Estimating entanglement of unknown states

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The experimental determination of entanglement is a major goal in the quantum information field. In general the knowledge of the state is required in order to quantify its entanglement. Here we express a lower bound to the robustness of entanglement of a state based only on the measurement of the energy observable and on the calculation of a separability energy. This allows the estimation of entanglement dismissing the knowledge of the state in question.

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Entanglement - purely quantum correlations - has been viewed as the main resource that allows several practical tasks such as quantum cryptography [1], teleportation [2], and quantum computation [3]. Despite the great efforts that have resulted in important advances in the area of Quantum Information, this resource is still to be completely understood. One important question is how to properly quantify entanglement and how to measure the amount of entanglement present in a system in real experiments, especially in the multipartite scenario.

One of the difficulties in dealing with the quantification of entanglement is the fact that most of the proposed quantifiers use mathematical operations without clear physical interpretations. Furthermore an extra difficulty appears when we do not know precisely the state we are dealing with, since the majority of the quantifiers are based on the knowledge of the density matrix of the system [23].

One alternative approach is just to detect entanglement, without quantifying it. A powerful method that can be used for this aim is the use of entanglement witnesses [4]. An entanglement witness for the entangled state \( \rho \) is given by a Hermitian operator \( W \) such that

\[
\text{Tr}(W \rho) < 0, \quad (1)
\]

while \( \text{Tr}(W \sigma) \geq 0, \forall \sigma \in \mathcal{S} \), \( (2) \)

where \( \mathcal{S} \) denotes the set of separable states (i.e., states with classically described correlations [5]). This is a simple consequence of the structure of the set of separable states: \( \mathcal{S} \) is a convex closed set. An important point about entanglement witnesses appears in the multipartite case: they can be used to detect different kinds of entanglement, just defining \( \mathcal{S} \) as the set of states which do not have such a kind of entanglement. Another important advantage is that, as \( W \) is a Hermitian operator, it can be seen as an observable, and can be directly measured [6]. Through this road, one goal was already achieved: experimental detection of entanglement without the previous determination of the quantum state.

In this Letter we aim to achieve another goal: we show a method for estimating how much entanglement a quantum state has, without the need of knowing it. More precisely, we show how to obtain a lower bound of the generalized robustness [7] of an unknown state by measuring only the expected value of an observable: energy. One must appreciate the importance of describing relevant physical properties of the system by a small amount of numbers (in the present case, just one), instead of a detailed “microscopic” description of the system (here, the knowledge of its state), as is done by Thermodynamics. One should remember that even the most simple system to show entanglement (two qubits) needs 15 numbers to be completely characterized [24], which makes the state reconstruction highly inefficient.

Recently, it has been shown that this thermodynamical analogy can be made much more deep. In fact, some thermodynamical properties can be regarded as true entanglement witnesses for some systems [8, 9, 10, 11, 12, 13, 14]. Specifically, it was shown that if the ground state of a Hamiltonian is entangled, a measurement of energy can directly show that the system is entangled [13]. This can be understood by noting that in this case there will be a lower bound to the energy of all separable states, denoted by \( E_{\text{sep}} \), which is higher than the ground state energy. Consequently, if the system is found with less energy than \( E_{\text{sep}} \) it is automatically known to be entangled.

The value of \( E_{\text{sep}} \) was determined for many interesting systems such as Heisenberg spin chains [13, 14] and the Bose-Hubbard model [13].

Following Tóth [13], one can rephrase the above discussion in terms of entanglement witnesses. One can define the energy-based entanglement witness \( W = H - E_{\text{sep}} I \), where \( H \) is the Hamiltonian, \( I \) is the identity operator, and \( E_{\text{sep}} = \min \langle \psi | H | \psi \rangle \), where \( S \) denotes the set of pure separable states. This definition guarantees the relation \( (2) \) to hold and, if \( E_{\text{sep}} \) is different from the ground

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state energy, all states with $\langle E \rangle < E_{\text{sep}}$ will be detected by $W$. In this case, one calls $E_{\text{sep}} - E_0$ the entanglement gap ($E_0$ being the ground state energy), as it gives the range in energy of the states which entanglement is unveiled by such a witness \[14\].

Up to this point, entanglement witnesses were said to only detect entanglement. However, they can also be used in quantifying it as well \[13\], \[14\], \[17\]. Following ref. \[17\], one can define a witnessed entanglement of the state $\rho$ as

$$E(\rho) = \max \left\{ 0, - \min_{W \in \mathcal{M}} \text{Tr}(W\rho) \right\}, \quad (3)$$

where $\mathcal{M}$ is a restricted set of entanglement witnesses. Depending on this restriction, different entanglement quantifiers appear (including some well known ones as special cases). In this Letter we shall only deal with the generalized robustness \[8\], which corresponds to the case with $\mathcal{M}$ given by the restriction $W \leq I$.

The generalized robustness of $\rho$ ($R_g(\rho)$) is defined as the minimum amount of mixing with another state (by means of a convex combination) such that this mixture loses its entanglement. More precisely, it is given by the minimum $s$ such that $\frac{E_{\text{sep}} - s}{1 + s}$ is a separable state, with $s$ representing any (not necessarily separable) state. It was found to have a direct information theoretical interpretation as the best fidelity of teleportation one can achieve using $\rho$ as a quantum channel (in the 2-qubit case \[18\]), or the improvement $\rho$ causes to the teleportation process if it is used as an ancillary state (for general bipartite states) \[19\]. It is thus important to estimate this entanglement quantifier from an operational point of view. We now set a scheme to obtain a lower bound for the generalized robustness of an unknown state. For an entangled state $\rho$, the above discussion implies

$$R_g(\rho) = - \text{Tr}(W_{\text{opt}}\rho) = - \langle W_{\text{opt}} \rangle, \quad (4)$$

where $W_{\text{opt}}$ is an optimal entanglement witness for $\rho$ in the sense of the minimization procedure in Eq. \[3\]. The crucial point is that for an unknown state one cannot determine an optimal witness.

Suppose now we are dealing with a bounded Hamiltonian $H$ (e.g., a finite dimensional Hilbert space). Set $A = \sup \{|H| - E_{\text{sep}}|\}$, where the supremum is taken over all quantum states, and define

$$W = \frac{H - E_{\text{sep}}I}{A}. \quad (5)$$

This $W$ is an entanglement witness for all states with $\langle H \rangle < E_{\text{sep}}$ and obeys $W \leq I$. Hence, if $\langle H \rangle < E_{\text{sep}}$, we have

$$R_g(\rho) \geq \frac{E_{\text{sep}} - \langle H \rangle}{A}, \quad (6)$$

since one can not guarantee that this witness is optimal.

In the multipartite scenario, the situation is even better: the measurement of one observable, the energy, can estimate various kinds of entanglement. For this, one only needs to be able to calculate the various values of $E_{\text{sep}}$, one for each kind of entanglement. Whenever $E_{\text{sep}}$ is greater than the measured energy, Eq. \[6\] gives a lower bound for that kind of entanglement. The quantity $E_{\text{sep}}$ can be obtained by the techniques in Refs. \[14\]. Naturally, this scheme works better for states with low energy, like low temperature thermal equilibrium states. However, it is also important that the scheme can be used for any state, with the only usual condition of reproducibility: i.e., one must prepare a whole ensemble of copies characterized by $\rho$ to get a good evaluation of $\langle E \rangle$, and from this obtain the estimates of entanglement.

As a matter of illustration we exemplify our results using a two-qubit system coupled through a Heisenberg $X X X$ interaction, with coupling constant $J > 0$, subjected to a magnetic field $B$:

$$H_{X X X} = J\sigma_x \cdot \sigma_x + B (\sigma_x \sigma_z + \sigma_z \sigma_x), \quad (7)$$

using $E_{\text{sep}}$ as calculated in Ref. \[13\]. In Figure 1 we have displayed the lower bound \[6\] for the state of thermal equilibrium $\rho(T) = Z^{-1}\exp(-\beta H)$, where $Z = \text{Tr}\exp(-\beta H)$ is the partition function and $\beta = (k_B T)^{-1}$, $k_B$ denoting the Boltzmann constant and $T$ the absolute temperature. The behavior at the picture is qualitatively in accordance with our intuition. For $B = 0$ the entanglement is greater, but even in this case, for a finite temperature it will vanish, in an entangled-disentangled transition \[20\]. As Eq. \[6\] only gives a lower bound to entanglement, the precise situation is a little bit different. For example, for $B = 0$ and $T = 0$, we have nothing more than the singlet, and its generalized robustness is 1, despite the energy-based witness only imply $R_g \geq 0.33$. Also the transition truly occurs at $T_c = 3.65$, while here the estimate only says that $T_c \geq 1.82$.

It is important to stress that the critical temperature $T_c$ below which the system is certainly entangled was already estimated for real systems and can give a notion
of the limits within our approach works. In Ref. [10] the authors found $T_{\text{sep}} \approx 5.6 K$ for the cooper nitrate (CN). Although it is expected that our method works better for low temperatures, some systems can show a separability temperature as high as $T_{\text{sep}} \approx 365 K$, as is the case of the nanotubular system Na$_2$V$_3$O$_7$ [11].

Some related previous work deserves mention. Ref. [21] shows how to implement experimentally maps related to positive but not completely positive maps on $n$ copies of a given state. This strategy permits the detection of entanglement in the bipartite scenario. Ref. [22] reports the measurement of concurrence on hyperentangled states, in which two logical copies of a qubit are encoded in just one physical system. Our approach works in the so-called one-copy regime, which usually is simpler from the experimental point of view, and is directly applicable for the estimation of multipartite entanglement as well.

Summarizing, we have presented a way of estimating the entanglement of a system without having previous knowledge of it. This method relies on a lower bound to the generalized robustness of entanglement which is given through the measurement of the mean value of the energy only. This special entanglement quantifier has an operational interpretation in terms of the best fidelity of teleportation. It is important to emphasize the advantage of measuring, or at least estimating, entanglement quantifiers for practical applications. We hope the present discussion help in this task and also add flavor on the experimental quantification of entanglement.

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23. This assumption is very natural from a theoretical point of view. However, in practical applications, it implies the necessity of a tomographic-like process to completely determine the state, prior to quantify its entanglement.
24. The independent 15 real parameters that determines a density matrix. Of course, if one make additional assumptions, as considering the global state pure, this number lowers.