}(x) = N . \quad (A.7)$$

Let η be an arbitrary positive number. Since $v_{\mu s}$ converges uniformly to

v_μ there exists $s_0(\mu) \in \mathbb{N}$ such that

$$|v_{\mu s}(\vec{x}, \vec{x}') - v_\mu(\vec{x}, \vec{x}')| \leq \eta \quad (A.8)$$

is valid for $\vec{x}, \vec{x}' \in S_1, \mu > 0$ and $s \geq s_0(\mu)$.

From (A.4), (A.7) and (A.8) we deduce

$$\begin{aligned} -\left(\frac{1}{x} + \eta\right) N &\leq - \int_0^1 dx' 4\pi x'^2 \rho_{\mu s}(x') \left\{ \frac{\theta(x-x')}{x} + \frac{\theta(x'-x)}{x'} \right\} - N\eta \\ &\leq - \int_{S_1} d^3x' \rho_{\mu s}(x') v_\mu(\vec{x}, \vec{x}') - N\eta \leq U_{\mu s}(x) \leq N\eta \end{aligned} \quad (A.9)$$

for $\mu > 0$ and $s \geq s_0(\mu)$.

We define α_0 by solving

$$\int_0^1 dx 4\pi x^2 2 \int_0^\infty \frac{d\varepsilon \sqrt{\varepsilon}}{\sqrt{2} \pi^2} \frac{1}{1 + e^{\beta(\varepsilon + N\eta) + \alpha_0}} = N \quad (A.10)$$

so that, as a consequence of (2.6), (A.6), (A.9) and (A.10) we obtain

$$\alpha_{\mu s} \geq \alpha_0 \quad (s \geq s_0(\mu)) \quad (A.11)$$

as (μ, s) -independent bound.

Substituting (A.9) and (A.11) into (A.6) we find another bound

$$0 \leq \rho_{\mu s}(x) \leq 2 \int_0^{\infty} \frac{d\varepsilon \sqrt{\varepsilon}}{\sqrt{2} \pi^2} \frac{1}{1 + e^{\beta(\varepsilon - x^{-1}N - N\eta) + \alpha_0}} = \rho_0(x) \quad . \quad (A.12)$$

$\rho_0(x)$ is continuous in $(0,1]$, and since $\lim_{x \rightarrow 0} x^{3/2} \rho_0(x)$ exists we deduce

$$0 \leq x^{3/2} \rho_{\mu s}(x) \leq c_0 \quad (A.13)$$

for $x \in [0,1]$ and with (μ, s) -independent c_0 (if $s \geq s_0(\mu)$).

By inserting (A.13) into (A.4) one obtains

$$-N\eta - 8\pi c_0 \leq u_{\mu s}(x) \leq N\eta \quad , \quad (A.14)$$

and from this result, (A.11), and (A.6)

$$c_1 \leq \rho_{\mu s}(x) \leq c_2 \quad (A.15)$$

with c_1 and c_2 also independent on μ and s (if $s \geq s_0(\mu)$).

The potentials $v_{\mu s}: S_1 \times S_1 \rightarrow \mathbb{R}$ are continuous and converge with $s \rightarrow \infty$ towards v_μ in the supremum norm. According to Ascoli's theorem ⁽¹⁰⁾

the family $v_{\mu s}$ (μ fixed, $s \in \mathbb{N}$) is then equi- and uniformly continuous. Because of (A.15) the $U_{\mu s}$ are also equicontinuous. With (A.14) it follows, again according to Ascoli's theorem, that we can choose a uniformly converging sequence out of the family $U_{\mu s}$ (μ fixed) the limit of which we shall call U_μ . The corresponding sequence $\alpha_{\mu s}$ (cf. equ. (2.6)) converges as well to an α_μ .

To these converging sequences of potentials and α 's corresponds a sequence of mass distributions which converges to a ρ_μ since (A.6) is a continuous mapping $\mathcal{C}_R(S_1) \rightarrow \mathcal{C}_R(S_1)$. The limiting functions satisfy the counterpart of (A.4), namely

$$u_\mu(x) = - \int d^3x' v_\mu(\vec{x}, \vec{x}') \rho_\mu(x') \quad . \quad (A.16)$$

If the family $U_{\mu s}$ has more than one accumulation point it is irrelevant, for the purpose of calculating the free energy, which we choose. They will

give rise to the same free energy since $F_{U_{\mu s}}$ decreases with increasing s (11).

That $\lim_{s \rightarrow \infty} F_{U_{\mu s}}$ is equal to $F_{\lim_{s \rightarrow \infty} U_{\mu s}}$ follows from the fact that the convergence of the potentials is uniform and S_1 is compact.

Because the limit q_{μ} also satisfies (A.15) we deduce

$$\begin{aligned} |\vec{\nabla} U_{\mu}(x)| &\leq c_2 \int_{S_1} d^3 x' \frac{1 - (1 + \mu |\vec{x} - \vec{x}'|) e^{-\mu |\vec{x} - \vec{x}'|}}{|\vec{x} - \vec{x}'|^2} \\ &\leq c_2 \int_{S_2} \frac{d^3 z}{z^2} \leq 8\pi c_2, \end{aligned} \quad (\text{A.17})$$

i.e. the U_{μ} are equicontinuous. Together with (A.14) this implies (using again Ascoli's theorem) that the family of potentials U_{μ} has an accumulation point which we shall call U (12). Likewise an accumulation point q exists such that U is the potential generated by q as defined with (2.5). By the same reasoning as for $s \rightarrow \infty$ we arrive at the following conclusion:

$$\lim_{\mu \rightarrow \infty} \lim_{s \rightarrow \infty} F_{U_{\mu s}} = F_U \quad (\text{A.18})$$

where U satisfies (2.9) with α and q being defined by (2.5) and (2.6). If (2.9) allows many solutions only those for which F_U is smallest are to be taken into account since the remaining solutions cannot be accumulation points of U .

Footnotes and references

- (1) We choose units $\hbar = 1$, $m = \text{fermion mass} = 1$, and $\kappa = \text{gravitation constant} = 1$.
- (2) P. Hertel and W. Thirring, preprint TH 1330 CERN (1971)
- (3) The factor 2 in front of the last term of equation (2.4) and in front of the right-hand side of (2.6) takes into account that we shall deal with fermions of spin = $\frac{1}{2}$, e.g. neutrons.
- (4) J.-M. Lévy-Leblond, J.Math.Phys. 10(1969)806
- (5) This can be understood from the uncertainty principle: one has $\Delta p \cdot (R^3/N)^{1/3} \sim 1$, hence $E \sim N(N^{2/3}/R^2) = N^2/R$ which assumes a minimum $E_0 \sim N^{7/3}$ for $R \sim N^{-1/3}$.
- (6) Cf. V.F. Weisskopf's lecture given at Erice-Summer-School 1969.
- (7) W. Thirring, Z. Physik 235(1970)339
 P. Hertel and W. Thirring, Annals of Physics 63(1971)
 See also D. Lynden-Bell, Brandeis lectures 1969
- (8) Although we did not demonstrate that the Thomas-Fermi equation becomes exact in the microcanonical ensemble as well we have no reason to doubt it.
- (9) We assume $R = 1$ in the appendix. This is no loss of generality because of (2.14). Thus the volume within which the particles are confined to move is the threedimensional unit ball S_1 .
- (10) J. Dieudonné, Eléments d'analyse, Tom I, p. 143,
 Paris, Gauthier-Villars 1969
- (11) It has been shown in ref. 2 that $F_{U_{\mu s}} = \inf_{K^s} F_U$ holds where F_U is

calculated with v_μ and $K^s = \{ U(x) = \sum_{a=1}^s u_a \varphi_a(\vec{x}), u_a \in \mathbb{R}^s \}$.

For $s' \geq s$ we have $K^{s'} \supset K^s$ and consequently $F_{U, \mu s'} \leq F_{U, \mu s}$.

(12) Again F_U is the same for possibly different accumulation points since

$$\frac{\partial}{\partial \mu} F_{U, \mu} = - \frac{1}{2} \int_{S_1} d^3x \int_{S_1} d^3x' \rho_\mu(x) e^{-\mu|\vec{x}-\vec{x}'|} \rho_\mu(x') \leq 0.$$

It turns out that at the phase transition point indeed two different solutions U with equal and minimal free energy exist.

Figure captions

Figure 1 Kinetic energy of homogeneously distributed noninteracting fermions vs. inverse radius for various temperatures. The straight line is $-\frac{1}{2}$ of the potential energy.

Figure 2 Free energy/temperature vs. inverse temperature

Figure 3 Total energy vs. inverse temperature

Figure 4 Entropy vs. total energy

Figure 5 Mass distribution vs. radial distance. The two curves for $\beta = 0.165 \text{ MEV}^{-1}$ correspond to the limits from both sides to the transition temperature.

Figure 6 Fermi degeneracy vs. radial distance.

FIG.1

FIG. 2

FIG. 3

FIG. 4

FIG.5

FIG. 6