Analogue spacetime based on 2-component Bose–Einstein condensates

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Summary. Analogue spacetimes are powerful models for probing the fundamental physical aspects of geometry — while one is most typically interested in ultimately reproducing the pseudo–Riemannian geometries of interest in general relativity and cosmology, analogue models can also provide useful physical probes of more general geometries such as pseudo–Finsler spacetimes. In this chapter we shall see how a 2-component Bose–Einstein condensate can be used to model a specific class of pseudo–Finsler geometries, and after suitable tuning of parameters, both bi-metric pseudo–Riemannian geometries and standard single metric pseudo–Riemannian geometries, while independently allowing the quasi-particle excitations to exhibit a “mass”. Furthermore, when extrapolated to extremely high energy the quasi-particles eventually leave the phononic regime and begin to act like free bosons. Thus this analogue spacetime exhibits an analogue of the “Lorentz violation” that is now commonly believed to occur at or near the Planck scale defined by the interplay between quantum physics and gravitational physics. In the 2-component Bose–Einstein analogue spacetime we will show that the mass generating mechanism for the quasi-particles is related to the size of the Lorentz violations. This relates the “mass hierarchy” to the so-called “naturalness problem”. In short the analogue spacetime based on 2-component Bose–Einstein condensates exhibits a very rich mathematical and physical structure that can be used to investigate many issues of interest to the high-energy physics, cosmology, and general relativity communities.
1 Introduction and motivation

Analogue models of curved spacetime are interesting for a number of reasons [1]: Sometimes the analogue spacetime helps us understand an aspect of general relativity, sometimes general relativity helps us understand the physics of the analogue spacetime, and sometimes we encounter somewhat unusual mathematical structures not normally part of the physics mainstream, with the payoff that one might now develop new opportunities for exploiting the traditional cross-fertilization between theoretical physics and mathematics [2, 3, 4].

Specifically, in this chapter we will discuss an analogue spacetime based on the propagation of excitations in a 2-component Bose–Einstein condensate (BEC) [5, 6, 7, 8, 9, 10]. This analogue spacetime has a very rich and complex structure. In certain portions of parameter space the most natural interpretation of the geometry is in terms of a specific class of pseudo–Finsler spacetimes, and indeed we will see how more generally it is possible to associate a pseudo–Finsler spacetime with the leading symbol of a wide class of hyperbolic partial differential equations. In other parts of parameter space, the most natural interpretation of the geometry is in terms of a bi-metric spacetime, where one has a manifold that is simultaneously equipped with two distinct pseudo-Riemannian metric tensors. Further specialization in parameter space leads to a region where a single pseudo-Riemannian metric tensor is encountered — this mono-metric regime corresponds to Lorentzian spacetimes of the type encountered in standard general relativity and cosmology [11, 12, 13, 14]. Thus the analogue spacetime based on 2-component BECs provides models not just for standard general relativistic spacetimes, but also for the more general bi-metric, and even more general pseudo–Finsler spacetimes.

Additionally, the 2-BEC system permits us to provide a mass-generating mechanism for the quasi-particle excitations [5, 6]. The specific mass-generating mechanism arising herein is rather different from the Higgs mechanism of the standard model of particle physics, and provides an interesting counterpoint to the more usual ways that mass-generation is achieved. Furthermore, at short distances, where the “quantum pressure” term can no longer be neglected, then even in the mono-metric regime one begins to see deviations from “Lorentz invariance” — and these deviations are qualitatively of the type encountered in “quantum gravity phenomenology”, with the interesting property that the Lorentz violating physics is naturally suppressed by powers of the quasi-particle mass divided by the mass of the fundamental bosons that form the condensate [7, 8, 9, 10]. So in these analogue systems the mass-generating mechanism is related to the “hierarchy problem” and the suppression of Lorentz-violating physics. The 2-BEC model also allows us to probe the “universality” (or lack thereof) in the Lorentz violating sector [7, 8, 9, 10]. More generally, as one moves beyond the hydrodynamic limit in generic pseudo–Finsler parts of parameter space, one can begin to see hints
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of geometrical structure even more general than the pseudo–Finsler geometries.

While we do not wish to claim that the 2-BEC analogue spacetime of this chapter is necessarily a good model for the real physical spacetime arising from the putative theory of “quantum gravity” (be it string-model, loop-variable, or lattice based), it is clear that the 2-BEC analogue spacetime is an extraordinarily rich mathematical and physical structure that provides many interesting hints regarding the sort of kinematics and dynamics that one might encounter in a wide class of models for “quantum gravity phenomenology”.

This is the fundamental reason for our interest in this model, and we hope we can likewise interest the reader in this system and its relatives.

2 Theory of the 2-component BEC

The basis for our analogue model is an ultra-cold dilute atomic gas of \( N \) bosons, which exist in two single-particle states \( |A\rangle \) and \( |B\rangle \). For example, we consider two different hyperfine states, \( |F = 1, m_F = -1\rangle \) and \( |F = 2, m_F = 2\rangle \) of \(^{87}\text{Rb} \) [15, 16]. They have different total angular momenta \( F \) and therefore slightly different energies. That permits us, from a theoretical point of view, to keep \( m_A \neq m_B \), even if they are very nearly equal (to about one part in \( 10^{16} \)). At the assumed ultra-cold temperatures and low densities the atoms interact only via low-energy collisions, and the 2-body atomic potential can be replaced by a contact potential. That leaves us with with three atom-atom coupling constants, \( U_{AA}, U_{BB}, \) and \( U_{AB} \), for the interactions within and between the two hyperfine states. For our purposes it is essential to include an additional laser field, that drives transition between the two single-particle states.

In Fig. 1 the energy levels for different hyperfine states of \(^{87}\text{Rb} \), and possible transitions involving three-level processes are schematically explained. A more detailed description on how to set up an external field driving the required transitions can be found in [17].

2.1 Gross–Pitaevskii equation

The rotating-frame Hamiltonian for our closed 2-component system is given by:

\[
\hat{H} = \int \! dr \left\{ \sum_{i=A,B} \left( -\hat{\psi}_i^\dagger \frac{\hbar^2 \nabla^2}{2m_i} \hat{\psi}_i + \hat{\psi}_i^\dagger V_{\text{ext},i}(r) \hat{\psi}_i \right) + \frac{1}{2} \sum_{i,j=A,B} \left( U_{ij} \hat{\psi}_i^\dagger \hat{\psi}_j^\dagger \hat{\psi}_j \hat{\psi}_i + \lambda \hat{\psi}_i^\dagger (\sigma_x)_{ij} \hat{\psi}_j \right) \right\},
\]

In general, it is possible that the collisions drive coupling to other hyperfine states. Strictly speaking the system is not closed, but it is legitimate to neglect this effect [18].
with the transition rate $\lambda$ between the two hyperfine states. Here $\hat{\Psi}_i(r)$ and $\hat{\Psi}_i^\dagger(r)$ are the usual boson field annihilation and creation operators for a single-particle state at position $r$, and $\sigma_x$ is the usual Pauli matrix. For temperatures at or below the critical BEC temperature, almost all atoms occupy the spatial modes $\Psi_A(r)$ and $\Psi_B(r)$. The mean-field description for these modes,

$$i\hbar \partial_t \Psi_i = \left[ -\frac{\hbar^2}{2m_i} \nabla^2 + V_i - \mu_i + U_{ii} |\Psi_i|^2 + U_{ij} |\Psi_j|^2 \right] \Psi_i + \lambda \Psi_j, \quad (2)$$

are a pair of coupled Gross–Pitaevskii equations (GPE): $(i, j) \rightarrow (A, B)$ or $(i, j) \rightarrow (B, A)$.

### 2.2 Dynamics

In order to use the above 2-component BEC as an analogue model, we have to investigate small perturbations (sound waves) in the condensate cloud.\(^5\)

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\(^5\) The perturbations amplitude have to be small compared to the overall size of the condensate cloud, so that the system remains in equilibrium.
The excitation spectrum is obtained by linearizing around some background densities $\rho_0$ and phases $\theta_0$, using:

$$\Psi_i = \sqrt{\rho_0 + \varepsilon \rho_i} e^{i(\theta_0 + \varepsilon \theta_i)} \quad \text{for} \quad i = A, B.$$  \hspace{1cm} (3)

To keep the analysis as general as possible, we allow the two initial background phases to be independent from each other, and define

$$\delta_{AB} \equiv \theta_{A0} - \theta_{B0},$$ \hspace{1cm} (4)

as their difference.

A tedious calculation [5, 6, 7] shows that it is convenient to introduce the following $2 \times 2$ matrices: An effective coupling matrix,

$$\hat{\Xi} = \Xi + \hat{X},$$ \hspace{1cm} (5)

where we introduced the energy-independent matrix

$$\Xi = \frac{1}{\hbar} \left[ \hat{U}_{AA} \hat{U}_{AB} \right].$$ \hspace{1cm} (6)

This matrix contains the quantities

$$\hat{U}_{AA} \equiv U_{AA} - \frac{\lambda \cos \delta_{AB} \sqrt{\rho_{A0} \rho_{B0}}}{2} \frac{1}{\rho_{A0}^2},$$ \hspace{1cm} (7)

$$\hat{U}_{BB} \equiv U_{BB} - \frac{\lambda \cos \delta_{AB} \sqrt{\rho_{A0} \rho_{B0}}}{2} \frac{1}{\rho_{B0}^2},$$ \hspace{1cm} (8)

$$\hat{U}_{AB} \equiv U_{AB} + \frac{\lambda \cos \delta_{AB} \sqrt{\rho_{A0} \rho_{B0}}}{2} \frac{1}{\rho_{A0} \rho_{B0}}.$$ \hspace{1cm} (9)

A second matrix, denoted $\hat{X}$, contains differential operators $\hat{Q} X_1$ — these are the second-order differential operators obtained from linearizing the quantum potential:

$$V_Q(\rho_X) \equiv - \frac{\hbar^2}{2m_X} \left( \frac{\nabla^2 \sqrt{\rho_X}}{\sqrt{\rho_X}} \right) = - \frac{\hbar^2}{2m_X} \left( \frac{\nabla^2 \sqrt{\rho_{X0} + \varepsilon \rho_{X1}}}{\sqrt{\rho_{X0} + \varepsilon \rho_{X1}}} \right)$$ \hspace{1cm} (10)

$$= - \frac{\hbar^2}{2m_X} \left( \hat{Q}_{X0}(\rho_{X0}) + \varepsilon \hat{Q}_{X1}(\rho_{X1}) \right).$$ \hspace{1cm} (11)

The quantity $\hat{Q}_{X0}(\rho_{X0})$ corresponds to the background value of the quantum pressure, and contributes only to the background equations of motion — it does not affect the fluctuations. Now in a general background

$$\hat{Q}_{X1}(\rho_{X1}) = \frac{1}{2} \left\{ \frac{(\nabla \rho_{X0})^2}{\rho_{X0}^2} - \frac{(\nabla^2 \rho_{X0}) \rho_{X0}}{\rho_{X0}^2} \frac{\nabla}{\rho_{X0} \nabla} + \frac{1}{\rho_{X0}} \nabla^2 \right\} \rho_{X1},$$ \hspace{1cm} (12)
and we define the matrix $\hat{X}$ to be

$$\hat{X} \equiv -\frac{\hbar}{2} \begin{bmatrix} \frac{Q_{A1}}{m_A} & 0 \\ 0 & \frac{Q_{B1}}{m_B} \end{bmatrix}. \quad (13)$$

Given the background homogeneity that will be appropriate for later parts of the current discussion, this will ultimately simplify to

$$\hat{Q}_{X1}(\rho_{X1}) = \frac{1}{2\rho_{X0}} \nabla^2 \rho_{X1}, \quad (14)$$

in which case

$$\hat{X} = -\frac{\hbar}{4} \begin{bmatrix} \frac{1}{m_A \rho_{A0}} & 0 \\ 0 & \frac{1}{m_B \rho_{B0}} \end{bmatrix} \nabla^2 = -X \nabla^2. \quad (15)$$

Without transitions between the two hyperfine states, when $\lambda = 0$, the matrix $\Xi$ only contains the coupling constants $\Xi_{jj} \rightarrow U_{ij}/\hbar$. While $\Xi$ is independent of the energy of the perturbations, the importance of $\hat{X}$ increases with the energy of the perturbation. In the so-called hydrodynamic approximation $\hat{X}$ can be neglected, effectively $\hat{X} \rightarrow 0$ and $\hat{\Xi} \rightarrow \Xi$.

Besides the interaction matrix, we also introduce a transition matrix,

$$A \equiv -\frac{2\lambda_0 \cos \delta_{AB} \sqrt{\rho_{A0} \rho_{B0}}}{\hbar} \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} \quad (16)$$

and a mass-density matrix,

$$D \equiv \begin{bmatrix} 0 & 0 \\ \rho_{A0} & \rho_{B0} \end{bmatrix} \equiv \begin{bmatrix} d_A & 0 \\ 0 & d_B \end{bmatrix}. \quad (17)$$

The final step is to define two column vectors,

$$\vec{\theta} \equiv [\theta_{A1}, \theta_{B1}]^T, \quad (18)$$

and

$$\vec{\rho} \equiv [\rho_{A1}, \rho_{B1}]^T. \quad (19)$$

We then obtain two compact equations for the perturbation in the phases and densities:

$$\dot{\vec{\theta}} = -\hat{\Xi} \vec{\rho} - \nabla \vec{\theta} + \Theta \vec{\theta}, \quad (20)$$

$$\dot{\vec{\rho}} = -\nabla \cdot (D \nabla \vec{\theta} + \vec{\rho} \nabla \vec{\theta}) - A \vec{\theta} - \Theta^T \vec{\rho}. \quad (21)$$

Here the background velocity matrix simply contains the two background velocities of each condensate,

$$\mathbf{V} = \begin{bmatrix} v_{A0} & 0 \\ 0 & v_{B0} \end{bmatrix}. \quad (22)$$
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with two possibly distinct background velocities,

\[ v_{A0} = \frac{\hbar}{m_A} \nabla \theta_{A0}, \]

\[ v_{B0} = \frac{\hbar}{m_B} \nabla \theta_{B0}. \]

Additionally we also introduce the matrix \( \Theta \), which depends on the difference of the initial phases and is defined as

\[ \Theta \equiv \frac{\lambda \sin \delta_{AB}}{\hbar} \left[ + \sqrt{\rho_{B0}} - \sqrt{\rho_{A0}} \right. \left. + \sqrt{\rho_{A0}} - \sqrt{\rho_{B0}} \right]. \]

Now combine these two equations into one:

\[ \partial_t (\hat{\Xi}^{-1} \dot{\theta}) = - \partial_t \left( \hat{\Xi}^{-1} \mathbf{V} \cdot \nabla \dot{\theta} \right) - \nabla (\mathbf{V} \hat{\Xi}^{-1} \dot{\theta}) + \nabla \cdot \left[ \left( \mathbf{D} - \mathbf{V} \hat{\Xi}^{-1} \mathbf{V} \right) \nabla \dot{\theta} \right] + \Lambda \dot{\theta} \]

\[ + K \dot{\theta} + \frac{1}{2} \left\{ \Gamma^a \partial_a \dot{\theta} + \partial_a (\Gamma^a \dot{\theta}) \right\}, \]

where the index \( a \) runs from 0–3 (that is, over both time and space), and we now define

\[ \Gamma^t = \hat{\Xi}^{-1} \Theta - \Theta^T \hat{\Xi}^{-1}, \]

\[ \Gamma^i = \mathbf{V} \hat{\Xi}^{-1} \Theta - \Theta^T \hat{\Xi}^{-1} \mathbf{V}, \]

and

\[ K = \Theta^T \hat{\Xi}^{-1} \Theta + \frac{1}{2} \partial_t \left( \hat{\Xi}^{-1} \Theta + \Theta^T \hat{\Xi}^{-1} \right) + \frac{1}{2} \nabla \left( \mathbf{V} \hat{\Xi}^{-1} \Theta + \Theta^T \hat{\Xi}^{-1} \mathbf{V} \right). \]

Note that the \( \Gamma^a \) matrices are antisymmetric in field-space (A ↔ B), while the matrix \( K \) is symmetric. Also, both \( \Gamma^a \to 0 \) and \( K \to 0 \) as \( \delta_{AB} \to 0 \).

Our first goal is to show that equation (25), which fundamentally describes quasi-particle excitations interacting with a condensed matter system in the mean-field approximation, can be given a physical and mathematical interpretation in terms of a classical background geometry for massless and massive particles propagating through an analogue spacetime [2, 3, 4, 19]. This analogy only holds (at least in its cleanest form) in the so-called hydrodynamic limit \( \hat{\Xi} \to \Xi \), which limit is directly correlated with the healing length which we shall now introduce.

2.3 Healing length

The differential operator \( \hat{Q}_{X1} \) that underlies the origin of the \( \hat{X} \) contribution above is obtained by linearizing the quantum potential
\[ V_Q(\rho_X) = -\frac{\hbar^2}{2m_X} \left( \nabla^2 \sqrt{\rho_X} \right) \]  
(29)

which appears in the Hamilton–Jacobi equation of the BEC flow. This quantum potential term is suppressed by the smallness of \( \hbar \), the comparative largeness of \( m_X \), and for sufficiently uniform density profiles. But of course in any real system the density of a BEC must go to zero at the boundaries of its electro-magnetic trap (given that \( \rho_X = |\psi_X(x,t)|^2 \)). In a 1-component BEC the healing length characterizes the minimal distance over which the order parameter goes from zero to its bulk value. If the condensate density grows from zero to \( \rho_0 \) within a distance \( \xi \) the quantum potential term (non local) and the interaction energy (local) are respectively 

\[ E_{\text{kinetic}} \sim \frac{\hbar^2}{2m\xi^2} \]  
and 

\[ E_{\text{interaction}} \sim 4\pi\hbar^2a\rho_0/m. \]

These two terms are comparable when

\[ \xi = (8\pi\rho_0a)^{-1/2}, \]

(30)

where \( a \) is the \( s \)-wave scattering length defined as

\[ a = \frac{mU_0}{4\pi\hbar^2}. \]

(31)

Note that what we call \( U_0 \) in the above expression is just the coefficient of the non-linear self-coupling term in the Gross–Pitaevskii equation, i.e., just \( U_{AA} \) or \( U_{BB} \) if we completely decouple the 2 BECs \( (U_{AB} = \lambda = 0) \).

Only for excitations with wavelengths much larger than the healing length is the effect of the quantum potential negligible. This is called the hydrodynamic limit because the single–BEC dynamics is then described by the continuity and Hamilton–Jacobi equations of a super-fluid, and its excitations behave like massless phononic modes. In the case of excitations with wavelengths comparable with the healing length this approximation is no longer appropriate and deviations from phononic behaviour will arise.

Such a simple discrimination between different regimes is lost once one considers a system formed by two coupled Bose–Einstein condensates. One is forced to introduce a generalization of the healing \( \xi \) length in the form of a “healing matrix”. If we apply the same reasoning used above for the definition of the “healing length” to the 2-component BEC system we again find a functional form like that of equation (30) however we now have the crucial difference that both the density and the scattering length are replaced by matrices. In particular, we generalize the scattering length \( a \) to the matrix \( \mathcal{A} \):

\[ \mathcal{A} = \frac{1}{4\pi\hbar^2} \left[ \begin{array}{cc} \sqrt{m_A} & 0 \\ 0 & \sqrt{m_B} \end{array} \right] \left[ \begin{array}{cc} \dot{U}_{AA} & \dot{U}_{AB} \\ \dot{U}_{AB} & \dot{U}_{BB} \end{array} \right] \left[ \begin{array}{cc} \sqrt{m_A} & 0 \\ 0 & \sqrt{m_B} \end{array} \right]. \]

(32)

Furthermore, from (30) a healing length matrix \( Y \) can be defined by

\[ Y^{-2} = \frac{2}{\hbar^2} \left[ \begin{array}{cc} \sqrt{\rho_0m_A} & 0 \\ 0 & \sqrt{\rho_0m_B} \end{array} \right] \left[ \begin{array}{cc} \ddot{U}_{AA} & \ddot{U}_{AB} \\ \ddot{U}_{AB} & \ddot{U}_{BB} \end{array} \right] \left[ \begin{array}{cc} \sqrt{\rho_0m_A} & 0 \\ 0 & \sqrt{\rho_0m_B} \end{array} \right]. \]

(33)
That is, in terms of the matrices we have so far defined:

\[ Y^{-2} = \frac{1}{2} X^{-1/2} \Xi X^{-1/2}; \quad Y^2 = 2 X^{1/2} \Xi^{-1} X^{1/2}. \] (34)

Define “effective” scattering lengths and healing lengths for the 2-BEC system as

\[ a_{\text{eff}} = \frac{1}{2} \text{tr}[A] = \frac{m_A \tilde{U}_{AA} + m_B \tilde{U}_{BB}}{8\pi\hbar^2}, \] (35)

and

\[ \xi_{\text{eff}}^2 = \frac{1}{2} \text{tr}[Y^2] = \text{tr}[X \Xi^{-1}] = \frac{\hbar^2 [\tilde{U}_{BB}/(m_A \rho_{A0}) + \tilde{U}_{AA}/(m_B \rho_{B0})]}{4(U_{AA}U_{BB} - U^2_{AB})}. \] (36)

That is

\[ \xi_{\text{eff}}^2 = \frac{\hbar^2 [m_A \rho_{A0} \tilde{U}_{AA} + m_B \rho_{B0} \tilde{U}_{BB}]}{4m_A m_B \rho_{A0} \rho_{B0} (U_{AA}U_{BB} - U^2_{AB})}. \] (37)

Note that if the two components are decoupled and tuned to be equivalent to each other, then these effective scattering and healing lengths reduce to the standard one-component results.

### 3 Emergent spacetime at low energies

The basic idea behind analogue models is to re-cast the equation for excitations in a fluid into the equation describing a massless or massive scalar field embedded in a pseudo–Riemannian geometry. Starting from a two component superfluid we are going to show that it is not only possible to obtain a massive scalar field from such an analogue model, in addition we are also able to model much more complex geometries. In Fig. 2 we illustrate how excitations in a 2-component BEC are associated with various types of emergent geometry.

Most generally, we show that excitations in a 2-component BEC (in the hydrodynamic limit) can be viewed as propagating through a specific class of pseudo–Finsler geometry. As additional constraints are placed on the BEC parameter space, the geometry changes from pseudo–Finsler, first to bi-metric, and finally to mono-metric (pseudo–Riemannian, Lorentzian) geometry. This can be accomplished by tuning the various BEC parameters, such as the transition rate \( \lambda \), the background velocities \( v_A, v_B \), the background densities \( \rho_A, \rho_B \), and the coupling between the atoms \( U_{AA}, U_{BB} \) and \( U_{AB} \).

At first, it might seem to be quite an artificial thing to impose such constraints onto the system. But if one considers that the two macroscopic wave functions represent two interacting classical fields, it is more or less obvious that this is the only way in which to enforce physical constraints onto the fields themselves, and on the way they communicate with each other.
3.1 Pseudo-Finsler geometry

In the hydrodynamic limit ($\Xi \to \Xi$), it is possible to simplify equation (25) — without enforcing any constraints on the BEC parameters — if we adopt a (3+1)-dimensional “spacetime” notation by writing $x^a = (t, x^i)$, with $i \in \{1, 2, 3\}$ and $a \in \{0, 1, 2, 3\}$. Then equation (25) can be very compactly rewritten as [2, 3]:

$$\partial_a \left( f^{ab} \partial_b \bar{\theta} \right) + (\Lambda + K) \bar{\theta} + \frac{1}{2} \{ \Gamma^a \partial_a \bar{\theta} + \partial_a (\Gamma^a \bar{\theta}) \} = 0. \quad (38)$$

The object $f^{ab}$ is a 4 $\times$ 4 spacetime matrix (actually a tensor density), each of whose components is a 2 $\times$ 2 matrix in field-space — equivalently this can be viewed as a 2 $\times$ 2 matrix in field-space each of whose components is a 4 $\times$ 4 spacetime tensor density. By inspection this is a self-adjoint second-order linear system of PDEs. The spacetime geometry is encoded in the leading-symbol of the PDEs, namely the $f^{ab}$, without considering the other subdominant terms. That this is a sensible point of view is most easily seen by considering the usual curved-spacetime d’Alembertian equation for a charged particle interacting with a scalar potential in a standard pseudo–Riemannian geometry

$$\frac{1}{\sqrt{-g}}[\partial_a - iA_a] \left( \sqrt{-g} g^{ab} [\partial_b - iA_b] \bar{\theta} \right) + V \theta = 0 \quad (39)$$

from which it is clear that we want to make the analogy.
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\[ f^{ab} \sim \sqrt{-g} \ g^{ab} \]  

as the key quantity specifying the geometry. In addition

\[ \Gamma^a \sim i A^a \quad \text{and} \quad A + K \sim V - g^{ab} A_a A_b \]

so that \( \Gamma^a \) is analogous to a vector potential and \( A \) (plus corrections) is related to the scalar potential \( V \) — in a translation invariant background this will ultimately provide a mass term.

Specifically in the current 2-BEC system we have

\[ f^{ab} = \begin{pmatrix} -\Xi^{-1} & -(V \Xi^{-1})^T \\ -V \Xi^{-1} & D - V \Xi^{-1} V^T \end{pmatrix}, \]

where

\[ V^T = \begin{bmatrix} v^T_A & 0 \\ 0 & v^T_B \end{bmatrix} \]

is a 2 \times 2 matrix in field space that is also a row vector in physical 3-space. Overall, this does look like a rather complicated object. However, it is possible to re-write the 4 \times 4 geometry containing 2 \times 2 matrices as its elements, in form of a single (2 \cdot 4) \times (2 \cdot 4) matrix. Explicitly we have

\[ f^{ab} = \begin{pmatrix} -\Xi^{-1} & -(V \Xi^{-1})^T \\ -V \Xi^{-1} & D - V \Xi^{-1} V^T \end{pmatrix}, \]

which we can re-write as

\[ f = \begin{pmatrix} -\Xi^{-1} V_1 V_1^T + D_{11} h & -\Xi^{-1} V_1 V_2^T + D_{12} h \\ -\Xi^{-1} V_2 V_1^T + D_{21} h & -\Xi^{-1} V_2 V_2^T + D_{22} h \end{pmatrix}, \]

where

\[ V_1^a := (1, v^i_A), \quad V_2^a := (1, v^i_B), \]

and

\[ h^{ab} := \text{diag}(0, 1, 1, 1). \]

\[ \Xi_{12} = \Xi_{21}, \quad \Xi_{12} = \Xi_{21}. \]
Even simpler is the form
\[
f = \begin{bmatrix}
-\Xi^{-1}_{11} V_1 V_1^T & -\Xi^{-1}_{12} V_1 V_2^T \\
-\Xi^{-1}_{21} V_2 V_1^T & -\Xi^{-1}_{22} V_2 V_2^T
\end{bmatrix} + D \otimes h.
\]

The key point is that this allows us to write
\[
f^{ab} = \begin{bmatrix}
f^{ab}_{11} & f^{ab}_{12} \\
f^{ab}_{21} & f^{ab}_{22}\end{bmatrix},
\]

where
\[
\begin{align*}
f^{ab}_{11} &= -\Xi^{-1}_{11} V_1^a V_1^b + D^{11} h^{ab}, \\
f^{ab}_{12} &= -\Xi^{-1}_{12} V_1^a V_2^b, \\
f^{ab}_{21} &= -\Xi^{-1}_{21} V_2^a V_1^b, \\
f^{ab}_{22} &= -\Xi^{-1}_{22} V_2^a V_2^b + D^{22} h^{ab}.
\end{align*}
\]

It is also possible to separate the representation of \(f^{ab}\) into field space and position space as follows
\[
f^{ab} = \begin{bmatrix}
\Xi^{-1}_{11} 0 & 0 \\
0 & 0 \\
0 & \Xi^{-1}_{22} \\
0 & 0
\end{bmatrix} V_1^a V_1^b + \begin{bmatrix}
0 & 0 \\
0 & \Xi^{-1}_{21} \\
0 & 0 \\
0 & 0
\end{bmatrix} V_2^a V_2^b
\]
\[
+ \begin{bmatrix}
0 & \Xi^{-1}_{12} \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix} V_1^a V_2^b + \begin{bmatrix}
0 & 0 \\
0 & \Xi^{-1}_{21} \\
0 & 0 \\
0 & 0
\end{bmatrix} V_2^a V_1^b + D h^{ab}.
\]

Why do we assert that the quantity \(f^{ab}\) defines a pseudo–Finsler geometry? (Rather than, say, simply a \(2 \times 2\) matrix of ordinary Lorentzian geometries?) To see the reason for this claim, recall the standard result [20] that the leading symbol of a system of PDEs determines the “signal speed” (equivalently, the characteristics, or the causal structure) [3]. Indeed if we consider the eikonal approximation (while still remaining in the realm of validity of the hydrodynamic approximation) then the causal structure is completely determined by the leading term in the Fresnel equation
\[
\text{det}[f^{ab} k_a k_b] = 0,
\]
where the determinant is taken in field space. (The quantity \(f^{ab} k_a k_b\) is exactly what is called the leading symbol of the system of PDEs, and the vanishing of this determinant is the statement that high-frequency modes can propagate with wave vector \(k_a\), thereby determining both characteristics and causal structure.) In the 2-BEC case we can explicitly expand the determinant condition as
\[
(f^{ab}_{11} k_a k_b)(f^{cd}_{22} k_c k_d) - (f^{ab}_{12} k_a k_b)(f^{cd}_{21} k_c k_d) = 0.
\]

Define a completely symmetric rank four tensor
\[
Q^{abcd} \equiv f^{(ab}_{11} f^{cd)}_{22} - f^{(ab}_{12} f^{cd)}_{21},
\]
then the determinant condition is equivalent to

\[ Q^{abcd} k_a k_b k_c k_d = 0, \quad (56) \]

which now defines the characteristics in terms of the vanishing of the pseudo-co-Finsler structure

\[ Q(k) = Q^{abcd} k_a k_b k_c k_c, \quad (57) \]

defined on the cotangent bundle. As explained in appendix A, this pseudo-co-Finsler structure can be Legendre transformed to provide a pseudo-Finsler structure, a Finslerian notion of distance

\[ ds^4 = g_{abcd} \, dx^a \, dx^b \, dx^c \, dx^d. \quad (58) \]

Here the completely symmetric rank 4 tensor \( g_{abcd} \) determines the “sound cones” through the relation \( ds = 0 \). It is interesting to note that a distance function of the form

\[ ds = \sqrt[4]{g_{abcd} \, dx^a \, dx^b \, dx^c \, dx^d} \quad (59) \]

first made its appearance in Riemann’s inaugural lecture of 1854 [21], though he did nothing further with it, leaving it to Finsler to develop the branch of geometry now bearing his name [22]. The present discussion is sufficient to justify the use of the term “pseudo–Finsler” in the generic 2-BEC situation, but we invite the more mathematically inclined reader to see appendix A for a sketch of how much further these ideas can be taken.

The pseudo–Finsler geometry implicit in (50) is rather complicated compared with the pseudo-Riemannian geometry we actually appear to be living in, at least as long as one accepts standard general relativity as a good description of reality. To mimic real gravity, we need to simplify our model. It is now time to use the major advantage of our analogue model, the ability to tune the BEC parameters, and with it the 2-field background configuration. The first order of business is to decouple \( f^{ab} \) in field space.

### 3.2 Bi-metric geometry

The reduction of equation (52) to a diagonal representation in field space (via an orthogonal rotation on the fields),

\[ f^{ab} \rightarrow \text{diag} \left[ f^{ab}_{11}, f^{ab}_{22} \right] = \text{diag} \left[ \sqrt{-g_{11}} \, g^{ab}_{11}, \sqrt{-g_{22}} \, g^{ab}_{22} \right], \quad (60) \]

enforces a bi-metric structure onto the condensate. There are two ways to proceed.

**Distinct background velocities**

For

\[ \mathcal{V}_1 \neq \mathcal{V}_2, \quad (61) \]
we require all five $2 \times 2$ matrices appearing in (52) to commute with each other. This has the unique solution $\Xi_{12}^{-1} = 0$, whence

$$\tilde{U}_{AB} = 0.$$  \hspace{1cm} (62)

We then get

$$f^{ab} = \left[ \begin{array}{cc} \Xi_{11}^{-1} & 0 \\ 0 & 0 \end{array} \right] \nu^a_1 \nu^b_1 + \left[ \begin{array}{cc} 0 & 0 \\ 0 & \Xi_{22}^{-1} \end{array} \right] \nu^a_2 \nu^b_2 + Dh^{ab}. \hspace{1cm} (63)$$

Since $D$ is a diagonal matrix this clearly represents a bi-metric geometry. The relevant parameters are summarized in Table 1.

**Equal background velocities**

For

$$\nu_1 = \nu_2 \equiv \nu, \hspace{1cm} (64)$$

we are still dealing with a pseudo–Finsler geometry, one which is now independently symmetric in field space ($f^{ab} = [f^T]^{ab}$), and position space $f^{ab} = f^{ba}$. In terms of the BEC parameters that means we must set equal the two background velocities, $v_{A0} = v_{B0} \equiv v_0$, and equation (52) is simplified to:

$$f^{ab} = -\Xi^{-1} \nu^a \nu^b + D h^{ab}. \hspace{1cm} (65)$$

From the above, diagonalizability in field space now additionally requires the commutator of the interaction and mass-density matrix to vanish:

$$[\Xi, D] = 0 \implies \tilde{U}_{AB}(d_A - d_B) = 0. \hspace{1cm} (66)$$

Here, we have a choice between two tuning conditions that do the job:

$$\tilde{U}_{AB} = 0 \quad \text{or} \quad d_A = d_B. \hspace{1cm} (67)$$

Under the first option, where $\tilde{U}_{AB} = 0$, the two off-diagonal elements in equation (65) are simply zero, and we get the desired bi-metricity in the form

$$f^{ab} = \left[ \begin{array}{cc} \Xi_{11}^{-1} & 0 \\ 0 & \Xi_{22}^{-1} \end{array} \right] \nu^a \nu^b + D h^{ab}. \hspace{1cm} (68)$$

Under the second option, for $d_A = d_B \equiv d$, we have $D = d I$. The situation is now a bit trickier, in the sense that one has to diagonalize $\Xi^{-1}$:

$$\tilde{\Xi}^{-1} = O^T \Xi^{-1} O$$

$$= \text{diag}\left[ \frac{\nu_1 \nu_2 + \sqrt{(\nu_1 \nu_2 - \nu_{12})^2 + 4 \nu_{12}^2}}{2(\nu_1 \nu_2 - \nu_{12})}, \frac{\nu_1 \nu_2 - \sqrt{(\nu_1 \nu_2 - \nu_{12})^2 + 4 \nu_{12}^2}}{2(\nu_1 \nu_2 - \nu_{12})} \right]. \hspace{1cm} (69)$$
Once this is done, the way to proceed is to use the elements of $\tilde{\Xi}^{-1}$ instead of $\Xi^{-1}$ in equation (68). The relevant parameters are summarized in Table 1.

<table>
<thead>
<tr>
<th>Bi-metric tuning scenarios</th>
<th>$v_{A0} \neq v_{B0}$</th>
<th>$v_{A0} = v_{B0}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U_{AB} = 0$</td>
<td>$U_{AB} = 0$</td>
<td>$d_A = d_B$</td>
</tr>
</tbody>
</table>

$\begin{array}{l}
\begin{align*}
&f_{11}^b \propto \left( -\frac{1}{v_{A0}^T v_{A0}} - \frac{v_{A0}^T}{v_{A0}} \right) \\
&f_{22}^b \propto \left( -\frac{1}{v_{B0}^T v_{B0}} - \frac{v_{B0}^T}{v_{B0}} \right)
\end{align*}
\end{array}$

$\begin{array}{l}
\begin{align*}
&g_{11,ab} \propto \left( -\frac{(c_{11}^2 - v_{A0}^2)}{v_{A0}^T} - \frac{v_{A0}^T}{v_{A0}} \right) \\
&g_{22,ab} \propto \left( -\frac{(c_{22}^2 - v_{B0}^2)}{v_{B0}^T} - \frac{v_{B0}^T}{v_{B0}} \right)
\end{align*}
\end{array}$

$\begin{array}{l}
\begin{align*}
&c_{11}^2 = \tilde{U}_{AA} d_A = \frac{v_{A0}^T v_{A0} - |U_{ABB}|^2}{m_A} \\
&c_{22}^2 = \tilde{U}_{BB} d_B = \frac{v_{B0}^T v_{B0} - |U_{ABB}|^2}{m_B}
\end{align*}
\end{array}$

Table 1. If the pseudo–Finsler geometry decouples into two independent Lorentzian geometries $f_{11} = \sqrt{-g_{11}g_{11}}$ and $f_{11}^b = \sqrt{-g_{11}g_{11}}$, with two distinct speed of sounds $c_{11}$ and $c_{22}$, we are effectively dealing with a bi-metric Lorentzian metric. The table shows the results from three different tuning scenarios, that are sufficient to drive the 2-component BEC from Finsler to bi-Lorentzian spacetime. The rightmost column $d_A = d_B$ is addressed in [12] where the authors analyze cosmic inflation in such a bi-metric system.

Once this is done, the way to proceed is to use the elements of $\tilde{\Xi}^{-1}$ instead of $\Xi^{-1}$ in equation (68). The relevant parameters are summarized in Table 1.

There is a subtlety implicit in setting the background velocities equal that should be made explicit. If $V_1 = V_2$ so that $v_{A0} = v_{B0}$, then since the masses appear in the relationship between phase and velocity we deduce

$$m_B \theta_{A0}(t, x) - m_A \theta_{B0}(t, x) = f(t). \quad (70)$$

If $m_A \neq m_B$, and if the background velocity is nonzero, we must deduce that $\delta_{AB}(t, x)$ will be at the very least be position dependent, and we will be unable to set it to zero. Alternatively, if we demand $\delta_{AB} = 0$, and have $\nabla \theta_{A0}(t, x) = \nabla \theta_{B0}(t, x) \neq 0$, then we cannot set $v_{A0} = v_{B0} \neq 0$. Fortunately this will not seriously affect further developments.

---

8 The most general pseudo–Finsler geometry is symmetric under simultaneous exchange of field space and position space: $\Gamma^a_{bc} = [\Gamma^T]_{ba}$.

9 We would like to stress that this constraint can be easily fulfilled, at least in the special case $\delta_{AB} = 0$, by tuning the transition rate $\lambda$, see equation (9).
Last, but certainly not least, we present the conditions for a mono-metric geometry in a 2-component BEC.

### 3.3 Mono-metric geometry

Despite the fact that there are three different routes to bi-metricity, once one demands mono-metricity, where

\[ f_{ab} = \text{diag} \left[ f_{11}^{ab}, f_{22}^{ab} \right] = \text{diag} \left[ \sqrt{-g_{11}}, \sqrt{-g_{11}} \right]. \tag{71} \]

then one ends up with one set unique of constraints to reduce from pseudo–Finsler to a single-metric Lorentzian geometry, namely:

\[
\begin{align*}
    v_{A0} &= v_{B0} = v_0; \\
    \tilde{U}_{AB} &= 0; \\
    \tilde{U}_{AA} &= \tilde{U}_{BB} = \tilde{U}; \\
    d_A &= d_B = d.
\end{align*}
\]

This tuning completely specifies the spacetime geometry, in that

\[ f_{11}^{ab} = f_{22}^{ab} \propto \left( \frac{-1}{v_{0}} \begin{pmatrix} -v_0^T \\ -v_0 \end{pmatrix} \right) \] \tag{73}

and after a small calculation we get

\[ g_{11}^{ab} = g_{22}^{ab} \propto \left( \frac{-\left(c^2 - v_0^2\right)}{v_0} \begin{pmatrix} -v_0^T \\ -v_0 \end{pmatrix} \right) \] \tag{74}

where we have defined

\[ c^2 = \tilde{U} d, \] \tag{75}

as the speed of sound.\(^\text{10}\)

Throughout the preceding few pages we have analyzed in detail the first term in equation (38), and identified different condensate parameters with different emergent geometries. Since there is more then one term in the wave equation describing excitations in a two-component system, this is not the end of the story. The remaining terms in equation (38), which we might generically view as “mass” and “vector potential” terms, do not directly affect the space-time geometry as such. But when an excitation propagates through a specific analogue spacetime geometry, these terms will contribute to the kinematics. It then becomes useful to consider the “mass eigenmodes” in field-space.

\(^{10}\) The speed of sound for quasi-particle excitations is of course our analogue for the speed of light in real gravity.
3.4 Merging spacetime geometry with mass eigenmodes

The eigenmodes we are interested in are eigenmodes of the field-space matrices occurring in the sub-dominant terms of the wave equation. These eigenmodes (when they exist) do not notice the presence of multiple fields — in our specific case a 2-field system — and therefore propagate nicely through the effective curved spacetime. As promised in the abstract and the motivation, we are striving for an analogue model representing a massive scalar field in a monometric Lorentzian structure. By using the results from section 3.3 we are able to decouple the first term of equation (38).

In the following we are focusing on two issues: First, we decouple the remaining terms in equation (38), and subsequently we check that these eigenmodes do not recouple the geometric term. There is however one more (technical) problem, and that is the fact that the terms we want to associate with the effective mass of the scalar field still contain partial derivatives in time and space, which ultimately implies a dependence on the energy of the propagating modes.\footnote{This can be easily be seen by going to the eikonal approximation where $\partial_x \to ik$ and $\partial_t \to i\omega$.} Luckily, this problem can be easily circumvented, for equal background phases,\footnote{Note that $\delta_{AB} = 0$ plus mono-metricity implies either $m_A = m_B$ with arbitrary $v_0 \neq 0$, or $m_A \neq m_B$ with zero $v_0 = 0$. These are exactly the two situations we shall consider below.}

$$\theta_{A0} = \theta_{B0}, \quad (76)$$

in which case

$$K = \Gamma^t = \Gamma^i = 0. \quad (77)$$

This has the effect of retaining only the matrix $A$ among the sub-dominant terms, so that the wave equation becomes

$$\partial_a (\Gamma^{ab} \bar{\partial}_b \bar{\theta}) + A \bar{\theta} = 0. \quad (78)$$

Due to the fact that the structure of the coupling matrix $A$ cannot be changed, its eigenmodes determine the eigenmodes of the overall wave equation. The eigenvectors of $A$ are given by

$$\text{EV1} := [+1, +1]$$
$$\text{EV2} := [-1, +1] \quad (79)$$

The final step is to make sure that our spacetime geometry commutes with the eigenvectors of $A$, that is

$$[\Gamma^{ab}, A] = 0. \quad (80)$$

This constraint is only fulfilled in the mono-metric case, where we are dealing with two identical classical fields, that effectively do not communicate with each other.\footnote{While $\tilde{U}_{AB} = 0$, $U_{AB} \neq 0$.} That is, all field matrices are proportional to the identity matrix.
3.5 Special case: $Ξ = \text{constant}$.

There is one specific class of geometries we are particularly interested in, and that is when $Ξ$ is a position independent and time independent constant. In the next section we will focus exclusively on this case, and apply it to quantum gravity phenomenology. This case is however, also of interest as an example of an alternate interplay between fine tuning and emergent geometry. Under the assumption that $Ξ$ is position and time independent, we are able to directly manipulate the overall wave equation for the excitations and as a consequence obtain slightly milder tuning conditions for mono-metricity.

Let us define
\[ \tilde{\theta} = Ξ^{-1/2} \theta, \]
and multiply the whole wave equation (38) with $Ξ^{1/2}$ from the left. What we are doing is a transformation in field space onto a new basis $\tilde{\theta}$, and in the new basis the wave equation is given by,
\[ \partial_a \left( \tilde{\Gamma}^a_{\;b} \partial_b \tilde{\theta} \right) + \left( \tilde{\Lambda} + \tilde{K} \right) \tilde{\theta} + \frac{1}{2} \left\{ \tilde{\Gamma}^a_{\;b} \partial_b \tilde{\theta} + \partial_a (\tilde{\Gamma}^a_{\;b} \tilde{\theta}) \right\} = 0, \]
where the matrices in field space transform as: $\tilde{\Lambda} = Ξ^{1/2} \Lambda Ξ^{1/2}$, $\tilde{K} = Ξ^{1/2} K Ξ^{1/2}$, $\tilde{\Gamma}^a_{\;b} = Ξ^{1/2} \Gamma^a_{\;b} Ξ^{1/2}$, and the tensor-density as
\[ \tilde{f}^{ab} = Ξ^{1/2} f^{ab} Ξ^{1/2}. \]
In general, the transformation matrix $Ξ^{1/2}$ is a non-diagonal, though always symmetric: \[ Ξ^{1/2} = \begin{pmatrix} \frac{Ξ + \sqrt{\det{Ξ}}}{\sqrt{\text{tr}[Ξ] + 2\sqrt{\det{Ξ}}}} & \mathbf{I} \\ \end{pmatrix}. \]

A close look at equation (52), now using the tensor-density $\tilde{f}^{ab}$, makes it obvious that for
\[ \tilde{U}_{AB} = 0, \]
the geometry reduces from pseudo-Finsler to bi-metric. For the sake of keeping the discussion short and easy to follow, we set the background velocities equal, and now get
\[ \tilde{f}^{ab} = \nu^a \nu^b + \tilde{D} h^{ab}. \]
In view of the tuning, $\tilde{U}_{AB} = 0$, we see
\[ \tilde{D} = \text{diag}(\tilde{U}_{AA} d_A, \tilde{U}_{BB} d_B). \]

The new mass-density matrix, and therefore the overall geometry is diagonal in field space, hence we are now dealing with the required bi-metric structure.

---

14 See appendix B.
Analogue spacetime based on 2-component Bose–Einstein condensates

So far we are in complete agreement with what we have obtained in our previous analysis, see Fig. 2. However, if we now ask for mono-metricity, we obtain a slightly milder constraint:

$$ U_{AA} d_A = U_{BB} d_B. $$

(88)

Last but not least, we show in detail the results we obtain for this tuning scenario when including the $\Lambda$ term (the mass term). To avoid confusion, we re-define a few matrices,

$$ C_0^2 = \Xi^{1/2} D \Xi^{1/2}; \quad \text{and} \quad \Omega^2 = \Xi^{1/2} \Lambda \Xi^{1/2}. $$

(89)

Both $C_0^2$ and $\Omega^2$ are symmetric matrices. If $[C_0^2, \Omega^2] = 0$, which is equivalent to the matrix equation $D \Xi \Lambda = \Lambda \Xi D$, and is certainly satisfied in view of the above constraint, then they have common eigenvectors. Decomposition onto the eigenstates of the system results in a pair of independent Klein–Gordon equations

$$ \frac{1}{\sqrt{-g_{I/II}}} \partial_a \left\{ \sqrt{-g_{I/II}} (g_{I/II})^{ab} \partial_b \hat{\theta}_{I/II} \right\} + \omega_{I/II}^2 \hat{\theta}_{I/II} = 0, $$

(90)

where the “acoustic metrics” are given by

$$ (g_{I/II})_{ab} \propto \left[ \begin{array}{c|c} - (c^2 - v_0^2) & -v_0 \tau^T \\ \hline -v_0 & 1_{d \times d} \end{array} \right]. $$

(91)

The metric components depend only on the background velocity $v_0$ and the common speed of sound $c$. It is also possible to calculate the eigenfrequencies of the two phonon modes,

$$ \omega_I^2 = 0; \quad \omega_{I/II}^2 = \text{tr}[\Omega^2]. $$

(92)

A zero/ non-zero eigenfrequency corresponds to a zero/ non-zero mass for the phonon mode.

In the eikonal limit we see that the in-phase perturbation will propagate with the speed of sound,

$$ v_s = v_0 + k c, $$

(93)

while the anti-phase perturbations propagates with a lower group velocity given by:

$$ v_g = \frac{\partial \omega}{\partial k} = v_0 + k \frac{c^2}{\sqrt{\omega_{II}^2 + c^2 k^2}}. $$

(94)

Here $k$ is the usual wave number. The dispersion relation we obtain for the mono-metric structure is Lorentz invariant.

The fact that we have an analogue model representing both massive and massless particles is promising for quantum gravity phenomenology if we now extend the analysis to high-energy phonon modes where the quantum pressure term is significant, and where we consequently expect a breakdown of
Lorentz invariance. For the following, we concentrate on the generalization of flat Minkowski spacetime, which implies a constant $\Xi$ and zero background velocities, $v_0$. In the language of condensed matter physics, we are thinking of a uniform condensate at rest.

4 Application to quantum gravity phenomenology

In using this 2-BEC model to probe issues of interest to the “quantum gravity phenomenology” community it behooves us to simplify as much as possible the parts of the model not of direct interest for current considerations. Specifically, we wish to use the “quantum pressure” term as a model for the type of Lorentz violating physics that might occur in the physical universe at or near the Planck scale [23]. Since we are then interested in high energies, and consequently short distances, one might expect the average spacetime curvature to be negligible — that is, we will be interested in looking for “quantum pressure” induced deviations from special relativity, and can dispense with the notion of curved spacetimes for now. (“Flat” pseudo-Finsler spaces are already sufficiently complicated to lead to interesting physics.) In terms of the BEC condensates this means that in this section of the chapter we will concentrate on a spatially-homogeneous time-independent background, so that in particular all the matrices $f^{ab}$ will be taken to be position-independent. (And similarly, $\Xi, \Lambda, D, \text{ etc.}$ are taken to be position independent and we set $v_0 = 0$, so the background is at rest.) This greatly simplifies the calculations (though they are still relatively messy), but without sacrificing the essential pieces of the physics we are now interested in.

Now the purpose of quantum gravity phenomenology is to analyze the physical consequences arising from various models of quantum gravity. One hope for obtaining an experimental grasp on quantum gravity is the generic prediction arising in many (but not all) quantum gravity models that ultraviolet physics at or near the Planck scale, $M_{\text{Planck}} = 1.2 \times 10^{19}$ GeV/c$^2$, (or in some models the string scale), typically induces violations of Lorentz invariance at lower scales [24, 25]. Interestingly most investigations, even if they arise from quite different fundamental physics, seem to converge on the prediction that the breakdown of Lorentz invariance can generically become manifest in the form of modified dispersion relations

$$\omega^2 = \omega_0^2 + (1 + \eta_2) c^2 k^2 + \eta_4 \left( \frac{\hbar}{M_{\text{Lorentz violation}}} \right)^2 k^4 + \ldots , \quad (95)$$

where the coefficients $\eta_n$ are dimensionless (and possibly dependent on the particle species considered), and we have restricted our expansion to CPT invariant terms (otherwise one would also get odd powers in $k$). The particular inertial frame for these dispersion relations is generally specified to be the frame set by cosmological microwave background, and $M_{\text{Lorentz violation}}$ is the
scale of Lorentz symmetry breaking which furthermore is generally assumed to be of the order of $M_{\text{Planck}}$.

Although several alternative scenarios have been considered in the literature in order to justify the modified kinematics discussed above, to date the most commonly explored avenue is an effective field theory (EFT) approach. In the present chapter we focus on the class of non-renormalizable EFTs with Lorentz violations associated to dispersion relations like equation (95). Relaxing our CPT invariance condition this class would include the model developed in [26], and subsequently studied by several authors, where an extension of quantum electrodynamics including only mass dimension five Lorentz-violating operators was considered. (That ansatz leads to order $k^3$ Lorentz and CPT violating terms in the dispersion relation.) Very accurate constraints have been obtained for this model using a combination of experiments and observations (mainly in high energy astrophysics). See e.g. [25, 27, 28, 29]. In spite of the remarkable success of this framework as a “test theory”, it is interesting to note that there are still significant open issues concerning its theoretical foundations. Perhaps the most pressing one is the so called naturalness problem which can be expressed in the following way: Looking back at our ansatz (95) we can see that the lowest-order correction, proportional to $\eta_2$, is not explicitly Planck suppressed. This implies that such a term would always be dominant with respect to the higher-order ones and grossly incompatible with observations (given that we have very good constraints on the universality of the speed of light for different elementary particles). Following the observational leads it has been therefore
often assumed either that some symmetry (other than Lorentz invariance) enforces the $\eta_2$ coefficients to be exactly zero, or that the presence of some other characteristic EFT mass scale $\mu \ll M_{\text{Planck}}$ (e.g., some particle physics mass scale) associated with the Lorentz symmetry breaking might enter in the lowest order dimensionless coefficient $\eta_2$ — which will be then generically suppressed by appropriate ratios of this characteristic mass to the Planck mass: $\eta_2 \propto (\mu/M_{\text{Planck}})^\sigma$ where $\sigma \geq 1$ is some positive power (often taken as one or two). If this is the case then one has two distinct regimes: For low momenta $p/(M_{\text{Planck}}c) \ll (\mu/M_{\text{Planck}})^\sigma$ the lower-order (quadratic in the momentum) deviations in (95) will dominate over the higher-order ones, while at high energies $p/(M_{\text{Planck}}c) \gg (\mu/M_{\text{Planck}})^\sigma$ the higher order terms will be dominant.

The naturalness problem arises because such a scenario is not well justified within an EFT framework; in other words there is no natural suppression of the low-order modifications in these models. In fact we implicitly assumed that there are no extra Planck suppressions hidden in the dimensionless coefficients $\eta_n$ with $n > 2$. EFT cannot justify why only the dimensionless coefficients of the $n \leq 2$ terms should be suppressed by powers of the small ratio $\mu/M_{\text{Planck}}$. Even worse, renormalization group arguments seem to imply that a similar mass ratio, $\mu/M_{\text{Planck}}$ would implicitly be present also in all the dimensionless $n > 2$ coefficients — hence suppressing them even further, to the point of complete undetectability. Furthermore it is easy to show [30] that, without some protecting symmetry, it is generic that radiative corrections due to particle interactions in an EFT with only Lorentz violations of order $n > 2$ in (95) for the free particles, will generate $n = 2$ Lorentz violating terms in the dispersion relation, which will then be dominant. Observational evidence [24] suggests that for a variety of standard model particles $|\eta_2| \lesssim 10^{-21}$. Naturalness in EFT would then imply that the higher order terms are at least as suppressed as this, and hence beyond observational reach.

A second issue is that of “universality”, which is not so much a “problem”, as an issue of debate as to the best strategy to adopt. In dealing with situations with multiple particles one has to choose between the case of universal (particle-independent) Lorentz violating coefficients $\eta_n$, or instead go for a more general ansatz and allow for particle-dependent coefficients; hence allowing different magnitudes of Lorentz symmetry violation for different particles even when considering the same order terms (same $n$) in the momentum expansion. The two choices are equally represented in the extant literature (see e.g. [31] and [27] for the two alternative ansätze), but it would be interesting to understand how generic this universality might be, and what sort of processes might induce non-universal Lorentz violation for different particles.

4.1 Specializing the wave equation

For current purposes, where we wish to probe violations of Lorentz invariance in a flat analogue spacetime, we start with our basic wave equation (25) and
make the following specializations: $\delta_{AB} \to 0$ (so that $\Gamma^a \to 0$ and $K \to 0$). We also set all background fields to be homogeneous (space and time independent), and use the formal operators $\hat{\Xi}^{1/2}$ and $\hat{\Xi}^{-1/2}$ to define a new set of variables

$$\tilde{\theta} = \hat{\Xi}^{-1/2} \bar{\theta},$$

(96)

in terms of which the wave equation becomes

$$\partial_t^2 \tilde{\theta} = \left( \hat{\Xi}^{1/2} \left[D\nabla^2 - A\right] \hat{\Xi}^{1/2} \right) \tilde{\theta},$$

(97)

or more explicitly

$$\partial_t^2 \tilde{\theta} = \left( \left[\Xi - Xk^2\right]^{1/2} \left[D\nabla^2 - A\right] \left[\Xi - Xk^2\right]^{1/2} \right) \tilde{\theta}.$$

(98)

This is now a (relatively) simple PDE to analyze. The objects $\hat{\Xi}^{1/2}$ and $\hat{\Xi}^{-1/2}$ are $2 \times 2$ matrices whose elements are pseudo-differential operators, but to simplify things it is computationally efficient to go directly to the eikonal limit where

$$\hat{\Xi} \to \Xi + Xk^2.$$

(99)

This finally leads to a dispersion relation of the form

$$\det \left\{ \omega^2 I - \left[ \Xi + Xk^2 \right]^{1/2} \left[ Dk^2 + A \right] \left[ \Xi + Xk^2 \right]^{1/2} \right\} = 0,$$

(100)

and “all” we need to do for the purposes of this chapter, is to understand this quasiparticle excitation spectrum in detail.

### 4.2 Hydrodynamic approximation

The hydrodynamic limit consists of formally setting $\tilde{X} \to 0$ so that $\hat{\Xi} \to \Xi$. (That is, one is formally setting the healing length matrix to zero: $Y \to 0$. More precisely, all components of the healing length matrix are assumed small compared to other length scales in the problem.) The wave equation (98) now takes the form:

$$\partial_t^2 \tilde{\theta} = \left( \Xi^{1/2} \left[D\nabla^2 - A\right] \Xi^{1/2} \right) \tilde{\theta}.$$

(101)

Since this is second-order in both space and time derivatives, we now have at least the possibility of obtaining an exact “Lorentz invariance”. We can now define the matrices

$$\Omega^2 = \Xi^{1/2} A \Xi^{1/2}; \quad C_0^2 = \Xi^{1/2} D \Xi^{1/2};$$

(102)

\[15\] Once we are in the eikonal approximation the pseudo-differential operator $\hat{\Xi}^{1/2} \to \sqrt{\Xi + k^2X}$ can be given a simple and explicit meaning in terms of the Hamilton–Cayley theorems of appendix B.
so that after Fourier transformation
\[
\omega^2 \tilde{\theta} = \{ C_0^2 k^2 + \Omega^2 \} \tilde{\theta} \equiv H(k^2) \tilde{\theta},
\]
leading to the Fresnel equation
\[
\det\{\omega^2 I - H(k^2)\} = 0.
\]
That is
\[
\omega^4 - \omega^2 \text{tr}[H(k^2)] + \text{det}[H(k^2)] = 0,
\]
whence
\[
\omega^2 = \frac{\text{tr}[H(k^2)] \pm \sqrt{\text{tr}[H(k^2)]^2 - 4 \text{det}[H(k^2)]}}{2}.
\]
Note that the matrices \( \Omega^2, C_0^2 \), and \( H(k^2) \) have now carefully been arranged to be symmetric. This greatly simplifies the subsequent matrix algebra. Also note that the matrix \( H(k^2) \) is a function of \( k^2 \); this will forbid the appearance of odd powers of \( k \) in the dispersion relation — as should be expected due to the parity invariance of the system.

Masses

We read off the “masses” by looking at the special case of space-independent oscillations for which
\[
\partial_t^2 \tilde{\theta} = -\Omega^2 \tilde{\theta},
\]
allowing us to identify the “mass” (more precisely, the natural oscillation frequency) as
\[
\text{“masses”} \propto \text{eigenvalues of } (\Xi^{1/2} A \Xi^{1/2}) = \text{eigenvalues of } (\Xi A).
\]
Since \( A \) is a singular 2 \times 2 matrix this automatically implies
\[
\omega^2_I = 0; \quad \omega^2_{II} = \text{tr}(\Xi A).
\]
So we see that one mode will be a massless phonon while the other will have a non zero mass. Explicitly, in terms of the elements of the underlying matrices
\[
\omega^2_I = 0; \quad \omega^2_{II} = -\frac{2\sqrt{\rho A_0 \rho B_0} \lambda}{\hbar^2} \{ \hat{U}_{AA} + \hat{U}_{BB} - 2\hat{U}_{AB} \}
\]
so that (before any fine-tuning or decoupling)
\[
\omega^2_{II} = -\frac{2\sqrt{\rho A_0 \rho B_0} \lambda}{\hbar^2} \times \left\{ \hat{U}_{AA} + \hat{U}_{BB} - 2\hat{U}_{AB} - \frac{\lambda}{2\sqrt{\rho A_0 \rho B_0}} \left[ \sqrt{\rho A_0} + \sqrt{\rho B_0} \right]^2 \right\}.
\]
It is easy to check that this quantity really does have the physical dimensions of a frequency.
Psycho-social conditions

In order for our system to be a perfect analogue of special relativity:

- we want each mode to have a quadratic dispersion relation;
- we want each dispersion relation to have the same asymptotic slope.

Let us start by noticing that the dispersion relation (106) is of the form

$$\omega^2 = [\text{quadratic}_1] \pm \sqrt{[\text{quartic}]}.$$  (112)

The first condition implies that the quartic must be a perfect square

$$[\text{quartic}] = [\text{quadratic}_2]^2,$$  (113)

but then the second condition implies that the slope of this quadratic must be zero. That is

$$[\text{quadratic}_2](k^2) = [\text{quadratic}_2](0),$$  (114)

and so

$$[\text{quartic}](k^2) = [\text{quartic}](0)$$  (115)

must be constant independent of $k^2$, so that the dispersion relation is of the form

$$\omega^2 = [\text{quadratic}_1](k^2) \pm [\text{quadratic}_2](0).$$  (116)

Note that this has the required form (two hyperbolae with the same asymptotes, and possibly different intercepts). Now let us implement this directly in terms of the matrices $C_2^0$ and $M^2$.

**Step 1:** Using the results of the appendix, specifically equation (255):

$$\det[H^2(k)] = \det[\Omega^2 + C_2^0 k^2]$$

$$= \det[\Omega^2] - \text{tr}\{\Omega^2 \bar{C}_0^2\} k^2 + \det[C_0^2] (k^2)^2.$$  (117)

(This holds for any linear combination of $2 \times 2$ matrices. Note that we apply trace reversal to the squared matrix $C_0^2$, we do not trace reverse and then square.) Since in particular $\det[\Omega^2] = 0$, we have:

$$\det[H^2(k)] = -\text{tr}\{\Omega^2 \bar{C}_0^2\} k^2 + \det[C_0^2] (k^2)^2.$$  (118)

**Step 2:** Now consider the discriminant (the quartic)

$$\text{quartic} = 4 \det[H(k^2)] - 4 \det[H(k^2)]^2$$

$$= (\text{tr}[\Omega^2] + \text{tr}[C_0^2] k^2)^2 - 4\left[ - \text{tr}\{\Omega^2 \bar{C}_0^2\} k^2ight.$$  

$$+ \det[C_0^2] (k^2)^2]$$

$$= \text{tr}[\Omega^2]^2 + 2\text{tr}[\Omega^2]\text{tr}[C_0^2] + 4\text{tr}\{\Omega^2 \bar{C}_0^2\} k^2$$

$$+ \{\text{tr}[C_0^2]^2 - 4\det[C_0^2]\} (k^2)^2$$  (120)

$$= \text{tr}[\Omega^2]^2 + 2\{2\text{tr}[\Omega^2][\text{tr}[C_0^2] - \text{tr}[\Omega^2]\text{tr}[C_0^2]]k^2$$

$$+ \{\text{tr}[C_0^2]^2 - 4\det[C_0^2]\} (k^2)^2.$$  (121)
So in the end the two conditions above for mono-metricity take the form

\[
\text{mono-metricity} \iff \begin{cases} 
\text{tr}[C_0^2]^2 - 4 \det[C_0^2] = 0; \\
2\text{tr} \{ \Omega^2 C_0^2 \} - \text{tr}[\Omega^2] \text{tr}[C_0^2] = 0.
\end{cases}
\] (124)

Once these two conditions are satisfied the dispersion relation is

\[
\omega^2 = \frac{\text{tr}[H(k^2)] \pm \text{tr}[\Omega^2]}{2} = \frac{\text{tr}[\Omega^2] \pm \text{tr}[\Omega^2] + \text{tr}[C_0^2] k^2}{2}
\] (125)

whence

\[
\omega_1^2 = \frac{1}{2} \text{tr}[C_0^2] k^2 = \tilde{c}_0^2 k^2 \quad \omega_2^2 = \text{tr}[\Omega^2] + \frac{1}{2} \text{tr}[C_0^2] k^2 = \omega_{1I}^2 + \tilde{c}_0^2 k^2,
\] (126)

as required. One mode is massless, one massive with exactly the “mass” previously deduced. One can now define the quantity

\[
m_{1I} = \hbar \omega_{1I}/\tilde{c}_0^2,
\] (127)

which really does have the physical dimensions of a mass.

**Interpretation of the mono-metricity conditions**

But now we have to analyse the two simplification conditions

\[
C1 : \quad \text{tr}[C_0^2]^2 - 4 \det[C_0^2] = 0; \quad (128)
\]
\[
C2 : \quad 2\text{tr} \{ \Omega^2 C_0^2 \} - \text{tr}[\Omega^2] \text{tr}[C_0^2] = 0; \quad (129)
\]

to see what they tell us. The first of these conditions is equivalent to the statement that the $2 \times 2$ matrix $C_0^2$ has two identical eigenvalues. But since $C_0^2$ is symmetric this then implies $C_0^2 = \tilde{c}_0^2 I$, in which case the second condition is automatically satisfied. (In contrast, condition C2 does not automatically imply condition C1.) Indeed if $C_0^2 = \tilde{c}_0^2 I$, then it is easy to see that (in order to make $C_0^2$ diagonal)

\[
\tilde{U}_{AB} = 0,
\] (130)

(which is sufficient, by itself, to imply bi-metricity) and furthermore that

\[
\frac{\tilde{U}_{AA} \rho_{A0}}{m_A} = \frac{\tilde{c}_0^2}{m_A} = \frac{\tilde{U}_{BB} \rho_{B0}}{m_B}.
\] (131)

Note that we can now solve for $\lambda$ to get

\[
\lambda = -2\sqrt{\rho_{A0} \rho_{B0}} U_{AB},
\] (132)

whence

\[
\tilde{c}_0^2 = \frac{U_{AA} \rho_{A0} + U_{AB} \rho_{B0}}{m_A} = \frac{U_{BB} \rho_{B0} + U_{AB} \rho_{A0}}{m_B},
\] (133)
Analogue spacetime based on 2-component Bose–Einstein condensates

\[ \omega_{II}^2 = \frac{4\rho_A \rho_B U_{AB}}{\hbar^2} \left\{ U_{AA} + U_{BB} - 2U_{AB} + \frac{\rho_A}{\rho_B} + \frac{\rho_B}{\rho_A} \right\}^2. \]  
(134)

Note that (134) is equivalent to (112) with (132) enforced. But this then implies

\[ \omega_{II}^2 = \frac{4\rho_A \rho_B U_{AB}}{\hbar^2} \left\{ U_{AA} + U_{BB} + U_{AB} \left[ \frac{\rho_A}{\rho_B} + \frac{\rho_B}{\rho_A} \right] \right\}^2. \]  
(135)

**Interpretation:** Condition C2 forces the two low-momentum “propagation speeds” to be the same, that is, it forces the two \( O(k^2) \) coefficients to be equal. Condition C1 is the stronger statement that there is no \( O(k^4) \) (or higher order) distortion to the relativistic dispersion relation.

### 4.3 Beyond the hydrodynamical approximation

At this point we want to consider the deviations from the previous analogue for special relativity. Our starting point is again equation (98), now retaining the quantum pressure term, which we Fourier transform to get:

\[ \omega^2 \tilde{\theta} = \left\{ \sqrt{\Xi + X k^2} \left[ D k^2 + \Lambda \right] \sqrt{\Xi + X k^2} \right\} \tilde{\theta} \equiv H(k^2) \tilde{\theta}. \]  
(136)

This leads to the Fresnel equation

\[ \det\{\omega^2 I - H(k^2)\} = 0. \]  
(137)

That is

\[ \omega^4 - \omega^2 \text{tr}[H(k^2)] + \text{det}[H(k^2)] = 0, \]  
(138)

whence

\[ \omega^2 = \frac{\text{tr}[H(k^2)] \pm \sqrt{\text{tr}[H(k^2)]^2 - 4 \text{det}[H(k^2)]}}{2}, \]  
(139)

which is now of the form

\[ \omega^2 = [\text{quartic}] \pm \sqrt{[\text{octic}]}. \]  
(140)

**Masses**

The “masses”, defined as the zero momentum oscillation frequencies, are again easy to identify. Just note that the \( k \)-independent term in the Fresnel equation is exactly the same mass matrix \( \Omega^2 = \Xi^{1/2} \Lambda \Xi^{1/2} \) that was present in the hydrodynamical limit. (That is, the quantum potential term \( X \) does not influence the masses.)
Dispersion relations

Differently from the previous case, when the hydrodynamic approximation held, we now have that the discriminant of (139) generically can be an eighth-order polynomial in $k$. In this case we cannot hope to recover an exact analogue of special relativity, but instead can at best hope to obtain dispersion relations with vanishing or suppressed deviations from special relativity at low $k$; possibly with large deviations from special relativity at high momenta. From the form of our equation it is clear that the Lorentz violation suppression should be somehow associated with the masses of the atoms $m_{A/B}$. Indeed we will use the underlying atomic masses to define our “Lorentz breaking scale”, which we shall then assume can be identified with the “quantum gravity scale”. The exact form and relative strengths of the higher-order terms will be controlled by tuning the 2–BEC system and will eventually decide the manifestation (or not) of the naturalness problem and of the universality issue.

Our approach will again consist of considering derivatives of (139) in growing even powers of $k^2$ (recall that odd powers of $k$ are excluded by the parity invariance of the system) and then setting $k \to 0$. We shall compute only the coefficients up to order $k^4$ as by simple dimensional arguments one can expect any higher order term will be further suppressed with respect to the $k^4$ one.

We can greatly simplify our calculations if before performing our analysis we rearrange our problem in the following way. First of all note that by the cyclic properties of trace

\[
\text{tr}[H(k^2)] = \text{tr}[(Dk^2 + \Lambda)(\Xi + k^2X)]
\]

(141)

\[
= \text{tr}[A\Xi + k^2(D\Xi + \Lambda X) + (k^2)^2DX]
\]

(142)

\[
= \text{tr}[\Xi^{1/2}A\Xi^{1/2} + k^2(\Xi^{1/2}D\Xi^{1/2} + X^{1/2}\Lambda X^{1/2})
\]

\[
+ (k^2)^2X^{1/2}DX^{1/2}].
\]

(143)

Putting this all together, we can now define symmetric matrices

\[
\Omega^2 = \Xi^{1/2}A\Xi^{1/2};
\]

(144)

\[
C_0^2 = \Xi^{1/2}D\Xi^{1/2}; \quad \Delta C^2 = X^{1/2}\Lambda X^{1/2};
\]

(145)

\[
C^2 = C_0^2 + \Delta C^2 = \Xi^{1/2}D\Xi^{1/2} + X^{1/2}\Lambda X^{1/2};
\]

(146)

\[
Z^2 = 2X^{1/2}DX^{1/2} = \frac{\hbar^2}{2}M^{-2}.
\]

(147)

With all these definitions we can then write

\[
\text{tr}[H(k^2)] = \text{tr}\left[\Omega^2 + k^2(C_0^2 + \Delta C^2) + \frac{1}{2}(k^2)^2Z^2\right],
\]

(148)

where everything has been done inside the trace. If we now define

\[
H_s(k^2) = \Omega^2 + k^2(C_0^2 + \Delta C^2) + \frac{1}{2}(k^2)^2Z^2,
\]

(149)
then $H_s(k^2)$ is by definition both polynomial and symmetric and satisfies
\[ \text{tr}[H(k^2)] = \text{tr}[H_s(k^2)], \quad (150) \]
while in contrast,
\[ \det[H(k^2)] \neq \det[H_s(k^2)]. \quad (151) \]
But then
\[ \omega^2 = \frac{1}{2} \left[ \text{tr}[H_s(k^2)] \pm \sqrt{\text{tr}[H_s(k^2)]^2 - 4\det[H(k^2)]} \right], \quad (152) \]
Whence
\[ \frac{d\omega^2}{dk^2} = \frac{1}{2} \left[ \frac{\text{tr}[H_s'(k^2)]}{\sqrt{\text{tr}[H_s(k^2)]^2 - 4\det[H(k^2)]}} \right], \quad (153) \]
and at $k = 0$
\[ \frac{d\omega^2}{dk^2} \bigg|_{k \to 0} = \frac{1}{2} \left[ \frac{\text{tr}[C^2] - 2\det'[H(k^2)]_{k \to 0}}{\text{tr}[\Omega^2]} \right]. \quad (154) \]
But now let us consider
\[
\det[H(k^2)] = \det[(Dk^2 + A)(\Xi + k^2X)] = \det[Dk^2 + A] \det[\Xi + k^2X] = \det[\Xi^{1/2}(Dk^2 + A)\Xi^{1/2}] \det[I + k^2\Xi^{-1/2}X\Xi^{-1/2}] \quad (155)
\]
where we have repeatedly used properties of the determinant. Furthermore
\[
\det[I + k^2\Xi^{-1/2}X\Xi^{-1/2}] = \det[I + k^2\Xi^{-1}X] = \det[I + k^2X^{1/2}\Xi X^{1/2}] = \det[I + k^2Y^2/2], \quad (156-160)
\]
so that we have
\[ \det[H(k^2)] = \det[\Omega^2 + C_0^2k^2] \det[I + k^2Y^2/2]. \quad (161) \]
Note the the matrix $Y^2$ is the “healing length matrix” we had previously defined, and that the net result of this analysis is that the full determinant is the product of the determinant previously found in the hydrodynamic limit with a factor that depends on the product of wavenumber and healing length.
But now, given our formula (255) for the determinant, we see
\[
\det'[H(k^2)] = (-\text{tr}[(\Omega^2C_0^2) + 2k^2\det[C_0^2]]) \det[I + k^2Y^2/2] + \det[\Omega^2 + C_0^2k^2] (-\text{tr}[\hat{Y}^2] + k^2\det[Y^2])/2, \quad (162)
\]
\[
\det'[H(k^2)]_{k\to0} = -\text{tr}(\Omega^2 C_0^2),
\]

and so
\[
\frac{d\omega^2}{dk^2} \bigg|_{k\to0} = \frac{1}{2} \left[ \text{tr}[C^2] + \frac{\text{tr}[\Omega^2][\text{tr}[C^2] + 2\text{tr}(\Omega^2 C_0^2)]}{\text{tr}[\Omega^2]} \right].
\]

That is:
\[
\frac{d\omega^2}{dk^2} \bigg|_{k\to0} = \frac{1}{2} \left[ \text{tr}[C^2] \pm \left\{ \text{tr}[C^2] + 2\frac{\text{tr}(\Omega^2 C_0^2)}{\text{tr}[\Omega^2]} \right\} \right].
\]

Note that all the relevant matrices have been carefully symmetrized. Also note the important distinction between \(C_0^2\) and \(C^2\). Now define
\[
c^2 = \frac{1}{2} \text{tr}[C^2],
\]
then
\[
\frac{d\omega^2}{dk^2} \bigg|_{k\to0} = c^2(1 \pm \eta_2),
\]
with
\[
\eta_2 = \left\{ \frac{\text{tr}[C^2][\text{tr}[\Omega^2] + 2\text{tr}(\Omega^2 C_0^2)]}{\text{tr}[\Omega^2][\text{tr}[C^2]]} \right\} = \left\{ 1 + \frac{\text{tr}(\Omega^2 C_0^2)}{\omega_{II}^2 c^2} \right\}. \tag{166}
\]

Similarly, consider the second derivative:
\[
\frac{d^2\omega^2}{d(k^2)^2} = \frac{1}{2} \left[ \text{tr}[H_s''(k^2)] \right.
\]
\[
\pm \frac{\text{tr}[H_s(k^2)][\text{tr}[H_s''(k^2)] + \text{tr}[H_s'(k^2)][\text{tr}[H_s'(k^2)] - 2\text{det}''[H(k^2)]}{\sqrt{\text{tr}[H_s(k^2)]^2 - 4\text{det}[H(k^2)]}}
\]
\[
\left. \pm \frac{(\text{tr}[H_s(k^2)][\text{tr}[H_s'(k^2)] - 2\text{det}''[H(k^2)])^2]}{\left(\text{tr}[H_s(k^2)]^2 - 4\text{det}[H(k^2)])^3/2 \right) \right], \tag{169}
\]

whence
\[
\frac{d^2\omega^2}{d(k^2)^2} \bigg|_{k\to0} = \frac{1}{2} \left[ \text{tr}[Z^2] \pm \frac{\text{tr}[\Omega^2][\text{tr}[Z^2] + \text{tr}[C^2]^2 - 2\text{det}''[H(k^2)]_{k\to0}}{\text{tr}[\Omega^2]} \right.
\]
\[
\left. \pm \frac{(\text{tr}[\Omega^2][\text{tr}[C^2] - 2\text{det}''[H(k^2)]_{k\to0})^2]}{\text{tr}[\Omega^2]^3} \right]. \tag{170}
\]

The last term above can be related to \(d\omega^2/dk^2\), while the determinant piece is evaluated using
\[
\text{det}''[H(k^2)] = (2\text{det}[C_0^2]) \det[I + k^2 Y^2/2] \tag{171}
\]
\[
+ (-\text{tr}(\Omega^2 C_0^2) + 2k^2\text{det}[C_0^2]) (-\text{tr}[Y^2] + k^2\text{det}[Y^2])/2
\]
\[
+ \text{det}[\Omega^2 + C_0^2 k^2] \text{det}[Y^2]/2
\]
\[
+ (-\text{tr}(\Omega^2 C_0^2) + 2k^2\text{det}[C_0^2]) (-\text{tr}[Y^2] + k^2\text{det}[Y^2])/2.
\]
Therefore

\[
\det''[H(k^2)]_{k \to 0} = (2\det[C^2_0]) + (-\text{tr}(\Omega^2 \tilde{C}^2_0)) (\text{tr}[Y^2])/2 + \det[\Omega^2] (\text{det}[Y^2])/2 \\
+ (-\text{tr}(\Omega^2 \tilde{C}^2_0)) (\text{tr}[Y^2])/2.
\]

(172)

That is, (recalling \(\text{tr}[\tilde{A}] = -\text{tr}[A]\)),

\[
\det''[H(k^2)]_{k \to 0} = (2\det[C^2_0]) - (\text{tr}(\Omega^2 \tilde{C}^2_0)) (\text{tr}[Y^2]),
\]

(173)

or

\[
\det''[H(k^2)]_{k \to 0} = -\text{tr}[C^2_0 \tilde{C}^2_0] - \text{tr}[\Omega^2 \tilde{C}^2_0] \text{tr}[Y^2].
\]

(174)

Now assembling all the pieces, a little algebra yields

\[
\frac{d^2 \omega^2}{d(k^2)^2} \bigg|_{k \to 0} = \frac{1}{2} \left[ \text{tr}[Z^2] \pm \text{tr}[\tilde{Z}^2] \pm 2\frac{\text{tr}[\Omega^2 \tilde{C}^2]}{\text{tr}[\Omega^2]} \text{tr}[Y^2] \pm \frac{\text{tr}[C^2]^2 - 4\det[C^2]}{\text{tr}[\Omega^2]} \right].
\]

(175)

With the above formula we have completed our derivation of the lowest-order terms of the generic dispersion relation of a coupled 2-BEC system — including the terms introduced by the quantum potential at high wavenumber — up to terms of order \(k^4\). From the above formula it is clear that we do not generically have Lorentz invariance in this system: Lorentz violations arise both due to mode-mixing interactions (an effect which can persist in the hydrodynamic limit where \(Z \to 0\) and \(Y \to 0\)) and to the presence of the quantum potential (signaled by \(Z \neq 0\) and \(Y \neq 0\)). While the mode-mixing effects are relevant at all energies the latter effect characterizes the discrete structure of the effective spacetime at high energies. It is in this sense that the quantum potential determines the analogue of quantum gravity effects in our 2-BEC system.

4.4 The relevance for quantum gravity phenomenology

Following this physical insight we can now easily identify a regime that is potentially relevant for simulating the typical ansatzes of quantum gravity phenomenology. We demand that any violation of Lorentz invariance present should be due to the microscopic structure of the effective spacetime. This implies that one has to tune the system in order to cancel exactly all those violations of Lorentz invariance which are solely due to mode-mixing interactions in the hydrodynamic limit.

We basically follow the guiding idea that a good analogue of quantum-gravity-induced Lorentz violations should be characterized only by the ultraviolet physics of the effective spacetime. In the system at hand the ultraviolet
physics is indeed characterized by the quantum potential, whereas possible violations of the Lorentz invariance in the hydrodynamical limit are low energy effects, even though they have their origin in the microscopic interactions. We therefore start by investigating the scenario in which the system is tuned in such a way that no violations of Lorentz invariance are present in the hydrodynamic limit. This leads us to again enforce the conditions $C_1$ and $C_2$ which corresponded to “mono-metricity” in the hydrodynamic limit.

In this case (165) and (175) take respectively the form

$$\frac{d^2 \omega^2}{dk^2} \bigg|_{k \to 0} = \frac{1}{2} \left( \text{tr}[C_0^2] + (1 \pm 1) \text{tr}[\Delta C^2] \right) = c_0^2 \pm \frac{1}{2} \text{tr}[\Delta C^2],$$  \hspace{1cm} (176)$$

and

$$\frac{d^2 \omega^2}{d(k^2)^2} \bigg|_{k \to 0} = \frac{\text{tr}[Z^2] \pm \text{tr}[Z^2]}{2} \pm \text{tr}[C_0^2]\text{tr}[Y^2]$$

$$\pm \frac{1}{2} \frac{\text{tr}[\Delta C^2]^2 + 2\text{tr}[C_0^2]\text{tr}[\Delta C^2]}{\text{tr}[Z^2]} \pm \frac{1}{2} \frac{\text{tr}[\Delta C^2]^2}{\text{tr}[Z^2]}$$

$$= \frac{\text{tr}[Z^2] \pm \text{tr}[Z^2]}{2} \pm \text{tr}[C_0^2] \left( -\text{tr}[Y^2] + \frac{\text{tr}[\Delta C^2]}{\text{tr}[Z^2]} \right).$$  \hspace{1cm} (177)$$

Recall (see section 4.2) that the first of the physical conditions $C_1$ is equivalent to the statement that the $2 \times 2$ matrix $C_0^2$ has two identical eigenvalues. But since $C_0^2$ is symmetric this then implies $C_0^2 = c_0^2 I$, in which case the second condition is automatically satisfied. This also leads to the useful facts

$$\tilde{U}_{AB} = 0 \quad \Rightarrow \quad \lambda = -2\sqrt{\rho_{A0} \rho_{B0}} U_{AB};$$  \hspace{1cm} (178)$$

$$c_0^2 = \frac{\tilde{U}_{AA} \rho_{A0}}{m_A} = \frac{\tilde{U}_{BB} \rho_{B0}}{m_B}. \hspace{1cm} (179)$$

Now that we have the fine tuning condition for the laser coupling we can compute the magnitude of the effective mass of the massive phonon and determine the values of the Lorentz violation coefficients. In particular we shall start checking that this regime allows for a real positive effective mass as needed for a suitable analogue model of quantum gravity phenomenology.

**Effective mass**

Remember that the definition of $m_{II}$ reads

$$m_{II}^2 = \hbar^2 \omega_{II}^2/c_0^4.$$  \hspace{1cm} (180)$$

Using equation (178) and equation (179) we can rewrite $c_0^2$ in the following form

$$c_0^2 = [m_B \rho_{A0} U_{AA} + m_A \rho_{B0} U_{BB} + U_{AB}(\rho_{A0} m_A + \rho_{B0} m_B)]/(2m_A m_B).$$  \hspace{1cm} (181)$$
Similarly equation (178) and equation (179) when inserted in equation (135) give
\[
\omega^2_{II} = \frac{4U_{AB}(\rho_{A0}m_B + \rho_{B0}m_A)c_0^2}{\hbar^2}.
\] (182)

We can now estimate \(m_{II}\) by simply inserting the above expressions in equation (180) so that
\[
m^2_{II} = \frac{8U_{AB}(\rho_{A0}m_A + \rho_{B0}m_B)m_Am_B}{[m_B\rho_{AB}U_{AA} + m_A\rho_{B0}U_{BB} + U_{AB}(\rho_{A0}m_A + \rho_{B0}m_B)]}. \tag{183}
\]

This formula is still a little clumsy but a great deal can be understood by doing the physically reasonable approximation \(m_A \approx m_B = m\) and \(\rho_A \approx \rho_B\). In fact in this case one obtains
\[
m^2_{II} \approx m^2 \frac{8U_{AB}}{[U_{AA} + 2U_{AB} + U_{BB}]}.
\] (184)

This formula now shows clearly that, as long as the mixing term \(U_{AB}\) is small compared to the “direct” scattering \(U_{AA} + U_{BB}\), the mass of the heavy phonon will be “small” compared to the mass of the atoms. Though experimental realizability of the system is not the primary focus of the current article, we point out that there is no obstruction in principle to tuning a 2-BEC system into a regime where \(|U_{AB}| \ll |U_{AA} + U_{BB}|\). For the purposes of this paper it is sufficient that a small effective phonon mass (small compared to the atomic masses which set the analogue quantum gravity scale) is obtainable for some arrangement of the microscopic parameters. We can now look separately at the coefficients of the quadratic and quartic Lorentz violations and then compare their relative strength in order to see if a situation like that envisaged by discussions of the naturalness problem is actually realized.

**Coefficient of the quadratic deviation**

One can easily see from (176) that the \(\eta_2\) coefficients for this case take the form
\[
\eta_{2,I} = 0;
\]
\[
\eta_{2,II} c_0^2 = \text{tr}[\Delta C^2] = \text{tr}[X^{1/2}AX^{1/2}] = \text{tr}[X] = -\frac{1}{2} \frac{\lambda}{m_A m_B} \left( \frac{m_A \rho_{A0} + m_B \rho_{B0}}{\sqrt{\rho_{A0} \rho_{B0}}} \right). \tag{186}
\]

So if we insert the fine tuning condition for \(\lambda\), equation (178), we get
\[
\eta_{2,II} = -\frac{U_{AB}(m_A \rho_{A0} + m_B \rho_{B0})}{m_A m_B c_0^2}. \tag{187}
\]
Remarkably we can now cast this coefficient in a much more suggestive form by expressing the coupling $U_{AB}$ in terms of the mass of the massive quasi-particle $m_I^2$. In order to do this we start from equation (182) and note that it enables us to express $U_{AB}$ in (187) in terms of $\omega_{II}^2$, thereby obtaining

$$\eta_{2,II} = \frac{\hbar^2}{4c_0} \frac{\rho_{A0} m_A + \rho_{B0} m_B}{\rho_{A0} m_B + \rho_{B0} m_A} \frac{\omega_{II}^2}{m_A m_B}. \quad (188)$$

Now it is easy to see that

$$\frac{\rho_{A0} m_A + \rho_{B0} m_B}{\rho_{A0} m_B + \rho_{B0} m_A} \approx O(1), \quad (189)$$

and that this factor is identically unity if either $m_A = m_B$ or $\rho_{A0} = \rho_{B0}$. All together we are left with

$$\eta_{2,II} = \bar{\eta} \left( \frac{m_{II}}{\sqrt{m_A m_B}} \right)^2, \quad (190)$$

where $\bar{\eta}$ is a dimensionless coefficient of order unity.

The product in the denominator of the above expression can be interpreted as the geometric mean of the fundamental bosons masses $m_A$ and $m_B$. These are mass scales associated with the microphysics of the condensate — in analogy with our experience with a 1-BEC system where the “quantum gravity scale” is set by the mass of the BEC atoms. It is then natural to define an analogue of the scale of the breakdown of Lorentz invariance as $M_{\text{eff}} = \sqrt{m_A m_B}$. (Indeed this “analogue Lorentz breaking scale” will typically do double duty as an “analogue Planck mass”.)

Using this physical insight it should be clear that equation (190) effectively says

$$\eta_{2,II} \approx \left( \frac{m_{II}}{M_{\text{eff}}} \right)^2, \quad (191)$$

which, given that $m_I = 0$, we are naturally lead to generalize to

$$\eta_{2,X} \approx \left( \frac{m_X}{M_{\text{eff}}} \right)^2 = \left( \frac{\text{mass scale of quasiparticle}}{\text{effective Planck scale}} \right)^2; \quad X = I, II. \quad (192)$$

The above relation is exactly the sort of dimensionless ratio $(\mu/M)^\sigma$ that has been very often conjectured in the literature on quantum gravity phenomenology in order to explain the strong observational constraints on Lorentz violations at the lowest orders. (See earlier discussion.) Does this now imply that this particular regime of our 2-BEC system will also show an analogue version of the naturalness problem? In order to answer this question we need to find the dimensionless coefficient for the quartic deviations, $\eta_4$, and check if it will or won’t itself be suppressed by some power of the small ratio $m_{II}/M_{\text{eff}}$. 
Coefficients of the quartic deviation

Let us now consider the coefficients of the quartic term presented in equation (177). For the various terms appearing in (177) we get

\[
\text{tr}[Z^2] = 2\text{tr}[DX] = \frac{\hbar^2}{2} \left( \frac{m_A^2 + m_B^2}{m_A m_B} \right); \quad (193)
\]

\[
\text{tr}[\Delta C^2] = \text{tr}[XA] = -\frac{\lambda m_A \rho_{A0} + m_B \rho_{B0}}{2} \sqrt{\rho_{A0} \rho_{B0}} m_A m_B = \frac{U_{AB} m_A \rho_{A0} + m_B \rho_{B0}}{m_A m_B}; \quad (194)
\]

\[
\text{tr}[Y^2] = 2\text{tr}[X \Xi^{-1}] = \frac{\hbar^2}{2} \rho_{A0} m_A \bar{U}_{AA} + \rho_{B0} m_B \bar{U}_{BB}; \quad (195)
\]

where in the last expression we have used the fact that in the current scenario \( \bar{U}_{AB} = 0 \). Now by definition

\[
\eta_4 = \frac{1}{2} \left( \frac{M_{\text{eff}}^2}{\hbar^2} \right) \left[ \frac{d^2 \omega^2}{(d^2 k)^2} \right]_{k=0} \quad (196)
\]

is the dimensionless coefficient in front of the \( k^4 \). So

\[
\eta_4 = \frac{M_{\text{eff}}^2}{2\hbar^2} \left[ \frac{\text{tr}[Z^2]}{2} \right] \pm \text{tr}[C_0^2] \left( \frac{\text{tr}[Y^2]}{2} \right) + \frac{\text{tr}[\Delta C^2]}{\text{tr}[\Omega^2]}; \quad (197)
\]

\[
\eta_{4,\text{I}} = \frac{M_{\text{eff}}^2 c_0^2}{\hbar^2} \left[ \frac{\text{tr}[Z^2]}{\text{tr}[C_0^2]} \right] + \left( \frac{\text{tr}[Y^2]}{2} \right) + \frac{\text{tr}[\Delta C^2]}{\text{tr}[\Omega^2]}; \quad (199)
\]

\[
\eta_{4,\text{II}} = \frac{M_{\text{eff}}^2 c_0^2}{\hbar^2} \left( \frac{\text{tr}[Y^2]}{2} - \frac{\text{tr}[\Delta C^2]}{\text{tr}[\Omega^2]} \right). \quad (200)
\]

Let us compute the two relevant terms separately:

\[
\frac{\text{tr}[Z^2]}{\text{tr}[C_0^2]} = \frac{\hbar^2}{4c_0^2} \left( \frac{m_A^2 + m_B^2}{m_A m_B} \right) = \frac{\hbar^2}{4c_0^2 M_{\text{eff}}^2} \left( \frac{m_A^2 + m_B^2}{m_A m_B} \right); \quad (201)
\]

\[
-\frac{\text{tr}[Y^2]}{2} + \frac{\text{tr}[\Delta C^2]}{\text{tr}[\Omega^2]} = -\frac{\hbar^2}{4M_{\text{eff}}^2} \left[ \frac{\rho_{A0} m_A \bar{U}_{AA}^2 + \rho_{B0} m_B \bar{U}_{BB}^2}{\rho_{A0} \rho_{B0} \bar{U}_{AA} \bar{U}_{BB} \left( \bar{U}_{AA} + \bar{U}_{BB} \right)} \right] = -\frac{\hbar^2}{4M_{\text{eff}}^2 c_0^2} \left[ \frac{m_A^2 \bar{U}_{AA} + m_B^2 \bar{U}_{BB}}{m_A m_B \left( \bar{U}_{AA} + \bar{U}_{BB} \right)} \right]; \quad (202)
\]
where we have used $\rho X_0 \tilde{U}_{XX} = m_X c_0^2$ for $X = A, B$ as in equation (179). Note that the quantity in square brackets in the last line is dimensionless. So in the end:

$$
\eta_{4,I} = 1/4 \left[ \frac{m_A^2 + m_B^2}{m_A m_B} - \frac{m_A^2 \tilde{U}_{AA} + m_B^2 \tilde{U}_{BB}}{m_A m_B (\tilde{U}_{AA} + \tilde{U}_{BB})} \right] \tag{203}
$$

$$
= 1/4 \left[ \frac{m_A^2 \tilde{U}_{BB} + m_B^2 \tilde{U}_{AA}}{m_A m_B (\tilde{U}_{AA} + \tilde{U}_{BB})} \right] \tag{204}
$$

$$
\eta_{4,II} = 1/4 \left[ \frac{m_A^2 \tilde{U}_{AA} + m_B^2 \tilde{U}_{BB}}{m_A m_B (\tilde{U}_{AA} + \tilde{U}_{BB})} \right]. \tag{205}
$$

**Note:** In the special case $m_A = m_B$ we recover identical quartic deviations $\eta_{4,I} = \eta_{4,II} = 1/4$, indicating in this special situation a “universal” deviation from Lorentz invariance. Indeed we also obtain $\eta_{4,I} = \eta_{4,II}$ if we demand $\tilde{U}_{AA} = \tilde{U}_{BB}$, even without fixing $m_A = m_B$.

Thus in the analogue spacetime we have developed the issue of universality is fundamentally related to the complexity of the underlying microscopic system. As long as we keep the two atomic masses $m_A$ and $m_B$ distinct we generically have distinct $\eta_4$ coefficients (and the $\eta_2$ coefficients are unequal even in the case $m_A = m_B$). However we can easily recover identical $\eta_4$ coefficients, for instance, as soon as we impose identical microphysics for the two BEC systems we couple.

**Avoidance of the naturalness problem**

We can now ask ourselves if there is, or is not, a naturalness problem present in our system. Are the dimensionless coefficients $\eta_{4,I/II}$ suppressed below their naive values by some small ratio involving $M_{\text{eff}} = \sqrt{m_A m_B}$? Or are these ratios unsuppressed? Indeed at first sight it might seem that further suppression is the case, since the square of the “effective Planck scale” seems to appear in the denominator of both the coefficients (204) and (205). However, the squares of the atomic masses also appear in the numerator, rendering both coefficients of order unity.

It is perhaps easier to see this once the dependence of (204) and (205) on the effective coupling $\bar{U}$ is removed. We again use the substitution $\bar{U}_{XX} = m_X c_0^2 / \rho X_0$ for $X = A, B$, so obtaining:

$$
\eta_{4,I} = 1/4 \left[ \frac{m_A \rho_0 + m_B \rho_0}{m_A \rho_0 + m_B \rho_0} \right] \tag{206}
$$

$$
\eta_{4,II} = 1/4 \left[ \frac{m_A^2 \rho_0 + m_B^2 \rho_0}{m_A m_B (m_A \rho_0 + m_B \rho_0)} \right]. \tag{207}
$$
From these expressions is clear that the $\eta_{4,1/11}$ coefficients are actually of order unity.

That is, if our system is set up so that $m_{II} < m_A/B$ — which we have seen in this scenario is equivalent to requiring $U_{AB} < U_{AA/BB}$ — no naturalness problem arises as for $p > m_{II}c_0$ the higher-order, energy-dependent Lorentz-violating terms ($n \geq 4$) will indeed dominate over the quadratic Lorentz-violating term.

It is quite remarkable that the quadratic coefficients (192) are exactly of the form postulated in several works on non-renormalizable EFT with Lorentz invariance violations (see e.g. [25]). They are indeed the squared ratio of the particle mass to the scale of Lorentz violation. Moreover we can see from (204) and (205) that there is no further suppression — after having pulled out a factor $(h/M_{\text{Lorentz violation}})^2$ — for the quartic coefficients $\eta_{4,1/11}$. These coefficients are of order one and generically non-universal, (though if desired they can be forced to be universal by additional and specific fine tuning).

The suppression of $\eta_2$, combined with the non-suppression of $\eta_4$, is precisely the statement that the “naturalness problem” does not arise in the current model. We stress this is not a “tree level” result as the dispersion relation was computed directly from the fundamental Hamiltonian and was not derived via any EFT reasoning. Moreover avoidance of the naturalness problem is not directly related to the tuning of our system to reproduce special relativity in the hydrodynamic limit. In fact our conditions for recovering special relativity at low energies do not a priori fix the the $\eta_2$ coefficient, as its strength after the “fine tuning” could still be large (even of order one) if the typical mass scale of the massive phonon is not well below the atomic mass scale. Instead the smallness of $\eta_2$ is directly related to the mass-generating mechanism.

The key question is now: Why does our model escape the naive predictions of dominant lowest-dimension Lorentz violations? (In fact in our model for any $p \gg m_{II}$ the $k^4$ Lorentz violating term dominates over the order $k^2$ one.) We here propose a nice interpretation in terms of “emergent symmetry”: Non-zero $\lambda$ simultaneously produces a non-zero mass for one of the phonons, and a corresponding non-zero Lorentz violation at order $k^2$. (Single BEC systems have only $k^4$ Lorentz violations as described by the Bogoliubov dispersion relation.) Let us now drive $\lambda \to 0$, but keep the conditions $C1$ and $C2$ valid at each stage. (This also requires $U_{AB} \to 0$.) One gets an EFT which at low energies describes two non-interacting phonons propagating on a common background. (In fact $\eta_2 \to 0$ and $c_I = c_{II} = c_0$.) This system possesses a $SO(2)$ symmetry. Non-zero laser coupling $\lambda$ softly breaks this $SO(2)$, the mass degeneracy, and low-energy Lorentz invariance. Such soft Lorentz violation is then characterized (as usual in EFT) by the ratio of the scale of the symmetry breaking $m_{II}$, and that of the scale originating the Lorentz violation in first place $M_{\text{Lorentz violation}}$. We stress that the $SO(2)$ symmetry is an “emergent symmetry” as it is not preserved beyond the hydrodynamic limit: the $\eta_4$ coefficients are in general different if $m_A \neq m_B$, so $SO(2)$ is generi-
cally broken at high energies. Nevertheless this is enough for the protection of the lowest-order Lorentz violating operators. The lesson to be drawn is that emergent symmetries are sufficient to minimize the amount of Lorentz violation in the lowest-dimension operators of the EFT. In this regard, it is intriguing to realise that an interpretation of SUSY as an accidental symmetry has indeed been considered in recent times [32], and that this is done at the cost of renouncing attempts to solve the hierarchy problem in the standard way. It might be that in this sense the smallness of the particle physics mass scales with respect to the Planck scale could be directly related to smallness of Lorentz violations in renormalizable operators of the low-energy effective field theory we live in. We hope to further investigate these issues in future work.

5 Outlook, summary and discussion

So where can (and should) we go from here? If 2-component BECs provide such a rich mathematical and physical structure, are 3-component BECs, or general multi-component BECs even better? That depends on what you are trying to do:

- If one wishes to actually build such an analogue spacetime in the laboratory, and perform actual experiments, then iteration through 1-BEC and 2-BEC systems seems the most promising route in terms of our technological capabilities.

- For n-component BECs we sketch the situation in figure 4. The key point is that due to overall translation invariance one again expects to find one massless quasi-particle, with now $n-1$ distinct massive modes. Unfortunately the matrix algebra is now considerably messier — not intrinsically difficult (after all we are only dealing with $n \times n$ matrices in field space) — but extremely tedious. Physical insight remains largely intact, but (except in some specific particularly simple cases), computations rapidly become lost in a morass of technical detail.

- However, if one wishes to draw general theoretical lessons from the analogue spacetime programme, then multi-component systems are definitely the preferred route — though in this case it is probably better to be even more abstract, and to go beyond the specific details of BEC-based systems to deal with general hyperbolic systems of PDEs.

- In appendix A we have sketched some of the key features of the pseudo–Finsler spacetimes that naturally emerge from considering the leading symbol of a hyperbolic system of PDEs. While it is clear that much more could be done based on this, and on extending the field theory “normal modes” of [2, 3], such an analysis would very much move outside the scope of the COSLAB programme.
In short the 2-BEC system is a good compromise between a system complex enough to exhibit a mass-generating mechanism, and still simple enough to be technologically tractable, with good prospects for laboratory realization of this system in the not too distant future.

Fig. 4. The figure captures the key features of possible eigenmodes for a small perturbation (circles) in a 1 (left side), 2, 3, 4, and 5-component (right side) BEC. In a 1-component system only one kind of perturbation is allowed, which corresponds to a massless particle propagating through an effective curved spacetime, while in the 2-component case two different kinds of mode appear, the one in-phase (massless particle) and one in anti-phase (massive particle). For a three-component system we again expect to find one massless particle, when all perturbations are in phase, and now in addition to that two massive particles.

The key features we have emphasised in this chapter have been:

- A general analysis of the 2-BEC system to see how perturbations on a 2-BEC background lead to a system of coupled wave equations.
- Extraction of the geometric notion of pseudo–Finsler spacetime from this wave equation, coupled with an analysis of how to specialize pseudo–Finsler geometry first to a bi-metric Lorentzian geometry and finally to the usual mono-metric Lorentzian geometry of most direct interest in general relativity and cosmology.
- The mass-generating mechanism we have identified in suitably coupled 2-component BECs is an essential step in making this analogue spacetime more realistic; whatever one’s views on the ultimate theory of “quantum gravity”, any realistic low-energy phenomenology must contain some mass-generating mechanism.
- Use of the “quantum pressure” term in the 2-BEC system to mimic the sort of Lorentz violating physics that (based on the relatively young field of “quantum gravity phenomenology”) is widely expected to occur at or near the Planck scale.
- Intriguingly, we have seen that in our specific model the mass-generating mechanism interacts with the Lorentz violating mechanism, naturally leading to a situation where the Lorentz violations are suppressed by powers of the quasi-particle mass scale divided by the analogue of the Planck scale.

In summary, while we do not personally believe that the real universe is an analogue spacetime, we are certainly intrigued by the fact that so much of what is normally viewed as being specific to general relativity and/or particle...
physics can be placed in this much wider context. We should also be forthright about the key weakness of analogue models as they currently stand: As we have seen, obtaining an analogue spacetime geometry (including spacetime curvature) is straightforward — but what is not straightforward is obtaining the Einstein equations. The analogue models are currently analogue models of quantum field theory on curved spacetime, but not (yet?) true analogue models of Einstein gravity. Despite this limitation, what can be achieved through the analogue spacetime programme is quite impressive, and we expect interest in this field, both theoretical and hopefully experimental, to continue unabated.

A Finsler and co–Finsler geometries

Finsler geometries are sufficiently unusual that a brief discussion is in order — especially in view of the fact that the needs of the physics community are often somewhat at odds with what the mathematical community might view as the most important issues. Below are some elementary results, where we emphasise that for the time being we are working with ordinary “Euclidean signature” Finsler geometry. For general references, see [33].

A.1 Basics

Euler theorem: If $H(z)$ is homogeneous of degree $n$ then

$$z^i \frac{\partial H(z)}{\partial z^i} = n \, H(z).$$

(208)

Finsler function: Defined on the “slit tangent bundle” $T_{\neq 0}(M)$ such that $F : T_{\neq 0}(M) \to [0, +\infty)$ where

$$F(x, t) : \ F(x, \lambda t) = \lambda \, F(x, t),$$

(209)

and

$$T_{\neq 0}(M) = \bigcup_{x \in M} \left[ T_x - \{0\} \right].$$

(210)

That is, the Finsler function is a defined only for nonzero tangent vectors $t \in [T_x - \{0\}]$, and for any fixed direction is linear in the size of the vector.

Finsler distance:

$$d_{\gamma}(x, y) = \int_{x}^{y} F(x(\tau), dx/d\tau) \, d\tau; \quad \tau = \text{arbitrary parameter.}$$

(211)

Finsler metric:

$$g_{ij}(x, t) = \frac{1}{2} \frac{\partial^2 [F^2(x, t)]}{\partial u^i \partial u^j}.$$  

(212)
The first slightly unusual item is the introduction of co–Finsler structure: *co–Finsler function*: Define a co–Finsler structure on the cotangent bundle by Legendre transformation of $F^2(x,t)$. That is:

$$G^2(x,p) = t^j(p) p_j - F^2(x,t(p))$$  \hspace{1cm} (213)

where $t(p)$ is defined by the Legendre transformation condition

$$\frac{\partial[F^2]}{\partial t^j}(x,t) = p_j.$$  \hspace{1cm} (214)

Note

$$\frac{\partial p_j}{\partial t^k} = \frac{\partial[F^2]}{\partial t^j \partial t^k} = 2g_{jk}(x,t),$$  \hspace{1cm} (215)

which is why we demand the Finsler metric be nonsingular.

**Lemma**: $G(x,p)$ defined in this way is homogeneous of degree 1.

**Proof**: Note

$$z^i \frac{\partial H(z)}{\partial z^i} = n H(z)$$  \hspace{1cm} (216)

implies

$$z^i \frac{\partial}{\partial z^i} \left[ \frac{\partial^m}{(\partial z)^m} H(z) \right] = (n - m) \left[ \frac{\partial^m}{(\partial z)^m} H(z) \right].$$  \hspace{1cm} (217)

In particular:

- $F^2$ is homogeneous of degree 2.
- $g_{ij}$ is homogeneous of degree 0.
- $\frac{\partial[F^2]}{\partial t}$ is homogeneous of degree 1.
- Therefore $p(t)$ is homogeneous of degree 1
  and $t(p)$ is homogeneous of degree 1.
- Therefore $t(p)p - F^2(t(p))$ is homogeneous of degree 2.
- Therefore $G(p)$ is homogeneous of degree 1.

Thus from a Finsler function $F(x,t)$ we can always construct a co–Finsler function $G(x,p)$ which is homogeneous of degree 1 on the cotangent bundle.

From the way the proof is set up it is clearly reversible — if you are given a co–Finsler function $G(x,p)$ on the cotangent bundle this provides a natural way of extracting the corresponding Finsler function:

$$F^2(x,t) = t p(t) - G^2(x,p(t)).$$  \hspace{1cm} (218)

**A.2 Connection with the quasi-particle PDE analysis**

From the PDE-based analysis we obtain the second-order system of PDEs

$$\partial_a \left( f^{ab}_{AB} \partial_b \theta^B \right) + \text{lower order terms} = 0.$$  \hspace{1cm} (219)

We are now generalizing in the obvious manner to any arbitrary number $n$ of interacting BECs, but the analysis is even more general than that — it
applies to any field-theory normal-mode analysis that arises from a wide class of Lagrangian based systems [2, 3].

Going to the eikonal approximation this becomes

\[ f_{AB}^{ab} p_a p_b \epsilon^B + \text{lower-order terms} = 0, \quad (220) \]

which leads (neglecting lower order terms for now) to the Fresnel-like equation

\[ \det[f_{AB}^{ab} p_a p_b] = 0. \quad (221) \]

But by expanding the \( n \times n \) determinant (\( n \) is the number of fields, not the dimension of spacetime) we have

\[ \det[f_{AB}^{ab} p_a p_b] = Q^{abcd\ldots} p_a p_b p_c p_d \ldots \quad (222) \]

where if there are \( n \) fields there will be \( 2n \) factors of \( p \).

Now define

\[ Q(x, p) = Q^{abcd\ldots} p_a p_b p_c p_d \ldots, \quad (223) \]

and

\[ G(x, p) = \sqrt[n]{Q(x, p)} = [Q(z, p)]^{1/(2n)}, \quad (224) \]

then

- \( Q(x, p) \) is homogeneous of degree \( 2n \).
- \( G(x, p) \) is homogeneous of degree 1, and hence is a co–Finsler function.
- We can now Legendre transform \( G \rightarrow F \), providing a chain

\[ Q(x, p) \rightarrow G(x, p) \rightarrow F(x, t). \quad (225) \]

Can this route be reversed?

**Step 1:** We can always reverse \( F(x, t) \rightarrow G(x, p) \) by Legendre transformation.

**Step 2:** We can always define

\[ g^{ab}(x, p) = \frac{1}{2} \frac{\partial}{\partial p_a} \frac{\partial}{\partial p_b} [G(x, p)^2], \quad (226) \]

this is homogeneous of degree 0, but is generically not smooth at \( p = 0 \).

In fact, if \( g^{ab}(x, p) \) is smooth at \( p = 0 \) then there exits a limit

\[ g^{ab}(x, p \rightarrow 0) = \bar{g}^{ab}(x), \quad (227) \]

but since \( g^{ab}(x, p) \) is homogeneous of degree 0 this implies

\[ g^{ab}(x, p) = \bar{g}^{ab}(x) \quad \forall p, \quad (228) \]

and so the geometry simplifies Finsler \( \rightarrow \) Riemann.

This observation suggests the following definition.

**Definition:** A co–Finsler function \( G(x, p) \) is \( 2n \)-smooth iff the limit
Analogue spacetime based on 2-component Bose–Einstein condensates

\[
\frac{1}{(2n)!} \lim_{p \to 0} \left( \frac{\partial}{\partial p} \right)^{2n} G(x, p)^{2n} = \tilde{Q}^{abcd...} \tag{229}
\]

exists independent of the direction \( p \) in which you approach zero.

**Lemma:** If \( G(x, p) \) is \( 2n \)-smooth then

\[
G(x, p)^{2n} = \tilde{Q}^{abcd...} p_a p_b p_c p_d \ldots, \tag{230}
\]

and indeed

\[
G(x, p) = \sqrt[n]{\tilde{Q}^{abcd...} p_a p_b p_c p_d \ldots} \tag{231}
\]

**Proof:** \( G^{2n} \) is homogeneous of degree \( 2n \), so \((\partial/\partial p)^{2n} G^{2n}\) is homogeneous of degree 0. Therefore if the limit

\[
\frac{1}{(2n)!} \lim_{p \to 0} \left( \frac{\partial}{\partial p} \right)^{2n} G(x, p)^{2n} = \tilde{Q}^{abcd...} \tag{232}
\]

exists, it follows that

\[
\frac{1}{(2n)!} \left( \frac{\partial}{\partial p} \right)^{2n} G(x, p)^{2n} = \tilde{Q}^{abcd...} \quad \forall p, \tag{233}
\]

and so the result follows.

**Special case** \( n = 1 \): If \( G(x, p) \) is 2-smooth then

\[
\frac{1}{2} \frac{\partial^2}{\partial p^a \partial p_b} G(x, p)^2 = \tilde{Q}^{ab} = g^{ab}(x, p), \tag{234}
\]

and co–Finsler \( \to \) Riemann.

These observations have a number of implications:

- For all those co–Finsler functions that are \( 2n \) smooth we can recover the tensor \( Q^{abcd...} \).
- Not all co–Finsler functions are \( 2n \) smooth, and for those functions we cannot extract \( Q^{abcd...} \) in any meaningful way.
- But those specific co–Finsler functions that arise from the leading symbol of a 2nd-order system of PDEs are naturally \( 2n \)-smooth, and so for the specific co–Finsler structures we are physically interested in

\[
Q(x, p) \leftrightarrow G(x, p) \leftrightarrow F(x, t). \tag{235}
\]

- Therefore, in the physically interesting case the Finsler function \( F(x, t) \) encodes all the information present in \( Q^{abcd...} \).

**Special case** \( n = 2 \): For two fields (appropriate for our 2-BEC system), we can follow the chain

\[
f^{ab} \to Q(x, p) \leftrightarrow G(x, p) \leftrightarrow F(x, t) \tag{236}
\]

to formally write
\[ ds^4 = g_{abcd} \, dx^a dx^b dx^c dx^d, \quad (237) \]
or
\[ ds = \sqrt[4]{g_{abcd} \, dx^a dx^b dx^c dx^d}. \quad (238) \]

This is one of the “more general” cases Riemann alludes to in his inaugural lecture of 1854 [21].

This discussion makes it clear that the general geometry in our 2-BEC system is a 4-smooth Finsler geometry. It is only for certain special cases that the Finsler geometry specializes first to “multi-metric” and then to “mono-metric” Riemannian geometries.

### A.3 Lorentzian signature Finsler geometries

The distinction between Finsler and pseudo–Finsler geometries has to do with the distinction between elliptic and hyperbolic PDEs. Elliptic PDEs lead to ordinary Finsler geometries, hyperbolic PDEs lead to pseudo–Finsler geometries.

Remember that in special relativity we typically define
\[ d_\gamma(x, y) = \int_x^y \sqrt{g_{ab}(dx^a/d\tau)(dx^b/d\tau)} d\tau, \quad (239) \]
then
- \( d_\gamma(x, y) \in \mathbb{R}^+ \) for spacelike paths;
- \( d_\gamma(x, y) = 0 \) for null paths;
- \( d_\gamma(x, y) \in \mathbb{I}^+ \) for timelike paths;

The point is that even in special relativity (and by implication in general relativity) “distances” do not have to be real numbers. This is why physicists deal with pseudo–Riemannian [Lorentzian] geometries, not (strictly speaking) Riemannian geometries.

To see how this generalizes in a Finsler situation let us first consider a co–Finsler structure that is multi-metric, that is:
\[ Q(x, p) = \prod_{i=1}^n (g_{ab}^{i} p_a p_b), \quad (240) \]
where each one of these \( n \) factors contains a Lorentzian signature matrix and so can pass through zero. Then
\[ G(x, p) = \sqrt[2n]{\prod_{i=1}^n (g_{ab}^{i} p_a p_b)}, \quad (241) \]
and
\[ G(x, p) \in \exp \left( \frac{i\pi \ell}{2n} \right) \mathbb{R}^+, \quad (242) \]
where
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- \( \ell = 0 \rightarrow G(x, p) \in \mathbb{R}^+ \rightarrow \) outside all \( n \) signal cones;
- \( \ell = n \rightarrow G(x, p) \in i^{\ell/n} \rightarrow \) inside all \( n \) signal cones.

So we can now define

- Spacelike \( \leftrightarrow \) outside all \( n \) signal cones \( \leftrightarrow G \) real;
- Null \( \leftrightarrow \) on any one of the \( n \) signal cones \( \leftrightarrow G \) zero;
- Timelike \( \leftrightarrow \) inside all \( n \) signal cones \( \leftrightarrow G \) imaginary;
- plus the various “intermediate” cases:

  \[
  \text{“intermediate”} \leftrightarrow \text{inside } \ell \text{ of } n \text{ signal cones} \leftrightarrow G \in i^{\ell/n} \times \mathbb{R}^+.
  \]

Now this basic idea survives even if we do not have a multi-metric theory. The condition \( Q(x, p) = 0 \) defines a polynomial of degree \( 2n \), and so defines \( n \) nested sheets (possibly crossing in places). Compare with Courant and Hilbert’s discussion of the Monge cone [20].

That is:

\[
Q(x, p) = 0 \Leftrightarrow Q(x, (E, p)) = 0;
\]

\( \Leftrightarrow \) polynomial of degree \( 2n \) in \( E \) for any fixed \( p \);

\( \Leftrightarrow \) in each direction \( \exists 2n \) roots in \( E \);

\( \Leftrightarrow \) corresponds to \( n \) [topological] cones.

(These are topological cones, not geometrical cones, and the roots might happen to be degenerate.)

**Question:** Should we be worried by the fact that the co-metric \( g^{ab} \) is singular on the signal cone? (In fact on all \( n \) of the signal cones.) Not really. We have

\[
G(x, p) = 2^n \sqrt[2n]{Q_{abcd...} p_a p_b p_c p_d ...},
\]

so

\[
g^{ab}(x, p) = \frac{1}{2} \frac{\partial^2}{\partial p^a \partial p^b} \left( \sqrt[2n]{Q(x, p)} \right) = \frac{1}{2n} \frac{\partial}{\partial p_b} \left( Q^{\frac{1}{2n}} Q_{abcd...} p_b p_c p_d ... \right),
\]

whence

\[
g^{ab}(x, p) = \frac{1}{2n} Q^{\frac{1}{2n}-1} Q_{abcd...} p_c p_d ...
\]

\[
+ \frac{1}{2n} \left( \frac{1}{n} - 1 \right) Q^{\frac{1}{2n}-2} [Q_{acde...} p_c p_d p_e ...] [Q_{bfgh...} p_f p_g p_h ...],
\]

which we can write as

\[
g^{ab}(x, p) = \frac{1}{2n} Q^{-(n-1)/n} Q_{abcd...} p_c p_d ...
\]

\[
- \frac{1}{2n} \left( \frac{n-1}{n} \right) Q^{-(2n-1)/n} [Q_{acde...} p_c p_d p_e ...] [Q_{bfgh...} p_f p_g p_h ...].
\]
Yes, this naively looks like it’s singular on the signal cone where $Q(x, p) = 0$. But no, this is not a problem: Consider

$$g^{ab} p_a p_b = \frac{1}{2n} Q^{-(2n-1)/n} Q - \frac{1}{2n} \frac{n-1}{n} Q^{-(2n-1)/n} Q^2, \quad (248)$$

then

$$g^{ab} p_a p_b = \frac{1}{2n} \left( 1 - \frac{n-1}{n} \right) Q^{1/n} = \frac{1}{2n^2} Q^{1/n} = 0, \quad (249)$$

and this quantity is definitely non-singular.

### A.4 Summary

In short:

- pseudo–Finsler functions arise naturally from the leading symbol of hyperbolic systems of PDEs;
- pseudo–Finsler geometries provide the natural “geometric” interpretation of a multi-component PDE before fine tuning;
- In particular the natural geometric interpretation of our 2-BEC model (before fine tuning) is as a 4-smooth pseudo–Finsler geometry.

### B Some matrix identities

To simplify the flow of argument in the body of the paper, here we collect a few basic results on $2 \times 2$ matrices that are used in our analysis.

#### B.1 Determinants

**Theorem:** For any two $2 \times 2$ matrix $A$:

$$\det(A) = \frac{1}{2} \left\{ \text{tr}[A]^2 - \text{tr}[A^2] \right\}. \quad (250)$$

This is best proved by simply noting

$$\det(A) = \lambda_1 \lambda_2 = \frac{1}{2} \left[ (\lambda_1 + \lambda_2)^2 - (\lambda_1^2 + \lambda_2^2) \right] = \frac{1}{2} \left\{ \text{tr}[A]^2 - \text{tr}[A^2] \right\}. \quad (251)$$

If we now define $2 \times 2$ “trace reversal” (in a manner reminiscent of standard GR) by

$$\bar{A} = A - \text{tr}[A] I; \quad \bar{A} = A; \quad (252)$$

then this looks even simpler

$$\det(A) = -\frac{1}{2} \text{tr}[A \bar{A}] = \det(\bar{A}). \quad (253)$$
A simple implication is now:

**Theorem:** For any two $2 \times 2$ matrices $A$ and $B$:

$$\det(A + \lambda B) = \det(A) + \lambda \{\text{tr}[A]\text{tr}[B] - \text{tr}[A B]\} + \lambda^2 \det(B). \quad (254)$$

which we can also write as

$$\det(A + \lambda B) = \det(A) - \lambda \text{tr}[A \tilde{B}] + \lambda^2 \det(B). \quad (255)$$

Note that $\text{tr}[A \tilde{B}] = \text{tr}[\tilde{A} B]$.

**B.2 Hamilton–Cayley theorems**

**Theorem:** For any two $2 \times 2$ matrix $A$:

$$A^{-1} = \frac{\text{tr}[A]}{\det[A]} I - \frac{\tilde{A}}{\det[A]} \quad (256)$$

**Theorem:** For any two $2 \times 2$ matrix $A$:

$$A^{1/2} = \pm \left\{ \begin{array}{c} A \pm \sqrt{\det A} I \\ \sqrt{\text{tr}[A]} \pm 2\sqrt{\det A} \end{array} \right\}. \quad (257)$$

**References**

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