Underdetermination of Bohmian Mechanics and quantum typicality

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Abstract

Bohmian mechanics represents the universe as set of continuous paths on the configuration space, with a probability measure defined on it. Such a structure corresponds to a stochastic process. Since there is just one universe, the probability measure is used to define typicality rather than relative frequencies. In the general language of stochastic processes, a precise criterion for the empirical equivalence of a process representing the universe to standard quantum mechanics is proposed. Such a criterion does not fix univocally the probability measure, which remains underdetermined. This corresponds to the well known underdetermination of Bohmian mechanics. In order to solve this ambiguity, the proposal is made to replace the probability measure with a less detailed mathematical structure, namely a quantum expression called typicality distance. Such a replacement requires a new rule, analogous to the Born rule, which derives typicality directly from the typicality distance, without the intermediation of a probability measure.

1 Introduction

Bohmian mechanics (references will be given in section 3) is a complete and coherent formulation of non relativistic quantum mechanics. In spite of this, such a formulation is far to be universally accepted. One of the criticisms is, for instance, the underdetermination of the theory [22], which is connected to the unobservability of Bohmian trajectories.

According to Bohmian mechanics, the particles of the universe follow definite trajectories satisfying a differential equation, the guidance equation. The set of these trajectories is endowed with a probability measure derived from the universal wave function. Such a structure makes Bohmian mechanics mathematically equivalent to a stochastic process.

A stochastic process is a mathematical tool which is commonly used to represent systems subjected to random evolution, like particles in a liquid subjected to the Brownian
motion. However, from the mathematical point of view, a stochastic process is a very general and powerful tool, and it allows to represent also a deterministic theory like Bohmian mechanics. In this case, the relevant feature is that there is a set of paths with a probability measure defined on it.

When a stochastic process represents the universe, like in the case of Bohmian mechanics, there are two conceptual differences with respect to the case in which it represents an ensemble of open systems. The first one is that, since we have just a single universe and not an ensemble, the probability measure is used to define typicality instead of relative frequencies. The second one refers to the definition of empirical equivalence, being the usual definition, which corresponds to mathematical equivalence, no more applicable.

In this paper a precise criterion for empirical equivalence is proposed, in the form of a criterion for the empirical equivalence of a stochastic process with standard quantum mechanics. According to the proposed criterion, it results that any stochastic process representing the universe is underdetermined.

In order to solve this ambiguity, the proposal is made to represent the universe by means of a less detailed structure than a stochastic process. Such a structure is a general set of paths endowed with a *typicality distance* in place of a probability measure. The typicality distance is a quantum expression which is present in the proposed criterion for empirical equivalence to quantum mechanics. Such a proposal require the important conceptual assumption that typicality could be derived directly from a quantum expression, without the intermediation of a probability measure. There is a strong analogy between this assumption and the Born rule: the first one connects a quantum expression to typicality, whilst the second one connects a quantum expression to probability.

In the literature, Bohmian mechanics is not explicitly presented as a stochastic process, also if in [4] it is presented as the set of all the continuous paths on the configuration space endowed with a probability measure concentrated on the subset of the solutions of the guidance equation. The problem of empirical equivalence of a theory with standard quantum mechanics has been discussed by many authors, for instance [4] [8] [9] [13] [23] [25] [12]. However, to our knowledge, it has never been examined in the formal language of stochastic processes, and no precise mathematical formulation of a criterion for such an equivalence has ever been proposed. Also the possibility to derive typicality from the quantum formalism without the need of a probability measure has never been proposed, also if the difference between the notions of probability and typicality has been emphasized in [18].

The plan of the paper is the following: in section 2 there is a short review of stochastic processes and of their usual interpretation. In section 3 the representation of the universe as a stochastic process is examined. The section is mainly devoted to Bohmian mechanics, since it is the best studied theory of this kind, both from the mathematical and from the conceptual point of view; however other theories are considered. In section 4 the typicality distance is defined, and a criterion for empirical equivalence of a stochastic processes with standard quantum mechanics is proposed. In section 5 the unavoidable underdetermination of any stochastic process representing the universe is pointed out, and the proposal is made to replace the probability measure with the typicality distance.
2 Stochastic processes and their usual interpretation

Let \((M, B)\) a measurable space and \(T\) an index set. A stochastic process with index set \(T\) and state space \(M\) is the quadruple \((\Omega, \mathcal{F}, \mu, \{z_t\}_{t \in T})\), where \((\Omega, \mathcal{F}, \mu)\) is a probability space and \(\{z_t : \Omega \rightarrow M\}_{t \in T}\) is a set of random variables indexed by \(T\). In this paper we shall always assume that \(M = \mathbb{R}^{3N}\) is the configuration space of an \(N\)-particles system, and \(T = \mathbb{R}^+\) is the positive time axis.

Given the finite sequence \(\{(t_1, \Delta_1), \ldots, (t_n, \Delta_n)\}\), where \(t_i \in T\) and \(\Delta_i \in B\), the cylinder set \(C[(t_1, \Delta_1) \ldots (t_n, \Delta_n)] \subseteq \Omega\) is defined as

\[C[(t_1, \Delta_1) \ldots (t_n, \Delta_n)] := \{\omega \in \Omega : z_{t_1}(\omega) \in \Delta_1, \ldots, z_{t_n}(\omega) \in \Delta_n\}.
\]

A cylinder set of the type \(C[(t, \Delta)]\) will be said a single-time cylinder set. For general cylinder sets and single-time cylinder sets the shorthand notation \(\{(t_1, \Delta_1) \ldots (t_n, \Delta_n)\}\) and \((t, \Delta)\) will be used.

The value of the measure \(\mu\) on the cylinder sets is said the time law of the process, while its value on the single-time cylinder sets is said the marginal distribution. Two stochastic processes with the same index set and space state are said equivalent if they have the same time law.

The canonical process. Any element \(\omega \in \Omega\) defines the path \(z_{(\cdot)}(\omega) : T \rightarrow M\), which is said a sample path. Consider the map \(\tau : \Omega \ni \omega \mapsto z_{(\cdot)}(\omega) \in M^T\), where \(M^T\) is the set of all the maps \(\lambda : T \rightarrow M\). The map \(\tau\) is measurable if \(M^T\) is endowed with the product \(\sigma\)-algebra \(\mathcal{B}^T\), which is the \(\sigma\)-algebra generated by the sets of the type \((t, \Delta) = \{\lambda \in M^T : \lambda(t) \in \Delta\}\). The process \((M^T, \mathcal{B}^T, \tau(\mu), \{y_t\}_{t \in T})\), where \(y_t(\lambda) = \lambda(t)\), is equivalent to the process \((\Omega, \mathcal{F}, \mu, \{z_t\}_{t \in T})\), and it is said the canonical process associated with \((\Omega, \mathcal{F}, \mu, \{z_t\}_{t \in T})\). Two equivalent processes have the same canonical process. According to the Kolmogorov reconstruction theorem, given any time law satisfying only formal requirements, there exists a canonical process with that time law.

Some remarks on the common usage of stochastic processes. Usually they are utilized to represent ensembles of identical open systems subjected to random evolution, like for instance particles in a liquid subjected to Brownian motion. A natural assumption is that all and only the possible measurements on the systems are finite sequence of positions measurements at different times. Any of such measurements corresponds to a cylinder set, and the relative frequency of the “yes” outcomes of the measurements corresponds to the measure of the cylinder set. Thus two processes are empirically equivalent iff they are mathematically equivalent, i.e. if they have the same time law.

Let us examine the possibility for a stochastic process to represent an ensemble of...
open quantum systems. One could try to define the following “quantum” time law:

$$\mu_Q[\{(t_1, \Delta_1) \ldots (t_n, \Delta_n)\}] := ||E(\Delta_n)U(t_n-t_{n-1})E(\Delta_{n-1}) \ldots E(\Delta_1)U(t_1)\Psi_0||^2,$$

where \(\Psi_0 \in L^2(M)\) is the quantum state of the systems at the time \(t = 0\), \(U(t)\) is the time evolution operator and \(E(\Delta)\) is the spatial projector on the region \(\Delta \subseteq M\). This choice corresponds to the usual interpretation according to which (1) is the quantum mechanical probability to find the particles in the regions \(\Delta_i\) at the time \(t_i\), for \(i = 1, \ldots, n\). The problem is that the set function (1) is not additive, i.e.

$$\mu_Q[\{(t_1, \Delta_1) \ldots (t_i, \Delta_i \cup \Delta'_i) \ldots (t_n, \Delta_n)\}] \neq \mu_Q[\{(t_1, \Delta_1) \ldots (t_i, \Delta_i) \ldots (t_n, \Delta_n)\}] + \mu_Q[\{(t_1, \Delta_1) \ldots (t_i, \Delta'_i) \ldots (t_n, \Delta_n)\}],$$

and thus it cannot be a time law. This is the paradoxical aspect of the superposition principle of quantum mechanics, which prevents an ensemble of open quantum systems to be represented by a stochastic process. As we shall see, the situation changes when the system is the universe.

### 3 The universe as a stochastic process

The most famous and the best studied theory which models the universe as a stochastic process is probably Bohmian mechanics. Most of the work made to show how Bohmian mechanics explains the observed phenomena of the universe can be used to show how a stochastic process in general can explain such phenomena. Let us quote from “What is Bohmian Mechanics”, by V. Allori, N. Zanghì [11]:

Bohmian mechanics (also called “pilot-wave theory” or “causal interpretation”) was discovered in 1927 by De Broglie [24] and soon abandoned. It was rediscovered, extensively extended, and for the first time fully understood, in 1952 by David Bohm [8]. During the sixties, seventies and eighties, John Bell was his principal proponent; his book [5] contains yet unsurpassable introductions to Bohmian mechanics. Other standard references are the books of Bohm and Hiley [10] and that of Holland [19]. The approach we are following here is that of the “Rutgers-München-Genova” group (quite in line with the approach of Bell), see, e.g., [13], [14], [16], [17], [15].

Also in this paper we follow the approach of the “Rutgers-München-Genova” group. According to this approach, Bohmian mechanics applies first of all to the universe. Both in this presentation of Bohmian mechanics and in the rest of the paper, the universe will always be considered as an idealized non-relativistic universe composed of \(N\) distinguishable spinless particles. According to Bohmian mechanics, the state of such a universe at the time \(t\) is represented by the pair \((x(t), \Psi(t))\), where \(x(t) \in R^{3N}\) is the position of the particles and \(\Psi(t) \in L^2(R^{3N})\) is the universal wave function. The universal wave function evolves in time according to the Schrödinger equation

$$i\hbar \frac{\partial \Psi}{\partial t} = -\sum_{k=1}^{N} \frac{\hbar^2}{2m_k} \Delta_k \Psi + V \Psi,$$
with the initial condition $\Psi(0) = \Psi_0$, $||\Psi_0||^2 = 1$. The time $t = 0$ is considered as the initial time of the universe. The position $x(t)$ evolves according to the guidance equation

$$\frac{dx_k}{dt} = \frac{\hbar}{m_k} \text{Im} \frac{\nabla_k \Psi}{\Psi}, \quad k = 1, \ldots, N. \quad (3)$$

The position of the particles at the initial time is distributed according to $|\Psi_0(x)|^2$. The combined action of the initial distribution and of the guidance equation determines the equivariance of the probability density of the particles $\rho(t, x)$, that is

$$\rho(t, x) = |\Psi(t, x)|^2 \quad (4)$$

for all $t$.

The way to derive a theory for the subsystems on which people make experiments from the theory for the whole universe is shown in [13], and is based on the following two steps: (1) the definition of a wave function for the subsystems, which is said the effective wave function; (2) the proof of the quantum equilibrium hypothesis, which claims that if we prepare $M$ subsystems with the same effective wave functions $\psi$, the empirical spatial distribution of the particles in the $M$ subsystems is given exactly by $|\psi(x)|^2$. The quantum equilibrium hypothesis has been proved by using the notion of typicality, as explained soon.

Since Bohmian mechanics is a theory for the universe, other than the statistical results of the quantum experiments it must also explain the macroscopic quasi-classicality of evolution. In other words, it must be proved that Bohmian trajectories appear quasi-classic at the macroscopic level. There are some studies in this sense [2] [3].

In spite of the fact that Bohmian mechanics is a deterministic theory, it is easy to recognize that it corresponds to the stochastic process $(\Lambda, \mathcal{F}, \mu, \{z_t\}_{t \in T})$, where $\Lambda$ is the set of the trajectories $\lambda: R^+ \rightarrow R^{3N}$ satisfying the guidance equation, $z_t(\lambda) = \lambda(t)$, $\mathcal{F}$ is the $\sigma$-algebra generated by the sets $\{z_t^{-1}(\Delta) : \Delta \in \mathcal{B}, t \in T\}$ and $\mu$ is defined by

$$\mu(\Gamma) := ||E[z_0(\Gamma)]\Psi_0||^2, \quad (5)$$

where $\Gamma \in \mathcal{F}$. The $\sigma$-algebra $\mathcal{F}$ has been defined such in a way that the maps $\{z_t\}_{t \in T}$ are measurable.

Another possible representation of Bohmian mechanics as a stochastic process is $(M_0^T, \tilde{\mu}, \tilde{\mathcal{F}}, \{\tilde{z}_t\}_{t \in T})$, where $M_0^T$ is the set of all the continuous paths from $T$ to $M$, and $\tilde{\mu}$ is defined by $\tilde{\mu}(\Gamma) = \mu(\Gamma \cap \Lambda)$. $\tilde{\mathcal{F}}$ and $\{\tilde{z}_t\}_{t \in T}$ are defined as in the previous case. This representation consists of a general set of continuous paths with a probability measure defined on it. The whole structure of the model is concentrated in the measure $\tilde{\mu}$, which can be considered as the fundamental law of the universe.

We are now in the position to explain how typicality works. In order to prove the quantum equilibrium hypothesis, it has been shown [13] that the subset $A$ of the Bohmian trajectories satisfying such a hypothesis is typical in the set $\Lambda$ of all the Bohmian trajectories, that is the overwhelming majority of the trajectories of $\Lambda$ also belong to $A$. The
The overwhelming majority is defined by means of the probability measure \( \mu \). Thus \( A \) is typical if \( \mu(\Lambda \setminus A) \ll 1 \). The typicality of \( A \) explains the fact that the actual trajectory of our universe, which is supposedly chosen at random from \( \Lambda \), satisfies such a hypothesis. See for instance [18] for a discussion on the power of such an explanation. The use of typicality in place of probability is one of the important differences we encounter when use a stochastic process to model the universe instead of an ensemble of open systems, and the reason for this difference is obvious: probability predicts relative frequencies of outcomes of measurements on identical systems, but here we have just one universe and just one outcome, i.e. the actual trajectory of the universe.

Let us return to the “Bohmian process”. Due to the equivariance property of Bohmian mechanics, equation (5) holds true for any time \( t \):

\[
\mu(\Gamma) = ||E[z_t(\Gamma)]\Psi(t)||^2,
\]

where \( \Psi(t) = U(t)\Psi_0 \). Thus, the marginal distribution of the Bohmian process is

\[
\mu[(t, \Delta)] = ||E(\Delta)\Psi(t)||^2.
\]

In general, we say that a stochastic process is equivariant if its marginal distribution is (7). Of course, Bohmian mechanics is not the only theory corresponding to an equivariant process. Another well known example is Nelson’s stochastic mechanics [20] [21]. In [23] it is shown that both Bohmian mechanics and Nelson’s stochastic mechanics are particular cases of a class of equivariant processes which can be derived by the same parametric stochastic equation. Other examples are given in [12]. The simplest and most “radical” equivariant process is the canonical process whose time law is

\[
\mu[\{(t_1, \Delta_1) \ldots (t_n, \Delta_n)\}] = ||E(\Delta_1)\Psi(t_1)||^2 \ldots ||E(\Delta_n)\Psi(t_n)||^2.
\]

This process was introduced, although in a less formal way, by Bell [6] [7]. This process is very unphysical, because it has no dynamics, i.e. no law connecting configurations at different times, and it defines no physical trajectory.\(^2\)

A last consideration on the use of a stochastic process as a model for the universe. In the case of Brownian motion, the stochastic process is considered as an approximate description of an evolution governed by more fundamental laws, in this case the differential equations of classical mechanics. When a stochastic process is used to represent the universe, the fundamental law is the stochastic process itself. In particular, one would have to accept that a measure on a set of paths could be a fundamental law, like differential equations are fundamental laws in classical physics.

\(^2\)Bell claims at the same time that this process cannot be taken seriously but that, at least from the logical point of view, it could explain our perception of a definite past evolution. The reasoning of Bell is based on the assumption that memories are part of the present configuration. We think however that this reasoning is not correct neither from the logical point of view. A study on this subject will be presented in a future paper.
4 Empirical equivalence to standard quantum mechanics

Assume that a suitable stochastic process \((\Omega, \mathcal{F}, \mu, \{z_t\}_{t \in T})\) represents the universe. In this section we shall try to express in a precise formal way the conditions under which the process is quantum equivalent, i.e. equivalent to standard quantum mechanics. Actually, we expect that a quantum equivalent process also predicts the correct evolution at the macroscopic level, to which quantum mechanics in its standard formulation does not apply; this requirement is added to our definition of quantum equivalence.

The conditions for quantum equivalence which are usually considered in the literature are equivariance \([4]\) and the effective collapse of the wave function \([25]\). The last phenomenon is also well described in the original Bohm’s papers \([8, 9]\) (in which however the term “effective collapse” does not appear). See also \([23, 12]\).

As to equivariance, an accurate examination of the proof of the quantum equilibrium hypothesis leads to the conclusion that it is based only on the equivariance of the Bohmian process, and not on the particular form of the Bohmian trajectories. In other words, any equivariant process satisfies the quantum equilibrium hypothesis, and thus reproduces the correct results for the quantum experiments. Actually, the requirement of equivariance can be weakened, because the proof of the quantum equilibrium hypothesis is based on typicality. This means that any process for which

\[
||E(\Delta)\Psi(t)||^2 \ll 1 \Rightarrow \mu[(t, \Delta)] \ll 1 \tag{9}
\]

satisfies the quantum equilibrium hypothesis. For convenience reasons, we substitute condition \([9]\) with the condition

\[
||E(\Delta)\Psi(t)|| \ll 1 \Rightarrow \sqrt{\mu[(t, \Delta)]} \ll 1. \tag{10}
\]

This corresponds to say that a set \(A\) is typical if \(\sqrt{\mu(A)} \ll 1\). Since typicality is an approximate notion, this definition can be acceptable, also if it appears unjustified. Its justification will be given later, when typicality will be defined by means of a distance between sets instead of by means of a probability measure.

As to the effective collapse of the wave function, suppose that at a suitable time \(t_A\) the universal wave function could be expressed as the sum of two branches, i.e.

\[
\Psi(t_A) = \psi + \Psi_\perp,
\]

with \(\psi(t_B) := U(t_B - t_A)\psi\) and \(\Psi_\perp(t_B) := U(t_B - t_A)\Psi_\perp\) non-overlapping for \(t_B \geq t_A\). Effective wave function collapse means that the process restricted to the support of \(\psi\) must be independent from the existence of the branch \(\Psi_\perp\).

We can be more precise. Let us start with the following definition:

**Definition:** a wave packet \(\psi \in L^2(M)\) is said **non-overlapping at the time** \(t\) if there exists a set \(\Delta \subseteq M\) such that

\[
\frac{||\psi - E(\Delta)\Psi(t)||}{||\psi||} \ll 1; \tag{11}
\]
such a set is said a support of $\psi$.

This definition of non-overlapping wave packet and of support of a wave packet are necessary since the corresponding rigorous mathematical definitions are not applicable.

Let us pose $\psi = E(\Delta_A)\Psi(t_A)$ by definition. The condition that $\psi(t_B)$ is non-overlapping at the time $t_B$ can be expressed by requiring that

$$\exists \Delta_B \subseteq M \text{ such that } \frac{|E(\Delta_B)\Psi(t_B) - U(t_B - t_A)E(\Delta_A)\Psi(t_A)|}{|E(\Delta_A)\Psi(t_A)|} \ll 1; \quad (12)$$

The condition that the process restricted to the support of $\psi$ is independent from the existence of the branch $\Psi_\perp$ can be expressed as follows:

$$\frac{|E(\Delta)U(t_B - t_A)E(\Delta_A)\Psi(t_A)|^2}{|E(\Delta_A)\Psi(t_A)|^2} = \frac{\mu[(t_B, \Delta) \cap (t_A, \Delta_A)]}{\mu[(t_A, \Delta_A)]}$$

for any $\Delta \subseteq M$ and $t_B \geq t_A$. Due to typicality, also condition (13) can be weakened to the condition

$$\frac{|E(\Delta)U(t_B - t_A)E(\Delta_A)\Psi(t_A)|}{|E(\Delta_A)\Psi(t_A)|} \ll 1 \Rightarrow \left(\frac{\mu[(t_A, \Delta_A) \setminus (t_B, \Delta)]}{\mu[(t_A, \Delta_A)]}\right)^{\frac{1}{2}} \ll 1, \quad \forall \Delta \subseteq M, \quad (14)$$

analogously to condition (10). Note that condition (14) includes (10) as a particular case.

Thus the criterion for for quantum equivalence is that implication (14) holds whenever condition (12) holds. It is possible to express such a criterion in an equivalent but more synthetic form. Let us introduce some notation. Let $A, B, \ldots$ denote the single-time cylinder sets $(t_A, \Delta_A), (t_B, \Delta_B), \ldots$. Let us define $F(A) = U^\dagger(t_A)E(\Delta_A)U(t_A)$ and $\Psi(A) = F(A)\Psi_0$. Moreover, we give the following formal definition:

**Definition (typicality distance):** given two single-time cylinder sets $A$ and $B$, the set function

$$\nu(A, B) = \frac{||\Psi(B) - \Psi(A)||}{\max\{|||\Psi(A)||, ||\Psi(B)||\}} \quad (15)$$

is said the typicality distance between $A$ and $B$.

The reason for this name will be given in the next section. The value $\max\{|||\Psi(A)||, ||\Psi(B)||\}$ is a normalization factor. Other possible choices for the normalization factor give rise to the following definitions:

$$\nu_2(A, B) = \frac{2||\Psi(B) - \Psi(A)||}{||\Psi(A)|| + ||\Psi(B)||}; \quad \nu_3(A, B) = \frac{||\Psi(B) - \Psi(A)||}{\min\{|||\Psi(A)||, ||\Psi(B)||\}}.$$

It is easy to verify that

$$\nu(A, B) \leq \nu_2(A, B) \leq \nu_3(A, B) \leq \frac{\nu(A, B)}{1 - \nu(A, B)} = \nu(A, B) + o[\nu(A, B)].$$
Thus the typicality distance \( \nu \) condition \( (12) \) and implication \( (14) \) are equivalent to the implication

\[
\text{Theorem:} \quad \text{condition } (12) \text{ and implication } (14) \text{ are equivalent to the implication}
\]

\[

\nu(A, B) \ll 1 \Rightarrow \left( \frac{\mu(A \setminus B)}{\mu(A)} + \frac{\mu(B \setminus A)}{\mu(B)} \right)^{\frac{1}{2}} < 1. \tag{17}
\]

**Proof.** In the proof it is assumed that \( a \ll 1 \) and \( b \ll 1 \Rightarrow a + b \ll 1 \).

Part 1. Assume \( (14) \) and \( \nu(A, B) \ll 1 \). From relations \( (16) \) it descends that

\[
\frac{|E(\Delta_B)\Psi(t_B) - U(t_B - t_A) E(\Delta_A)\Psi(t_A)|}{|E(\Delta_A)\Psi(t_A)|} = \frac{|\Psi(B) - \Psi(A)|}{|\Psi(A)|} \ll 1,
\]

thus condition \( (12) \) is satisfied with respect to \( \Delta_A \). Due to the equality

\[
||\Psi(B) - \Psi(A)||^2 = ||F(B)\Psi_0 - F(B)F(A)\Psi_0||^2 + ||F(A)\Psi_0 - F(B)F(A)\Psi_0||^2,
\]

we have that

\[
\frac{|E(\Delta_B)U(t_B - t_A) E(\Delta_A)\Psi(t_A)|}{|E(\Delta_A)\Psi(t_A)|} \leq \frac{|F(A)\Psi_0 - F(B)F(A)\Psi_0|}{|F(A)\Psi_0|} \ll 1.
\]

From implication \( (14) \) one obtains \( \sqrt{\mu(A \setminus B)/\mu(A)} \ll 1 \). By interchanging \( A \) and \( B \) one obtains \( \sqrt{\mu(B \setminus A)/\mu(B)} \ll 1 \), and thus the right hand of implication \( (17) \) is derived.

Part 2. Assume \( (14) \) and

\[
\frac{|E(\Delta)U(t_B - t_A) E(\Delta_A)\Psi(t_A)|}{|E(\Delta_A)\Psi(t_A)|} \ll 1.
\]

\[\text{Actually we also must require that } t_A \text{ and } t_B \text{ are "not too close", in the following sense: consider the function } f(|t_B - t_A|) = |\Psi(t_A)\Psi(t_B)| = |\Psi_0U(t_B - t_A)\Psi_0|]. \text{ We have that } f(0) = 1, \text{ and we expect that } f(\tau) \text{ goes quickly and stays around 0 when } \tau \text{ is greater then a suitable value } \tau_0. \text{ Thus "not too close" means } |t_B - t_A| \geq \tau_0.\]

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Assume moreover that there exists a set $\Delta_B$ for which
\[
\frac{||E(\Delta_B)\Psi(t_B) - U(t_B - t_A)E(\Delta_A)\Psi(t_A)||}{||E(\Delta_A)\Psi(t_A)||} \ll 1.
\]

By posing $B' = (t_B, \Delta)$, we have
\[
||F(A)\Psi_0 - F[(t_B, \Delta \cap \Delta_B)]\Psi_0|| = ||F(A)\Psi_0 - F(B)F(B')\Psi_0|| =
||F(A)\Psi_0 - F(B')F(A)\Psi_0 + F(B')F(A)\Psi_0 - F(B')F(B)\Psi_0|| \leq
||E(\Delta)U(t_B - t_A)E(\Delta_A)\Psi(t_A)|| + ||E(\Delta_B)\Psi(t_B) - U(t_B - t_A)E(\Delta_A)\Psi(t_A)||.
\]

Thus $\nu[A, (t_B, \Delta_B \cap \Delta)] \ll 1$. By applying (17) we obtain
\[
1 \gg \left( \frac{\mu[A \setminus (t_B, \Delta_B \cap \Delta)]}{\mu(A)} + \frac{\mu[(t_B, \Delta_B \cap \Delta) \setminus A]}{\mu[(t_B, \Delta_B \cup \Delta)]} \right)^{\frac{1}{2}} \geq \left( \frac{\mu[(t_B, \Delta_B) \setminus (t_B, \Delta)]}{\mu[(t_B, \Delta_B)]} \right)^{\frac{1}{2}}.
\]
q.e.d.

In conclusion, we can formulate the following:

**Criterion for quantum equivalence:** a stochastic processes $(\Omega, \mathcal{F}, \mu, \{z_t\}_{t \in T})$ representing the universe is empirically equivalent to standard quantum mechanics if, for any pair of single-time cylinder sets $A$ and $B$, we have
\[
\nu(A, B) \ll 1 \Rightarrow \left( \frac{\mu(A \setminus B)}{\mu(A)} + \frac{\mu(B \setminus A)}{\mu(B)} \right)^{\frac{1}{2}} \ll 1.
\] (18)

In order to obtain a time symmetric criterion, in (18) the condition $t_A \leq t_B$ has not been required. Note moreover that (18) include (10) as a particular case.

We have seen that if $\nu(A, B) \ll 1$, then $\Delta_A$ and $\Delta_B$ are the supports of a branch of the universal wave function at the times $t_A$ and $t_B$ respectively. The right hand side of (18) implies that the overwhelming majority of the paths belonging to $A$ also belong to $B$, and vice-versa. In conclusion, condition (18) constraints the paths to follows approximately the branches of the universal wave function. This result follows from the requirement of effective wave function collapse, which is not satisfied, for instance, by the Bell’s process (8). In order to obtain a quasi-classical evolution at the macroscopic level, it is necessary that the branches of the universal wave functions present such a evolution. We do not address this problem here.

## 5 Underdetermination of the time law and quantum typicality

It is easy to recognize that the criterion for quantum equivalence does not fix univocally the time law, i.e. there are many processes with different time laws satisfying the criterion.
This means that the time law is *underdetermined* by the quantum equivalence criterion. This conclusion is another important difference with respect to the case of a stochastic process representing an ensemble of open systems, where the criterion of empirical equivalence determines exactly the time law. Of course, there is a strict connection between the underdetermination of the time law and the unobservability of Bohmian trajectories, and the underdetermination of Bohmian Mechanics is one of the main criticisms which are made to this theory [22].

There are essentially two ways to solve this ambiguity. The first one is finding some criteria of simplicity and invariance in order to select a particular process among those satisfying the quantum equivalence criterion, and surely Bohmian mechanics is the best candidate for this choice. The second way, the one we propose in this paper, is to realize that the time law become unnecessary if we admit that typicality can derive directly from the typicality distance $\nu$, without the intermediation of a probability measure. Let us explain.

Let us introduce the following straightforward generalization of the notion of typicality: two single-time cylinder sets $A$ and $B$ are *mutually typical* if the overwhelming majority of the elements of $A$ also belongs to $B$ and vice-versa. Since $A$ and $B$ are subsets of a probability space, it is natural to introduce the set function

$$\eta(A, B) := \frac{\mu(A \setminus B)}{\mu(A)} + \frac{\mu(B \setminus A)}{\mu(B)},$$

and to state the implication

$$\sqrt{\eta(A, B)} \ll 1 \Rightarrow A \text{ and } B \text{ are mutually typical.} \quad (19)$$

Here the square root has been introduced for the same reasons of relation (10). According to the quantum equivalence criterion, we have the following implication chain:

$$\nu(A, B) \ll 1 \Rightarrow \sqrt{\eta(A, B)} \ll 1 \Rightarrow A \text{ and } B \text{ are mutually typical.} \quad (20)$$

The goal of the quantum equivalence criterion it to guarantee that $A$ and $B$ are mutually typical whenever $\nu(A, B) \ll 1$. The role of the set function $\eta$, i.e. of the probability measure $\mu$, is to define the mutual typicality of $A$ and $B$. The proposal is than to admit that mutual typicality could be directly defined by the typicality distance $\nu$, without the intermediation of the probability measure $\mu$. In our knowledge, the possibility to derive typicality independently from probability has never been considered before in the literature, also if Goldstein writes [18]:

While typicality is usually defined [...] in terms of a probability measure, the basic concept is not genuinely probabilistic, but rather a less detailed concept. A measure $\mu$ of typicality need not be countably additive, nor even finitely additive. Moreover, for any event $E$, if $\mu$ is merely a measure of typicality, there is no point worrying about, nor any sense to, the question as to the real meaning of say $'\mu(E) = 1/2'$. Distinctions such as between $'\mu(E) = 1/2'$ and $'\mu(E) = 3/4'$ are distinctions without a difference.
Thus we propose to remove the intermediate step from chain (20), and to state the following rule:

**Quantum typicality rule:** \( \nu(A, B) \ll 1 \Rightarrow A \) and \( B \) are mutually typical.

There is a strong analogy between the quantum typicality rule and the Born rule: the Born rule connects a quantum expression with probability, whilst the quantum typicality rule connects a quantum expression, namely the typicality distance \( \nu \), with typicality. Note moreover that we have no difficulty in accepting the quantum typicality rule in the case in which, for instance, \( B = \Omega \). Indeed in this case we have \( \nu(A, B) = \|E(\Delta A)\Psi(t_A)\| \ll 1 \Rightarrow \|E(\Delta A)\Psi(t_A)\|^2 \ll 1 \Rightarrow A \) is typical \( \Rightarrow \Omega \) and \( A \) are mutually typical. This happens because the expression \( \|E(\Delta A)\Psi(t_A)\|^2 \), according to the Born rule, corresponds to the probability measure of \( A \).

We explain now the reason for the name “typicality distance” and for the preference of \( \nu(A, B) \ll 1 \) instead of \( \nu^2(A, B) \ll 1 \) as a condition for mutual typicality (this choice has lead to the use of the square roots in expressions (10), (14) and in the quantum equivalence criterion). The reason is that the set function \( \nu \) can be considered as a “normalized distance”, in the following sense: let \( S \) denote the set of the single-time cylinder sets. On \( S \) the natural distance

\[
 d(A, B) = \|\Psi(B) - \Psi(A)\|
\]

is defined, and \( \nu(A, B) \) can be expressed as

\[
 \nu(A, B) = \frac{d(A, B)}{\max\{d(A, \emptyset), d(B, \emptyset)\}}.
\]

According to the quantum typicality rule, two sets are mutually typical when their normalized distance is close to zero.

In conclusion, according to the above theory, a model for the universe more adequate than a stochastic process is the triple

\[
 (M^T, S, \nu), \tag{21}
\]

which can be considered as a canonical stochastic process with the typicality distance \( \nu : S \times S \rightarrow \mathbb{R}^+ \) in place of the more detailed probability measure \( \mu \).

Remember that, according to the results of the previous section, the typicality distance \( \nu \) has two effects: it constraints the paths to follow approximately the branches of the universal wave function, thus giving rise to a quasi-classical macroscopic evolution. Moreover, it defines the typicality which is necessary to explain the statistical results of the quantum experiments.

According to representation (21), the trajectories of the universe are not required to be continuous. However such a requirement can be added by simply replacing \( M^T \) with the set \( M^T_0 \) of the continuous paths, also if such a replacement has no empirical relevance.

Two final remarks. The first one is that the (spatial) branching structure of the wave function enters at the mathematical level in the theory, being the typicality distance \( \nu \) a
way to mathematically define the branches. This is not the case, for instance, of Bohmian mechanics, whose mathematical theory does not require this branching structure; however such a structure is widely assumed in the conceptual interpretation of the theory. The definition of a branch is approximate, and this approximation corresponds to the approximation introduced by typicality. In other words, the replacement of the exact notion of probability with the approximate notion of typicality allows to introduce the approximate notion of branch of the universal wave function as a fundamental entity of the theory, and the two notions of typicality and of branch are intrinsically tied in the typicality distance $\nu$.

The second remark refers to the form of the physical laws. We have seen that in the case of Bohmian mechanics one must admit a probability measure as a fundamental physical law. According to the representation $\mathbf{21}$, one must admit the typicality distance $\nu$ as a fundamental law.

6 Summary

The main steps of the reasoning are the following: (1) Bohmian mechanics and other formulations of quantum mechanics represent the universe as a set of continuous paths with a probability measure defined on it. Such a mathematical structure corresponds to a stochastic process. (2) The probability measure should be considered as the fundamental law of the universe. Since there is just one universe, the probability measure is used to define typicality instead of relative frequencies, and the proof that Bohmian mechanics explains the observed results of the statistical experiments is based on typicality. (3) In the language of stochastic processes, a criterion for quantum equivalence, i.e. for the empirical equivalence of a process to standard quantum mechanics, has been proposed; such a criterion derives from the requirement of equivariant typicality and of the effective collapse of the wave function. A quantum expression called typicality distance is an element of such a criterion. (4) The criterion for quantum equivalence does not fix univocally the probability measure of the process, which remains unavoidably underdetermined. (5) In order to solve this ambiguity, the proposal is made that the typicality measure itself is the fundamental law of the universe, and that the probability measure is unnecessary. (6) Such a proposal requires a rule, analogous to the Born rule, according to which typicality derives directly from the typicality distance, without the intermediation of a probability measure.

References


