Second-Order Gauge Invariant Cosmological Perturbation Theory

--- Einstein Equations in Terms of Gauge Invariant Variables ---

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Following the general framework of the gauge invariant perturbation theory developed in the papers [K. Nakamura, Prog. Theor. Phys. 110 (2003), 723; ibid. 113 (2005), 481], we formulate second-order gauge invariant cosmological perturbation theory in a four-dimensional homogeneous isotropic universe. We consider perturbations both in the universe dominated by a single perfect fluid and in that dominated by a single scalar field. We derive all the components of the Einstein equations in the case that the first-order vector and tensor modes are negligible. All equations are derived in terms of gauge invariant variables without any gauge fixing. These equations imply that second-order vector and tensor modes may be generated due to the mode-mode coupling of the linear-order scalar perturbations. We also briefly discuss the main progress of this work through comparison with previous works.

§1. Introduction

The general relativistic cosmological linear perturbation theory has been developed to a high degree of sophistication during the last 25 years. One of the motivations of this development is to clarify the relation between the scenarios of the early universe and cosmological data, such as the cosmic microwave background (CMB) anisotropies. Recently, the first-order approximation of our universe from a homogeneous isotropic one was revealed through the observation of the CMB by the Wilkinson Microwave Anisotropy Probe (WMAP). This observation suggests that the fluctuations are adiabatic and Gaussian at least to a first order approximation. One of the next important theoretical studies is to clarify the accuracy of these results by studying the non-Gaussian behavior, non-adiabatic behaviors, and so on. These will be goals of future satellite missions. To estimate the accuracy of the first-order approximation, theoretically, it is necessary to investigate second-order cosmological perturbations. From the observational point of view, also, with the increase of precision of the CMB data, the study of relativistic cosmological perturbations beyond linear order is becoming a topical subject, especially in regard to studying the generation of the primordial non-Gaussian behavior in inflationary scenarios and the non-Gaussian component in the CMB anisotropy.

In the literature, the second-order general relativistic perturbation theory has been investigated by many researchers. For the pioneering work, Tomita investigated general relativistic second-order perturbations in the Einstein-de Sitter model (vanishing model), and his treatment is in the synchronous gauge. His second-order perturbation theory was later extended to the general relativistic Zel’dovich approximation. Recently, nonlinear gauge transformations and the concept of gauge
invariance have been studied by Bruni et al.\(^9\) As the by-product of their research, Sonego and Bruni\(^{10}\) obtained a representation of the higher-order Taylor expansion of tensors on a manifold in a quite generic form, and the second-order gauge transformation from the synchronous gauge to the Poisson gauge has been performed by Matarrese et al.\(^{11}\) More recently, Noh and Hwang\(^{12}\) derived second-order perturbation equations in the Friedmann universe. Further, Tomita\(^{13}\) also extended his original works to second-order perturbations of nonzero-\(\Lambda\) cosmological models and studied the CMB anisotropy and its non-Gaussian nature. However, their treatments of the general relativistic second-order perturbation is very complicated. Hence, to avoid this complicated formulation, there have also been several attempts to investigate the nonlinear effects of general relativistic perturbations.\(^{14}\)

In this paper, we present a very clear formulation of the general relativistic second-order cosmological perturbations in a homogeneous isotropic universe. This paper is the complete version of the previous short paper\(^{15}\) by the present author. Our formulation in this paper is one of the applications of the gauge invariant formulation of the second-order perturbation theory on a generic background spacetime developed in two papers by the present author.\(^{16,17}\) These papers are referred to in this paper as KN2003\(^{16}\) and KN2005.\(^{17}\) This formulation is a by-product of investigations of the oscillatory behavior of a self-gravitating Nambu-Goto membrane\(^{18}\) and was first applied to a comparison between the oscillatory behavior of a gravitating Nambu-Goto string and that of a test string.\(^{19}\) This was a trivial application of the general formulation developed in KN2003 and KN2005, while the second-order cosmological perturbation carried out in this paper is the first non-trivial application of the formulation presented in KN2003 and KN2005.

The formulation developed in KN2003 and KN2005 is an extension of the works of Bruni et al.\(^9\) The gauge transformation rules of the perturbations formulated by Bruni et al.\(^9\) are extended to those in multi-parameter perturbation theory.\(^{16,20}\) Based on these gauge transformation rules, in KN2003, we proposed a procedure to find gauge invariant variables on a generic background spacetime to third-order perturbations, assuming that we already know the procedure to find gauge invariant variables for the linear-order metric perturbations. We also showed in KN2005 that the proposal of the gauge invariant variables in KN2003 provides a self-consistent second order perturbation theory in the generic background spacetime. It is straightforward to apply this general formulation to cosmological perturbations. In the cosmological perturbation case, there are some proposals of gauge invariant formulations of the second-order perturbation. For example, Mukhanov et al.\(^{21}\) proposed a gauge invariant second-order perturbation to evaluate the back reaction effect of the inhomogeneities in the universe on the effective expansion law of the universe. However, we should distinguish our formulation presented in this paper from the proposal of Mukhanov et al.\(^{21}\) These are quite different approaches.

To develop the gauge invariant perturbation theory, we start by explaining the concept of the “gauge” in general relativistic perturbation theory to avoid any misunderstanding of our formulation. General relativity is based on the concept of general covariance. Intuitively, the principle of general covariance states that there is no preferred coordinate system in nature, though the notion of general covariance
is mathematically included in the definition of a spacetime manifold in a trivial way. This is based on the philosophy that coordinate systems are originally chosen by us, and natural phenomena have nothing to do with our coordinate systems. If we apply a peculiar coordinate system to investigate natural phenomena, we will see peculiar behavior of that natural phenomena due to this peculiar coordinate system. This is an intuitive explanation of general covariance. Due to this general covariance, the \textit{gauge degree of freedom}, which is the unphysical degree of freedom of perturbations, arises in general relativistic perturbations. To obtain physically meaningful results, we have to fix this gauge degree of freedom or to extract the \textit{gauge invariant part of perturbations}.

As reviewed in detail in §2 of this paper, the developments in KN2003 and KN2005 are based on the understanding of the “gauge” in the perturbation theory which was first proposed by Stewart et al.\textsuperscript{22}) and developed by Bruni et al.\textsuperscript{9,20}) Based on this formulation, we define the complete set of gauge invariant variables of the second-order cosmological perturbations in the Friedmann-Robertson-Walker universe. We consider two cases of the Friedmann-Robertson-Walker universe, one in which the universe is filled with a single perfect fluid and one in which the universe is filled with a single scalar field. We also derive the second-order Einstein equations of cosmological perturbations in terms of these gauge invariant variables without any gauge fixing in these two cases.

The organization of this paper is as follows. In §2, we review the general framework of the second-order gauge invariant perturbation theory developed in KN2003 and KN2005. This review also includes additional explanation not given in those papers. In §3 we summarize the Einstein equations in the case of a background homogeneous isotropic universe, which are used in the derivation of the first- and second-order Einstein equations. In §4 we define the first- and second-order gauge invariant variables for the cosmological perturbations. The first-order perturbation of the Einstein equations is reviewed in §5. This perturbation is used in the derivation of the second-order Einstein equations. Then, the derivation of the second-order Einstein equation is given in §6. The final section, §7, is devoted to a summary and discussions concerning the relation between this and previous works.

We employ the notation of KN2003 and KN2005 and use abstract index notation.\textsuperscript{23}) We also employ natural units in which Newton’s gravitational constant is denoted by $G$ and the velocity of light satisfies $c = 1$.

\section*{§2. General framework of the gauge invariant perturbation theory}

In this section, we briefly review the general framework of the gauge invariant perturbation theory developed in KN2003 and KN2005 by the present author. To explain the \textit{gauge degree of freedom} in perturbation theories, we have to recall what we are doing when we consider perturbations. Further, we comment on the Taylor expansion of tensors on a manifold, at first, in §2.1 because any perturbation theory is based on the Taylor expansion. Next, in §2.2 we review the basic understanding of the gauge degree of freedom in perturbation theory based on the work of Stewart et al.\textsuperscript{22}) and Bruni et al.\textsuperscript{9}) When we consider perturbations in the theory with general
covariance, we have to exclude these gauge degrees of freedom in the perturbations. To accomplish this, gauge invariant variables of perturbations are useful, and these are regarded as physically meaningful quantities. In §2.3, we review the procedure for finding gauge invariant variables of perturbations, which was developed by the present author in KN2003. Then, in §2.4, we briefly review the general issue of the gauge invariant formulation for the second-order perturbation of the Einstein equation developed in KN2005. We emphasize that the ingredients of this section do not depend on the background spacetime, and they are applicable not only to cosmological perturbations but also to any other general relativistic perturbations.

2.1. Taylor expansion of tensors on a manifold

Here, we comment on the general form of the Taylor expansion of tensors on a manifold $M$. We first consider the Taylor expansion of a scalar function $f: M \to \mathbb{R}$, which can be extended to any tensor field on a manifold.

The Taylor expansion of a function $f$ is an approximated form of $f(q)$ at $q \in M$ in terms of the variables at $p \in M$, where $q$ is in the neighborhood of $p$. To consider the Taylor expansion of a function $f$, we introduce a one-parameter family of diffeomorphisms $\Phi_\lambda: M \to M$, where $\Phi_\lambda(p) = q$ and $\Phi_{\lambda=0}(p) = p$. One example of a diffeomorphisms $\Phi_\lambda$ is an exponential map. However, we consider a more general class of diffeomorphisms.

In terms of the diffeomorphism $\Phi_\lambda$, the Taylor expansion of the function $f(q)$ is given by

$$f(q) = f(\Phi_\lambda(p)) = (\Phi^*_\lambda f)(p) = f(p) + \frac{\partial}{\partial \lambda} (\Phi^*_\lambda f) \bigg|_p \lambda + \frac{1}{2} \frac{\partial^2}{\partial \lambda^2} (\Phi^*_\lambda f) \bigg|_p \lambda^2 + O(\lambda^3). \quad (2.1)$$

Since this expression hold for an arbitrary smooth function $f$, we may regard the Taylor expansion to be the expansion of the pull-back $\Phi^*_\lambda$ of the diffeomorphism $\Phi_\lambda$, rather than the expansion of the function $f$.

Further, as shown by Sonego and Bruni,\(^{10}\) there exist vector fields $\xi^1_a$ and $\xi^2_a$ such that the expansion (2.1) is given by

$$f(q) = (\Phi^*_\lambda f)(p) = f(p) + (\mathcal{L}_{\xi^1} f) \bigg|_p \lambda + \frac{1}{2} (\mathcal{L}_{\xi^2} + \mathcal{L}_{\xi^1}^2) f \bigg|_p \lambda^2 + O(\lambda^3), \quad (2.2)$$

without loss of generality. In the representation (2.2) of the Taylor expansion, $\xi^1_a$ and $\xi^2_a$ are the generators of the one-parameter family of diffeomorphisms $\Phi_\lambda$ and these represent the direction along which the Taylor expansion is carried out. The generator $\xi^1_a$ is the first-order approximation of the flow of the diffeomorphism $\Phi_\lambda$, and the generator $\xi^2_a$ is the second-order correction to this flow.

When the generator $\xi^2_a$ is proportional to the generator $\xi^1_a$, the representation (2.2) of the Taylor expansion reduces to that of the pull-back of an exponential map. Therefore, we may regard that the Taylor expansion (2.2) is the generalization of an exponential map (one-parameter group of diffeomorphisms). However, as shown by Sonego and Bruni,\(^{10}\) the Taylor expansion of a $C^m$ one-parameter family of diffeomorphisms can always be represented in the form of (2.2). In general, the generator $\xi^2_a$ may not be proportional to the generator $\xi^1_a$. Hence, we regard the generators $\xi^1_a$ and $\xi^2_a$ to be independent.
Further, \( \Phi_\lambda \) can be extended to diffeomorphisms acting on tensor fields of all types. Thus, the Taylor expansion of a tensor field \( Q \) of any type on a manifold \( M \) is given by

\[
Q(q) = (\Phi_\lambda^* Q)(p) = Q(p) + (\mathcal{L}_{\xi_1} Q)|_p \lambda + \frac{1}{2} (\mathcal{L}_{\xi_2} + \mathcal{L}_{\xi_1}^2) Q|_p \lambda^2 + O(\lambda^3), \tag{2.3}
\]

and we conclude that the representation of the Taylor expansion \(2.3\) is quite general.

2.2. Gauge degree of freedom in perturbation theory

Now, we explain the concept of gauge in general relativistic perturbation theory. To explain this, we first point out that, in any perturbation theory, we always treat two spacetime manifolds. One is the physical spacetime \( M \), which we attempt to describe in forms of perturbations, and the other is the background spacetime \( M_0 \), which is a fictitious manifold which we prepare for perturbative analyses. We emphasize that these two spacetime manifolds \( M \) and \( M_0 \) are distinct. Let us denote the physical spacetime by \((M, \bar{g}_{ab})\) and the background spacetime by \((M_0, g_{ab})\), where \( \bar{g}_{ab} \) is the metric on the physical spacetime manifold, \( M \), and \( g_{ab} \) is the metric on the background spacetime manifold, \( M_0 \). Further, we formally denote the spacetime metric and the other physical tensor fields on the physical spacetime by \( Q \) and its background value on the background spacetime by \( Q_0 \).

Second, in any perturbation theories, we always write equations for the perturbation of the physical variable \( Q \) in the form

\[
Q("p") = Q_0(p) + \delta Q(p). \tag{2.4}
\]

Usually, this equation is simply regarded as a relation between the physical variable \( Q \) and its background value \( Q_0 \), or as the definition of the deviation \( \delta Q \) of the physical variable \( Q \) from its background value \( Q_0 \). However, Eq. \(2.4\) has deeper implications. Keeping in our mind that we always treat two different spacetimes, \((M, \bar{g}_{ab})\) and \((M_0, g_{ab})\), in perturbation theory, Eq. \(2.4\) is a rather strange equation in the following sense: The variable on the left-hand side of Eq. \(2.4\) is a variable on the physical spacetime \((M, \bar{g}_{ab})\), while the variables on the right-hand side of Eq. \(2.4\) are variables on the background spacetime, \((M_0, g_{ab})\). Hence, Eq. \(2.4\) gives a relation between variables on two different manifolds.

Further, through Eq. \(2.4\), we have implicitly identified points in these two different manifolds. More specifically, \( Q("p") \) on the left-hand side of Eq. \(2.4\) is a field on the physical spacetime, \( M \), and \( "p" \in M \). Similarly, we should regard the background value \( Q_0(p) \) of \( Q("p") \) and its deviation \( \delta Q(p) \) of \( Q("p") \) from \( Q_0(p) \), which are on the right-hand side of Eq. \(2.4\), as fields on the background spacetime, \( M_0 \), and \( p \in M_0 \). Because Eq. \(2.4\) is regarded as an equation for field variables, it implicitly states that the points \( "p" \in M \) and \( p \in M_0 \) are same. This represents the implicit assumption of the existence of a map \( M_0 \to M : p \in M_0 \mapsto "p" \in M \), which is usually called a gauge choice in perturbation theory.\(^{22}\)

It is important to note that the correspondence between points on \( M_0 \) and \( M \), which is established by such a relation as Eq. \(2.4\), is not unique to the theory in which general covariance is imposed. Rather, Eq. \(2.4\) involves the degree of
freedom corresponding to the choice of the map \( \mathcal{X} : \mathcal{M}_0 \to \mathcal{M} \). This is called the \textit{gauge degree of freedom}. Such a degree of freedom always exists in perturbations of a theory in which we impose general covariance. “General covariance” intuitively means that there is no preferred coordinate system in the theory. If general covariance is not imposed on the theory, there is a preferred coordinate system, and we naturally introduce this coordinate system onto both \( \mathcal{M}_0 \) and \( \mathcal{M} \). Then, we can choose the identification map \( \mathcal{X} \) using this coordinate system. However, there is no such coordinate system in general relativity, due to its general covariance, and we have no guiding principle to choose the identification map \( \mathcal{X} \). Indeed, we could identify \( p \in \mathcal{M} \) with \( q \in \mathcal{M}_0 \) \( (q \neq p) \) instead of \( p \in \mathcal{M}_0 \). In the above understanding of the concept of “gauge” in general relativistic perturbation theory, a gauge transformation is simply a change of the map \( \mathcal{X} \).

These are the basic ideas necessary to understand \textit{gauge degree of freedom} in the general relativistic perturbation theory proposed by Stewart and Walker.\textsuperscript{22} This understanding has been developed by Bruni et al.,\textsuperscript{9} and by the present author.\textsuperscript{16,17} We briefly review this development.

To formulate the above understanding in more detail, we introduce an infinitesimal parameter \( \lambda \) for the perturbation. Further, we consider the 4 + 1-dimensional manifold \( \mathcal{N} = \mathcal{M} \times \mathbb{R} \), where \( 4 = \dim \mathcal{M} \) and \( \lambda \in \mathbb{R} \). The background spacetime \( \mathcal{M}_0 = \mathcal{N}|_{\lambda=0} \) and the physical spacetime \( \mathcal{M} = \mathcal{M}_\lambda = \mathcal{N}|_{\mathbb{R}=\lambda} \) are also submanifolds embedded in the extended manifold \( \mathcal{N} \). Each point on \( \mathcal{N} \) is identified by a pair, \((p, \lambda)\), where \( p \in \mathcal{M}_\lambda \), and each point in the background spacetime \( \mathcal{M}_0 \) in \( \mathcal{N} \) is identified by \( \lambda = 0 \).

Through this construction, the manifold \( \mathcal{N} \) is foliated by four-dimensional submanifolds \( \mathcal{M}_\lambda \) of each \( \lambda \), and these are diffeomorphic to the physical spacetime \( \mathcal{M} \) and the background spacetime \( \mathcal{M}_0 \). The manifold \( \mathcal{N} \) has a natural differentiable structure consisting of the direct product of \( \mathcal{M} \) and \( \mathbb{R} \). Further, the perturbed spacetimes \( \mathcal{M}_\lambda \) for each \( \lambda \) must have the same differential structure with this construction. In other words, we require that perturbations be continuous in the sense that \((\mathcal{M}, \bar{g}_{ab})\) and \((\mathcal{M}_0, g_{ab})\) are connected by a continuous curve within the extended manifold \( \mathcal{N} \). Hence, the changes of the differential structure resulting from the perturbation, for example the formation of singularities and singular perturbations in the sense of fluid mechanics, are excluded from consideration in this paper.

Let us consider the set of field equations

\[ \mathcal{E}[Q_\lambda] = 0 \tag{2.5} \]

on the physical spacetime \( \mathcal{M}_\lambda \) for the physical variables \( Q_\lambda \) on \( \mathcal{M}_\lambda \). The field equation \( \tag{2.5} \) formally represents the Einstein equation for the metric on \( \mathcal{M}_\lambda \) and the equations for matter fields on \( \mathcal{M}_\lambda \). If a tensor field \( Q_\lambda \) is given on each \( \mathcal{M}_\lambda \), \( Q_\lambda \) is automatically extended to a tensor field on \( \mathcal{N} \) by \( Q(p, \lambda) := Q_\lambda(p) \), where \( p \in \mathcal{M}_\lambda \). In this extension, the field equation \( \tag{2.5} \) is regarded as an equation on the extended manifold \( \mathcal{N} \). Thus, we have extended an arbitrary tensor field and the field equations \( \tag{2.5} \) on each \( \mathcal{M}_\lambda \) to those on the extended manifold \( \mathcal{N} \).

Tensor fields on \( \mathcal{N} \) obtained through the above construction are necessarily “tangent” to each \( \mathcal{M}_\lambda \), i.e., their normal component to each \( \mathcal{M}_\lambda \) identically vanishes. To
consider the basis of the tangent space of $\mathcal{N}$, we introduce the normal form and its
dual, which are normal to each $\mathcal{M}_\lambda$ in $\mathcal{N}$. These are denoted by $(d\lambda)_a$ and $(\partial/\partial \lambda)^a$,
respectively, and they satisfy

$$(d\lambda)_a \left( \frac{\partial}{\partial \lambda} \right)^a = 1. \quad (2.6)$$

The form $(d\lambda)_a$ and its dual, $(\partial/\partial \lambda)^a$, are normal to any tensor field extended from
the tangent space on each $\mathcal{M}_\lambda$ through the above construction. The set consisting
of $(d\lambda)_a$, $(\partial/\partial \lambda)^a$ and the basis of the tangent space on each $\mathcal{M}_\lambda$ is regarded as the
basis of the tangent space of $\mathcal{N}$.

To define the perturbation of an arbitrary tensor field $Q$, we compare $Q$ on the
physical spacetime $\mathcal{M}_\lambda$ with $Q_0$ on the background spacetime, and it is necessary
to identify the points of $\mathcal{M}_\lambda$ with those of $\mathcal{M}_0$. This point identification map is the
so-called gauge choice in the context of perturbation theories, as mentioned above.
The gauge choice is made by assigning a diffeomorphism $\mathcal{X}_\lambda : \mathcal{N} \rightarrow \mathcal{N}$ such that $\mathcal{X}_\lambda : \mathcal{M}_0 \rightarrow \mathcal{M}_\lambda$. Following the paper of Bruni et al.,
we introduce a gauge choice $\mathcal{X}_\lambda$ as one of the one-parameter groups of diffeomorphisms, i.e., an exponential map, for
simplicity. We denote the generator of this exponential map by $\mathcal{X}_\eta^a$. This generator
$\mathcal{X}_\eta^a$ is decomposed by the basis on the tangent space of $\mathcal{N}$ which are constructed
above. Though the generator $\mathcal{X}_\eta^a$ should satisfy some appropriate conditions, the
arbitrariness of the gauge choice $\mathcal{X}_\lambda$ is represented by the tangential component of
the generator $\mathcal{X}_\eta^a$ to the tangent space of $\mathcal{M}_\lambda$.

The pull-back $\mathcal{X}_\lambda^* Q$, which is induced by the exponential map $\mathcal{X}_\lambda$, maps a tensor
field $Q$ on the physical manifold $\mathcal{M}_\lambda$ to a tensor field $\mathcal{X}_\lambda^* Q$ on the background
spacetime. In terms of this generator $\mathcal{X}_\eta^a$, the pull-back $\mathcal{X}_\lambda^* Q$ is represented by the
Taylor expansion

$$Q(r) = Q(\mathcal{X}_\lambda(p)) = \mathcal{X}_\lambda^* Q(p) = Q(p) + \lambda \mathcal{L}_{\mathcal{X}_\eta} Q|_p + \frac{1}{2} \lambda^2 \mathcal{L}_{\mathcal{X}_\eta}^2 Q|_p + O(\lambda^3), \quad (2.7)$$

where $r = \mathcal{X}_\lambda(p) \in \mathcal{M}_\lambda$. Because $p \in \mathcal{M}_0$, we may regard the equation

$$\mathcal{X}_\lambda^* Q(p) = Q_0(p) + \lambda \mathcal{L}_{\mathcal{X}_\eta} Q|_{\mathcal{M}_0} (p) + \frac{1}{2} \lambda^2 \mathcal{L}_{\mathcal{X}_\eta}^2 Q|_{\mathcal{M}_0} (p) + O(\lambda^3) \quad (2.8)$$

as an equation on the background spacetime $\mathcal{M}_0$, where $Q_0 = Q|_{\mathcal{M}_0}$ is the back-
ground value of the physical variable of $Q$. Once the definition of the pull-back of
the gauge choice $\mathcal{X}_\lambda$ is given, the perturbation $\Delta^\mathcal{X} Q_\lambda$ of a tensor field $Q$ under
the gauge choice $\mathcal{X}_\lambda$ is simply defined as

$$\Delta^\mathcal{X} Q_\lambda := \mathcal{X}_\lambda^* Q|_{\mathcal{M}_0} - Q_0. \quad (2.9)$$

We note that all variables in this definition are defined on $\mathcal{M}_0$. Expanding the first
term on the right-hand side of (2.9) as

$$\mathcal{X}_\lambda^* Q|_{\mathcal{M}_0} = Q_0 + \lambda \mathcal{X}^{(1)} Q + \frac{1}{2} \lambda^2 \mathcal{X}^{(2)} Q + O(\lambda^3), \quad (2.10)$$
we define the first- and the second-order perturbations of a physical variable $Q_\lambda$ under the gauge choice $\mathcal{X}_\lambda$ by

$$\langle^{(1)} Q \rangle = \left. \mathcal{L}_\lambda \eta^\lambda \right|_{\mathcal{M}_0}, \quad \langle^{(2)} Q \rangle = \left. \mathcal{L}_{\lambda}^2 \eta^\lambda \right|_{\mathcal{M}_0}. \quad (2.11)$$

Now, we consider two **different gauge choices** based on the above understanding of the gauge choice in the perturbation theory. Suppose that $\mathcal{X}_\lambda$ and $\mathcal{Y}_\lambda$ are two exponential maps with the generators $\mathcal{X}^a$ and $\mathcal{Y}^a$ on $\mathcal{N}$, respectively. In other words, $\mathcal{X}_\lambda$ and $\mathcal{Y}_\lambda$ are two gauge choices. Then, the integral curves of each $\mathcal{X}^a$ and $\mathcal{Y}^a$ in $\mathcal{N}$ are the orbits of the actions of the gauge choices $\mathcal{X}_\lambda$ and $\mathcal{Y}_\lambda$, respectively. Since we choose the generators $\mathcal{X}^a$ and $\mathcal{Y}^a$ so that these are transverse to each $\mathcal{M}_\lambda$ everywhere on $\mathcal{N}$, the integral curves of these vector fields intersect with each $\mathcal{M}_\lambda$. Therefore, points lying on the same integral curve of either of the two are to be regarded as *the same point* within the respective gauges. When these curves are not identical, i.e., the tangential components to each $\mathcal{M}_\lambda$ of $\mathcal{X}^a$ and $\mathcal{Y}^a$ are different, these point identification maps $\mathcal{X}_\lambda$ and $\mathcal{Y}_\lambda$ are regarded as **two different gauge choices**.

We next introduce the concept of *gauge invariance*. Following the paper by Bruni et al.\(^{20}\) we consider the concept of **gauge invariance up to order $n$**. Suppose that $\mathcal{X}_\lambda$ and $\mathcal{Y}_\lambda$ are two different gauge choices which are generated by the vector fields $\mathcal{X}^a$ and $\mathcal{Y}^a$, respectively. These gauge choices also pull back a generic tensor field $Q$ on $\mathcal{N}$ to two other tensor fields, $\mathcal{X}^\lambda Q$ and $\mathcal{Y}^\lambda Q$, for any given value of $\lambda$. In particular, on $\mathcal{M}_0$, we now have three tensor fields associated with a tensor field $Q$: one is the background value $Q_0$ of $Q$, and the other two are the pulled-back variables of $Q$ from $\mathcal{M}_\lambda$ to $\mathcal{M}_0$ by the two different gauge choices,

$$\langle^{(1)} Q \rangle := \left. \mathcal{X}^\lambda Q \right|_{\mathcal{M}_0} = Q_0 + \lambda^{(1)} Q + \frac{1}{2} \lambda^{(2)} Q + O(\lambda^3) = Q_0 + \Delta^{(1)} Q_\lambda, \quad (2.12)$$

$$\langle^{(2)} Q \rangle := \left. \mathcal{Y}^\lambda Q \right|_{\mathcal{M}_0} = Q_0 + \lambda^{(1)} Q + \frac{1}{2} \lambda^{(2)} Q + O(\lambda^3) = Q_0 + \Delta^{(2)} Q_\lambda. \quad (2.13)$$

Here, we have used Eqs. (2.9) and (2.10). Because $\mathcal{X}_\lambda$ and $\mathcal{Y}_\lambda$ are gauge choices that map the background spacetime $\mathcal{M}_0$ into the physical spacetime $\mathcal{M}_\lambda$, $\mathcal{X}^\lambda Q$ and $\mathcal{Y}^\lambda Q$ are the representations on $\mathcal{M}_0$ of the perturbed tensor field $Q$ in the two different gauges. The quantities $\langle^{(k)} Q \rangle$ and $\langle^{(k)} Q \rangle$ in Eqs. (2.12) and (2.13) are the perturbations of $O(k)$ in the gauges $\mathcal{X}_\lambda$ and $\mathcal{Y}_\lambda$, respectively. We say that $Q$ is **gauge invariant up to order $n$** iff for any two gauge choices $\mathcal{X}_\lambda$ and $\mathcal{Y}_\lambda$ the following holds:

$$\langle^{(k)} Q \rangle = \langle^{(k)} Q \rangle \quad \forall k, \quad \text{with} \quad k < n. \quad (2.14)$$

From this definition, we can prove that the $n$th-order perturbation of a tensor field $Q$ is gauge invariant up to order $n$ iff in a given gauge $\mathcal{X}_\lambda$ we have $\mathcal{L}_\xi Q = 0$ for any vector field $\xi^a$ defined on $\mathcal{M}_0$ and for any $k < n$. As a consequence, the $n$th-order perturbation of a tensor field $Q$ is gauge invariant up to order $n$ iff $Q_0$ and all its perturbations of lower than $n$th order are, in any gauge, either vanishing or constant scalars, or a combination of Kronecker deltas with constant coefficients.\(^9\),\(^{20}\),\(^{22}\)
Further, even if its lower order perturbations are not trivial, we can decompose any perturbation of $Q$ into the gauge invariant and gauge variant parts, as shown in KN2003. This will be also explained in the next subsection.

Now, we consider the gauge transformation rules between different gauge choices. In general, the representation $XQ_{\lambda}$ on $M_0$ of the perturbed variable $Q$ on $M_{\lambda}$ depends on the gauge choice $X_{\lambda}$. If we employ a different gauge choice, the representation of $Q_{\lambda}$ on $M_0$ may change. Suppose that $X_{\lambda}$ and $Y_{\lambda}$ are different gauge choices, which are the point identification maps from $M_0$ to $M_{\lambda}$, and the generators of these gauge choices are given by $X_{\eta}^a$ and $Y_{\eta}^a$, respectively. Then, the change of the gauge choice from $X_{\lambda}$ to $Y_{\lambda}$ is represented by the diffeomorphism

$$
\Phi_{\lambda} := (X_{\lambda})^{-1} \circ Y_{\lambda}. \tag{2.15}
$$

This diffeomorphism $\Phi_{\lambda}$ is the map $\Phi_{\lambda} : M_0 \rightarrow M_0$ for each value of $\lambda \in \mathbb{R}$. The diffeomorphism $\Phi_{\lambda}$ does change the point identification, as expected from the understanding of the gauge choice discussed above. Therefore, the diffeomorphism $\Phi_{\lambda}$ is regarded as the gauge transformation $\Phi_{\lambda} : X_{\lambda} \rightarrow Y_{\lambda}$.

The gauge transformation $\Phi_{\lambda}$ induces a pull-back from the representation $XQ_{\lambda}$ of the perturbed tensor field $Q$ in the gauge choice $X_{\lambda}$ to the representation $XQ_{\lambda}$ in the gauge choice $Y_{\lambda}$. Actually, the tensor fields $XQ_{\lambda}$ and $YQ_{\lambda}$, which are defined on $M_0$, are connected by the linear map $\Phi_{\lambda}$ as

$$
YQ_{\lambda} = XQ_{\lambda}|_{M_0} = ((X_{\lambda})^* (X_{\lambda}^{-1})*Q)|_{M_0} = (X_{\lambda}^{-1}Y_{\lambda})* (X_{\lambda}^* Q)|_{M_0} = \Phi_{\lambda}^* XQ_{\lambda}. \tag{2.16}
$$

According to generic arguments concerning the Taylor expansion of the pull-back of a tensor field on the same manifold, given in §2.1, it should be possible to express the gauge transformation $\Phi_{\lambda}^* XQ_{\lambda}$ in the form

$$
\Phi_{\lambda}^* XQ = XQ + \lambda L_{\xi_1} XQ + \frac{\lambda^2}{2} \{ L_{\xi_2} + L_{\xi_1}^2 \} XQ + O(\lambda^3), \tag{2.17}
$$

where the vector fields $\xi_1^a$ and $\xi_2^a$ are the generators of the gauge transformation $\Phi_{\lambda}$.

Comparing the representation (2.17) of the Taylor expansion in terms of the generators $\xi_1^a$ and $\xi_2^a$ of the pull-back $\Phi_{\lambda}^* XQ$ and that in terms of the generators $X_{\eta}^a$ and $Y_{\eta}^a$ of the pull-back $Y_{\lambda}^* (X_{\lambda}^{-1})^* XQ = \Phi_{\lambda}^* XQ$, we readily obtain explicit expressions for the generators $\xi_1^a$ and $\xi_2^a$ of the gauge transformation $\Phi = X_{\lambda}^{-1} \circ Y_{\lambda}$ in terms of the generators $X_{\eta}^a$ and $Y_{\eta}^a$ of the gauge choices as follows:

$$
\xi_1^a = Y_{\eta}^a - X_{\eta}^a, \quad \xi_2^a = [Y_{\eta}, X_{\eta}]^a. \tag{2.18}
$$

Further, because the gauge transformation $\Phi_{\lambda}$ is a map within the background space-time $M_0$, the generator should consist of vector fields on $M_0$. This can be satisfied by imposing some appropriate conditions on the generators $Y_{\eta}^a$ and $X_{\eta}^a$.

We can now derive the relation between the perturbations in the two different gauges. Up to second order, these relations are derived by substituting (2.12) and
into (2.17):

\begin{align}
\gamma Q - \chi Q &= \mathcal{L}_{\xi(1)} Q_0, \\
\hat{\gamma} Q - \hat{\chi} Q &= 2\mathcal{L}_{\xi(1)} \chi Q + \left\{ \mathcal{L}_{\xi(2)} + \mathcal{L}_{\xi(1)}^2 \chi \right\} Q_0.
\end{align}

These results are, of course, consistent with the concept of gauge invariance up to order $n$, as introduced above. Further, inspecting these gauge transformation rules, we can define the gauge invariant variables at each order, as shown below.

Here, we should comment on the gauge choice in the above explanation. We have introduced an exponential map $\mathcal{X}_{\lambda}$ (or $\mathcal{Y}_{\lambda}$) as the gauge choice, for simplicity. However, this simplified introduction of the gauge choice $\mathcal{X}_{\lambda}$ as an exponential map is not essential to the gauge transformation rules (2.19) and (2.20). Indeed, we can generalize the diffeomorphism $\mathcal{X}_{\lambda}$ from an exponential map. If we generalize the diffeomorphism $\mathcal{X}_{\lambda}$, the representation (2.8) of the pulled-back variable $\mathcal{X}_{\lambda} Q(p)$, the representations of the perturbations (2.11), and the relations (2.18) between generators of $\Phi_{\lambda}$, $\mathcal{X}_{\lambda}$, and $\mathcal{Y}_{\lambda}$ will be changed. However, the gauge transformation rules (2.19) and (2.20) are direct consequences of the Taylor expansion (2.17) of $\Phi_{\lambda}$. As commented in §2.1, the representation of the Taylor expansion (2.17) of $\Phi_{\lambda}$ is quite general. Therefore, the gauge transformation rules (2.19) and (2.20) do not change, even if we generalize the choice of $\mathcal{X}_{\lambda}$. Further, the relations (2.18) between generators also imply that, even if we consider a simple exponential map as the gauge choice, both of the generators $\xi_1^a$ and $\xi_2^a$ are naturally induced by the generators of the original gauge choices. Hence, we conclude that the gauge transformation rules (2.19) and (2.20) are quite general and irreducible. In this paper, we develop a second-order gauge invariant cosmological perturbation theory based on the above understanding of the gauge degree of freedom only through the gauge transformation rules (2.19) and (2.20). Hence, the development of the cosmological perturbation theory presented below is not changed if we generalize the gauge choice $\mathcal{X}_{\lambda}$ from a simple exponential map.

2.3. **Gauge invariant variables**

Inspecting the gauge transformation rules (2.19) and (2.20), we can define the gauge invariant variables for a metric perturbation and for arbitrary matter fields. Employing the idea of gauge invariance up to order $n$ for $n$th-order perturbations, we proposed a procedure to construct gauge invariant variables of higher-order perturbations. This proposal is as follows. First, we decompose a linear-order metric perturbation into its gauge invariant and variant parts. The procedure for decomposing linear-order metric perturbations is extended to second-order metric perturbations, and we can decompose the second-order metric perturbations through a procedure similar to that for the linear-order metric perturbation. Then, we define the gauge invariant variables for the first- and second-order perturbations of an arbitrary field other than the metric by using the gauge variant parts of the first- and second-order metric perturbations. Though the procedure for finding gauge invariant variables for linear-order metric perturbations is highly non-trivial, once we know this procedure, we can easily find the gauge invariant part of a higher-
order perturbation through a simple extension of the procedure for the linear-order perturbations.

To consider a metric perturbation, we expand the metric on the physical spacetime \( \mathcal{M} \), which is pulled back to the background spacetime \( \mathcal{M}_0 \) using a gauge choice in the form given in (2.10),

\[
X^*_\lambda g_{ab} = g_{ab} + \lambda \mathcal{A}_{ab} + \frac{\lambda^2}{2} \mathcal{A}_a^b + O(\lambda^3),
\]

(2.21)

where \( g_{ab} \) is the metric on the background spacetime \( \mathcal{M}_0 \). Of course, the expansion (2.21) of the metric depends entirely on the gauge choice \( X^*_\lambda \). Nevertheless, henceforth, we do not explicitly express the index of the gauge choice \( X^*_\lambda \) in an expression if there is no possibility of confusion.

Our starting point to construct gauge invariant variables is the assumption that we already know the procedure for finding gauge invariant variables for the linear metric perturbations. Then, a linear metric perturbation \( h_{ab} \) is decomposed as

\[
h_{ab} =: \mathcal{H}_{ab} + \mathcal{L}_X g_{ab},
\]

(2.22)

where \( \mathcal{H}_{ab} \) and \( X^a \) are the gauge invariant and variant parts of the linear-order metric perturbations,\(^{16}\) i.e., under the gauge transformation (2.19), these are transformed as

\[
y \mathcal{H}_{ab} - \mathcal{X} \mathcal{H}_{ab} = 0, \quad y X^a - \mathcal{X} X^a = \xi^a_{(1)}.
\]

(2.23)

As emphasized in KN2003 and KN2005, the above assumption is quite strong and it is not simple to carry out the systematic decomposition (2.22) on an arbitrary background spacetime, since this procedure depends completely on the background spacetime \( (\mathcal{M}_0, g_{ab}) \). However, we show that this procedure exists in the case of cosmological perturbations of a homogeneous and isotropic universe in §4.1.

Once we accept this assumption for linear-order metric perturbations, we can always find gauge invariant variables for higher-order perturbations.\(^{16}\) As shown in KN2003, at second order, the metric perturbations are decomposed as

\[
l_{ab} =: \mathcal{L}_{ab} + 2 \mathcal{L}_X h_{ab} + \{ \mathcal{L}_Y - \mathcal{L}_X^2 \} g_{ab},
\]

(2.24)

where \( \mathcal{L}_{ab} \) and \( Y^a \) are the gauge invariant and variant parts of the second order metric perturbations, i.e.,

\[
y \mathcal{L}_{ab} - \mathcal{X} \mathcal{L}_{ab} = 0, \quad y Y^a - \mathcal{X} Y^a = \xi^a_{(2)} + [\xi^a_{(1)}, X]^a.
\]

(2.25)

The details of the derivation of this gauge invariant part of the second-order metric perturbation are explained in the context of cosmological perturbations in §4.2.

Furthermore, as shown in KN2003, using the first- and second-order gauge variant parts, \( X^a \) and \( Y^a \), of the metric perturbations, the gauge invariant variables for an arbitrary field \( Q \) other than the metric are given by

\[
^{(1)}Q := (^{(1)}Q - \mathcal{L}_X Q_0),
\]

(2.26)

\[
^{(2)}Q := (^{(2)}Q - 2 \mathcal{L}_X (^{(1)}Q - \{ \mathcal{L}_Y - \mathcal{L}_X^2 \} Q_0).
\]

(2.27)
It is straightforward to confirm that the variables \((p)Q\) defined by (2.26) and (2.27) are gauge invariant under the gauge transformation rules (2.19) and (2.20), respectively. Equations (2.26) and (2.27) have several more important implications. To see this, we represent these equations as

\[
(1) Q = (1) Q + L_X Q_0, \quad (2.28)
\]

\[
(2) Q = (2) Q + 2L_X (1) Q + \left\{ L_Y - L_X^2 \right\} Q_0. \quad (2.29)
\]

These equations imply that any perturbation of first and second order can always be decomposed into gauge invariant and gauge variant parts as Eqs. (2.28) and (2.29), respectively. In §4.3, we see that these formulae for the decomposition into gauge invariant and variant parts of each order perturbation are very important.

### 2.4. Perturbations of the Einstein tensor and the Einstein equations

Now, we review the formulae for the perturbative Einstein tensor at each order that are presented in KN2005. The relation between the curvatures associated with the metrics on the physical spacetime \(\mathcal{M}_\lambda\) and the background spacetime \(\mathcal{M}_0\) is given by the relation between the pulled-back operator \(X_\lambda^* \nabla_a (X_\lambda^{-1})^*\) of the covariant derivative \(\nabla_a\) associated with the metric \(g_{ab}\) on \(\mathcal{M}_0\) and the covariant derivative \(\nabla_a\) associated with the metric \(g_{ab}\) on \(\mathcal{M}_0\). The pulled-back covariant derivative \(X_\lambda^* \nabla_a (X_\lambda^{-1})^*\) depends on the gauge choice \(X_\lambda\). The property of the derivative operator \(X_\lambda^* \nabla_a (X_\lambda^{-1})^*\) as the covariant derivative on the physical spacetime \(\mathcal{M}_\lambda\) is given by

\[
X_\lambda^* \nabla_a \left( (X_\lambda^{-1})^* X_\lambda^* g_{ab} \right) = 0, \quad (2.30)
\]

where \(X_\lambda^* g_{ab}\) is the pull-back of the metric on the physical spacetime \(\mathcal{M}_\lambda\), which is expanded as Eq. (2.21). In spite of the gauge dependence of the operator \(X_\lambda^* \nabla_a (X_\lambda^{-1})^*\), we simply denote this operator by \(\nabla_a\), because our calculations are carried out only on the background spacetime \(\mathcal{M}_0\) in the same gauge choice \(X_\lambda\). Further, we denote the pulled-back metric \(X_\lambda^* g_{ab}\) of the physical spacetime \(\mathcal{M}_\lambda\) by \(\bar{g}_{ab}\), as mentioned above. Though we have to keep in our mind that we are treating perturbations in the single gauge choice when we treat the derivative operator \(\nabla_a\) and the pulled-back physical metric \(\bar{g}_{ab}\) on the background spacetime \(\mathcal{M}_0\), there is no confusion in the development of the perturbation theory if we treat perturbations only in the single gauge choice \(X_\lambda\).

Since the derivative operator \(\nabla_a \left( = X_\lambda^* \nabla_a (X_\lambda^{-1})^* \right)\) may be regarded as a derivative operator on the background spacetime that satisfies the property (2.30), there exists a tensor field \(C^c_{ab}\) on the background spacetime \(\mathcal{M}_0\) such that

\[
\nabla_a \omega_b = \nabla_a \omega_b - C^c_{ab} \omega_c, \quad (2.31)
\]

where \(\omega_c\) is an arbitrary one-form on the background spacetime \(\mathcal{M}_0\). From the property (2.30) of the covariant derivative operator \(\nabla_a\) on \(\mathcal{M}\), the tensor field \(C^c_{ab}\) on \(\mathcal{M}_0\) is given by

\[
C^c_{ab} = \frac{1}{2} \bar{g}^{cd} \left( \nabla_a \bar{g}_{db} + \nabla_b \bar{g}_{da} - \nabla_d \bar{g}_{ab} \right), \quad (2.32)
\]
where $\bar{g}^{ab}$ is the inverse metric of $\bar{g}_{ab}$, i.e., $\bar{g}_{ac}\bar{g}^{cb} = \delta^b_a$. We note that the gauge dependence of the derivative $\nabla_a$ as a derivative operator on $\mathcal{M}_0$ is included only in this tensor field $C^{c}_{ab}$. The Riemann curvature $\bar{R}_{abc}^d$ on the physical spacetime $\mathcal{M}_\lambda$, which is pulled back to the background spacetime $\mathcal{M}_0$, is given by the Riemann curvature $R_{abc}^d$ on the background spacetime $\mathcal{M}_0$ and the tensor field $C^{c}_{ab}$ as follows:

$$
\bar{R}_{abc}^d = R_{abc}^d - 2\nabla_{[a}C^d_{b]c} + 2C^{e}_{[a}C^d_{b]e}.
$$

(2.33)

The perturbative expression for the curvatures are obtained from the perturbative expansion of Eq. (2.33) through the perturbative expansion of the tensor $C^{c}_{ab}$ defined by Eq. (2.32).

The first- and the second-order perturbations of the Riemann, the Ricci, the Weyl curvatures, and the Einstein tensors on the general background spacetime are summarized in KN2005. We also derived the perturbative form of the divergence of an arbitrary tensor field of second rank to check the perturbative Bianchi identities in KN2005. In this paper, we only present the perturbative expression for the Einstein tensor.

We expand the Einstein tensor $\bar{G}^b_a$ on the physical spacetime $\mathcal{M}_\lambda$ as

$$
\bar{G}^b_a = G^b_a + \lambda^{(1)}G^b_a + \frac{1}{2}\lambda^2(2)G^b_a + O(\lambda^3).
$$

(2.34)

As shown in KN2005, each order perturbation of the Einstein tensor is given by

$$(1)\bar{G}^b_a = (1)G^b_a [\mathcal{H}] + \mathcal{L}_XG^b_a,$$

(2.35)

$$(2)\bar{G}^b_a = (1)G^b_a [\mathcal{L}] + (2)G^b_a [\mathcal{H}, \mathcal{H}] + 2\mathcal{L}_X(1)\bar{G}^b_a + \{\mathcal{L}_Y - \mathcal{L}_X^2\}G^b_a,$$

(2.36)

where

$$(1)G^b_a [A] := (1)\Sigma^b_a [A] - \frac{1}{2}\delta^b_a(1)\Sigma^c_c [A],$$

(2.37)

$$(2)G^b_a [A, B] := (2)\Sigma^b_a [A, B] - \frac{1}{2}\delta^b_a(2)\Sigma^c_c [A, B],$$

(2.38)

$$(1)\Sigma^b_a [A] := -2\nabla_{[a}H^b_{d]} [A] - A^{cb}R_{ac},$$

(2.39)

$$\Sigma^b_a [A, B] := 2R_{ad}B^b_c (A^{d)c} + 2H^a_{[a}H^b_{e]} [B]a + 2H^a_{a}H^b_{e} [B]H^d_{e} [A]$$

$$+ 2A^d_e \nabla_{[a}H^b_{d]} [B] + 2B^c_e \nabla_{[a}H^d_{e]} [B] + 2B^b_e \nabla_{[a}H^d_{e]} [A],$$

(2.40)

and

$$H^c_{ab} [A] := \nabla_{(a}A^c_{b)} - \frac{1}{2}\nabla^cA_{ab},$$

(2.41)

$$H^b_{abc} [A] := g_{cd}H^d_{ab} [A], \quad H^c_{a} [A] := g^{bd}H^c_{ad} [A],$$

(2.42)

$$H^b_{a c} [A] := g_{cd}H^b_{a d} [A].$$

We note that $(1)G^b_a [\star]$ and $(2)G^b_a [\star, \star]$ in Eqs. (2.35) and (2.36) are the gauge invariant parts of the perturbative Einstein tensors, and Eqs. (2.35) and (2.36) have the same forms as Eqs. (2.26) and (2.29), respectively.
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We also note that \((1)G_a^b [s]\) and \((2)G_a^b [*, s]\) defined by Eqs. (2.37)–(2.40) satisfy the identities

\[
\nabla_a (1)G_b^a [A] = -H_{ca}^a [A] G_b^c + H_{ba}^c [A] G_c^a, \quad (2.43)
\]

\[
\nabla_a (2)G_b^a [A, B] = -H_{ca}^a [A] (1)G_b^c [B] - H_{ba}^e [B] (1)G_e^a [A]
+ H_{ba}^e [A] (1)G_c^a [B] + H_{ca}^e [A] (1)G_e^a [B]
- \left( H_{bad} [B] A^{dc} + H_{bad} [A] B^{dc} \right) G_c^a
+ \left( H_{cad} [B] A^{ad} + H_{cad} [A] B^{ad} \right) G_e^c, \quad (2.44)
\]

for arbitrary tensor fields \(A_{ab}\) and \(B_{ab}\), respectively. We can directly confirm these identities, and these identities guarantee the first-order and second-order perturbations of the Bianchi identity \(\bar{\nabla}_b \bar{G}_b^a = 0\), respectively, as shown in KN2005. These identities are also useful when we check whether the derived components of \((1)G_a^b [s]\) and \((2)G_a^b [*, s]\) are correct.

Finally, we consider perturbations of the Einstein equation of first and second order. First, we expand the energy-momentum tensor as

\[
\bar{T}_{ab} = T_{ab} + \lambda (1)T_{ab} + \frac{1}{2} \lambda^2 (2)T_{ab} + O(\lambda^3). \quad (2.45)
\]

Following the definitions (2.26) and (2.27) of gauge invariant variables, the gauge invariant variables \((1)T_a^b\) and \((2)T_a^b\) for the perturbations \((1)\bar{T}_{ab}\) and \((2)\bar{T}_{ab}\) of the energy-momentum tensor are defined by

\[
(1)T_a^b := (1)\bar{T}_a^b - \mathcal{L}X T_a^b, \quad (2.46)
\]

\[
(2)T_a^b := (2)\bar{T}_a^b - 2 \mathcal{L}_X (1)\bar{T}_a^b - \{ \mathcal{L}_Y - \mathcal{L}_X^2 \} T_a^b. \quad (2.47)
\]

In §4.3, we show that these definitions of the gauge invariant part of the first- and second-order perturbation of the energy-momentum tensor are appropriate in the cases of both a perfect fluid and a single scalar field. Further, we impose the perturbed Einstein equation of each order,

\[
(1)G_a^b = 8\pi G (1)T_a^b, \quad (2)G_a^b = 8\pi G (2)T_a^b. \quad (2.48)
\]

Then, the perturbative Einstein equation is given by

\[
(1)G_a^b [\mathcal{H}] = 8\pi G (1)T_a^b \quad (2.49)
\]

at linear order and

\[
(1)G_a^b [\mathcal{L}] + (2)G_a^b [\mathcal{H}, \mathcal{H}] = 8\pi G (2)T_a^b \quad (2.50)
\]

at second order. These explicitly show that, order by order, the Einstein equations are necessarily given in terms of gauge invariant variables only. Therefore, we do not have to consider the gauge degree of freedom, at least at the level where we concentrate only on the perturbed Einstein equations.

We have reviewed the general outline of the second-order gauge invariant perturbation theory. Within this general framework, we develop a second-order cosmological perturbation theory in terms of the gauge invariant variables.
§3. Cosmological background spacetime

Here, we consider the background spacetime for cosmological perturbation theory. The background spacetime considered here is a homogeneous, isotropic universe that is foliated by the three-dimensional hypersurface \( \Sigma(\eta) \), which is parameterised by \( \eta \). Each hypersurface of \( \Sigma(\eta) \) is a maximally symmetric three-space, and the spacetime metric of this universe is given by

\[
g_{ab} = a^2(\eta) \left( -(d\eta)_a (d\eta)_b + \gamma_{ij}(dx^i)_a (dx^j)_b \right),
\]

where \( a = a(\eta) \) is the scale factor, \( \gamma_{ij} \) is the metric on the maximally symmetric 3-space with curvature constant \( K \), and the indices \( i, j, k, \ldots \) for the spatial components run from 1 to 3. Depending on the behavior of the scale factor \( a \), this metric (3.1) can represent a Friedmann-Robertson-Walker universe or a de Sitter spacetime. In terms of the coordinate system in which the spacetime metric is given by (3.1), the components of the Christoffel symbol of this background spacetime are given by

\[
\Gamma^\eta_{\eta\eta} = H, \quad \Gamma^\eta_{ij} = \Gamma^j_{\eta i} = 0, \quad \Gamma^\eta_{ij} = \mathcal{H}\gamma_{ij}, \quad \Gamma^k_{ij} = (3)\Gamma^k_{ij},
\]

where \( \mathcal{H} = \partial_\eta a/a \), \( (3)\Gamma^k_{ij} \) is the Christoffel symbol associated with the three metric \( \gamma_{ij} \), and \( \gamma^i_j = \gamma^j_k \gamma^k_i \) is the three-dimensional Kronecker delta. These components of the Christoffel symbol are useful when we write down the components of the four-dimensional covariant derivative of tensors in terms of the derivative with respect to \( \eta \) and the three-dimensional covariant derivative associated with the metric \( \gamma_{ij} \).

Since \( \gamma_{ij} \) is the metric on the maximally symmetric 3-space with the curvature constant \( K \), the curvatures associated with the metric \( \gamma_{ij} \) are given by

\[
(3)R_{ijkl} = 2K \gamma_{k[i} \gamma_{j]l}, \quad (3)R_{ij} = 2K \gamma_{ij}, \quad (3)R = 6K.
\]

These are useful when we calculate the components of the perturbative curvatures in terms of the 3+1 decomposition, as in the metric (3.1). The four-dimensional background curvature tensors are also necessary to calculate the components of the perturbative curvatures. These are given by

\[
R_{ab} = -3\partial_\eta \mathcal{H} (d\eta)_a (d\eta)_b + \left( \partial_\eta \mathcal{H} + 2\mathcal{H}^2 + 2K \right) \gamma_{ab}, \quad (3.4)
\]

\[
R^b_a = \frac{3}{a^2} \partial_\eta \mathcal{H} (d\eta)_a \left( \frac{\partial}{\partial \eta} \right)^b + \frac{1}{a^2} \left( \partial_\eta \mathcal{H} + 2\mathcal{H}^2 + 2K \right) \gamma^b_a, \quad (3.5)
\]

\[
R := R^a_a = \frac{6}{a^2} \{ \partial_\eta \mathcal{H} + \mathcal{H}^2 + K \}, \quad (3.6)
\]

\[
G^b_a := R^b_a - \frac{1}{2} \delta^b_a R
\]

\[
= -\frac{3}{a^2} \left[ \mathcal{H}^2 + K \right] (d\eta)_a \left( \frac{\partial}{\partial \eta} \right)^b - \frac{1}{a^2} \left[ 2\partial_\eta \mathcal{H} + \mathcal{H}^2 + K \right] \gamma^b_a, \quad (3.7)
\]

where \( \gamma_{ab} = \gamma_{ij}(dx^i)_a (dx^j)_b \) and \( \gamma^b_a = \gamma^i_j (dx^i)_a (\partial/\partial x^j)^b \).
To study the Einstein equation for this background spacetime, we introduce the energy-momentum tensor for a perfect fluid,
\[ T_{ab}^b = \epsilon u_a u^b + p(\delta_a^b + u_a u^b) \]  (3.8)
\[ = -\epsilon (d\eta)_a \left( \frac{\partial}{\partial \eta} \right)^b + p\gamma_a^b, \]  (3.9)
where we have used
\[ u_a = -a(d\eta)_a, \quad u^a = \frac{1}{a} \left( \frac{\partial}{\partial \eta} \right)^a, \quad \delta_a^b = (d\eta)_a \left( \frac{\partial}{\partial \eta} \right)^b + \gamma_a^b. \]  (3.10)

We also consider the energy-momentum tensor for the scalar field, which is given by
\[ T_{ab}^b = \nabla_a \varphi \nabla^b \varphi - \frac{1}{2} \delta_a^b (\nabla_c \varphi \nabla^c \varphi + 2V(\varphi)) \]  (3.11)
\[ = - \left( \frac{1}{2a^2} (\partial_\eta \varphi)^2 + V(\varphi) \right) (d\eta)_a \left( \frac{\partial}{\partial \eta} \right)^b \]
\[ + \left( \frac{1}{2a^2} (\partial_\eta \varphi)^2 - V(\varphi) \right) \gamma_a^b, \]  (3.12)
where we have assumed that the scalar field \( \varphi \) is homogeneous, i.e., \( \varphi = \varphi(\eta) \). Comparing (3.12) with (3.9), the energy density and the pressure for the homogeneous scalar field are given by
\[ \epsilon = \frac{1}{2a^2} (\partial_\eta \varphi)^2 + V(\varphi), \quad p = \frac{1}{2a^2} (\partial_\eta \varphi)^2 - V(\varphi). \]  (3.13)

The Einstein equations \( G_{ab}^a = 8\pi GT_{ab}^b \) for this background spacetime filled with a perfect fluid are given by
\[ \mathcal{H}^2 + K = \frac{8\pi G}{3} a^2 \epsilon, \quad 2\partial_\eta \mathcal{H} + \mathcal{H}^2 + K = -8\pi G a^2 p. \]  (3.14)

In the derivation of the perturbative Einstein equations, the equation
\[ \mathcal{H}^2 + K - \partial_\eta \mathcal{H} = 4\pi G a^2 (\epsilon + p) \]  (3.15)
is also useful. Of course, there is an equation for the energy conservation of the matter fields, and this equation gives the behavior of the energy density in the scale factor if we apply an appropriate equation of state for the matter field. This equation is consistent with the two equations in (3.14).

Further, in the case of the single scalar field model, the Einstein equations are given by
\[ \mathcal{H}^2 + K = \frac{8\pi G}{3} a^2 \left( \frac{1}{2a^2} (\partial_\eta \varphi)^2 + V(\varphi) \right), \]  (3.16)
\[ 2\partial_\eta \mathcal{H} + \mathcal{H}^2 + K = -8\pi G a^2 \left( \frac{1}{2a^2} (\partial_\eta \varphi)^2 - V(\varphi) \right), \]  (3.17)
through the relations (3.13). We also note that the equations (3.16) and (3.17) lead to

\[ \mathcal{H}^2 + K - \partial_\eta \mathcal{H} = 4\pi G (\partial_\eta \varphi)^2. \]  

(3.18)

Equation (3.18) is also useful when we derive the perturbative Einstein equations. Equations (3.16) and (3.17) are often used to investigate the inflationary scenario. Actually, in the situation that the potential term of the scalar field is sufficiently larger than its kinetic term, the spacetime is approximately a de Sitter spacetime, and this situation may be realized in the very early universe.\(^3\) Hence, the background spacetime described by Eqs. (3.16) and (3.17) also includes inflationary universes, and the second-order perturbation theory developed below is also applicable to the inflationary universe.

§4. Gauge invariant variables of cosmological perturbations

Now, we develop the second-order perturbation theory with the cosmological background spacetime in §3 within the general framework of the gauge invariant perturbation theory reviewed in §2. The important step when we apply the above general framework of the gauge invariant perturbation theory is to confirm that the assumption for the decomposition (2.22) of the linear-order metric perturbation is correct. This confirmation is accomplished in §4.1. Hence, the general framework reviewed in §2 is applicable. Applying this framework, we define the second-order gauge invariant variable of the metric perturbation in §4.2 and of the matter perturbations in §4.3.

4.1. First-order metric perturbations

On the background spacetime discussed in §3 we consider the metric perturbation to be that given in Eq. (2.21). To show that the assumption for the decomposition (2.22) of the linear-order metric perturbation is correct, we first consider the components of the linear-order metric perturbation in the coordinate system (3.1),

\[ h_{ab} = h_{\eta\eta}(d\eta)_a(d\eta)_b + 2h_{\eta i}(d\eta)(d\gamma^i)_b + h_{ij}(dx^i)_a(dx^j)_b. \]  

(4.1)

Because components belonging to different groups are coupled through contraction with the metric tensor and the covariant derivatives in the Einstein equations, the grouping \(\{h_{\eta\eta}, h_{\eta i}, h_{ij}\}\) is not so useful. Instead, using the fact that the three manifold \((\Sigma(\eta), a^2\gamma_{ab})\) is maximally symmetric, we further decompose the vector \(h_{\eta i}\) and tensor \(h_{ij}\) as

\[ h_{\eta i} = D_i h_{(V)\eta}, \quad D^i h_{(V)i} = 0, \]  

(4.2)

\[ h_{ij} = a^2h_{(L)ij} + a^2h_{(T)ij}, \quad h_{(T)i} := \gamma^{ij} h_{(T)ij} = 0, \]  

(4.3)

\[ h_{(T)ij} = \left( D_i D_j - \frac{1}{3} \gamma_{ij} \Delta \right) h_{(TL)} + 2D_i h_{(TV)j} + h_{(TT)ij}, \]  

(4.4)

\[ D^i h_{(TV)i} = 0, \quad D^i h_{(TT)ij} = 0, \]  

(4.5)
where \( D_i \) is the covariant derivative associated with the metric \( \gamma_{ij} \), and \( \Delta = D_i D^i \). It is well-known that the linearized Einstein equations on a homogeneous isotropic universe, whose metric is given by Eq. (3.1), can be decomposed into groups that each contains only variables belonging to one of the three sets \( \{h_{\eta\eta}, h_{(VL)}, h_{(L)}, h_{TL}\} \), \( \{h_{(V)i}, h_{(TV)i}\} \), and \( h_{(TT)ij} \). Variables belonging to these sets are called scalar-type, vector-type and tensor-type variables, respectively. This segregation of variables is due to the fact that the metric tensor \( a^2 \gamma_{ab} \) is the only non-trivial tensor on the maximally symmetric space, and as a consequence, the tensorial operations on \( h_{ab} \) to construct the linearized Einstein tensors preserve this decomposition.

In the linear perturbation theory, the covariant derivatives are always combined into the Laplacian \( \Delta \) in the linearized Einstein equations after the decompositions (4.2)–(4.4), because the metric tensor \( a^2 \gamma_{ab} \) is the only non-trivial tensor on the maximally symmetric space. Thus, the harmonic expansion of the perturbation variables with respect to the Laplacian is also useful in the linear perturbation theory. However, in the second-order perturbation theory, mode-mode coupling occurs due to the non-linearity of the Einstein equations. For this reason, we do not apply the harmonic expansion of the perturbation variables with respect to the Laplacian in this paper, though the harmonic expansion should be also useful after the nonlinear terms in the second-order Einstein equations are clarified. Instead, we assume the existence of some Green functions, as explained below.

To clarify the uniqueness of the decompositions (4.2)–(4.4), we consider the inverse relations of Eqs. (4.2)–(4.4). To do this, we first note that the commutator of the divergence and the Laplacian is given by

\[
D_i \Delta t^i - \Delta D^i t_i = D^i \left( (3)^{R_{ij}t^j} \right) = 2KD^i t_i, \tag{4.6}
\]

\[
D^i \Delta t^i_{ij} - \Delta D^i t^i_{ij} = D^i \left((3)^{R_{i}^k e_k} + (3)^{R_{ij}^k} t^k_{ij} + (3)^{R_{ikjl}} D^k t^i_{ij}\right) = 4K \left(D^i t^i_{(ij)} - \frac{1}{2} D^j t^i_{i}\right), \tag{4.7}
\]

for any tensor fields \( t_i \) and \( t_{ij} \), where we have used Eqs. (3.3). These relations are also useful when we write down the components of the perturbative Einstein tensor. They show that on the maximally symmetric space, the Laplacian preserves the transverse condition

\[
D^j \Delta h_{(V)j} = (\Delta + 2K) D^j h_{(V)j} = 0. \tag{4.8}
\]

The inverse relations of the decompositions (4.2)–(4.4) are given by

\[
h_{(VL)} = \Delta^{-1} D^i h_{\eta i}, \tag{4.9}
\]

\[
h_{(V)i} = h_{\eta i} - D_i \Delta^{-1} D^j h_{\eta j}, \tag{4.10}
\]

\[
h_{(L)} = \frac{1}{3a^2} h_{i i}, \tag{4.11}
\]

\[
h_{(TT)ij} = \frac{1}{a^2} \left(h_{ij} - \frac{1}{3} h_{k} \gamma_{ij}\right), \tag{4.12}
\]
\[ h_{(TL)} = \frac{3}{2} (\Delta + 3K)^{-1} \Delta^{-1} D^i D^j h_{(T)ij}, \quad (4.13) \]
\[ h_{(TV)} = (\Delta + 2K)^{-1} D^k h_{(T)ik} - (\Delta + 2K)^{-1} D_i \Delta^{-1} D^k D^l h_{(T)kl}, \quad (4.14) \]
\[ h_{(TT)ij} = h_{(T)ij} - \frac{3}{2} \left( D_i D_j - \frac{1}{3} \gamma_{ij} \Delta \right) (\Delta + 3K)^{-1} \Delta^{-1} D^k D^l h_{(T)kl} \]
\[ -2D_i (\Delta + 2K)^{-1} D^k h_{(T)jk} \]
\[ + 2D_i (\Delta + 2K)^{-1} D^k D^l h_{(T)kl}. \quad (4.15) \]

Equations (4.9)–(4.15) show that there should exist Green functions of the operators \( \Delta, \Delta + 2K, \) and \( \Delta + 3K \) to guarantee the one to one correspondence of the decompositions of the set \( \{ h_{\eta \eta}, h_{\eta i}, h_{ij} \} \) and the sets \( \{ h_{\eta \eta}, h_{(V) L}, h_{(L)}, h_{(T) L} \}, \{ h_{(V) i}, h_{(T) Vi} \}, h_{(TT)ij} \} \). Actually, these Green functions exist if we specify the domain of the perturbations, for example \( L^2 \)-space on \( \Sigma(\eta) \) with appropriate boundary condition. Therefore, we assume the existence of these Green functions in this paper.

By this assumption, any tensor that belongs to the kernel of any of the operators \( \Delta, \Delta + 2K, \) and \( \Delta + 3K \) is excluded from consideration. For example, a Killing field \( v_i \) on the three-dimensional hypersurface \( \Sigma(\eta) \), which satisfies the Killing equation \( D_i v_j = 0 \), belongs to the kernel of the operator \( \Delta + 2K \), since we can easily confirm \( (\Delta + 2K)v_i = 0 \) from the Killing equation. If it is necessary to investigate such tensors as the perturbative mode, separate treatments are necessary. Because the treatment of these exceptional modes is beyond the scope of this paper, we ignore all modes that belong to the kernels of the operators \( \Delta, \Delta + 2K, \) and \( \Delta + 3K \).

Now, we consider the decomposition \( (2.22) \) of the linear-order metric perturbation and show that the decomposition \( (2.22) \) is valid in the case of cosmological perturbations, though this decomposition is merely an assumption in the general framework reviewed in \( (2.2) \).

To accomplish the decomposition \( (2.22) \), we consider the gauge transformation rule \( (2.19) \), which is given by
\[ \gamma h_{ab} - \chi h_{ab} = \mathcal{L}_\xi g_{ab} = 2 \nabla_a (\xi_b) \quad (4.16) \]
for linear-order metric perturbations. In Eq. \( (4.16) \), the generator \( \xi^a \) of the gauge transformation is an arbitrary vector field on the background spacetime \( \mathcal{M}_0 \). We decompose the generator \( \xi^a \) in terms of the 3+1 decomposition as
\[ \xi_a = \xi^i \eta_a + \xi^i (dx^i)_a, \quad (4.17) \]
and, further, the component \( \xi_i \) as
\[ \xi_i = D_i \xi_{(L)} + \xi_{(T)i}, \quad D^i \xi_{(T)i} = 0. \quad (4.18) \]

In terms of the 3+1 decomposition, the gauge transformation rules \( (4.16) \) are given by
\[ \gamma h_{\eta \eta} - \chi h_{\eta \eta} = 2 (\partial_\eta - \mathcal{H}) \xi_\eta, \quad (4.19) \]
\[ \gamma h_{\eta i} - \chi h_{\eta i} = D_i \xi_\eta + (\partial_\eta - 2\mathcal{H}) \xi_i, \quad (4.20) \]
\[ \gamma h_{ij} - \chi h_{ij} = 2 D_i \xi_j - 2\mathcal{H} \gamma_{ij} \xi_\eta. \quad (4.21) \]
Furthermore, following the decomposition (4.18) of the component $\xi_i$, the gauge transformation rules (4.20) and (4.21) are obtained in terms of $\xi_\eta$, $\xi(L)$, and $\xi(T)$ as

$$
\begin{aligned}
\gamma \eta_i - \lambda \eta_i &= D_i \{ (\partial_\eta - 2H) \xi(L) + \xi_\eta \} + (\partial_\eta - 2H) \xi(T), \\
\gamma \eta_{ij} - \lambda \eta_{ij} &= 2 \left( \frac{1}{3} \Delta \xi(L) - \mathcal{H} \xi_\eta \right) \gamma_{ij} + 2 \left( D_i D_j - \frac{1}{3} \gamma_{ij} \Delta \right) \xi(L) + 2 D_{i<j} \xi^{(T)},
\end{aligned}
$$

(4.22)

Since the tensorial decomposition (4.2)–(4.4) has the inverse relations (4.9)–(4.15), the gauge transformation rules (4.22) and (4.23) lead to those for the metric perturbations $h(V_L)$, $h(V)_i$, $h(L)$, $h(T_L)$, $h(T_V)_i$, and $h(T T)_{ij}$ as follows:

$$
\begin{aligned}
\gamma h(V_L) - \lambda h(V_L) &= \xi_\eta + (\partial_\eta - 2H) \xi(L), \\
\gamma h(V)_i - \lambda h(V)_i &= (\partial_\eta - 2H) \xi(T)_i, \\
a^2 \gamma h(L) - a^2 \lambda h(L) &= -2H \xi_\eta + \frac{2}{3} \Delta \xi(L), \\
a^2 \gamma h(T_L) - a^2 \lambda h(T_L) &= 2 \xi(L), \\
a^2 \gamma h(T_V)_i - a^2 \lambda h(T_V)_i &= \xi(T)_i, \\
a^2 \gamma h(T T)_{ij} - a^2 \lambda h(T T)_{ij} &= 0.
\end{aligned}
$$


Inspecting the gauge transformation rules (4.24)–(4.29), together with Eq. (4.19), we find gauge invariant and variant variables of the first-order metric perturbation. We first construct gauge invariant variables.

First, equation (4.29) shows that the transverse traceless part $h(T T)_{ij}$ is itself gauge invariant. We denote this transverse traceless part by

$$
\begin{aligned}
(1) \chi_{ij} := h(T T)_{ij}, \\
(1) \chi_{ij} = (1) \chi_{ji}, \\
(1) \chi^i_i = 0, \\
D^i (1) \chi_{ij} = 0.
\end{aligned}
$$

(4.30)

The transverse traceless tensor $(1) \chi_{ij}$ has two independent components and is called the “tensor mode” in the context of cosmological perturbations. This transverse traceless part of the metric perturbations is well known as gravitational waves on a homogeneous isotropic universe.

Next, we consider to the gauge transformation rules (4.25) and (4.28). From these gauge transformation rules, we can easily see that the variable defined by

$$
\begin{aligned}
a^2 \nu_i := h(V)_i - (\partial_\eta - 2H) (a^2 h(T V)_i) \\
= h(V)_i - a^2 \partial_\eta h(T V)_i
\end{aligned}
$$

(4.31)

is gauge invariant. The gauge invariant variable $\nu_i$ is called a “vector mode” in the context of cosmological perturbations. It satisfies the equation

$$
D^i (1) \nu_i = 0
$$

(4.32)
from the divergenceless property of the variables $h_{(V)i}$ and $h_{(TV)i}$. Equation (4.32) implies that the vector mode $^{(1)}\nu_i$ includes two independent components.

In addition to the vector and tensor mode of the perturbation, there are two scalar modes in the linear-order metric perturbations $h_{ab}$. To see this, we first consider the gauge transformation rules (4.24) and (4.27). From these transformation rules, the variable defined by

$$\bar{X}_\eta := h_{(VL)} - \frac{1}{2} (\partial_\eta - 2\mathcal{H}) \left(a^2 h_{(TL)}\right) = h_{(VL)} - \frac{1}{2} a^2 \partial_\eta h_{(TL)} \quad (4.33)$$

is transformed as

$$3\bar{X}_\eta - \chi \bar{X}_\eta = 3h_{(VL)} - \chi h_{(VL)} - \frac{1}{2} (\partial_\eta - 2\mathcal{H}) \left(a^2 \left(3h_{(TL)} - \chi h_{(TL)}\right)\right)$$

$$= \partial_\eta \xi_{(L)} + \xi_\eta - 2\mathcal{H} \xi_{(L)} - \frac{1}{2} (\partial_\eta - 2\mathcal{H}) \left(2\xi_{(L)}\right). \quad (4.34)$$

Using $\bar{X}_\eta$ and inspecting the gauge transformation rule (4.19), we easily find that the variable $^{(1)}\Phi$ defined by

$$-2a^2 \Phi := h_{\eta\eta} - 2(\partial_\eta - \mathcal{H}) \bar{X}_\eta \quad (4.35)$$

is gauge invariant. Further, from gauge transformation rules (4.26), (4.27), and (4.34), the variable $^{(1)}\Psi$ defined by

$$-2a^2 \Psi := a^2 \left(h_{(L)} - \frac{1}{3} \Delta h_{(TL)}\right) + 2\mathcal{H} \bar{X}_\eta \quad (4.36)$$

is gauge invariant. The two scalar functions $^{(1)}\Phi$ and $^{(1)}\Psi$ are called “scalar perturbations” in the context of cosmological perturbations.

Thus, we have six components of gauge invariant variables: two components of the tensor mode, two components of the vector mode, and two scalar modes. Since the metric perturbation $h_{ab}$ has ten components, there are four remaining components, which are the components of the gauge variant part of the metric perturbation.

Because we already have all gauge invariant variables, we can specify the variant part $X_a$ of the metric perturbation $h_{ab}$ as in Eq. (2.22). Using the gauge invariant variables $\Phi$, $\Psi$, $\nu_i$, and $\chi_{ij}$, the components of the metric perturbation $h_{ab}$ are given by

$$h_{\eta\eta} = -2a^2 \Phi + 2(\partial_\eta - \mathcal{H}) \bar{X}_\eta, \quad (4.37)$$

$$h_{\eta i} = a^2 \nu_i + a^2 \partial_\eta h_{(TV)i} + D_i h_{(VL)}, \quad (4.38)$$

$$h_{ij} = -2a^2 \Psi \gamma_{ij} + a^2 \chi_{ij} + a^2 D_i D_j h_{(TL)} - 2\mathcal{H} \bar{X}_\eta \gamma_{ij} + 2a^2 D_i h_{(TV)j}. \quad (4.39)$$
On the other hand, in terms of the 3+1 decomposition, the components of \( \mathbf{22} \) are given by

\[
\begin{align*}
    h_{\eta \eta} &= H_{\eta \eta} + 2 (\partial_{\eta} - \mathcal{H}) X_{\eta}, \\
    h_{\eta i} &= H_{\eta i} + D_i X_{\eta} + \partial_{\eta} X_i - 2 \mathcal{H} X_i, \\
    h_{ij} &= H_{ij} + 2 D_i X_j - 2 \mathcal{H} \gamma_{ij} X_{\eta}.
\end{align*}
\] (4.40)

Here, \( \mathcal{H}_{ab} \) is gauge invariant, and its components should be identified by

\[
\begin{align*}
    H_{\eta \eta} &:= -2a^2 (1) \Phi, \\
    H_{\eta i} &:= a^2 (1) \nu_i, \\
    H_{ij} &:= -2a^2 (1) \Psi \gamma_{ij} + a^2 (1) \chi_{ij}.
\end{align*}
\] (4.43)

Then, we obtain the equations for \( X_a \):

\[
\begin{align*}
    2 (\partial_{\eta} - \mathcal{H}) X_{\eta} &= 2 (\partial_{\eta} - \mathcal{H}) \bar{X}_{\eta}, \\
    D_i X_{\eta} + (\partial_{\eta} - 2 \mathcal{H}) X_i &= a^2 \partial_{\eta} h_{(TV)i} + D_i h_{(VL)}, \\
    2 D_i X_j - 2 \mathcal{H} \gamma_{ij} X_{\eta} &= a^2 D_i D_j h_{(TL)} - 2 \mathcal{H} \bar{X}_{\eta} \gamma_{ij} + 2a^2 D_i h_{(TV)j}.
\end{align*}
\] (4.44-4.46)

Equation (4.44) yields

\[
X_{\eta} = \bar{X}_{\eta} + a \bar{C}_{\eta},
\] (4.47)

where \( \bar{C}_{\eta} \) is a scalar function satisfying the equation \( \partial_{\eta} \bar{C}_{\eta} = 0 \). Substituting (4.47) and (4.33) into (4.45), we obtain

\[
X_i = a^2 \left( h_{(TV)i} + \frac{1}{2} D_i h_{(TL)} \right) - D_i \bar{C}_{\eta} a^2 \int \frac{d\eta}{a} + a^2 \bar{C}_i,
\] (4.48)

where \( \bar{C}_i \) is the vector field satisfying the condition \( \partial_{\eta} \bar{C}_i = 0 \). Substituting (4.47) and (4.48) into (4.46), we obtain

\[
a^2 D_i \bar{C}_j - D_i D_j C_{\eta} a^2 \int \frac{d\eta}{a} - \mathcal{H} \gamma_{ij} a \bar{C}_{\eta} = 0.
\] (4.49)

Thus, we have found that the gauge variant part of the metric perturbation \( X_a \) is given by

\[
\begin{align*}
    X_{\eta} &= \bar{X}_{\eta} + C_{\eta} = h_{(VL)} - \frac{1}{2} a^2 \partial_{\tau} h_{(TL)} + C_{\eta}, \\
    X_i &= a^2 \left( h_{(TV)i} + \frac{1}{2} D_i h_{(TL)} \right) + C_i,
\end{align*}
\] (4.50-4.51)

where \( C_{\eta} \) and \( C_i \) are defined by

\[
\begin{align*}
    C_{\eta} &:= a \bar{C}_{\eta}, \\
    C_i &:= -D_i \bar{C}_{\eta} a^2 \int \frac{d\eta}{a} + a^2 \bar{C}_i,
\end{align*}
\] (4.52-4.53)

and \( \partial_{\eta} \bar{C}_{\eta} = 0 = \partial_{\eta} \bar{C}_i \).
Through Eqs. (4.52) and (4.53), with the constraint (4.49), it is easy to confirm that the vector field defined by

\[
C_a := \xi_\eta (d\eta)_a + \xi_i (dx^i)_a
\]  

(4.54)
is a Killing vector on the background spacetime \( M_0 \). Actually, it is readily shown that

\[
2\nabla_\eta C_\eta = 2(\partial_\eta - \mathcal{H}) C_\eta = 0, 
\]  

(4.55)
due to the definition (4.52) and \( \partial_\eta \tilde{C}_\eta = 0 \). Further, from the definition (4.53), we can easily confirm that

\[
2\nabla_\eta C_i = \partial_\eta C_i + D_i C_\eta - 2\mathcal{H} C_i = 0. 
\]  

(4.56)

Finally, the constraint (4.49) leads to

\[
2\nabla_i C_j = 2D_i C_j - 2\mathcal{H} \gamma_{ij} C_\eta = 0. 
\]  

(4.57)

Thus, we have

\[
\nabla_a C_b = 0; 
\]  

(4.58)
i.e., \( C_a \) defined by Eq. (4.54) is a Killing vector. Hence, we have been able to specify the gauge variant part \( X_a \) of the linear-order metric perturbation as

\[
X_a := X_\eta (d\eta)_a + X_i (dx^i)_a. 
\]  

(4.59)

The relation between the components of the gauge variant part, \( X_a \), and the components of the linear-order metric perturbation, \( h_{(V\ell)} \), \( h_{(T\ell)} \), \( h_{(TV)} \), is determined up to the degree of freedom of the Killing vector field. Since \( X_a \) contributes to the metric perturbation as in Eq. (2.22), the Killing vector field \( C_a \) in Eqs. (4.50) and (4.51) does not contribute to the metric perturbation.

Further, because we do not consider the kernels of the operators \( \Delta \), \( \Delta + 2K \), and \( \Delta + 3K \) as the domain of the perturbations, the gauge variant part \( X_a \) of the metric perturbation does not have the degree of freedom of the Killing vector field; i.e., the gauge variant part \( X_a \) is determined without ambiguity. To satisfy Eq. (4.49) for any scale factor \( a \), the components \( \tilde{C}_\eta \) and \( \tilde{C}_i \) should satisfy the equations

\[
\tilde{C}_\eta = 0, \quad D_i \tilde{C}_j = 0. 
\]  

(4.60)

This implies that \( \tilde{C}_j \) is a Killing vector on a three-dimensional hypersurface \( \Sigma(\eta) \). As commented just after Eqs. (4.9)–(4.15), it is easily shown that \( (\Delta + 2K) \tilde{C}_i = 0 \). Therefore, the vector field \( \tilde{C}_i \) does not belong to the domain of perturbations considered here. Thus, we have \( \tilde{C}_j = 0 \). Hence, we conclude that we can determine the gauge variant part \( X_a \) without ambiguity. Of course, it might be possible to include the Killing fields in our consideration by extending the domain of the perturbations. Separate treatments are necessary to do this, as mentioned above.

Finally, we check the transformation rules for the vector field \( X_a \) under the gauge transformation \( \chi_\lambda \rightarrow \chi_\lambda \). Because the component \( X_\eta \) of the vector \( X_a \) is transformed as

\[
\chi X_\eta - \chi X_\eta = \xi_\eta, 
\]  

(4.61)
as noted in Eq. (4.34). From Eq. (4.51) with $C_i = 0$ and the gauge transformation rules (4.27) and (4.28), the gauge transformation rule of the component $X_i$ defined by Eq. (4.51) is given by

$$YX_i - \chi X_i = a^2\mathcal{h}_{(TV)i} - a^2\mathcal{h}_{(TV)i} + \frac{1}{2}D_i \left(a^2\mathcal{h}_{(TL)} - a^2\mathcal{h}_{(TL)}\right)$$

$$= \xi_{(T)i} + D_i \xi_{(L)} = \xi_i.$$  (4.62)

Together with the transformation rules (4.61) and (4.62), the vector field $X_a$ defined by Eq. (4.59) is transformed as

$$YX_a - \chi X_a = \xi_a.$$  (4.63)

This shows that $X_a$ is the gauge variant part of the metric perturbation in the decomposition (2.22).

Thus, we know the procedure to find the gauge invariant variable $\mathcal{H}_{ab}$ and the gauge variant variable $X_a$ in the case of cosmological perturbations. We have confirmed the important premise of the general framework of the second-order perturbation theory reviewed in §2. Hence, we can apply this general framework for the second-order gauge invariant perturbation theory presented in KN2003 and KN2005 to cosmological perturbations.

### 4.2. Second order metric perturbations

Here we consider second-order metric perturbations. We expand the metric $\bar{g}_{ab}$ on the physical spacetime $M_\lambda$ as Eq. (2.21). According to the gauge transformation rule (2.20), the second-order metric perturbation $l_{ab}$ is transformed as

$$Yl_{ab} - \chi l_{ab} = 2\mathcal{L}_{\xi(1)} l_{ab} + \left(\mathcal{L}_{\xi(2)} + \mathcal{L}_{\xi(1)}^2\right) g_{ab}$$  (4.64)

under the gauge transformation $\Phi_\lambda = (\mathcal{X}_\lambda)^{-1} \circ \mathcal{Y}_\lambda : \mathcal{X}_\lambda \rightarrow \mathcal{Y}_\lambda$. As shown in §4.1, the first-order metric perturbation $h_{ab}$ is decomposed in the form (2.22). Using this important fact, as shown in KN2003, the second-order metric perturbation $l_{ab}$ can be decomposed as Eq. (2.24). Here, we demonstrate this.

Inspecting the gauge transformation rule (4.64), we first introduce the variable $\hat{L}_{ab}$ defined by

$$\hat{L}_{ab} := l_{ab} - 2\mathcal{L}_{\chi} h_{ab} + \mathcal{L}_{\chi}^2 g_{ab}.$$  (4.65)

Under the gauge transformation $\Phi_\lambda = (\mathcal{X}_\lambda)^{-1} \circ \mathcal{Y}_\lambda : \mathcal{X}_\lambda \rightarrow \mathcal{Y}_\lambda$, the variable $\hat{L}_{ab}$ is transformed as

$$Y\hat{L}_{ab} - \chi \hat{L}_{ab} = Yl_{ab} - 2\mathcal{L}_{\chi} y_{ab} + \mathcal{L}_{\chi}^2 g_{ab}$$

$$- \chi l_{ab} + 2\mathcal{L}_{\chi} x l_{ab} + \mathcal{L}_{\chi}^2 x g_{ab}$$

$$= 2\mathcal{L}_{\xi(1)} h_{ab} + \left(\mathcal{L}_{\xi(2)} + \mathcal{L}_{\xi(1)}^2\right) g_{ab}$$

$$- 2\mathcal{L}_{\chi} x + \xi(1) \left(\mathcal{h}_{ab} + \mathcal{L}_{\xi(1)} g_{ab}\right) + \mathcal{L}_{\chi}^2 x + \xi(1) g_{ab}$$

$$- \chi l_{ab} + 2\mathcal{L}_{\chi} x l_{ab} - \mathcal{L}_{\chi}^2 x g_{ab}$$

$$= \mathcal{L}_{\sigma} g_{ab},$$  (4.67)
where
\[ \sigma^a := \xi^a_{(2)} + [\xi_{(1)}, X]^a. \] (4.68)

The gauge transformation rule (4.67) is identical to that for a linear metric perturbation. Therefore, we may apply the procedure to find the gauge invariant and variant variables of linear-order metric perturbations to the decomposition of the components of the variable \( \hat{L}_{ab} \). Arguments completely analogous to those used in the case of linear-order metric perturbation show that the variable \( \hat{L}_{ab} \) can be decomposed as
\[ \hat{L}_{ab} = \mathcal{L}_{ab} + \mathcal{L} g_{ab}, \] (4.69)

where \( \mathcal{L}_{ab} \) is the gauge invariant part of the variable \( \hat{L}_{ab} \), or equivalently, of the second-order metric perturbation \( l_{ab} \), and \( Y^a \) is the gauge variant part of \( \hat{L}_{ab} \), i.e., the gauge variant part of \( l_{ab} \). Under the gauge transformation \( \Phi_\lambda = (X_\lambda)^{-1} \circ \mathcal{Y}_\lambda \), the variables \( \mathcal{L}_{ab} \) and \( Y^a \) are transformed as
\[ \gamma \mathcal{L}_{ab} - \chi \mathcal{L}_{ab} = 0, \quad \gamma Y_a - \chi Y_a = \sigma_a, \] (4.70)

respectively. Thus, we have reached the decomposition (2.24) of the second-order metric perturbation \( l_{ab} \) into the gauge variant and gauge invariant parts. Following the same argument as in the linear case, the components of the gauge invariant variables \( \mathcal{L}_{ab} \) are given by
\[ \mathcal{L}_{ab} = -2a^2 \Phi^2 (d\eta)_a(d\eta)_b + 2a^2 \nu_i^{(2)} (d\eta)_a(dx^i)_b \]
\[ + a^2 \left( -2 \Psi^{(2)} \gamma_{ij} + \chi^{(2)}_{ij} \right) (dx^i)_a(dx^j)_b, \] (4.71)

where \( \nu_i^{(2)} \) and \( \chi^{(2)}_{ij} \) satisfy the equations
\[ D^i \nu_i^{(2)} = 0, \quad (2)^i \chi^{(2)}_{ij} = 0, \quad D^i \chi^{(2)}_{ij} = 0. \] (4.72)

The gauge invariant variables \( \Phi^{(2)} \) and \( \Psi^{(2)} \) are the scalar mode perturbations of second order, and \( \nu_i^{(2)} \) and \( \chi^{(2)}_{ij} \) are the second-order vector and tensor modes of the metric perturbations, respectively.

4.3. Matter perturbations

Since we have obtained the first- and the second-order gauge variant parts, \( X_a \) and \( Y_a \), of the metric perturbation, we can define the gauge invariant variables for an arbitrary field \( Q \), except for the metric by following the definitions (2.26) and (2.27). Here, we consider the first- and second-order gauge invariant variables for the perturbations of the perfect fluid components, the single scalar field, and their energy-momentum tensors.
4.3.1. Perfect fluid

First, we consider the perturbation of a perfect fluid. As shown in Eq. (3.8), the total energy-momentum tensor of the fluid is characterized by the energy density $\bar{\epsilon}$, the pressure $\bar{p}$, and the four-velocity $\bar{u}^a$:

$$\bar{T}^b_a = (\bar{\epsilon} + \bar{p})\bar{u}^a\bar{u}^b + \bar{p}\delta^b_a.$$  \hspace{1cm} (4.73)

Of course, this energy-momentum tensor is the representation on the physical spacetime $\mathcal{M}_\lambda$, but we can regard this equation to be the representation on the background spacetime $\mathcal{M}_0$ which is pulled back by an appropriate gauge choice $X^\lambda$. The background value of the energy momentum tensor (4.73) is given by Eqs. (3.8)–(3.10) in \textsection 3. In addition to the components of Eq. (4.73), we may also include the anisotropic stress as non-diagonal space-space components of the energy-momentum tensor, as phenomenology. Because we can always extend our arguments to those including the anisotropic stress, we ignore it here, for simplicity.

Now, we consider perturbations of the fluid components of the energy-momentum tensor (4.73):

$$\bar{\epsilon} := \epsilon + \lambda (^1\epsilon) + \frac{1}{2} \lambda^2 (^2\epsilon) + O(\lambda^3),$$ \hspace{1cm} (4.74)

$$\bar{p} := p + \lambda (^1p) + \frac{1}{2} \lambda^2 (^2p) + O(\lambda^3),$$ \hspace{1cm} (4.75)

$$\bar{u}_a := u_a + \lambda (^1u_a) + \frac{1}{2} \lambda^2 (^2u_a) + O(\lambda^3).$$ \hspace{1cm} (4.76)

The fluid four-velocities $\bar{u}_a$ on the physical spacetime and $u_a$ on the background spacetime satisfy the normalization condition

$$\bar{g}^{ab}\bar{u}_a\bar{u}_b = g^{ab}u_a u_b = -1. \hspace{1cm} (4.77)$$

The perturbative expansion of the normalization conditions (4.77) gives the constraints for the components of the first- and second-order perturbative four-velocities $(^1u_a)$ and $(^2u_a)$. The perturbative expansion of the normalization condition (4.77) to second order gives the normalization condition at each order:

$$u^a(^1u_a) = \frac{1}{2} h^{ab} u_a u_b,$$ \hspace{1cm} (4.78)

$$u^a(^2u_a) = h^{ab} u_a (^1u_b) - g^{ab} (^1u_a)(^1u_b) - h^{ac} h^b_c u_a u_b + \frac{1}{2} l^{ab} u_a u_b. \hspace{1cm} (4.79)$$

We also consider the perturbation of the four-velocity $\bar{u}^a$ as

$$\bar{u}^a = u^a + \lambda (^1u^a) + \frac{1}{2} \lambda^2 (^2u^a).$$ \hspace{1cm} (4.80)

The perturbative expansion of the equation $\bar{u}^a = \bar{g}^{ab}\bar{u}_b$ leads to

$$^1u^a = g^{ab} (^1u_b) - h^{ab} u_b, \hspace{1cm} (4.81)$$

$$^2u^a = g^{ab} (^2u_b) - 2h^{ab} (^1u_b) + (2h^{ac} h^b_c - l^{ab}) u_b. \hspace{1cm} (4.82)$$
Further, the first-order perturbation (4.78) of the normalization condition (4.77) is given by

\[(1) \quad (u^a) u_a + u^a (u_a) = 0. \quad (4.83)\]

Next, we define the gauge invariant variable for the perturbation of the fluid components \(\tilde{\epsilon}, \tilde{p}\), and \(\tilde{u}_a\). Following the definitions (2.26) and (2.27) of the gauge invariant variable for an arbitrary matter field\(^{16}\), we define the variables

\[
\begin{align*}
(1) \quad \mathcal{E} & := \epsilon - \mathcal{L}_X \epsilon, \\
(1) \quad \mathcal{P} & := p - \mathcal{L}_X p, \\
(1) \quad U_a & := (u_a) - \mathcal{L}_X u_a, \\
(2) \quad \mathcal{E} & := \epsilon - 2 \mathcal{L}_X \epsilon - \{ \mathcal{L}_Y - \mathcal{L}_X^2 \} \epsilon, \\
(2) \quad \mathcal{P} & := p - 2 \mathcal{L}_X p - \{ \mathcal{L}_Y - \mathcal{L}_X^2 \} p, \\
(2) \quad U_a & := (u_a) - 2 \mathcal{L}_X u_a - \{ \mathcal{L}_Y - \mathcal{L}_X^2 \} u_a,
\end{align*}
\]

where the vector fields \(X_a\) and \(Y_a\) are the gauge variant parts of the first- and second-order metric perturbations, respectively, which were defined in §4.2 and 4.1.

The first-order perturbation (4.78) of the normalization condition (4.77) is given by

\[
\begin{align*}
(1) \quad u^a U_a & = \frac{1}{2} \mathcal{H}_{ab} u^a u^b - \mathcal{L}_X \left( \frac{1}{2} u^a u_a \right) \\
& = \frac{1}{2} \mathcal{H}_{ab} u^a u^b, \quad (4.90)
\end{align*}
\]

while the second-order perturbation (4.79) of Eq. (4.77) is given by

\[
\begin{align*}
(2) \quad u^a U_a & = 2 \mathcal{H}_{ab} u^a g^{bc} U_c - g^{ab} U_a u_b - \mathcal{H}_{ac} \mathcal{H}_{db} g^{de} u_a u_b + \frac{1}{2} \mathcal{L}_{ab} u^a u^b \\
& \quad - 2 \mathcal{L}_X \left( u^a (u_a) - \frac{1}{2} \mathcal{H}_{ab} u^a u^b \right) - \{ \mathcal{L}_Y - \mathcal{L}_X^2 \} \left( \frac{1}{2} u^a u_a \right) \\
& = 2 \mathcal{H}_{ab} u^a g^{bc} U_c - g^{ab} U_a u_b - \mathcal{H}_{ac} \mathcal{H}_{db} g^{de} u_a u_b + \frac{1}{2} \mathcal{L}_{ab} u^a u^b, \quad (4.93)
\end{align*}
\]

where we have used Eqs. (4.77) and (4.78). We note that (4.90) and (4.92) have the same forms as definitions (2.26) and (2.27) of the first- and second-order gauge invariant variables for an arbitrary tensor field, respectively. These are natural results, because Eqs. (4.90) and (4.92) are the results of the first- and second-order perturbative expansions of the variable \(\frac{1}{2} \bar{u}^a \bar{u}_a\).

We also decompose the first- and second-order perturbations of the four-velocity \(\bar{u}^a\), which are given by (4.81) and (4.82), respectively, into gauge invariant and
variant parts:

\[(u^a) = g^{ab} \mathcal{U}_b - \mathcal{H}^{ab} u_b + \mathcal{L}_X u^a, \quad (4.94)\]

\[(u^a) = g^{ab} \mathcal{U}_b - 2\mathcal{H}^{ab} \mathcal{U}_b + 2\mathcal{H}^{ac} \mathcal{H}_{cb} u^b - \mathcal{L}_b^a u^b
+ 2\mathcal{L}_X \left( g^{ab} (u_b) - h^{ab} u_b \right) + \left( \mathcal{L}_Y - \mathcal{L}_X^2 \right) (u^a). \quad (4.95)\]

We note that these expressions have the same forms as Eqs. (2.28) and (2.29).

Next, we consider the expansion of the energy-momentum tensor (4.73). Substituting the expansion (4.74)–(4.76) of the fluid components \(\bar{\epsilon}, \bar{p},\) and \(\bar{u}_a\) into (4.73), we have the perturbative form of the energy-momentum tensor:

\[
\bar{T}_a^b =: T_a^b + \lambda^{(1)} \left( T_a^b \right) + \frac{1}{2} \lambda^{(2)} \left( T_a^b \right) + O(\lambda^3), \quad (4.96)
\]

where

\[
(1) \left( T_a^b \right) = \left( \epsilon + \frac{1}{p} \right) u_a u^b + (\epsilon + p) u_a \left( u^b \right) + (\epsilon + p) \left( u_a \right) u^b + \frac{1}{2} \lambda^{(2)} \left( T_a^b \right), \quad (4.97)
\]

\[
(2) \left( T_a^b \right) = \left( \epsilon + \frac{1}{p} \right) u_a u^b + (\epsilon + p) u_a \left( u^b \right) + (\epsilon + p) \left( u_a \right) u^b + \frac{1}{2} \lambda^{(2)} \left( T_a^b \right)
+ 2 \left( \epsilon + \frac{1}{p} \right) u_a \left( u^b \right) + 2 \left( \epsilon + \frac{1}{p} \right) \left( u_a \right) u^b
+ 2 (\epsilon + p) \left( u_a \right) u^b. \quad (4.98)
\]

Further, we consider the decomposition of the perturbed energy-momentum tensor into its gauge invariant and variant parts. Using the definitions (4.84)–(4.86) of the gauge invariant variables of the first-order perturbations and Eq. (4.94), the first-order perturbation of the energy-momentum tensor of a perfect fluid is given by

\[
(1) \left( T_a^b \right) =: \left( 1 \right) T_a^b + \mathcal{L}_X T_a^b, \quad (4.99)
\]

where the gauge invariant part of the energy-momentum tensor is given by

\[
(1) \left( T_a^b \right) := \left( \epsilon + \frac{1}{p} \right) u_a u^b + \mathcal{P} \left( \frac{1}{p} \right) \delta_a^b
+ \left( \epsilon + p \right) \mathcal{U}_b - \mathcal{H}^{bc} u_c u_a + \mathcal{U}_a u^b. \quad (4.100)
\]

Similarly, using the definitions (4.84)–(4.89) of the gauge invariant variables of the first- and second-order perturbations and Eqs. (4.94) and (4.95), the second-order perturbation of the energy-momentum tensor of a perfect fluid is given by

\[
(2) \left( T_a^b \right) =: \left( 2 \right) T_a^b + 2 \mathcal{L}_X \left( 1 \right) T_a^b + \left\{ \mathcal{L}_Y - \mathcal{L}_X^2 \right\} T_a^b, \quad (4.101)
\]
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\[
(2) \mathcal{T}_a^b := \left( \mathcal{E} + \mathcal{P} \right) u_a u^b + 2 \left( \mathcal{E} + \mathcal{P} \right) u_a \left( \mathcal{U}^b - \mathcal{H}^{bc} u_c \right)
+ (\epsilon + p) u_a \left( g^{bc} \mathcal{U}_c - 2 \mathcal{H}^{bc} \mathcal{U}_c + 2 \mathcal{H}^{bc} \mathcal{H}_{cd} u^d - g^{bc} \mathcal{L}_{cd} u^d \right)
+ 2 \left( \mathcal{E} + \mathcal{P} \right) \mathcal{U}_a u^b + 2 (\epsilon + p) u_a \left( g^{bc} \mathcal{U}_c - \mathcal{H}^{bc} u_c \right)
+ (\epsilon + p) \mathcal{U}_a u^b + \mathcal{P} \delta_a^b. \tag{4.102}
\]

Here, again, we have seen that the perturbative expressions (4.99) and (4.101) of the energy-momentum tensor have the same forms as Eqs. (2.28) and (2.29), as expected. We also note that in the derivation of the expressions (4.99)–(4.102), we did not explicitly use any background values of the fluid component and the metric. Therefore, the expressions (4.99)–(4.102) are valid for any background spacetime. This implies that the definitions (2.46) and (2.47) of the gauge invariant variables in §2.4 are appropriate in the case of a perfect fluid.

Finally, we consider the perturbation of the equation of state for a fluid. In the generic case, the equation of state of a fluid is given by

\[
\bar{p} = \bar{p}(\bar{\epsilon}, \bar{S}), \tag{4.103}
\]

which gives the relation between the pressure \(\bar{p}\), the energy density \(\bar{\epsilon}\), and the entropy \(\bar{S}\). In addition to the perturbative expansions (4.74) and (4.75) for the energy density and pressure, we consider the perturbative expansion of the entropy:

\[
\bar{S} = S + \lambda \frac{(1)}{S} + \frac{1}{2} \lambda^2 \frac{(2)}{S} + O(\lambda^3). \tag{4.104}
\]

Hence, the generic equation of state (4.103) is expanded as

\[
p + \lambda \frac{(1)}{p} + \frac{1}{2} \lambda^2 \frac{(2)}{p} = \bar{p} \left( \epsilon + \lambda \frac{(1)}{\epsilon} + \frac{1}{2} \lambda^2 \frac{(2)}{\epsilon}, S + \lambda \frac{(1)}{S} + \frac{1}{2} \lambda^2 \frac{(2)}{S} \right) \tag{4.105}
\]

\[
= p(\epsilon, S) + \lambda \left( \frac{\partial \bar{p}}{\partial \epsilon} \frac{(1)}{\epsilon} + \frac{\partial \bar{p}}{\partial S} \frac{(1)}{S} \right)
+ \frac{1}{2} \lambda^2 \left( \frac{(2)}{\epsilon} \frac{\partial^2 \bar{p}}{\partial \epsilon^2} (\epsilon, S) + \frac{(1)}{\epsilon} \frac{\partial^2 \bar{p}}{\partial \epsilon \partial S} (\epsilon, S) + 2 \frac{(1)}{\epsilon} \frac{\partial \bar{p}^2}{\partial \epsilon \partial S} (\epsilon, S) \right) + \frac{(2)}{S} \frac{\partial \bar{p}}{\partial S} (\epsilon, S) + \frac{(1)}{S} \frac{\partial^2 \bar{p}}{\partial S^2} (\epsilon, S). \tag{4.106}
\]

Thus, we obtain the equation of state of the first- and second-order perturbation of the fluid components:

\[
p = \frac{\partial \bar{p}}{\partial \epsilon} \frac{(1)}{\epsilon} + \frac{\partial \bar{p}}{\partial S} \frac{(1)}{S}, \tag{4.107}
\]
\[
\begin{aligned}
(2) \quad P &= (2) \frac{\partial \bar{p}}{\partial \epsilon} (\epsilon, S) + \left(1\right)^2 \frac{\partial^2 \bar{p}}{\partial \epsilon^2} (\epsilon, S) + 2 \left(1\right) \frac{\epsilon}{S} \frac{\partial^2 \bar{p}}{\partial \epsilon \partial S} (\epsilon, S) \\
&+ \left(2\right) \frac{\partial \bar{p}}{\partial S} (\epsilon, S) + \left(2\right)^2 \frac{\partial^2 \bar{p}}{\partial S^2} (\epsilon, S).
\end{aligned}
\]  
(4.108)

In addition to the definitions of the gauge invariant variables (4.84)–(4.89) for the first- and second-order perturbations of the fluid components, we also define the gauge invariant variables for the entropy perturbations:

\[
(1) \quad S := S - \mathcal{L}_X S, 
\]  
(4.109)

\[
(2) \quad S := S - 2 \mathcal{L}_X (1)^2 S - \{ \mathcal{L}_Y - \mathcal{L}_X^2 \} S.
\]  
(4.110)

Substituting Eqs. (4.84), (4.87), and (4.109) into the perturbations of the equation of state (4.107), we obtain the first-order perturbation of the equation of state of the fluid in terms of the gauge invariant variables:

\[
(1) \quad \frac{\partial \bar{p}}{\partial \epsilon} (\epsilon, S) + \mathcal{L}_X \left( \frac{\partial \bar{p}}{\partial \epsilon} + \frac{\partial \bar{p}}{\partial S} \left( \frac{\partial \bar{p}}{\partial S} + \mathcal{L}_X S \right) \right) = \frac{\partial \bar{p}}{\partial \epsilon} \left( \frac{\partial \bar{p}}{\partial \epsilon} \right) + \mathcal{L}_X \frac{\partial \bar{p}}{\partial S}.
\]  
(4.111)

The right-hand side of Eq. (4.111) has the same form as Eq. (2.28), as expected by considering the left-hand side of Eq. (4.111). Hence, we obtain the first-order perturbation of the equation of state in terms of the gauge invariant variables:

\[
(1) \quad P = c_s^2 (1)^2 + \tau (1) S,
\]  
(4.112)

where we have

\[
c_s^2 := \frac{\partial \bar{p}}{\partial \epsilon}, \quad \tau := \frac{\partial \bar{p}}{\partial S},
\]  
(4.113)

and \( c_s \) is interpreted as the sound velocity of the fluid. The equation (4.112) is the equation of state for the gauge invariant variables of the first-order perturbation of the fluid components.

Next, we consider the second-order perturbation of the equation of state of the fluid in terms of gauge invariant variables:

\[
(2) \quad \mathcal{L}_X \left( \frac{\partial \bar{p}}{\partial \epsilon} + \left(1\right)^2 \frac{\partial \bar{p}}{\partial \epsilon^2} (\epsilon, S) + 2 \left(1\right) \frac{\epsilon}{S} \frac{\partial \bar{p}}{\partial \epsilon \partial S} (\epsilon, S) + \left(2\right) \frac{\partial \bar{p}}{\partial S} (\epsilon, S) + \left(1\right)^2 \frac{\partial^2 \bar{p}}{\partial S^2} (\epsilon, S) \right) \\
+ 2 \mathcal{L}_X \left( \frac{\partial \bar{p}}{\partial \epsilon} (\epsilon, S) + \frac{\partial \bar{p}}{\partial S} (\epsilon, S) \right) + \mathcal{L}_Y p(\epsilon, S) - \mathcal{L}_X^2 p(\epsilon, S).
\]  
(4.114)

The right-hand side of Eq. (4.114) has the same form as expected by considering the left-hand side of Eq. (4.114). Then, we obtain the second-order perturbation of the
equation of state in terms of the gauge invariant variables:

\[
\mathcal{P} = c_s^2 \frac{(2)}{E^2} + \tau \frac{(2)}{S} + \frac{\partial}{\partial \tau} \frac{(2)}{E^2} + \frac{2}{\partial \tau} \frac{\partial^2}{\partial S^2} \frac{(1)}{E^2} + \frac{\partial}{\partial S} \frac{(1)}{E^2} ,
\]

(4-115)

where we have used Eqs. (4.113).

4.3.2. Scalar field

Here, we consider the energy-momentum tensor of the single scalar field \( \bar{\varphi} \),

\[
\bar{T}_a^b = \nabla_a \bar{\varphi} \nabla_b \bar{\varphi} - \frac{1}{2} \delta_a^b \left( \nabla_c \bar{\varphi} \nabla^c \bar{\varphi} + 2V(\bar{\varphi}) \right) ,
\]

(4-116)

where \( V(\bar{\varphi}) \) is the potential of \( \bar{\varphi} \). Because we are considering a perturbation theory on a homogeneous and isotropic universe, the scalar field must also be approximately homogeneous. Hence, the scalar field \( \bar{\varphi} \) can be expanded as

\[
\bar{\varphi} = \varphi + \lambda \hat{\varphi}_1 + \frac{1}{2} \lambda^2 \hat{\varphi}_2 + O(\lambda^3) ,
\]

(4-117)

where \( \varphi = \varphi(\eta) \) is a homogeneous function on the homogeneous isotropic universe. The background field \( \varphi \) is the homogeneous part of the scalar field which drives the background homogeneous isotropic model, and \(|\hat{\varphi}_1| \ll |\varphi| \) and \(|\hat{\varphi}_2| \ll |\hat{\varphi}_1| \) are the first- and second-order perturbations of the scalar field \( \varphi \), respectively. The energy-momentum tensor (4-116) can also be decomposed into the background, the first-order perturbation, and the second-order perturbation as

\[
\bar{T}_a^b = T_a^b + \lambda \left( T_a^b \right)^{(1)} + \frac{1}{2} \lambda^2 \left( T_a^b \right)^{(2)} + O(\lambda^3) ,
\]

(4-118)

where \( \left( T_a^b \right)^{(1)} \) is linear in the metric and the matter perturbations \( h_{ab} \) and \( \hat{\varphi}_1 \), and \( \left( T_a^b \right)^{(2)} \) includes the second-order metric and matter perturbations, \( l_{ab} \) and \( \hat{\varphi}_2 \), and the quadratic terms of the first-order perturbations, \( \hat{\varphi}_1 \) and \( h_{ab} \).

Expanding the metric as in Eq. (2.21) and the scalar field as in Eq. (4-117), the perturbations \( \left( T_a^b \right)^{(1)} \) and \( \left( T_a^b \right)^{(2)} \) of the energy-momentum tensor (4-116) are given by

\[
\left( T_a^b \right)^{(1)} := \nabla_a \varphi \nabla^c \hat{\varphi}_1 - \nabla_a \varphi h^{bc} \nabla_c \varphi + \nabla_a \hat{\varphi}_1 \nabla^b \varphi - \frac{1}{2} \delta_a^b \left( \nabla_c \varphi \nabla^c \hat{\varphi}_1 - \nabla_c \varphi h^{de} \nabla_d \varphi + \nabla_c \hat{\varphi}_1 \nabla^c \varphi + 2\hat{\varphi}_1 \frac{\partial V}{\partial \varphi} \right) ,
\]

(4-119)

\[
\left( T_a^b \right)^{(2)} := \nabla_a \varphi \nabla^b \hat{\varphi}_2 - 2\nabla_a \varphi h^{bc} \nabla_c \hat{\varphi}_1 + \nabla_a \varphi \left( 2h^{bd} h_d^c - l^{bc} \right) \nabla_c \varphi + 2\nabla_a \hat{\varphi}_1 \nabla^b \hat{\varphi}_1 - 2\nabla_a \hat{\varphi}_1 h^{bc} \nabla_c \varphi + \nabla_a \hat{\varphi}_2 g^{bc} \nabla_c \varphi - \frac{1}{2} \delta_a^b \left( \nabla_c \varphi \nabla^c \hat{\varphi}_2 - 2\nabla_c \varphi h^{de} \nabla_d \hat{\varphi}_1 + \nabla_c \varphi \left( 2h^{de} h_d^c - l^{de} \right) \nabla_d \varphi + 2\nabla_c \hat{\varphi}_1 \nabla^c \hat{\varphi}_1 - 2\nabla_c \hat{\varphi}_1 h^{de} \nabla_d \varphi + \nabla_c \hat{\varphi}_2 \nabla^c \varphi + 2\hat{\varphi}_2 \frac{\partial V}{\partial \varphi} + 2(\hat{\varphi}_1)^2 \frac{\partial^2 V}{\partial \varphi^2} \right) .
\]

(4-120)
According to the decompositions (2.28) and (2.29), the perturbations of the scalar field $\varphi$ at each order can be decomposed into the gauge invariant and variant parts as

$$\hat{\varphi}_1 =: \varphi_1 + L_X \varphi,$$

$$\hat{\varphi}_2 =: \varphi_2 + 2L_X \hat{\varphi}_1 + (L_Y - L_X^2) \varphi,$$

where $\varphi_1$ and $\varphi_2$ are the first-order and second-order gauge invariant perturbations of the scalar field. Through these gauge invariant variables, the perturbed energy-momentum tensor at each order can also be decomposed into the gauge invariant and variant parts. Substituting Eqs. (4.121) and (2.22) into Eq. (4.119), the first-order perturbation (4.119) of the scalar field is given by

$$^{(1)}(T^a{}_b) =: ^{(1)}T^a{}_b + L_X T^a{}_b,$$

where

$$^{(1)}T^a{}_b := \nabla^a \varphi \nabla^b \varphi_1 - \nabla^a \varphi \mathcal{H}^{bc} \nabla_c \varphi + \nabla^a \varphi_1 \nabla^b \varphi$$

$$-\frac{1}{2} \delta^a{}_b \left( \nabla_c \varphi \nabla^c \varphi_1 - \nabla_c \varphi \mathcal{H}^{dc} \nabla_d \varphi + \nabla_c \varphi_1 \nabla^c \varphi + 2 \varphi_1 \frac{\partial V}{\partial \varphi} \right),$$

is the gauge invariant part of the first-order perturbation of the energy-momentum tensor for the single scalar field. Through Eqs. (4.121), (4.122), (2.22), and (2.24), the second-order perturbation (4.120) of the energy-momentum tensor is given by

$$^{(2)}(T^a{}_b) =: ^{(2)}T^a{}_b + 2L_X (^{(1)}T^a{}_b) + (L_Y - L_X^2) T^a{}_b,$$

where

$$^{(2)}T^a{}_b = \nabla^a \varphi \nabla^b \varphi_2 - 2 \nabla^a \varphi \mathcal{H}^{bc} \nabla_c \varphi_1 + 2 \nabla^a \varphi \mathcal{H}^{bd} \mathcal{H}_{dc} \nabla^c \varphi - \nabla^a \varphi \mathcal{H}^{bd} \mathcal{L}_{dc} \nabla^c \varphi$$

$$+ 2 \nabla^a \varphi_1 \nabla^b \varphi_1 - 2 \nabla^a \varphi_1 \mathcal{H}^{bc} \nabla_c \varphi + \nabla^a \varphi_2 \nabla^b \varphi$$

$$-\frac{1}{2} \delta^a{}_b \left( \nabla_c \varphi \nabla^c \varphi_2 - 2 \nabla_c \varphi \mathcal{H}^{de} \nabla_d \varphi_1 + 2 \nabla^c \varphi \mathcal{H}^{de} \mathcal{H}_{dc} \nabla^c \varphi - \nabla_c \varphi \mathcal{L}_{dc} \nabla^d \varphi$$

$$+ 2 \nabla_c \varphi_1 \nabla^c \varphi_1 - 2 \nabla_c \varphi_1 \mathcal{H}^{de} \nabla_d \varphi + \nabla_c \varphi_2 \nabla^c \varphi$$

$$+ 2 \varphi_2 \frac{\partial V}{\partial \varphi} + 2 \varphi_1^2 \frac{\partial^2 V}{\partial \varphi^2} \right).$$

The tensor $^{(2)}T^a{}_b$ is the second-order gauge invariant part of the energy-momentum tensor for the single scalar field. Here again, we have seen that the perturbative expressions (4.123) and (4.125) of the energy-momentum tensor have the same forms as Eqs. (2.28) and (2.29), respectively, as expected. We also note that in the derivation of the expressions (4.123)–(4.126), we did not explicitly use any background values of the scalar field nor the metric. Therefore, the expressions (4.123)–(4.126) are valid for any background spacetime. This implies that the definitions (2.40) and (2.47) of the gauge invariant variables of the perturbed energy-momentum tensor in §2.4 are appropriate not only in the case of a perfect fluid but also in the case of a single scalar field.
§5. First-order Einstein equations

In this section, we consider the perturbed Einstein equations of linear order, \([2.49]\). To derive the components of the gauge invariant part of the linearized Einstein tensor \((1)G^a_b[H]\), which is defined by Eqs. \((2.37)\) and \((2.39)\), we first derive the components of the tensor \(H_{ab}^c[H]\), which is defined in Eq. \((2.41)\) with \(A_{ab} = H_{ab}\). Since the components of the gauge invariant part \(H_{ab}\) of the first-order metric perturbation are given by Eq. \((4.43)\), the components of the tensor \(H_{ab}^c[H]\) are as follows:

\[
H_{\eta\eta}^i[H] = \partial_\eta \eta^i, \tag{5.1}
\]

\[
H_{\eta\nu}^i[H] = D_i^\nu + \mathcal{H} \nu_i, \tag{5.2}
\]

\[
H_{ij}^\nu[H] = -\left(2\mathcal{H} \eta^{(1)} + \partial_\eta \eta^{(1)}\right) \gamma_{ij} - D_{(i}^\nu j)} + \frac{1}{2} \left(\partial_\eta + 2\mathcal{H}\right) \chi_{ij}, \tag{5.3}
\]

\[
H_{\eta j}^i[H] = D_i^\nu \eta^j + (\partial_\eta + \mathcal{H}) \nu_j^i, \tag{5.4}
\]

\[
H_{j\eta}^i[H] = -\partial_\eta \eta^{i} \eta^j + \frac{1}{2} \left(D_j \nu^i - D_i \nu^j\right) + \frac{1}{2} \partial_\eta \chi_i^j, \tag{5.5}
\]

\[
H_{jk}^i[H] = D_i^\nu \eta^k - 2\gamma^i_{(k} D_j^\nu \eta^l_{(j)} - \mathcal{H} \gamma^i_{k\eta} - D_{(j}^i \chi_{k)} - \frac{1}{2} D^j \chi_{k j}. \tag{5.6}
\]

The components of the tensors \(H_{abc}[\mathcal{H}], H_{a}^{bc}[\mathcal{H}], H_{a}^{b c}[\mathcal{H}],\) and \(H^{abc}[\mathcal{H}]\) are also useful when we derive the components of the gauge invariant parts \((1)G^a_b[H]\) and \((2)G^a_c[H, \mathcal{H}]\) of the perturbative Einstein tensor. These components are summarized in Appendix A.

Following to the definitions \((2.37)\) and \((2.39)\) of the gauge invariant part \((1)G^a_b[H]\) of the first-order perturbation of the Einstein tensor, its components are derived as follows:

\[
(1)G^\eta_{\eta i} [H] = \frac{-1}{a^2} \left\{(-6\mathcal{H}\partial_\eta + 2\Delta + 6\mathcal{K}) \eta^{(1)} \phi^{(1)} + 6\mathcal{H} \phi^{(1)}\right\}, \tag{5.7}
\]

\[
(1)G^\eta_{\nu i} [H] = \frac{-1}{a^2} \left(2\partial_\eta D_i^\nu + 2\mathcal{H} D_i^\nu \phi - \frac{1}{2} \left(\Delta + 2\mathcal{K}\right) \nu_i\right), \tag{5.8}
\]

\[
(1)G^i_{\eta j} [H] = \frac{1}{a^2} \left\{2\partial_\eta D_i^j \eta^{(1)} + 2\mathcal{H} D_i^j \eta^{(1)} + \frac{1}{2} \left(-\Delta + 2\mathcal{K} + 4\mathcal{H}^2 - 4\partial_\eta \mathcal{H}\right) \nu^j\right\}, \tag{5.9}
\]

\[
(1)G^j_{\nu i} [H] = \frac{1}{a^2} \left[D_i D^j \left(\eta^{(1)} \phi - \eta \phi\right) + \left(-\Delta + 2\mathcal{K} + 4\mathcal{H} \partial_\eta - 2\mathcal{K}\right) \eta^j + (2\mathcal{H} \partial_\eta + 4\partial_\eta \mathcal{H} + 2\mathcal{H}^2 + \Delta) \phi\right] \gamma_i^j - \frac{1}{2a^2 \partial_\eta} \left\{a^2 \left(D_i^{(1)} \nu^j + D_j^{(1)} \nu_i\right)\right\}
\]
Straightforward calculations show that these components of the first-order gauge invariant perturbation \( G_a^b [\mathcal{H}] \) of the Einstein tensor satisfies the first-order perturbation \( 2.43 \) of the Bianchi identity. This implies that we have derived the components \( 5.7 - 5.10 \) of \( G_a^b [\mathcal{H}] \) consistently.

Together with the components of the gauge invariant part \( T_a^b \) of the first-order perturbation of the energy-momentum tensor, the first-order Einstein equation \( 2.49 \) is given as equations for the gauge invariant variables \( \phi, \psi, \nu_i, \) and \( \chi_{ij} \). We consider these equations for two cases, that in which the energy-momentum tensor is dominated by the single perfect fluid and that in which it is dominated by the single scalar field.

5.1. Perfect fluid case

Here, we consider the linearized Einstein equation of a homogeneous isotropic universe filled with a perfect fluid.

We first consider the components of the gauge invariant part \( U_a \) of the perturbative four-velocity of the fluid. Taking into account the perturbation \( 4.91 \) of the normalization condition \( 4.77 \), the components of \( U_a \) are decomposed as

\[
(1) U_a = -a (1) \phi (d \eta)_a + \left( D_i (1) \nu^i + (1) \mathcal{V}_i \right) (d x^i)_a, \quad D_i (1) \mathcal{V}_i = 0, \tag{5.11}
\]

where the \( \eta \)-component of \( U_a \) are determined by Eq. \( 4.91 \). Here, we note that the divergenceless part of the spatial component of the four-velocity, \( (1) \mathcal{V}_i \), contributes to the vorticity perturbation. Substituting \( 3.10 \) and \( 5.11 \) into \( 4.100 \), the components of the gauge invariant part \( T_a^b \) of the first-order perturbation of the energy-momentum tensor are obtained as

\[
(1) T_{\eta \eta} = - (1) E, \tag{5.12}
\]

\[
(1) T_{\eta i} = (\epsilon + p) \left\{ (1) \nu^i - \left( D_i (1) \nu^i + (1) \mathcal{V}_i \right) \right\}, \tag{5.13}
\]

\[
(1) T_{i \eta} = (\epsilon + p) \left( D_i (1) \nu^i + (1) \mathcal{V}_i \right), \tag{5.14}
\]

\[
(1) T_{i j} = (1) \mathcal{P} \gamma_i^j. \tag{5.15}
\]

Through Eqs. \( 5.7 - 5.10 \) and \( 5.12 - 5.15 \), the linearized Einstein equations \( 2.49 \) are found to be

\[
4\pi G a^2 (1) E = (-3 \mathcal{H} \partial_{\eta} + \Delta + 3K) (1) \psi - 3 \mathcal{H}^2 (1) \phi, \tag{5.16}
\]
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\[ 4\pi Ga^2 (\epsilon + p) \left( D_i \frac{(1)}{\nu_i} + \frac{(1)}{\nu_i} \right) = -\partial_\eta D_i \frac{(1)}{\Psi} - \mathcal{H} D_i \frac{(1)}{\Phi} + \frac{1}{4} (\Delta + 2K) \frac{(1)}{\nu_i} , \quad (5.17) \]

\[ 4\pi Ga^2 \frac{(1)}{\mathcal{P}} \gamma_i^j = \frac{1}{2} D_i D^j \left( \frac{(1)}{\psi} - \frac{(1)}{\Phi} \right) \]

\[ + \left\{ \left( \partial_\eta^2 + 2\mathcal{H}\partial_\eta - K - \frac{1}{2}\Delta \right) \psi + \left( \mathcal{H}\partial_\eta + 2\partial_\eta\mathcal{H} + \mathcal{H}^2 + \frac{1}{2}\Delta \right) \Phi \right\} \gamma_i^j \]

\[ - \frac{1}{4a^2 \partial_\eta} \left\{ a^2 \left( D_i \frac{(1)}{\nu^j} + D^j \frac{(1)}{\nu_i} \right) \right\} \]

\[ + \frac{1}{4} \left( \partial_\eta^2 + 2\mathcal{H}\partial_\eta + 2K - \Delta \right) \chi_i^j , \quad (5.18) \]

where the component \( \frac{(1)}{G}_{\gamma_i^j} [\mathcal{H}] = 8\pi G \frac{(1)}{T}_{\gamma_i^j} \) is identical to Eq. \((5.17)\) by virtue of the background Einstein equation \((3.15)\).

We decompose Eqs. \((5.16) - (5.18)\) similarly to Eqs. \((4.2) - (4.4)\) for the metric perturbation \(h_{ab}\), whose inverse relations are given by Eqs. \((4.9) - (4.15)\). Then, we obtain the equations for the scalar mode perturbations as

\[ 4\pi Ga^2 \frac{(1)}{\mathcal{E}} = \left( -3\mathcal{H}\partial_\eta + \Delta + 3K \right) \frac{(1)}{\psi} - 3\mathcal{H}^2 \frac{(1)}{\Phi} , \quad (5.19) \]

\[ 4\pi Ga^2 (\epsilon + p) D_i \frac{(1)}{v} = -\partial_\eta D_i \frac{(1)}{\Psi} - \mathcal{H} D_i \frac{(1)}{\Phi} , \quad (5.20) \]

\[ 4\pi Ga^2 \mathcal{P} = \left( \partial_\eta^2 + 2\mathcal{H}\partial_\eta - K - \frac{1}{3}\Delta \right) \psi \]

\[ + \left( \mathcal{H}\partial_\eta + 2\partial_\eta\mathcal{H} + \mathcal{H}^2 + \frac{1}{3}\Delta \right) \Phi , \quad (5.21) \]

\[ \frac{1}{3}\Delta (\Delta + 3K) \left( \frac{(1)}{\psi} - \frac{(1)}{\Phi} \right) = 0 , \quad (5.22) \]

the equations for the vector-mode perturbation as

\[ 4\pi Ga^2 (\epsilon + p) \frac{(1)}{\nu_i} = \frac{1}{4} (\Delta + 2K) \frac{(1)}{\nu_i} , \quad (5.23) \]

\[ \partial_\eta \left\{ a^2 (\Delta + 2K) \frac{(1)}{\nu^j} \right\} = 0 , \quad (5.24) \]

and the equation for the tensor-mode perturbation as

\[ \left( \partial_\eta^2 + 2\mathcal{H}\partial_\eta + 2K - \Delta \right) \chi_{ij} = 0 . \quad (5.25) \]

Equation \((5.25)\) describes the evolution of gravitational waves.

Equation \((5.22)\) yields

\[ \frac{(1)}{\psi} = \frac{(1)}{\Phi} . \quad (5.26) \]
From this equation, the energy density perturbation \( \epsilon \), the velocity perturbation \( D_i v \), and the pressure perturbation \( P \) are found to satisfy

\[
4\pi G a^2 \epsilon = (\Delta - 3H \partial_\eta - 3(H^2 - K)) \Phi ,
\]
\( (5.27) \)

\[
4\pi G a^2 (\epsilon + p) D_i v = -\partial_\eta D_i \Phi - H D_i \Phi ,
\]
\( (5.28) \)

\[
4\pi G a^2 P = (\partial^2 \eta + 3H \partial_\eta + 2\partial_\eta H + H^2 - K) \Phi .
\]
\( (5.29) \)

In the Newtonian limit, Eq. (5.27) reduces to the usual Poisson equation for the gravitational potential \( \Phi \) induced by the energy-density perturbation \( \epsilon \). This supports the interpretation of \( \Phi \) as the relativistic generalization of the Newtonian gravitational potential. Equation (5.27) is the generalized form of the Poisson equation obtained by taking into account the expansion of the universe.

Next, we apply the equation of state \((4.112)\) for the first order perturbation. Then, from Eqs. (5.19) and (5.21), we obtain the well-known master equation for the scalar mode perturbation:\( (2,3) \)

\[
\{ \partial^2 \eta + 3H(1 + c_s^2) \partial_\eta - c_s^2 \Delta + 2\partial_\eta H + (1 + 3c_s^2)(H^2 - K) \} \Phi
\]
\( = 4\pi G a^2 \tau S . \)
\( (5.30) \)

The scalar mode perturbations are completely determined by this master equation \((5.30)\). If we obtain the solution \( \Phi \) to Eq. \((5.30)\), we can obtain another scalar perturbation \( \psi \) through Eq. \((5.26)\), and the energy density perturbation \( \epsilon \), the velocity perturbation \( D_i v \), and the pressure perturbation \( P \) are obtained from Eqs. \((5.27)\), \((5.28)\), and \((5.29)\), respectively. It is also well-known that Eq. \((5.30)\) is reduced to simpler equation through a change of variables.\( ^3 \)

Here, we comment on the contribution of the anisotropic stress, which is ignored in the above derivation of the linearized Einstein equation. If an anisotropic stress exists in the linear-order energy-momentum tensor, these can be formally decomposed into scalar, vector and tensor types in forms similar to Eqs. \((4.3)\) and \((4.4)\). If these exist, the anisotropic stress of scalar type contributes to the scalar mode of the perturbation and will appear on the right hand side of Eq. \((5.22)\). In this case, the equation \((5.26)\) is no longer valid. Instead, \( \psi - \Phi \) is proportional to the anisotropic stress of scalar type. As a result, the master equation \((5.30)\) will have a source term which is proportional to the anisotropic stress, in addition to the entropy perturbation. The anisotropic stress of vector type contributes as the source term of Eq. \((5.24)\), if it exists at linear order. Though the solution \( \nu_i \) to Eq. \((5.24)\) is only the decaying mode in the absence of the anisotropic stress of vector type, the vector perturbation \( \nu_i \) of the metric is generated by the anisotropic stress of vector type,
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if it exits, and the resulting vector perturbation, \( \nu_i \), of the metric directly generates the divergenceless part, \( \nu_i \), of the four-velocity of the fluid, which may contribute to the vorticity of the fluid. Finally, if the anisotropic stress of tensor type exists, it appears as the source term in Eq. (5.25), and it generates gravitational waves, \( \chi_{ij} \). Thus, the anisotropic stress may generate all types of perturbations. The situation is similar in the case of the second-order perturbations, as shown below.

5.2. Scalar field case

Here, we consider the linearized Einstein equation of a homogeneous isotropic universe filled with a single scalar field.

We note that the background scalar field \( \varphi \) is homogeneous, i.e., \( \varphi = \varphi(\eta) \), where \( \eta \) is the conformal time. Thus, the components of the gauge invariant part of the first-order energy-momentum tensor \( \mathcal{T}_a^b \) are given by

\[
\begin{align*}
\mathcal{T}_\eta^\eta &= -\frac{1}{a^2} \left( \partial_\eta \varphi_1 \partial_\eta \varphi - \Phi (\partial_\eta \varphi)^2 + a^2 \frac{dV}{d\varphi} \varphi_1 \right), \\
\mathcal{T}_i^\eta &= -\frac{1}{a^2} D_i \varphi_1 \partial_\eta \varphi, \\
\mathcal{T}_\eta^i &= \frac{1}{a^2} \partial_\eta \varphi \left( D_i \varphi_1 + (\partial_\eta \varphi) \nu^i \right), \\
\mathcal{T}_j^i &= \frac{1}{a^2} \gamma^j_i \left( \partial_\eta \varphi_1 \partial_\eta \varphi - \Phi (\partial_\eta \varphi)^2 - a^2 \frac{dV}{d\varphi} \varphi_1 \right).
\end{align*}
\]

Equation (5.35) shows that there is no anisotropic stress in the energy-momentum tensor of the single scalar field. Then, as in the case of a perfect fluid, we obtain Eq. (5.26). From Eqs. (5.7)–(5.10), (5.26), and (5.31)–(5.34), the components of the linearized Einstein equation (2.49) are obtained as

\[
\begin{align*}
\partial_\eta (\Phi) &= 4\pi G \frac{\partial_\eta \varphi_1}{\partial_\eta \varphi}, \\
\partial_\eta^2 (\Phi) + \Delta (\Phi) &= 4\pi G \partial_\eta \varphi_1 \partial_\eta \varphi, \\
\partial_\eta^2 \partial_\eta (\Phi) + \partial_\eta \partial_\eta (\Phi) &= 4\pi G \partial_\eta \varphi_1 \partial_\eta \varphi. 
\end{align*}
\]

In the derivation of Eqs. (5.35)–(5.37), we have used Eq. (3.15). We also note that only two of these equations are independent. Further, the vector part of the component \( \mathcal{G}_i^\eta [\mathcal{H}] = 8\pi G \mathcal{T}_i^\eta \) of the Einstein equations shows that there is no vector mode as an initial value constraint. The equation for the tensor mode \( \mathcal{T}_i^\eta \) is identical to Eq. (5.25).

Combining Eqs. (5.35) and (5.37), we eliminate the potential term of the scalar field and thereby obtain

\[
\begin{align*}
\partial_\eta^2 + \Delta + 4K \Phi &= 8\pi G \partial_\eta \varphi_1 \partial_\eta \varphi.
\end{align*}
\]
Further, using Eq. (5.36) to express $\partial_{\eta}\varphi_1$ in terms of $\Phi^{(1)}$ and $\phi^{(1)}$, we also eliminate $\partial_{\eta}\varphi_1$ in Eq. (5.38). Hence, we have

$$\left(\partial_{\eta}^2 + 2\left(\mathcal{H} - \frac{2\partial_{\eta}^2 \varphi}{\partial_{\eta} \varphi}\right)\partial_{\eta} - \Delta - 4K + 2\left(\partial_{\eta} \mathcal{H} - \frac{\mathcal{H} \partial_{\eta}^2 \varphi}{\partial_{\eta} \varphi}\right)\right)\Phi^{(1)} = 0. \quad (5.39)$$

This is the master equation for the scalar mode perturbation of the cosmological perturbation in universe filled with a single scalar field. It is also known that Eq. (5.39) reduces to a simple equation through a change of variables, and this equation has the same form as Eq. (5.30) with $c_s = 1$.

### §6. Second-order Einstein equations

In this section, we derive the second-order perturbation of the Einstein equation in the context of cosmological perturbations. In the generic case, the second order perturbation of the Einstein equation is given by Eq. (2.50). This equation is for the gauge invariant second-order metric perturbation $\mathcal{L}_{ab}$, whose components are given in Eq. (4.71). To derive the equations for each component of $\mathcal{L}_{ab}$, we have to evaluate $^{(1)} G_a^b [\mathcal{L}]$, $^{(2)} G_a^b [\mathcal{H}, \mathcal{H}]$, and $^{(2)} T_a^b$.

In this paper, we consider the simple situation in which the first-order vector and tensor modes are negligible:

$$^{(1)} \nu_i = 0, \quad ^{(1)} \chi_{ij} = 0. \quad (6.1)$$

In the linear-order perturbation, the vector mode $^{(1)} \nu_i$ decays as $a^{-2}$, which becomes smaller than the scalar perturbation. Further, as seen in §5.2, the vector mode $^{(1)} \nu_i$ is not generated in an inflationary universe driven by a single scalar field. For these reasons, it is reasonable to omit the vector mode $^{(1)} \nu_i$ for a wide class of scenarios of the evolution of the universe. By contrast, the tensor mode $^{(1)} \chi_{ij}$, i.e., gravitational waves, may be generated by quantum fluctuations during the inflation. The amplitude of these stochastic gravitational waves depends on the scenario of the inflation. However, from the observational result of CMB, $^{(1)}$ scalar mode fluctuations should be dominant, with the scalar-tensor ratio being less than unity. We can thus assume that the dominant contribution to the fluctuations in the universe is that of scalar type. Hence, we ignore the first-order tensor mode $^{(1)} \chi_{ij}$ in this paper.

Of course, it is possible to extend our formulation by taking into account of the vector- and tensor-mode contributions to the second-order perturbations, but we only consider the main contribution to the second-order fluctuations in this paper.

Because the components of the second-order gauge invariant metric perturbation $\mathcal{L}_{ab}$ are obtained through the replacements

$$\Phi^{(1)} \rightarrow \Phi^{(2)}, \quad ^{(1)} \nu_i \rightarrow ^{(2)} \nu_i, \quad ^{(1)} \psi \rightarrow ^{(2)} \psi, \quad ^{(1)} \chi_{ij} \rightarrow ^{(2)} \chi_{ij} \quad (6.2)$$

of the variables in the first-order gauge invariant metric perturbation $\mathcal{H}_{ab}$, the evaluation of $^{(1)} G_a^b [\mathcal{L}]$ is accomplished through these replacements in the first-order
gauge invariant Einstein tensor \((1)G^b_a[H]\), which appear in Eqs. \((5.7) - (5.10)\). Then, the components of \((1)G^b_a[L]\) are given by

\[
(1)G^\eta_\eta[L] = -\frac{1}{a^2} \left\{ (-6H\partial_\eta + 2\Delta + 6K) \psi^2 - 6H^2 \phi \right\},
\]

\[
(1)G^i_\eta[L] = -\frac{1}{a^2} \left( 2\partial_\eta D_i \psi^2 + 2\partial_\eta H \phi - \frac{1}{2} \Delta \right),
\]

\[
(1)G^i_j[L] = \frac{1}{a^2} \left\{ 2\partial_\eta D^i \psi^2 + 2\partial_\eta H \phi + \frac{1}{2} \left( -\Delta + 2K + 4H^2 - 4\partial_\eta H \right) \right\},
\]

\[
(1)G^i_j[L] = \frac{1}{a^2} \left[ D_i D^j \left( \psi^2 - \phi^2 \right) + \frac{1}{2} \left( \partial^2 \phi + 2\partial_\eta H \phi + 2\Delta - \Delta \right) \right].
\]

In the simple situation described by Eq. \((6.3)\), the components of the quadratic term \((2)G^b_a[H, H]\) of the linear order perturbations, which are defined by Eq. \((2.38)\), are given by

\[
(2)G^\eta_\eta = \frac{2}{a^2} \left\{ 12H\partial_\eta \left[ \frac{1}{2}(\psi^2 - \phi^2) - 12 \left( K \left( \frac{1}{2} \psi^2 - \frac{1}{2} \phi^2 \right) \right) - 3 \left( D_k \left( \frac{1}{2} \psi^2 + \partial_\eta \left( \frac{1}{2} \psi^2 \right) \right) \right) - 8 \Delta \right\},
\]

\[
(2)G^i_\eta = \frac{4}{a^2} \left\{ 2\partial_\eta \left( \frac{1}{2} \psi^2 - \phi^2 \right) + \partial_\eta \psi \left( \frac{1}{2} \psi^2 - \phi^2 \right) + 2 \left( \frac{1}{2} \psi^2 - \phi^2 \right) \right\},
\]

\[
(2)G^i_j = \frac{4}{a^2} \left\{ 4H \left( \frac{1}{2} \psi^2 - \phi^2 \right) \right\},
\]

\[
(2)G^i_j = \frac{2}{a^2} \left[ D_i \left( \frac{1}{2} \psi^2 - \phi^2 \right) - D_j \left( \frac{1}{2} \psi^2 - \phi^2 \right) - \frac{1}{2} \left( \partial^2 \phi + 2\partial_\eta H \phi + 2\Delta - \Delta \right) \right].
\]
\[-2D_k \Psi D^k \Psi - D_k \Phi D^k \Phi + 2 \left( \Psi - \Phi \right) \Delta \Phi \]

\[-4 \Psi \Delta \Psi \right\} \gamma_{ij} \right). \quad (6.10)\]

Through the components (3.7) of the background Einstein tensor \(G^a_b\), the components (5.1)–(5.6) of the tensor field \(H_{c}^{ab}\), and the components (5.7)–(5.10) of the gauge invariant part \((1)G^b_a[H]\) of the linear-order perturbation of the Einstein tensor, it is straightforward to confirm that \((2)G^b_a[H, H]\) of the components (6.7)–(6.10) satisfies the second-order perturbation of the Bianchi identity (2.44), i.e.,

\[\nabla_a (2)G^a_b [H, H] = -2H_{ca}^a [H] (1)G^b_c [H] + 2H_{ba}^c [H] (1)G^a_c [H] - 2H_{bad} [H] \mathcal{H}^{de}G^e_c + 2H_{cad} [H] \mathcal{H}^{ad}G^c_b. \quad (6.11)\]

This implies that we have consistently derived the components (6.7)–(6.10).

Further, we impose the relation (5.26), which can always be derived from the first-order Einstein equation when the anisotropic stress of scalar type is negligible. Then, the gauge invariant part of the first-order metric perturbation is given by

\[\mathcal{H}_{ab} = -2a^2 (1)\Phi (d\eta)_a (d\eta)_b - 2a^2 (1)\Phi \gamma_{ij} (dx^i)_a (dx^j)_b, \quad (6.12)\]

and the components (6.7)–(6.10) of the tensor \((2)G^b_a[H, H]\) are reduced as follows:

\[(2)G^\eta_\eta = -\frac{2}{a^2} \left\{ 3D_k (1)\Phi D^k (1)\Phi + 3 \left( \partial_\eta (1)\Phi \right)^2 + 8 (1)\Phi \Delta (1) \phi \right\} \gamma_{ij}, \quad (6.13)\]

\[(2)G^\eta_i = \frac{4}{a^2} \left( 4\mathcal{H} (1)\Phi D_i (1)\Phi + 4 (1)\Phi \partial_\eta (1)\Phi D_i (1) \phi \right), \quad (6.14)\]

\[(2)G^i_\eta = \frac{4}{a^2} \left( \partial_\eta (1)\Phi D^i (1)\Phi + 4 (1)\Phi \partial_\eta D^i (1) \phi \right), \quad (6.15)\]

\[(2)G^j_i = \frac{2}{a^2} \left[ 2D_i (1)\Phi D^j (1)\Phi + 4 (1)\Phi D_i D^j (1) \phi \right. \right.

\[\left. - \left( 3D_k (1)\Phi D^k (1)\Phi + 4 (1)\Phi \Delta (1) \phi + \left( \partial_\eta (1)\Phi \right)^2 \right) \right.

\[+ 8\mathcal{H} (1)\Phi \partial_\eta (1)\phi + 4 (2\partial_\eta \mathcal{H} + K + \mathcal{H}^2) \left( (1)\phi \right)^2 \right] \right\} \gamma_{ij}. \quad (6.16)\]

Thus, we have evaluated \((1)G^a_b[L]\) and \((2)G^a_b[H, H]\) in the second-order Einstein equation (2.50). Next, we evaluate \((2)T^{ab}\) separately in two cases, that of a universe filled with a single perfect fluid and that of the universe filled with a single scalar field.
6.1. Perfect fluid case

Here, we consider the second-order perturbation of the Einstein equation in the case of a universe filled with a single-component perfect fluid.

Because we concentrate only on the case in which the vector and tensor modes are negligible, as in Eq. (6.1), we should ignore the divergenceless part of the spatial velocity of the fluid, setting

$$V^i = 0,$$  \hspace{1cm} (6.17)

in accordance with the first-order Einstein equation (5.23). Then, the components of the gauge invariant first-order perturbation of the fluid four-velocity are given by

$$u^a = a (\Phi (d\eta)_a + aD_i v^i (dx^i)_a)$$  \hspace{1cm} (6.18)

In the simple situation that Eqs. (6.1) and (6.17) are satisfied, the normalization condition (4.93) of the second-order perturbation $u^a$ of the fluid four-velocity is given by

$$u^a u_a = \left(\frac{\phi}{\phi}\right)^2 - D_i v^i D^j v_j - \phi,$$  \hspace{1cm} (6.19)

and the components of $u_a$ are decomposed as

$$u^a u_a = a \left(\frac{\phi}{\phi}\right)^2 - D_i v^i D^j v_j - \phi,$$  \hspace{1cm} (6.19)

$$D^i \phi = 0.$$  \hspace{1cm} (6.21)

Then, the components of the gauge invariant part $(2)T^b_a$ of the second-order perturbation of the energy-momentum tensor are given by

$$(2)T^\eta_\eta = -2(\epsilon + p)D^j v^j D_i v^i - \mathcal{E},$$  \hspace{1cm} (6.22)

$$(2)T^\eta_i = (\epsilon + p) \left(D_i \phi (\nu_\eta + \nu_i - 2 \frac{\phi}{\phi} D^j v^j) + 2 \left(\frac{\phi}{\phi} + \frac{\mathcal{P}}{\mathcal{P}}\right) D_i \phi v^i\right),$$  \hspace{1cm} (6.23)

$$(2)T^i_\eta = (\epsilon + p) \left(\nu^i - D^j v^j D^i v^i - 6 \frac{\phi}{\phi} D^i D_i v^i\right) - 2 \left(\frac{\phi}{\phi} + \frac{\mathcal{P}}{\mathcal{P}}\right) D^i (\nu^i v^i),$$  \hspace{1cm} (6.24)

$$(2)T^j_i = 2(\epsilon + p) D_i v^i D_j v^j + \mathcal{P} \gamma_i^j.$$  \hspace{1cm} (6.25)

Hence, through the background Einstein equations (3.14) and (3.15), and the first-order perturbation of the Einstein equations (5.27)–(5.29), the Einstein equations
for the second-order perturbations are obtained as

\[
\begin{align*}
-3\mathcal{H}\partial_\eta + \Delta + 3K \right) \hspace{1cm} & \hspace{1cm} \Psi - 3\mathcal{H}^2 \Phi - 4\pi Ga^2 \varepsilon = \Gamma_0, \\
2\partial_\eta D_i \hspace{1cm} & \hspace{1cm} \Psi + 2\mathcal{H}D_i \Phi - \frac{1}{2} (\Delta + 2K) \nu_i \\
\hspace{1cm} & \hspace{1cm} + 8\pi Ga^2 (\epsilon + p) \left( D_i \hspace{1cm} \nu_i \right) = \Gamma_i,
\end{align*}
\]

(6.26)

\[
D_iD^j \hspace{1cm} \left( \frac{2}{\Psi} - \frac{2}{\Phi} \right) \\
\hspace{1cm} + \left\{ (2\partial_\eta^2 - \Delta + 4\mathcal{H}\partial_\eta - 2K) \hspace{1cm} \Psi + (2\mathcal{H}\partial_\eta + 4\partial_\eta \mathcal{H} + 2\mathcal{H}^2 + \Delta) \hspace{1cm} \Phi \right\} \gamma_i^j \\
\hspace{1cm} - \frac{1}{2a^2} \partial_\eta \left\{ a^2 \left( D_i \hspace{1cm} \nu_i + D^j \hspace{1cm} \nu_i^j \right) \right\} \\
\hspace{1cm} + \frac{1}{2} (\partial_\eta^2 + 2\mathcal{H}\partial_\eta + 2K - \Delta) \chi_i^j \\
\hspace{1cm} - 8\pi Ga^2 \hspace{1cm} \frac{2}{\mathcal{P}} \gamma_i^j = \Gamma_i^j,
\]

(6.27)

where

\[
\begin{align*}
\Gamma_0 := 8\pi Ga^2 (\epsilon + p) D^i \hspace{1cm} v D_i \hspace{1cm} v \\
& \hspace{1cm} - 3D_k \hspace{1cm} (\Phi D^k \hspace{1cm} (\Phi) - 3 \left( \partial_\eta \hspace{1cm} \Phi \right)^2 - 8 \hspace{1cm} (\Phi \hspace{1cm} \Delta \hspace{1cm} \Phi) - 12 \hspace{1cm} (\mathcal{H}^2) \hspace{1cm} (\Phi) - \Delta \hspace{1cm} \Phi) \hspace{1cm} (\Phi) - \frac{2}{D_i} \hspace{1cm} (\Phi), \\
\Gamma_i := -16\pi Ga^2 \hspace{1cm} \left( \frac{1}{\varepsilon} + \frac{1}{\mathcal{P}} \right) D_i \hspace{1cm} v \\
& \hspace{1cm} + 12\mathcal{H} \hspace{1cm} (\Phi D_i \hspace{1cm} (\Phi) - \frac{4}{D_i} \hspace{1cm} (\Phi \hspace{1cm} \partial_\eta \hspace{1cm} \Phi) - \frac{12}{D_i} \hspace{1cm} (\Phi \hspace{1cm} \partial_\eta \hspace{1cm} \Phi) - \frac{1}{D_i} \hspace{1cm} (\Phi) - \frac{1}{D_i} \hspace{1cm} (\Phi) D_i \hspace{1cm} (\Phi), \\
\Gamma_i^j := 16\pi Ga^2 \hspace{1cm} (\epsilon + p) D_i \hspace{1cm} v D^j \hspace{1cm} v - 4D_i \hspace{1cm} (\Phi \hspace{1cm} D^j \hspace{1cm} (\Phi) - 8 \hspace{1cm} \Phi \hspace{1cm} D_i \hspace{1cm} D^j \hspace{1cm} (\Phi) \\
& \hspace{1cm} + 2 \left( 3D_k \hspace{1cm} (\Phi D^k \hspace{1cm} (\Phi) + 4 \hspace{1cm} (\Phi \hspace{1cm} \Delta \hspace{1cm} \Phi) + \left( \partial_\eta \hspace{1cm} \Phi \right)^2 + 4 \hspace{1cm} (\Phi \hspace{1cm} \partial_\eta \hspace{1cm} \Phi) \right) \right) \gamma_i^j.
\end{align*}
\]

(6.29)

Equation (6.27) can be decomposed into scalar and vector parts. Taking the divergence of (6.27), we obtain

\[
8\pi Ga^2 (\epsilon + p) D_i \hspace{1cm} v = -2\partial_\eta D_i \hspace{1cm} \Psi - 2\mathcal{H} \hspace{1cm} D_i \hspace{1cm} \Phi + D_i \hspace{1cm} \Delta^{-1} \hspace{1cm} D^k \hspace{1cm} \Gamma_k.
\]

(6.32)

Then subtracting Eq. (6.32) from Eq. (6.27), we obtain

\[
8\pi Ga^2 (\epsilon + p) \hspace{1cm} \nu_i = \frac{1}{2} (\Delta + 2K) \hspace{1cm} \nu_i + \left( \Gamma_i - D_i \hspace{1cm} \Delta^{-1} \hspace{1cm} D^k \hspace{1cm} \Gamma_k \right).
\]

(6.33)
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Equation (6.28) can be decomposed into the trace part, and the traceless part. This traceless part of Eq. (6.28) can be decomposed into the scalar, vector, and tensor parts. The trace part of (6.28) is given by

\[
\left( \partial^2_\eta + 2H \partial_\eta - \frac{1}{3} \Delta - K \right) (\Psi) + \left( \mathcal{H} \partial_\eta + 2 \partial_\eta \mathcal{H} + \mathcal{H}^2 + \frac{1}{3} \Delta \right) (\Phi)
\]

\[
-4\pi G a^2 \frac{P}{6} = \frac{1}{6} \Gamma^k_k.
\]

The traceless part of Eq. (6.28) is given by

\[
\left( D_i D_j - \frac{1}{3} \gamma_{ij} \Delta \right) \left( \Psi - \Phi \right) - \frac{1}{a^2} \partial_\eta \left( a^2 D_{(i} \nu_{j)} \right)
\]

\[
+ \frac{1}{2} \left( \partial^2_\eta + 2H \partial_\eta + 2K - \Delta \right) \frac{\chi_{ij}}{\chi_{ij}} = \Gamma_{ij} - \frac{1}{3} \gamma_{ij} \Gamma^k_k,
\]

where \( \Gamma_{ij} = \gamma_{jk} \Gamma^k_i \). Taking the divergence of Eq. (6.35), we obtain

\[
\left( \frac{2}{3} D_i D_j + 2K D_i \right) \left( \Psi - \Phi \right) - \frac{1}{2a^2} \partial_\eta \left( a^2 (\Delta + 2K) \nu_i \right)
\]

\[
= D_j \Gamma^j_i - \frac{1}{3} D_i \Gamma^k_k.
\]

Further, taking the divergence of Eq. (6.36), we have

\[
\frac{2}{3} (\Delta + 3K) \Delta \left( \Psi - \Phi \right) = D^i D_j \Gamma^j_i - \frac{1}{3} \Delta \Gamma^k_k.
\]

Thus, we have extracted the scalar part in traceless part (6.35) of Eq. (6.28), which is given by

\[
\Psi - \Phi = \frac{3}{2} (\Delta + 3K)^{-1} \left( \Delta^{-1} D^i D_j \Gamma^j_i - \frac{1}{3} \Delta \Gamma^k_k \right).
\]

Substituting (6.38) into (6.36), we obtain the vector part of Eq. (6.28),

\[
\partial_\eta \left( a^2 \nu_i \right) = 2a^2 (\Delta + 2K)^{-1} \left\{ D_i \Delta^{-1} D^k D_l \Gamma_{k}^l - D_i \Gamma^k \right\},
\]

and substituting Eqs. (6.38) and (6.39), we obtain the tensor part of Eq. (6.28),

\[
\left( \partial^2_\eta + 2H \partial_\eta + 2K - \Delta \right) \chi_{ij}
\]

\[
= 2\Gamma_{ij} - \frac{2}{3} \gamma_{ij} \Gamma^k_k - 3 \left( D_i D_j - \frac{1}{3} \gamma_{ij} \Delta \right) (\Delta + 3K)^{-1} \left( \Delta^{-1} D^k D_l \Gamma_{k}^l - \frac{1}{3} \Gamma^k_k \right)
\]

\[
+ 4 \left( D_{(i} (\Delta + 2K)^{-1} D_{j)} \Delta^{-1} D^l D_k \Gamma_{l}^k - D_{(i} (\Delta + 2K)^{-1} D^k \Gamma_{j)k} \right).
\]

Equations (6.39) and (6.40) imply that the second-order vector and tensor modes may be generated by the scalar-scalar mode coupling of the first-order perturbation if accidental cancellation in the source term does not occur.
Further, the equations of the scalar mode perturbations (6.26) and (6.34) are reduced to a single equation for \( (2) \Phi \) as follows. Substituting (6.38) into (6.26), (6.32), and (6.34), the second-order perturbation of the energy density, the scalar part of the spatial velocity, and the pressure of the fluid are given by

\[
4\pi G a^2 \varepsilon^{(2)} = \left( -3H\partial_\eta + \Delta + 3K - 3H^2 \right) \Phi - \Gamma_0
\]

\[
+ \frac{3}{2} \left( \Delta^{-1} D^j D_j \Gamma_i^{(2)} - \frac{1}{3} \Gamma_k^k \right)
\]

\[
- \frac{9}{2} H\partial_\eta \left( \Delta + 3K \right)^{-1} \left( \Delta^{-1} D^j D_j \Gamma_i^{(2)} - \frac{1}{3} \Gamma_k^k \right),
\] (6.41)

\[
8\pi G a^2 (\epsilon + p) D_i^{(2)} v = -2\partial_\eta D_i \Phi - 2H D_i \left( \frac{2}{3} \Delta \right) \left( \Delta^{-1} D^j D_j \Gamma_i^{(2)} - \frac{1}{3} \Gamma_k^k \right)
\]

\[
- 3\partial_\eta D_i \left( \Delta + 3K \right)^{-1} \left( \Delta^{-1} D^j D_j \Gamma_i^{(2)} - \frac{1}{3} \Gamma_k^k \right),
\] (6.42)

\[
4\pi G a^2 P = \left( \frac{\partial_\eta^2}{2} + 3H\partial_\eta - K + 2\partial_\eta H + H^2 \right) \Phi
\]

\[
+ \frac{3}{2} \left( \partial_\eta^2 + 2H\partial_\eta \right) \left( \Delta + 3K \right)^{-1} \left( \Delta^{-1} D^j D_j \Gamma_i^{(2)} - \frac{1}{3} \Gamma_k^k \right)
\]

\[
- \frac{1}{2} \Delta^{-1} D^j D_j \Gamma_i^{(2)}
\] (6.43)

Through the second-order perturbation (4.115) of the equation of state, Eqs. (6.41) and (6.43) yield a single equation for \( (2) \Phi \):

\[
\left\{ \partial_\eta^2 + 3H(1 + c_s^2)\partial_\eta - c_s^2 \Delta + 2\partial_\eta H + (1 + 3c_s^2)(H^2 - K) \right\} \Phi
\]

\[= 4\pi G a^2 \left\{ \frac{1}{2} S + \frac{\partial c_s^2}{\partial \epsilon} \left( \frac{1}{\epsilon} \right)^2 + 2 \frac{\partial c_s^2}{\partial S} \left( \frac{1}{S} \right)^2 + \frac{\partial \tau}{\partial S} \left( \frac{1}{S} \right)^2 \right\}
\]

\[= -c_s^2 \left( \Gamma_0 + \frac{1}{2} \Gamma_k^k \right) + \frac{3}{2} \left( c_s^2 + 1 \right) \Delta^{-1} D^j D_j \Gamma_i^{(2)}
\]

\[= -\frac{3}{2} \left( \partial_\eta^2 + 2(1 + 3c_s^2)H\partial_\eta \right) \left( \Delta + 3K \right)^{-1} \left( \Delta^{-1} D^j D_j \Gamma_i^{(2)} - \frac{1}{3} \Gamma_k^k \right). \] (6.44)

This is the second-order extension of the master equation (5.30) of the scalar mode perturbations in the case of a universe filled with a perfect fluid. Actually, if the quadratic terms of the first-order perturbations on the right-hand side of Eq. (6.44) are absent, this equation coincides with Eq. (5.30).

To solve the system of second-order perturbations of the Einstein tensor, we have to carry out the following process. First, we solve the first-order master equation (5.30) for the perturbations. The solution to Eq. (5.30) gives the energy density, the velocity, and the pressure perturbation of the fluid through Eqs. (5.27)–(5.29). Next, we evaluate the source term of (6.44) and solve this equation. Since the homogeneous solutions to Eq. (6.44) coincide with the solutions to Eq. (5.30), which are known as
the growing and decaying modes of linear perturbation theory, the general solution to Eq. (6.44) is given by an inhomogeneous solution to Eq. (6.44), together with the growing and decaying modes of the linear-order scalar mode perturbation $\Phi^{(1)}$ with arbitrary coefficients. Once we obtain the solution to Eq. (6.44), we can also obtain the energy density, velocity, pressure perturbation at second order through Eqs. (6.41)–(6.44).

Further, we have equations for the vector and tensor modes (6.33), (6.39), and (6.40). Once we obtain the solution to the linearized Einstein equations, (5.30) and (5.27)–(5.29), we can evaluate the quadratic terms of the linear-order perturbations in Eqs. (6.33), (6.39), and (6.40). The evolution of the vector mode of the second-order metric perturbation is determined by Eq. (6.39), and the rotational part $\psi_i^{(2)}$ of the spatial velocity of the fluid is determined by Eq. (6.33). The tensor mode, i.e., the gravitational wave mode, at second order is determined by Eq. (6.40). Since the homogeneous solutions to Eq. (6.40) coincide with the solutions to Eq. (5.25), the general solution to Eq. (6.40) is also given by an inhomogeneous solution to Eq. (6.40), together with two independent solutions to (5.25) of linear order with arbitrary coefficients.

Of course, we need the additional information concerning the entropy perturbations $S^{(1)}$ and $S^{(2)}$ at each order to determine the first- and second-order perturbation. Once we obtain this information, all modes of the second-order perturbation are determined by the above second-order perturbation equations, (6.33), (6.39)–(6.44) of the Einstein equation. This is one of the main results of this paper.

6.2. Scalar field case

In the simple situation in which the first-order vector and tensor modes are negligible, the components of the second-order perturbation (1.126) of the energy-momentum tensor for a single scalar field are given by

\begin{align*}
^{(2)T}_{\eta \eta} & = -\frac{1}{a^2} \left( \partial_{\eta} \varphi \partial_{\eta} \varphi_2 - 4 \partial_{\eta} \varphi \ \Phi \ \partial_{\eta} \varphi_1 + 4(\partial_{\eta} \varphi)^2 \ \Phi \ \partial_{\eta} \varphi_1 + 4(\partial_{\eta} \varphi)^2 \ \Phi \ + (\partial_{\eta} \varphi_1)^2 \right) \\
& \quad + D_i \varphi_1 D_i \varphi_1 + a^2 \varphi_2 \frac{\partial V}{\partial \varphi} + a^2 (\varphi_1)^2 \frac{\partial^2 V}{\partial \varphi^2}, \ (6.45) \\
^{(2)T}_{\eta i} & = -\frac{1}{a^2} \left( D_i \varphi_2 \partial_{\eta} \varphi + 2 D_i \varphi_1 \partial_{\eta} \varphi_1 - 4 D_i \varphi_1 \ \Phi \ \partial_{\eta} \varphi \right), \ (6.46) \\
^{(2)T}_{\eta i} & = \frac{1}{a^2} \left( \partial_{\eta} \varphi D_i \varphi_2 + 4 \partial_{\eta} \varphi \ \Psi \ D_i \varphi_1 + (\partial_{\eta} \varphi)^2 \ \nu^i_2 + 2 \partial_{\eta} \varphi_1 D_i \varphi_1 \right), \ (6.47) \\
^{(2)T}_{\eta j} & = 2 \frac{1}{a^2} D_i \varphi_1 D_i \varphi_1 \\
& \quad + \frac{1}{a^2} \gamma_i \ \epsilon \left( + \partial_{\eta} \varphi \partial_{\eta} \varphi_2 - 4 \partial_{\eta} \varphi \ \Phi \ \partial_{\eta} \varphi_1 + 4(\partial_{\eta} \varphi)^2 \ \Phi \ \partial_{\eta} \varphi_1 + 4(\partial_{\eta} \varphi)^2 \ \Phi \ - (\partial_{\eta} \varphi)^2 \ \Phi \right) \\
\end{align*}
\[(\partial_\eta \varphi)^2 - D_k \varphi_1 D^k \varphi_1 - a^2 \varphi_2 \partial V \partial_\varphi - a^2 \varphi_1^2 \partial^2 V \partial_\varphi^2) \tag{6.48}\]

Through Eqs. (3.18) and (5.36), the components 
\[(1) G_\eta^i \eta [\mathcal{L}] + (2) G_\eta^i \eta [\mathcal{H}, \mathcal{H}] = 8\pi G \ (2) \mathcal{T}_\eta^i \eta \text{ and } (1) G_\eta^i \eta [\mathcal{L}] + (2) G_\eta^i \eta [\mathcal{H}, \mathcal{H}] = 8\pi G \ (2) \mathcal{T}_i \eta \eta \] of the second-order Einstein equation (2.50) give the single equation
\[2 \partial_\eta D_i \psi + 2 \mathcal{H} D_i \phi - \frac{1}{2} (\Delta + 2K) \psi = -8\pi GD_i \varphi_2 \partial_\eta \varphi = \Gamma_i \tag{6.49}\]
where
\[\Gamma_i := -4 \partial_\eta \phi D_i \phi + 8 \mathcal{H} \phi D_i \phi - 8 \phi \partial_\eta D_i \phi + 16\pi GD_i \varphi_1 \partial_\eta \varphi_1 \tag{6.50}\]

Taking the divergence of Eq. (6.49), we obtain the scalar part of Eq. (6.49),
\[2 \partial_\eta \psi + 2 \mathcal{H} \phi - 8\pi G \varphi_2 \partial_\eta \varphi = \Delta^{-1} \Gamma_k \tag{6.51}\]

Subtracting Eq. (6.51) from Eq. (6.49), we obtain the vector part of Eq. (6.49),
\[\psi_i = 2 (\Delta + 2K)^{-1} \left\{ D_i \Delta^{-1} \Gamma_k - \Gamma_i \right\} \tag{6.52}\]

Through Eqs. (3.18) and (5.38), the component 
\[(1) G_\eta \eta \eta [\mathcal{L}] + (2) G_\eta \eta \eta [\mathcal{H}, \mathcal{H}] = 8\pi G \ (2) \mathcal{T}_\eta \eta \eta \text{ of the second-order Einstein equation (2.50) gives}
\[-3\mathcal{H} \partial_\eta + \Delta + 3K \] \psi + (\partial_\eta \mathcal{H} - 2\mathcal{H}^2 + K) \phi

\[-4\pi G \left( \partial_\eta \varphi \partial_\eta \varphi_2 + a^2 \varphi_2^2 \partial V \partial_\varphi \right) = \Gamma_0 \tag{6.53}\]
where
\[\Gamma_0 := -2 \phi \partial_\eta^2 \phi - 3 \partial_\eta \phi \phi - 3D_k \phi D^k \phi - 10 \phi \Delta \phi - 4 \left( \partial_\eta \mathcal{H} + 4 \mathcal{H}^2 \right) \phi \right)^2

\[+ 4\pi G \left( (\partial_\eta \varphi_1)^2 + D_k \varphi_1 D^k \varphi_1 + a^2 \varphi_1^2 \partial^2 V \partial_\varphi^2 \right) \tag{6.54}\]

Similarly, through Eqs. (3.18) and (5.38), the component 
\[(1) G_i \eta \eta [\mathcal{L}] + (2) G_i \eta \eta [\mathcal{H}, \mathcal{H}] = 8\pi G \ (2) \mathcal{T}_i \eta \eta \text{ of the second-order Einstein equation (2.50) yields}
\[D_i D_j \left( \psi - \phi \right) \tag{6.55}\]

\[D_i \left\{ (-\Delta + 2\mathcal{H}^2 + 4 \mathcal{H} \partial_\eta - 2K) \psi + (2 \mathcal{H} \partial_\eta + 2 \partial_\eta \mathcal{H} + 4 \mathcal{H}^2 + \Delta + 2K) \phi \right\} \gamma_{ij} \tag{6.56}\]
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\[-\frac{1}{a^2} \partial_\eta \left( a^2 D_i \nu_{ij} \right) \]
\[+ \frac{1}{2} \left( \partial_\eta^2 + 2 \mathcal{H} \partial_\eta + 2 K - \Delta \right) \chi_{ij} \]
\[-8\pi G \left( \partial_\eta \varphi \partial_\eta \varphi_2 - a^2 \varphi_2 \frac{\partial V}{\partial \varphi} (\varphi) \right) \gamma_{ij} = \Gamma_{ij}, \tag{6.55} \]

where
\[\Gamma_{ij} := -4D_i \Phi D_j \Phi - 8 \Phi D_i D_j \Phi + 16\pi G D_i \varphi_1 D_j \varphi_1 \]
\[+ 2 \left( 8\mathcal{H} \Phi \partial_\eta \Phi - 2 \Phi \partial_\eta^2 \Phi + \left( \partial_\eta \Phi \right)^2 + 3D_k \Phi D^k \Phi \right) \]
\[+ 2 \Phi \Delta_\Phi + 4 \left( \partial_\eta \mathcal{H} + 2\mathcal{H}^2 \right) \left( \Phi^2 \right) \]
\[+ 4\pi G \left( \left( \partial_\eta \varphi \right)^2 - D_k \varphi_1 D^k \varphi_1 - a^2 (\varphi_1)^2 \frac{\partial^2 V}{\partial \varphi^2} \right) \gamma_{ij}. \tag{6.56} \]

As seen in the case of a perfect fluid in §6.1, Eq. (6.55) can be decomposed into the trace and the traceless parts. Further, the traceless part Eq. (6.55) is decomposed into the scalar, vector, and tensor parts. The trace part of Eq. (6.55) is given by
\[\left( \partial_\eta^2 + 2 \mathcal{H} \partial_\eta - \frac{1}{3} \Delta - K \right) \Phi + \left( \mathcal{H} \partial_\eta + \partial_\eta \mathcal{H} + 2\mathcal{H}^2 + \frac{1}{3} \Delta + K \right) \Phi \]
\[-4\pi G \left( \partial_\eta \varphi \partial_\eta \varphi_2 - a^2 \varphi_2 \frac{\partial^2 V}{\partial \varphi^2} (\varphi) \right) = \frac{1}{6} \Gamma^k_{ik}, \tag{6.57} \]

where \( \Gamma^k_{ik} = \gamma^{ij} \Gamma_{ij} \). The traceless scalar part of Eq. (6.55) is given by
\[\left( \partial_\eta^2 + 2 \mathcal{H} \partial_\eta - \frac{1}{3} \Delta - K \right) \Phi + \left( \mathcal{H} \partial_\eta + \partial_\eta \mathcal{H} + 2\mathcal{H}^2 + \frac{1}{3} \Delta + K \right) \Phi \]
\[-4\pi G \left( \partial_\eta \varphi \partial_\eta \varphi_2 - a^2 \varphi_2 \frac{\partial^2 V}{\partial \varphi^2} (\varphi) \right) = \frac{1}{6} \Gamma^k_{ik}, \tag{6.58} \]

where \( \Gamma^i_{ik} := \gamma^{kj} \Gamma_{ik} \). The vector part of Eq. (6.55) is given by
\[\partial_\eta \left( a^2 \nu_{ij} \right) = 2 a^2 (\Delta + 2 K)^{-1} \left\{ D_i \Delta^{-1} D^k D_k \Gamma^l_{ik} - D_k \Gamma^l_{ik} \right\}. \tag{6.59} \]

Finally, the tensor part of Eq. (6.55) is given by
\[\left( \partial_\eta^2 + 2 \mathcal{H} \partial_\eta + 2 K - \Delta \right) \chi_{ij} \]
\[= 2 \Gamma_{ij} - \frac{2}{3} \gamma_{ij} \Gamma^k_{ik} - 3 \left( D_i D_j - \frac{1}{3} \gamma_{ij} \Delta \right) \left( \Delta + 3 K \right)^{-1} \left( \Delta^{-1} D^k D_l \Gamma^l_{ik} - \frac{1}{3} \Gamma^k_{ik} \right) \]
\[+ 4 \left\{ D_{(i} (\Delta + 2 K)^{-1} D_{j)} \Delta^{-1} D^l D_k \Gamma^k_{lk} - D_{(i} (\Delta + 2 K)^{-1} D^k \Gamma_{lj)k} \right\}. \tag{6.60} \]

From Eqs. (6.51), (6.53), (6.57), and (6.58), we obtain a single equation for the second-order perturbation \( \Phi \) as Eq. (6.44) in the case of a perfect fluid considered
Combining Eqs. (6.53) and (6.51), we have
\[
\left( \partial^2_\eta - \mathcal{H} \partial_\eta + \frac{2}{3} \Delta + 2K \right)^{(2)} \Psi + \left( \mathcal{H} \partial_\eta + \frac{1}{3} \Delta + 2K \right)^{(2)} \Phi = -8\pi G \partial_\eta \varphi \partial_\eta \varphi_2 = \Gamma_0 + \frac{1}{6} \Gamma_k^k, \tag{6.61}
\]
and
\[
\left( -\partial^2_\eta - 5\mathcal{H} \partial_\eta + \frac{4}{3} \Delta + 4K \right)^{(2)} \Psi - \left( 2\partial_\eta \mathcal{H} + \mathcal{H} \partial_\eta + 4\mathcal{H}^2 + \frac{1}{3} \Delta \right)^{(2)} \Phi = -8\pi G a^2 \varphi_2 \frac{\partial V}{\partial \varphi} = \Gamma_0 - \frac{1}{6} \Gamma_k^k. \tag{6.62}
\]
Further, substituting Eq. (6.58) into Eq. (6.61), we obtain
\[
\left( \partial^2_\eta + \Delta + 4K \right)^{(2)} \Phi = -8\pi G \partial_\eta \varphi \partial_\eta \varphi_2
\]
\[
= \Gamma_0 + \frac{1}{6} \Gamma_k^k - \Delta^{-1} D_i D_j \Gamma_i^j + \frac{1}{3} \Gamma_k^k
\]
\[
- \frac{3}{2} \left( \partial^2_\eta - \mathcal{H} \partial_\eta \right) \left( \Delta + 3K \right)^{-1} \left\{ \Delta^{-1} D_i D_j \Gamma_i^j - \frac{1}{3} \Gamma_k^k \right\}. \tag{6.63}
\]
On the other hand, differentiating Eq. (6.51) with respect to the conformal time \( \eta \), we obtain
\[
- 8\pi G \partial_\eta \varphi_2 \partial_\eta \varphi = \left( \frac{2\partial^2_\eta \varphi}{\partial_\eta \varphi} \partial_\eta - 2 \partial^2_\eta \right)^{(2)} \Psi + \left( \frac{2\partial^2_\eta \varphi}{\partial_\eta \varphi} \mathcal{H} - 2\partial_\eta \mathcal{H} - 2\mathcal{H} \partial_\eta \right) \left( \partial^2_\eta \varphi \partial_\eta \varphi \right)^{(2)} \Phi
\]
\[
- \frac{\partial^2_\eta \varphi}{\partial_\eta \varphi} \Delta^{-1} D^k \Gamma_k + \partial_\eta \Delta^{-1} D^k \Gamma_k
\]
\[
= \left( -2\partial^2_\eta - 2\mathcal{H} \partial_\eta + \frac{2\partial^2_\eta \varphi}{\partial_\eta \varphi} \partial_\eta + \frac{2\partial^2_\eta \varphi}{\partial_\eta \varphi} \mathcal{H} - 2\partial_\eta \mathcal{H} \right) \left( \partial^2_\eta \varphi \partial_\eta \varphi \right)^{(2)} \Phi
\]
\[
+ \frac{3}{2} \left( \frac{2\partial^2_\eta \varphi}{\partial_\eta \varphi} \partial_\eta - 2 \partial^2_\eta \right) \left( \Delta + 3K \right)^{-1} \left\{ \Delta^{-1} D_i D_j \Gamma_i^j - \frac{1}{3} \Gamma_k^k \right\}
\]
\[
- \frac{\partial^2_\eta \varphi}{\partial_\eta \varphi} \Delta^{-1} D^k \Gamma_k + \partial_\eta \Delta^{-1} D^k \Gamma_k, \tag{6.64}
\]
where we have again used Eq. (6.58). Substituting (6.64) into (6.63), we obtain the master equation:
\[
\left\{ \partial^2_\eta + 2 \left( \mathcal{H} - \frac{\partial^2_\eta \varphi}{\partial_\eta \varphi} \right) \partial_\eta - \Delta - 4K + 2 \left( \partial_\eta \mathcal{H} - \frac{\partial^2_\eta \varphi}{\partial_\eta \varphi} \mathcal{H} \right) \right\} \left( \partial^2_\eta \varphi \partial_\eta \varphi \right)^{(2)} \Phi
\]
\[
= -\Gamma_0 - \frac{1}{2} \Gamma_k^k + \Delta^{-1} D_i D_j \Gamma_i^j + \left( \partial_\eta - \frac{\partial^2_\eta \varphi}{\partial_\eta \varphi} \right) \Delta^{-1} D^k \Gamma_k
\]
\[
- \frac{3}{2} \left\{ \partial^2_\eta - \left( \frac{2\partial^2_\eta \varphi}{\partial_\eta \varphi} - \mathcal{H} \right) \partial_\eta \right\} \left( \Delta + 3K \right)^{-1} \left\{ \Delta^{-1} D_i D_j \Gamma_i^j - \frac{1}{3} \Gamma_k^k \right\}. \tag{6.65}
\]
This is the second-order extension of Eq. (5.39), which is the master equation of scalar mode of the second-order cosmological perturbation in a universe filled with a single scalar field. This equation would coincide with Eq. (5.39) if the quadratic terms of the linear-order perturbations were absent.

Thus, we have a set of ten equations for the second-order perturbations of a universe filled with a single scalar field, Eqs. (6.51), (6.52), (6.58)–(6.60), (6.62), and (6.65). To solve this system, we have to solve the linear-order system first. Next, we evaluate the quadratic terms, $\Gamma_0$, $\Gamma_i$, and $\Gamma_{ij}$, of the linear-order perturbations. Then, using the information of the quadratic terms of the linear-order perturbation, we estimate the source term in Eq. (6.65). The general solution to Eq. (6.65) is given by an inhomogeneous solution to Eq. (6.65) and two independent homogeneous solutions of this equation with arbitrary constants. Since Eq. (6.65) is the same as Eq. (5.39), except for the source term, which consists of the quadratic terms of the linear-order perturbations, the homogeneous solutions to (6.65) coincide with the solutions to the linear-order perturbations, $\Phi^{(1)}$. Hence, we can construct the general solution to Eq. (6.65) if we obtain a special solution of this equation. After constructing the solution $\Phi^{(2)}$ to Eq. (6.65), we can obtain the second-order metric perturbation $\Psi^{(2)}$ through Eq. (6.58). Thus, we have obtained the second-order gauge invariant perturbation $\varphi_2$ of the scalar field through Eq. (6.51). Equation (6.62) is then used to check the consistency of the second-order perturbation of the Klein Gordon equation,

$$\bar{\nabla}^a \bar{\nabla}_a \varphi + \frac{\partial V}{\partial \varphi} = 0.$$ (6.66)

Evaluating the source terms in Eq. (6.60) through the evaluation of the quadratic terms $\Gamma_0$, $\Gamma_i$, and $\Gamma_{ij}$ of the linear-order perturbations, we can solve Eq. (6.60). We also note that Eq. (6.60) is identical to Eq. (6.40), except for the definition of the quadratic terms. As in the case of a perfect fluid, this equation implies that a scalar mode perturbation of linear order may generat the second-order tensor mode, i.e., the second-order gravitational waves if accidental cancellation in the source term does not occur.

For the vector-mode perturbation $\nu_i^{(1)}$, the situation is different from that in the case of a perfect fluid. Since there is no rotational component in a single scalar field system, there is no rotational spatial component of the velocity of the matter field, i.e., there is no vorticity. Instead, in addition to the evolution equation (6.59) of the vector mode, we have the initial value constraint (6.52) of the vector mode. However, the constraint (6.52) and the evolution equation (6.59) also imply that the second-order vector-mode perturbation may be generated by the scalar-scalar mode coupling of the linear order perturbations.

Thus, we have formulated a second-order perturbation theory of a universe filled with a single scalar field. All modes of the second-order perturbation are determined by the above second-order perturbation equations, (6.51), (6.52), (6.58)–(6.60), (6.62), and (6.65). This and the result in the case of a perfect fluid are
§7. Summary and discussions

In summary, we have confirmed that the general formulation of the second-order perturbation theory developed in KN2003 and KN2005 is applicable to cosmological perturbation theory. We have shown that the method for finding higher-order gauge invariant variables proposed in KN2003 does work in the case of cosmological perturbations. The key point of our method is the assumption that we already know the procedure for decomposing the first-order metric perturbation into gauge invariant and variant parts. In particular, to apply this method to higher-order perturbations, we have to find the gauge variant vector field $X_a$ in the first-order metric perturbation, which is defined by Eq. (2.22). As shown in §3.1, the vector field $X_a$ is found by restricting the domain of the perturbations to the space of functions in which the Green functions $\Delta^{-1}$, $(\Delta + 2K)^{-1}$, and $(\Delta + 3K)^{-1}$ can be defined, where $\Delta$ and $K$ are the Laplacian associated with the metric $\gamma_{ij}$ and the curvature constant of the maximally symmetric three-space, respectively. As a result, we found gauge invariant variables for the first-order metric perturbation that are discussed in the literature. The resulting gauge invariant metric perturbation has the same form as the metric perturbation described by the longitudinal gauge (the Newtonian gauge). This result for linear metric perturbations was then extended to second-order perturbations, as proposed in KN2003. In this way, we obtain second-order gauge invariant metric perturbations whose components are similar to those of linear metric perturbations in the longitudinal gauge. If we apply the gauge fixing so that $X_a = Y_a = 0$, where $X_a$ and $Y_a$ are the first- and second-order gauge variant parts of the metric perturbations studied in §4, this gauge fixing corresponds to the Poisson gauge in the literature, which is the higher-order extension of the longitudinal gauge. Further, as proposed in KN2003, we can also define the gauge invariant variables for the perturbations of an arbitrary field other than the metric, as shown in Eqs. (2.28) and (2.29).

We have also shown that the formulae (2.28) and (2.29) of the decomposition of the gauge invariant and variant parts for an arbitrary field other than the metric play crucial roles in the second-order gauge invariant perturbation theory. We have seen that the first- and second-order perturbative energy momentum-tensor of a perfect fluid [Eqs. (4.99)–(4.102)] and a single scalar field [Eqs. (4.123)–(4.126)] can be decomposed into gauge invariant and variant parts in the same forms as Eqs. (2.28) and (2.29). Here again, we note that no background values of the fluid components, the scalar field, nor the metric were explicitly used in the derivation of these expressions. We thus conclude that Eqs. (4.99)–(4.102) and Eqs. (4.123)–(4.126) are valid in the perturbation theory on any background spacetime.

Further, we have derived the perturbed Einstein equations of first and second order in terms of the above gauge invariant variables. As shown in KN2005, each order of the Einstein equations can be written in terms of only the gauge invariant variables, and we do not have to consider the gauge degree of freedom when we treat the perturbed Einstein equations. Though it is well known that the first-order met-
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ric perturbation in a homogeneous and isotropic universe can be decomposed into scalar, vector, and tensor types, we have shown that the second-order metric perturbation can also be decomposed into these three types. In the perturbation theory at linear order, these three types of perturbations are decoupled. However, at higher orders, these three types of perturbations are coupled. At second order, this mode-mode coupling appears as a source term which consists of quadratic terms of the linear-order perturbations. Because the scalar mode of the perturbations yields the dominant contribution in many cosmological contexts, we consider only the situation in which the first-order vector and the tensor modes are negligible. Even in this simple situation, the second-order Einstein equations imply that the second-order vector and tensor modes may be generated by the scalar-scalar mode coupling. Further, we have shown that the Einstein equations for the second-order perturbations have forms similar to those for the linear-order perturbations, but there are source terms due to the quadratic terms of the linear-order perturbations. Since we have developed the perturbation theory order by order, the Einstein equations for the second-order perturbations can be solved using techniques for linear-order perturbations. In particular, we have also seen that the second-order Einstein equations for the scalar mode perturbations are reduced to single equations in the cases of both a perfect fluid and a scalar field. The resulting equations have forms similar to those for the linear-order perturbations, but there is a source term which consists of the quadratic terms of the linear-order perturbations.

Now, we discuss the definitions of the gauge invariant variables found in the literature. It is well known that there are many definitions of the gauge invariant variables for density perturbation.\(^1\),\(^2\) Thus, there is no uniqueness in the definitions of the gauge invariant variables. This results from the fact that we can always construct new gauge invariant quantities from combinations of other gauge invariant variables. In many works, the interpretation of gauge invariant quantities is based on the coincidence of the perturbative variables in an appropriate gauge choice. For example, the gauge invariant variable \(\epsilon^{(1)}\) defined by Eq. (4.84) describes the density perturbation, because the variable \(\epsilon^{(1)}\) coincides with \(\epsilon^{(1)}\), defined in Eq. (4.74) in the gauge choice \(\eta = 0\). This criterion for the interpretation of a gauge invariant variable for density perturbations produces many different definitions of the density perturbations, as pointed out in the literature.\(^1\),\(^2\) However, we have to recall that the gauge choice is the point identification map between the physical spacetime \(M_\lambda\) and the background spacetime \(M_0\), as reviewed \(^2\). The concept of the gauge choice does not exist for the physical spacetime; it has meaning only if we introduce a reference manifold, \(M_0\). Moreover, because all variables on the physical spacetime, \(M_\lambda\), are pulled back to the background spacetime, \(M_0\), these pulled-back variables necessarily depend on the gauge choice, in general. This gauge dependence is due to the point identification map between \(M_\lambda\) and \(M_0\), and it is not due to the nature of \(M_\lambda\) itself. Thus, the density perturbation \(\epsilon^{(1)}\) in Eq. (4.74) is defined by the pull back of the gauge choice and depends on the gauge choice. However, this gauge dependence is not due to the nature of the physical spacetime nor the background spacetime
themselves. We should emphasize that the physical spacetime is described only by the gauge invariant quantities, and hence physical density perturbation is the gauge invariant variables \( \mathcal{E}^{(1)} \), not \( \epsilon^{(1)} \). Thus, it is meaningless to interpret the variables \( \mathcal{E}^{(1)} \) in terms of its coincidence with \( \epsilon^{(1)} \) in some gauge choice, because the physical meaning of the variable \( \epsilon^{(1)} \) may depend on the gauge choices, whereas the physical meaning of the gauge invariant variable \( \mathcal{E}^{(1)} \) does not depend on this choice. To understand why the gauge invariant variable \( \mathcal{E}^{(1)} \) has the meaning of the energy density, it is enough to point out two facts. First, note that the gauge variant part \( X_a \) of the pulled-back metric perturbation \( h_{ab} \) is not a variable on the physical spacetime, which arises from the gauge choice, i.e., the point identification map between the physical spacetime \( \mathcal{M}_\lambda \) and the background spacetime \( \mathcal{M}_0 \). The information regarding the gauge choice is provided by the gauge variant part, \( X_a \), of the metric perturbation, but there is no information regarding the physical spacetime in the gauge variant part of the metric perturbation. Second, \( \mathcal{E}^{(1)} \) consists of only the variables of the energy density, its perturbation, and the vector field \( X_a \) of the gauge choice. Hence, the only possible physical meaning of \( \mathcal{E}^{(1)} \) is that it is the energy density perturbation. Similarly, and more generally, the gauge invariant variables defined by Eqs. (2.26) and (2.27) for the first- and second-order perturbations of an arbitrary matter field \( Q \) have the physical meaning of the first- and second-order perturbation of the physical variable \( Q \), respectively.

Of course, we can ask which variable is useful when we have many definitions of the gauge invariant variables. This question is a different from the main point discussed above. To answer this question, we have to specify the problem which we want to clarify, and we have to specify for what the variables are useful. A partial answer to this question should be provided by the correspondence found between the gauge invariant variables and observables in observations and experiments. The gauge invariant variables which are useful to study physical processes should appear in these physical processes. Therefore, the question of which variable is useful is reduced to the question of which variable appears in the physical processes. Because observations and experiments are constructed from some physical processes, it would be interesting to investigate the correspondence between gauge invariant variables and observables in observations and experiments. Through such an investigation we could confirm that our formulation developed in KN2003, KN2005 and in this paper is relevant to physical processes.

Ten years ago, Mukhanov et al. proposed a gauge invariant treatment of second-order cosmological perturbations.\(^ {21} \) Their aim in investigating such perturbations was to clarify the back reaction effect on the expansion of the universe due to the inhomogeneities of the gravitational field. They also proposed an averaging procedure. In their papers, they also discussed the gauge issue in second-order general relativistic perturbations. In their proposal, the gauge transformation of a second-order perturbation should be given by an exponential map. From our understanding
of the “gauge” in general relativistic perturbations, which was reviewed in §2, this proposal corresponds to our gauge transformation with $\xi(2) = 0$. Moreover, in their treatment, the perturbative expansions of the metric and the matter fields take the form

$$\bar{Q} = Q_0 + \lambda \delta Q,$$

instead of Eq. (2.10) in this paper, which includes a term of $O(\lambda^2)$. They also discussed the back reaction effect due to the nonlinear effects of the Einstein equation through the substitution of the expansion (7.1) into the Einstein equation, and the evaluation of the quadratic terms of $\delta Q$. Thus, their treatment of the second-order perturbations is quite different from the formulation developed in this paper. Thus, we have to regard their treatment of second-order perturbations, including their arguments concerning the gauge, as being based on a perturbation scheme that is quite different from that given in this paper. Further, the correspondence between their works and ours is highly non-trivial. At this time, it is not clear that we should clarify the correspondence between their works and ours, because our formulation has not yet been constructed to treat the back reaction effect. If the formulations of the back reaction effect or some averaging procedures are formulated on the basis of our formulation, it will be worthwhile to compare their works and ours.

Recently, Noh and Hwang studied second-order cosmological perturbations\textsuperscript{12} on the basis of the ADM formulation. They investigated various gauge fixing methods, gauge invariance, and the second-order Einstein equations in a complicated manner. Contrastingly, in our formulation, all gauge invariant variables for all fields were prepared before the derivation of the perturbed Einstein equations. As shown in KN2005, the Einstein equation is necessarily given in terms of gauge invariant variables only. This is shown without assuming an explicit background spacetime metric. Therefore, we do not have to consider the gauge degree of freedom when we study perturbations of the Einstein equation in both the cosmological perturbation theory and any other general relativistic perturbation theory, as shown in KN2005. In this sense, we can conclude that the formulation developed in this paper is clearer than the formulation of Noh and Hwang.\textsuperscript{12} However, it would be interesting to compare their approach and ours.

In addition to the above works treating the formulation of second-order perturbation theory, there is a series of papers by Matarrese and his co-workers\textsuperscript{6,11,27} concerning non-Gaussian behavior generated by second-order general relativistic perturbations. They also considered gauge invariant variables, but they concentrated on only the conserved quantities which correspond to Bardeen’s parameter in the linear-order perturbation theory. By contrast, in this paper, we found gauge invariant variables for the first- and second-order perturbations of all quantities. The second-order perturbative Einstein equations on a homogeneous isotropic background universe were derived in terms of gauge invariant variables, without any gauge fixing. This is the main result of this paper. Hence, with regard to the gauge issue of second-order perturbation theory, we regard the formulation of second-order gauge invariant cosmological perturbations to be completed in this paper. Many parts of their works are based on the Poisson gauge explained above, and the Poisson gauge
is a complete gauge fixing method. Therefore, it would be interesting to follow their physical arguments based on the gauge invariant perturbation theory formulated in this paper.

Though we have ignore the first-order vector and tensor modes, it is, in principle, possible to include these modes. To do this, the long algebraic calculations are needed. Other than these long calculations, however, there is no technical problem to include them. If we include the first-order vector and tensor modes in our consideration, all types of mode coupling, i.e., scalar-vector, scalar-tensor, vector-vector, vector-tensor, and tensor-tensor mode couplings may occur, in addition to the scalar-scalar mode coupling discussed in this paper. These additional mode couplings are included in the source terms $\Gamma_0$, $\Gamma_i$, and $\Gamma_{ij}$ in the second-order Einstein equations, which consist of the quadratic terms of the first-order perturbations, as in the case of the scalar-scalar mode coupling studied in §6. Here, we emphasize that even if we consider the simple situation in which the first-order vector and tensor modes are negligible, the second-order vector and tensor modes may be generated by the scalar-scalar mode coupling if accidental cancellations in the quadratic terms of the linear perturbations do not occur.

Recently, Tomita has extended his pioneering works to a universe filled with dust and a cosmological constant, and he also discussed non-Gaussian behavior in CMB due to the nonlinear effect of the gravitational perturbations. His works are based on the synchronous gauge. He also claimed that there is no vorticity perturbation in this system even if we take into account the effects of second-order perturbations. Contrastingly, in our analyses, the divergenceless part of the spatial velocity of the fluid may be generated by the non-linear effects. The divergenceless part of the spatial velocity of the fluid contributes to the vorticity. Of course, in the model of a universe filled with a single scalar field, there is no vorticity, because the matter current of the scalar field is proportional to the gradient of the scalar field. In this case, the Einstein equation gives the constraint equation for the vector-mode metric perturbations. Because the Einstein equations constitute a first-class constrained system, the initial value constraint should be consistent with the evolution equation of the vector-mode metric perturbations. In this sense, the initial value constraint (6.52) for the vector mode should be consistent with the evolution equation (6.59). We can easily understand the reason for the absence of vorticity in a universe filled with a single scalar field, but the absence of vorticity in a universe filled with a dust field is not so trivial. Since the vorticity in the early universe is related to the generation of the magnetic field in the early universe through the Harrison mechanism, and to the generation of the B-mode polarization in CMB anisotropy, the existence of vorticity in the early universe is very important in cosmology. Because vorticity perturbations of the fluid velocity related to vector-mode perturbations, we conclude that the generation of the vector-mode perturbations is an important issue in the cosmological context.

In addition to the generation of vector-mode perturbations, the generation of tensor modes, which corresponds to gravitational waves, is also interesting for cosmology. The upper limit of the amplitude of the vorticity perturbations and the gravitational wave perturbations is constrained by the observational data of CMB.
However, it is known that the fluctuations of the scalar mode perturbations do exist in the early universe from the anisotropy of the CMB. Hence, the generation of vector and tensor modes due to the scalar-scalar mode coupling in second-order perturbations will give a lower limits on the vorticity and gravitational waves in the early universe from a theoretical point of view.

From the above discussion, we see that there are many issue which should be clarified using second-order cosmological perturbations. These are quite interesting not only from the theoretical point of view but also from the observational point of view. We have to clarify these issues one by one. To carry this out, the gauge invariant formulation developed in this paper should provide very powerful theoretical tools, and we hope that these issues will be clarified in terms of the gauge invariant variables defined in this paper. We leave these issues as future works.

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Appendix A

The components of $H_{abc}[\mathcal{H}]$, $H_{a}{}^{bc}[\mathcal{H}]$, $H_{a}{}^{b}{}_{c}[\mathcal{H}]$, and $H^{abc}[\mathcal{H}]$

In the derivation of the gauge invariant part of the perturbed Einstein tensor, the components $H_{abc}[\mathcal{H}]$, $H_{a}{}^{bc}[\mathcal{H}]$, $H_{a}{}^{b}{}_{c}[\mathcal{H}]$, and $H^{abc}[\mathcal{H}]$ defined by Eqs. (2.41) and (2.42) are useful. These components are given below.

- Components of $H_{abc}[\mathcal{H}]$:

\[
H_{\eta\eta\eta}[\mathcal{H}] = -a^2 \partial_\eta \Phi, \quad (A.1)
\]
\[
H_{\eta\eta\eta}[\mathcal{H}] = -a^2 (D_\eta \Phi + 2 H \nu_i), \quad (A.2)
\]
\[
H_{ij\eta}[\mathcal{H}] = a^2 \left\{ (2 \mathcal{H} (\Psi + \Phi) + \partial_\eta \Psi) \gamma_{ij} + D_{(i} \nu_{j)} - \frac{1}{2} (\partial_\eta + 2 \mathcal{H}) \chi_{ij} \right\}, \quad (A.3)
\]
\[
H_{\eta\eta i}[\mathcal{H}] = a^2 \left\{ D_\eta \Phi + (\partial_\eta + \mathcal{H}) \nu_i \right\}, \quad (A.4)
\]
\[
H_{j\eta i}[\mathcal{H}] = a^2 \left\{ -\partial_\eta \Psi \gamma_{ij} - D_{[i} \nu_{j]} + \frac{1}{2} \partial_\eta \chi_{ij} \right\}, \quad (A.5)
\]
\[
H_{jki}[\mathcal{H}] = a^2 \left\{ D_i \Psi \gamma_{kj} - 2 \gamma_{i(k} D_{j)} \Psi - \mathcal{H} \gamma_{kj} \nu_i + D_{(j} \chi_{k)i} - \frac{1}{2} D_i \chi_{kj} \right\}. \quad (A.6)
\]
- Components of $H_a^{bc} [\mathcal{H}]$:

\[
H_{\eta}^{\eta} [\mathcal{H}] = -\frac{1}{a^2} \partial_\eta \Phi, \quad (A.7)
\]
\[
H_i^{\eta} [\mathcal{H}] = -\frac{1}{a^2} (D_i \Phi + \mathcal{H} \nu_i), \quad (A.8)
\]
\[
H_{\eta}^i [\mathcal{H}] = \frac{1}{a^2} \{ D^i \Phi + \mathcal{H} \nu^i \}, \quad (A.9)
\]
\[
H_i^{j} [\mathcal{H}] = -\frac{1}{a^2} \left\{ (2\mathcal{H} (\Psi + \Phi) + \partial_\eta \Psi) \gamma_i^j + \frac{1}{2} (D_i \nu^j + D^j \nu_i) \right. \]
\[
\left. - \frac{1}{2} (\partial_\eta + 2\mathcal{H}) \chi_i^j \right\}, \quad (A.10)
\]
\[
H_{ij}^{\eta} [\mathcal{H}] = -\frac{1}{a^2} \{ D^i \Phi + (\partial_\eta + \mathcal{H}) \nu^i \}, \quad (A.11)
\]
\[
H_j^{\eta} [\mathcal{H}] = \frac{1}{a^2} \left\{ \partial_\eta \Psi \gamma_j^i + \frac{1}{2} (D^i \nu_j - D_j \nu^i) - \frac{1}{2} \partial_\eta \chi_j^i \right\}, \quad (A.12)
\]
\[
H_{ij}^{\eta} [\mathcal{H}] = \frac{1}{a^2} \left\{ -\partial_\eta \Psi \gamma^j_\eta + D^j \gamma^i_\eta \right\}, \quad (A.13)
\]
\[
H_j^{ki} [\mathcal{H}] = \frac{1}{a^2} \left\{ -\gamma^{ik} D_j \Psi + 2\gamma^k_j D^i \Psi - \mathcal{H} \gamma_k^j \nu^i \right. \]
\[
\left. + \frac{1}{2} D_j \chi^{ki} + D^k \chi_j^i \right\}. \quad (A.14)
\]

- Components of $H_a^{bc} [\mathcal{H}]$:

\[
H_{\eta}^{\eta} [\mathcal{H}] = \partial_\eta \Phi, \quad (A.15)
\]
\[
H_i^{\eta} [\mathcal{H}] = D_i \Phi + \mathcal{H} \nu_i, \quad (A.16)
\]
\[
H_{\eta}^i [\mathcal{H}] = -D^i \Phi - \mathcal{H} \nu^i, \quad (A.17)
\]
\[
H_i^{j} [\mathcal{H}] = \frac{1}{2} (D_i \nu^j + D^j \nu_i) + (2\mathcal{H} (\Psi + \Phi) + \partial_\eta \Psi) \gamma_i^j \]
\[
- \frac{1}{2} (\partial_\eta + 2\mathcal{H}) \chi_i^j, \quad (A.18)
\]
\[
H_{ij}^{\eta} [\mathcal{H}] = -D_i \Phi - (\partial_\eta + \mathcal{H}) \nu_i, \quad (A.19)
\]
\[
H_j^{\eta} [\mathcal{H}] = \partial_\eta \Psi \gamma^j_\eta + D_j \nu^j - \frac{1}{2} \partial_\eta \chi_j^i, \quad (A.20)
\]
\[
H_j^{i} [\mathcal{H}] = -\partial_\eta \Psi \gamma_j^i + \frac{1}{2} (D^i \nu_j - D_j \nu^i) + \frac{1}{2} \partial_\eta \chi_j^i, \quad (A.21)
\]
\[
H_j^{k} [\mathcal{H}] = -\gamma_{ij} D^k \Psi + 2\gamma^k_j D_i \Psi - \mathcal{H} \gamma_k^j \nu_i + \frac{1}{2} D^k \chi_{ji} + D^k \chi_j^i. \quad (A.22)
\]

- Components of $H^{abc} [\mathcal{H}]$:

\[
H^{\eta \eta \eta} [\mathcal{H}] = \frac{1}{a^4} \partial_\eta \Phi, \quad (A.23)
\]
\[
H^{\eta \eta} [\mathcal{H}] = -\frac{1}{a^4} (D^i \Phi + \mathcal{H} \nu^i), \quad (A.24)
\]
\[ H^{ij\eta} [\mathcal{H}] = -\frac{1}{a^4} \left\{ \left( 2\mathcal{H} (\Psi + \Phi) + \partial_\eta \Psi \right) \gamma^{ij} + D^{[i} \nu^{j]} \right. \\
\left. - \frac{1}{2} \left( \partial_\eta + 2\mathcal{H} \right) \chi^{ij} \right\}, \quad (A.25) \]

\[ H^{\eta i} [\mathcal{H}] = \frac{1}{a^4} \left\{ D^i \Phi + \left( \partial_\eta + \mathcal{H} \right) \nu^i \right\}, \quad (A.26) \]

\[ H^{\eta j} [\mathcal{H}] = \frac{1}{a^4} \left\{ \partial_\eta \Psi \gamma^{ij} - D^{[i} \nu^{j]} - \frac{1}{2} \partial_\eta \chi^{ij} \right\}, \quad (A.27) \]

\[ H^{jki} [\mathcal{H}] = \frac{1}{a^4} \left\{ -\gamma^{ik} D^j \Psi + 2\gamma^{j[k} D^{i]} \Psi - \mathcal{H} \gamma^{j} \nu^i \\
+ D^{(i} \chi^{jk)} - \frac{1}{2} D^i \chi^{jk} \right\}. \quad (A.28) \]

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