New ‘phase’ of quantum gravity

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Abstract

The emergence of loop quantum gravity over the past two decades has stimulated a great resurgence of interest in unifying general relativity and quantum mechanics. Amongst a number of appealing features of this approach are the intuitive picture of quantum geometry using spin networks and powerful mathematical tools from gauge field theory. However, the present form of loop quantum gravity suffers from a quantum ambiguity, due to the presence of a free (Barbero-Immirzi) parameter. Following recent progress on the conformal decomposition of gravitational fields, we present a new phase space for general relativity. In addition to spin-gauge symmetry, the new phase space also incorporates conformal symmetry making the description parameter free. The Barbero-Immirzi ambiguity is shown to occur only if the conformal symmetry is gauge-fixed prior to quantization. By withholding its full symmetries, the new phase space offers a promising platform for the future development of loop quantum gravity. This paper aims to provide an exposition, at a reduced technical level, of the above theoretical advances and their background developments. Further details are referred to cited references.
I. INTRODUCTION: QUANTIZATION OF GRAVITY

Unification is a very cherished concept in theoretical physics. A classical example is Maxwell’s unified electromagnetic theory. With the birth of special relativity, Einstein further integrated electromagnetism with the principle of inertia. The work of Dirac successfully brought together special relativity and quantum mechanics. A key common feature in all these achievements is that they rely on no additional physical constants, spacetime dimensions or material contents other than already available experimental facts. Nonetheless, these unified theories lead to experimentally testable predictions: The electromagnetic wave predicted by Maxwell propagates at the speed of light calculated from the known permittivity and permeability coefficients. Special relativity preserves the speed of light as a universal constant valid in any inertial frames. Dirac’s relativistic quantum theory of electrons predicts intrinsic spin in units of the measured Planck constant.

In the above examples, it is the new face of the same physical entity that the new theory portrays. Such theoretical advance demands tremendous insights into the interplay between the physical descriptions to be amalgamated. In this spirit, Einstein found an elegant reconciliation of Newton’s gravity and special relativity using general relativity, preserving the spacetime dimensions, gravitational constant and speed of light. What’s radically new is the different interpretation of gravity in terms of spacetime curvature. Later efforts to merge general relativity and quantum mechanics have unfortunately encountered stiff and persistent resistance, and have given rise to serious doubts on established physical principles and observed shape of the Universe. Countless ingenious ideas have been put forward as possible solutions, involving extra dimensions, super symmetries, strings, branes or combinations of them. (For a review, see [1].) While these intellectual yet experimentally unverified hypotheses may well lead to the ultimate theory of quantum gravity and indeed the ultimate theory of everything, the search for quantum gravity without the above postulated structures or contents of the Universe has continued – the direction this paper shall focus on.
II. DIRAC'S QUANTIZATION OF CONSTRAINED HAMILTONIAN SYSTEMS

Dirac expressed strong faith in canonical quantization [2] that has been so successfully applied to quantize elementary particles and force fields coupling them together, gravity being an exception. An intriguing aspect of the force fields in nature is that they all have one kind of symmetry or another. These are called gauge symmetries and hence the fields are called gauge fields. Each gauge field has an associated gauge symmetry group, the simplest example is the U(1) gauge group for Maxwell’s electromagnetic potential field. Gravity as described by general relativity has symmetries inherent from the general covariance. Physically this means that there is no preferred reference frame, either inertial or accelerating. However, there are at least two reasons for gravity to be so hard to quantize as a gauge field. First, the theory is highly nonlinear, and secondly the gauge transformation involves the change of time itself which is usually fixed in a quantum evolution. Dirac launched a systematic programme to investigate the canonical quantization of systems with general gauge symmetries. He found that the redundancy of the degrees of freedom due to gauge symmetries can be treated in an extended Hamiltonian formalism with constraints in a way similar to the Lagrange multiplier method. The analogue of the Lagrange constraint function is the constraint function depending on the canonical coordinates. Such systems are called constrained Hamiltonian systems. For example, a Hamiltonian system having a finite dimensional phase space with coordinates \((q^k, p_k) (k = 1, 2, \cdots)\) subject to constraints \(\chi_m(q^k, p_k) (m = 1, 2, \cdots)\) has a Hamiltonian function of the form:

\[
H(q^k, p_k, \lambda^m) = h(q^k, p_k) + \lambda^m \chi_m(q^k, p_k) \tag{1}
\]

where \(\lambda^m\) are Lagrange multipliers. Einstein’s summation convention for repeated indices is implied throughout this paper.

This extended Hamiltonian structure is so general that the redundancy does not even have to be of gauge origin. Nonetheless, the constraints of gauge origin have very simple interpretations and properties. The time evolution of a system with gauge symmetries includes gauge transformations generated by the constraint terms in the Hamiltonian. The constraint plays the role of the canonical generator of gauge symmetry. The collection of these constraints can be shown to be algebraically closed. Their respective Lagrange multipliers in can be arbitrarily specified reflecting the gauge freedom. In terms of Dirac’s terminology, these constraints are called first class constraints. Other constraints not having
the above properties are not due to gauge symmetries and are called second class constraints. When quantizing a system, first class constraints become quantum constraint operators that annihilate the physical quantum state. Due to certain consistency conditions, second class constraints cannot be treated this way and should in principle be eliminated before quantization.

III. QUANTIZATION AMBIGUITIES DUE TO GAUGE-FIXING

Given a Hamiltonian system with only first class constraints, i.e. with any possible second class constraints eliminated, one can in principle follow Dirac's quantization procedure as described by treating all constraints on the same footing. On the other hand, one might try and eliminate some of the constraints to end up with a Hamiltonian system of fewer degrees of freedom as well as less constraints. In doing so, some kind of gauge-fixing must be introduced to remove the respective redundancy in the original system. As a result, we have a reduced phase space. An interesting question to ask is whether or not the quantization of the gauge-fixed system yields an equivalent quantum system as the quantization of the original system. Not surprisingly, the answer is ‘no’ in general, as the classical gauge-fixing condition may violate the uncertainty principle associated with the quantum variables from the original phase space. Consequently, when a system is gauge-fixed with respect to a constraint, but in two different ways, two inequivalent quantizations may arise. This means that not all states from one quantization can be mapped to states from the other quantization unitarily. This is an important source of quantum ambiguity. To avoid it, Dirac's method of quantization should be performed without gauge fixing any first class constraints that would lead to a quantum ambiguity.

IV. GEOMETRODYNAMICS: BUILDING SPACETIME BY EVOLVING SPACE

As a continuum, spacetime described by special relativity has no absolute time which features in Newtonian dynamics. The Galilean inertial frames sharing the same time measurement are replaced by the Lorentz inertial frames, each having their own time. The underlying geometry is Minkowskian. General relativity goes further by assuming only local Lorentz frames. In order to interpret gravity as spacetime curvature, the pseudo-Riemannian
geometry is invoked. Consequently, global Lorentz frames are no longer available and general spacetime coordinates must be used. Spacetime now appears to be an elastic continuum that can be deformed due to massive objects like stars and admits propagating disturbances, i.e. gravitational waves. Indeed, general relativity can be shown to be the simplest metric theory of gravity to possess general covariance. The related equivalence principle has been experimentally verified to a few parts in a trillion [3].

However, for the purpose of canonically quantizing gravity using Dirac’s prescription, general covariance has to be formally broken. A somewhat artificial time coordinate must be chosen in order to define evolution. See Fig. 1. Nonetheless, one hopes that the general covariance will emerge via the time-slicing independent evolution of spacetime dynamics as per Dirac’s canonical formulation.

![Geometrodynamics](image)

**FIG. 1:** Geometrodynamics describes spacetime as a result of the canonical evolution of a spatial hypersurface with respect to a coordinate time.

Such a space-time split has been developed by Arnowitt, Deser and Misner (ADM) [4, 5]. Here, we shall use lower case Greek letters, e.g. $\alpha, \beta$, to denote spacetime indices ranging from 0 to 3, and use lower case Latin letters, e.g. $a, b$, to denote spatial indices ranging from 1 to 3. Starting from arbitrary spacetime coordinates $(x^\alpha) = (t = x^0, x^a)$, the spacetime metric $g_{\alpha\beta}$ with metric signature $(-, +, +, +)$ is decomposed into the time-time component $g_{00}$, space-time components $g_{0a}$ and space-space components $g_{ab}$ accordingly. The space-space components have an immediate interpretation as they constitute the 3-metric with metric signature $(+, +, +)$ on the spatial hypersurface with a constant coordinate time $t$. To interpret the time-time and space-time metric components, ADM introduced the lapse function $N = N(x^c, t)$ and the spatial shift vector $X^a = X^a(x^c, t)$ and wrote the squared
space time line element in a ‘3 + 1’ fashion as

\[ \text{ds}^2 = -N^2 \text{dt}^2 + g_{ab}(dx^a + X^a \text{dt})(dx^b + X^b \text{dt}). \]  

(2)

Equivalently, the spacetime metric and its space-time decomposition can be written as

\[
(g_{\alpha\beta}) = \begin{pmatrix}
g_{00} & g_{0a} \\
g_{0a} & g_{ab}
\end{pmatrix} = \begin{pmatrix}
-N^2 + X^cX_c & X_a \\
X_a & g_{ab}
\end{pmatrix}
\]  

(3)

which shows the clear correspondence between \((N, X^a)\) and \((g_{00}, g_{0a})\). From the expression (2), we see the following. Consider two events \((x^a, t)\) and \((x^a, t + \text{dt})\) belonging to two nearby spatial hypersurfaces with coordinate times \(t\) and \(t + \text{dt}\). With respect to these hypersurfaces the squared proper time and length separations between these two events are given, respectively, by \(N^2 \text{dt}^2\) and \(g_{ab}(X^a \text{dt})(X^b \text{dt})\). This justifies the above names for \(N\) and \(X^a\). (See Fig. 2).

FIG. 2: **The effects of the lapse function and shift vector** are illustrated by considering two constant time hypersurfaces intersecting event A \((x^a, t)\) and event D \((x^a + dx^a, t + \text{dt})\). The proper separation between these two events can be understood intuitively by considering first the proper time between event A and event B \((x^a - X^a \text{dt}, t + \text{dt})\) followed by considering the proper distance between event B and event C \((x^a, t + \text{dt})\) and finally the proper distance between events C and D. These considerations lead to the line element expression given in Eq. (2).

It is well-known that Einstein’s field equations can be generated by varying the Einstein-Hilbert action with respect to the spacetime metric \(g_{\alpha\beta}\) \([6]\). The passage to canonical formulation is via the Legendre transformation of the Einstein-Hilbert Lagrangian with respect to the coordinate time derivatives of the spatial metric \(g_{ab}\). Since the time derivatives of the lapse function \(N\) and shift vector \(X^a\) are absent in the Einstein-Hilbert Lagrangian,
they become Lagrange multipliers. This leads to the ADM Hamiltonian for gravity of the following form \(5\):

\[
H = \int (N\mathcal{H} + X^a\mathcal{D}_a) \, d^3x
\]  

(4)

where \(p^{ab}\) is the conjugate moment of the metric \(g_{ab}\), \(\mathcal{H} = \mathcal{H}[g_{ab}, p^{ab}]\) the Hamiltonian constraint and \(\mathcal{D} = \mathcal{D}[g_{ab}, p^{ab}]\) the diffeomorphism (or, historically, momentum) constraint. This Hamiltonian has a natural continuum extension of Dirac’s constrained Hamiltonian of the form \(1\). However, the gravitational Hamiltonian \(4\) constants of constraint terms only and is therefore ‘totally constrained’. This reflects the general covariance of the theory where no spacetime coordinates are preferred. Generally speaking, the constraint \(\mathcal{D}\) generates diffeomorphisms (translations) of ADM’s canonical variables \((g_{ab}, p^{ab})\) on the spatial hypersurface while \(\mathcal{H}\) generates their time evolution normal to the spatial hypersurface. Both \(\mathcal{H}\) and \(\mathcal{D}_a\) are first class and both \(N\) and \(X^a\) are arbitrarily specifiable.

The quantization follows formally from Dirac by turning the classical constraint equations \(N = 0\) and \(X^a = 0\) into the following quantum constraint equations \(5\)

\[
\hat{\mathcal{H}}\Psi = 0 \tag{5}
\]

\[
\hat{\mathcal{D}}_a\Psi = 0 \tag{6}
\]

for the quantum state \(\Psi = \Psi[g_{ab}]\), where the quantum operators are obtained from the substitution \(p^{ab} \rightarrow -i\delta / \delta g_{ab}\) in terms of the functional derivative.

The quantum operator \(\hat{\mathcal{D}}_a\) now generates the diffeomorphism of the quantum state \(\Psi\). Therefore, the quantum constraint \(5\) implies that \(\Psi\) is diffeomorphism invariant and hence depends on the spatial geometry instead of the metric. The quantum Hamiltonian constraint \(5\) is called the Wheeler-DeWitt equation and generates quantum evolution with respect to some geometric time to be isolated from the spatial geometry. The above quantization of gravity is conceptually appealing. However, the lack of suitable functional analytic techniques means that this approach can at best stay at a formal level. This difficulty has motivated the developments of alternative canonical formulations of general relativity in the hope that a naturally preferred phase space will be found in which the quantization of gravity is fully implementable.
V. SPIN-GAUGE VARIABLES OF GRAVITY

At present, the most promising approach to canonical general relativity is based on the use of spin connection variables that allows general relativity to be reformulated into a Yang-Mills like gauge field theory. Powerful background independent quantum field theoretical techniques may then be invoked and adapted for quantum gravity. Ashtekar discovered a particular set of spin connection variables in which the gravitational constraints take very simple polynomial forms. However, in order to yield real physical observables, certain reality conditions must be satisfied but their implementation became problematic. To resolve this issue, Barbero put forward an alternative set of spin connection variables based on the real spin gauge group SU(2) at the expense of losing polynomiality of the Hamiltonian constraint. At the first sight, the resulting Hamiltonian constraint appears to be too complicated to be quantized. Fortunately, the difficulty can be overcome by a regularization scheme developed by Thiemann. However, a further issue is a free parameter being introduced in Barbero’s construction of the real spin connection variables, pointed out by Immirzi. It is called the Barbero-Immirzi parameter and will result in a one-parameter ambiguity in the subsequent quantization. Nonetheless, an avenue has been opened to embrace the above-mentioned technical advantages by quantizing general relativity as a Yang-Mills like gauge theory.

FIG. 3: Triads can be used to specify the 3-metric over a spatial hypersurface.

The formulation can be briefly outlined as follows. Over the spatial hypersurface, a set of orthonormal vector fields \( e_i^a \), called a triad, is introduced. We use Latin indices staring from \( i \) with the range \( i, j, \cdots = 1, 2, 3 \) to label a member vector of the triad. In terms of the
triad, the contravariant spatial metric is simply

\[ g^{ab} = e_i^a e_i^b. \] (7)

There is an obvious redundancy in mapping triads to metrics due to an arbitrary 3-dimensional rotation of the triad. The idea is to replace the metric by the triad as gravitational variables having a spin-gauge symmetry with SU(2) as the gauge group. See Fig. 3. Furthermore, the general SU(2) spin connection \( A_i^a \) may be employed to complete a canonical transformation. The result is that, for any positive (Barbero-Immirzi) parameter \( \beta \), a rescaled densitized triad \( E_i^a \) can be defined as

\[ E_i^a = \beta^{-1} g^{1/2} e_i^a \] (8)

where \( g = \det g_{ab} \). Using this, the following transformation

\[ (g_{ab}, p^{ab}) \rightarrow (A_i^a, E_i^a) \] (9)

can be shown to be canonical \[ 10, 28 \]. Here \( E_i^a \) is regarded as the momentum of the spin connection \( A_i^a \) and hence \( E_i^a \) has been made to carry density weight one in \( 8 \). It is in complete analogy with the ‘electrical field’ of the standard SU(2) Yang-Mills gauge theory where the index \( i \) labels a base element in the associated su(2) Lie algebra.

Just as in Maxwell’s U(1) gauge theory and Yang-Mills’ SU(2) gauge theory, here the gravitational analogue of the electrical fields \( E_i^a \) also satisfies the ‘Gauss law’, i.e. the Gauss constraint equation \( \mathcal{G}_k = 0 \) where

\[ \mathcal{G}_k := D_a E_k^a \] (10)

is called the Gauss constraint and \( D_a \) is the covariant derivative associated with the connection \( A_i^a \). This constraint compensates the redundancy in the variables \( (A_i^a, E_i^a) \) due to the spin gauge.

In the spin gauge variables \( (A_i^a, E_i^a) \), all of the Hamiltonian constraint \( \mathcal{H} \), diffeomorphism constraint \( \mathcal{D} \) and Gauss constraint \( \mathcal{G}_k \) are first class and they enter into the gravitational Hamiltonian according to:

\[ H = \int \left( N \mathcal{H} + X^a \mathcal{D}_a + Y^k \mathcal{G}_k \right) d^3x \] (11)

with additional Lagrange multipliers \( Y^k = Y^k(x^a, t) \).
The Dirac quantization of gravity in these canonical variables follows formally as

\[ \hat{\mathcal{H}} \Psi = 0 \] (12)

\[ \hat{D}_a \Psi = 0 \] (13)

\[ \hat{G}_k \Psi = 0 \] (14)

where \( \Psi = \Psi[A^a_\alpha] \). Compared with the quantization in the ADM variables using (5) and (6), we have now an additional quantum Gaussian constraint (14). The quantization task appears to be more complicated. However, as stated earlier in this section, the justification for the spin-gauge formulation is to tap into powerful techniques for gauge field theories. Specifically, the Gauss constraint (14) can be solved exactly for Wilson loop states constructed from the holonomies of the spin connection \( A^a_\alpha \) [14], and hence the name ‘loop quantum gravity’. By extending Penrose’s original spin network concept [15], Rovelli and Smolin later generalized the loop states to spin network states that provide further mathematical advantages including the availability of a complete orthonormal basis for all spin gauge invariant states [16]. See Fig. 4. For recent reviews, see e.g. [7, 17, 18]. However, these exciting developments still do not address the quantum ambiguity due to the Barbero-Immirzi parameter.

FIG. 4: A specimen spin network consisting of 6 nodes and 11 links. Here 2 nodes are connected to 3 links and 4 nodes are connected to 4 links. Each link is labelled with a half integer as a spin quantum number. Each node has also an ‘intertwiner’ quantum number which is not shown.
VI. THE NEED FOR CONFORMAL SYMMETRY

The quantum ambiguity resulting from the above spin-gauge formulation is due to the different choice of the scaling parameter $\beta$. Classically, different choices of $\beta$ correspond to different sets of canonical coordinates for general relativity. These sets are merely related by canonical transformations and hence describe the same classical physics. However, they give rise to inequivalent quantum theories as demonstrated by Rovelli and Thiemann [19].

One view on this problem is that an alternative choice of the enlarged spin-gauge group $\text{SO}(4,\mathbb{C})$ could remove the free parameter $\beta$ [20, 21]. However, its implementation has been complicated by certain second class constraints. Some other authors see the Barbero-Immirzi parameter as a parity violation parameter in loop quantum gravity [22, 23]. A new viewpoint is based on the observation that the free parameter $\beta$ defines the scale of the densitized triad $E_i^a$. This signals a new conformal gauge symmetry associated with an underlying fundamental phase space of general relativity. If we work directly with this phase space and treat the respective constraint, which is ideally first class, using Dirac’s theory of quantization, then the Barbero-Immirzi ambiguity may be removed. On the other hand, if the conformal gauge is fixed by choosing a $\beta$ value, then this value may enter into the resulting quantization. This way, we may explain the Barbero-Immirzi ambiguity as a quantum ambiguity due to gauge-fixing at the classical level as discussed in section III.

Naturally, this new phase space may be obtained by extending the phase space of general relativity with conformal symmetry. Although the need for this symmetry is motivated from the spin-gauge formulation of gravity, the conformal gauge by itself is quite independent of the spin gauge. Furthermore, there are problems in general relativity where the role of conformal gauge is of primary importance [24]. It is therefore useful to consider first a conformally extended phase space from that of the geometrodynamics, i.e. the ADM phase space.

The problem is intimately related to the true dynamical degrees of gravity identified as the conformal three-geometry by York [25, 26]. This identification has provided power tools in the analytical initial value problems as well as numerical integrations of the gravitational field. Attempts have been made to apply York’s conformal decomposition in quantum gravity, but have been hammered by the absence of an appropriate Hamiltonian structure.

In a recent paper [27], the canonical evolution of conformal three-geometry for arbitrary
spacetime foliations is formulated using a new form of Hamiltonian for general relativity. It is achieved by extending the ADM phase space to that consisting of York’s mean extrinsic curvature time, conformal three-metric and their momenta. Accordingly, an additional constraint is introduced, called the conformal constraint. The complete set of the conformal, diffeomorphism and Hamiltonian constraints are shown to be of first class through the explicit construction of their Poisson brackets. The extended algebra of constraints has as a subalgebra the Lie algebra for the conformorphism transformations of the spatial hypersurface. See Fig. 5.

FIG. 5: The conformal decomposition of gravity can be thought of as separating the scaling part of spatial geometry from the shearing part. The former describes the expansion of the Universe, illustrated here as 3 expanding surfaces, while the latter describes gravitational waves, indicated here as ripples on the surfaces.

The above canonical framework has been developed into a parameter-free gauge formulation of general relativity in [28]. (For a review, see [29].) The result is a further enlarged set of first class gravitational constraints consisting of a reduced Hamiltonian constraint and the canonical generators for spin-gauge and conformorphism transformations. The formalism has most recently been simplified into a form more suitable for quantum implementation, which will form a basis for the following discussions. Details of these recent developments will be reported elsewhere [30, 31].
The new starting point is the canonical transformation of the gravitational variables of the following form \[30\]:

\[
(g_{ab}, p^{ab}) \rightarrow (\gamma_{ab}, \pi^{ab}; \phi, \pi)
\]  

(15)

where \(\gamma_{ab}\) and \(\pi^{ab}\) are rescaled from the ADM metric and momentum according to

\[
\gamma_{ab} = \phi^{-4} g_{ab}, \quad \pi^{ab} = \phi^{4} p^{ab}
\]  

(16)

using an arbitrary positive function \(\phi\) as the conformal factor, with the respective canonical momentum \(\pi\). The above construction obviously involves a local rescaling redundancy. Accordingly, an additional constraint

\[
C = \gamma_{ab} \pi^{ab} - \frac{1}{4} \phi \pi
\]  

(17)

is introduced to offset this redundancy, so that the number of physical degrees of freedom remains unchanged. It turns out that the constraint \(C\) is first class and is the canonical generator of the conformal transformations. It follows from Dirac's theory of constrained Hamiltonian systems that the gravitational Hamiltonian in terms of the variables \((\gamma_{ab}, \pi^{ab}; \phi, \pi)\) can be cast in the form:

\[
H = \int (N\mathcal{H} + X^a \mathcal{D}_a + ZC) d^3x
\]  

(18)

where \(Z = Z(x^a, t)\) is a new Lagrange multiplier. The proof of the above statement and further details can be found in \[30\].

FIG. 6: A conformal equivalence class of triads consists of triads with the same orientation but different scaling factors.
With the availability of the canonical formulation of general relativity in terms of conformal equivalence classes of metrics, we may further extend the phase space to accommodate conformal equivalence classes of triads (Fig. 6) and the corresponding equivalence classes of SU(2) connections. To this end, introduce the triad $\epsilon^a_i$ associated with the rescaled metric such that

$$\gamma^{ab} = \epsilon^a_i \epsilon^b_i$$  \hspace{1cm} (19)$$

By analogy with the densitized triad in (8), we introduce the rescaled densitized triad by

$$E^a_i = \gamma^{1/2} \epsilon^a_i = \phi^{-4} g^{1/2} \epsilon^a_i.$$  \hspace{1cm} (20)$$

Here, we see that the global scaling parameter in (8) has now become a local scaling coefficient according to $\beta \rightarrow \phi^{1/4}$. By regarding $E^a_i$ as the momentum of the SU(2) connection $A^i_a$ associated with the rescaled triad $\epsilon^a_i$, we can complete the following canonical transformation $^{31}$:

$$(\gamma_{ab}, \pi^{ab}; \phi, \pi) \rightarrow (A^i_a, E^a_i; \phi, \pi).$$  \hspace{1cm} (21)$$

We have now redundancies in the variables $(A^i_a, E^a_i; \phi, \pi)$ due to spin-gauge and conformal transformations, generated by the Gauss constraint $G$ and conformal constraint $C$ respectively. This leads to our final form of the gravitational Hamiltonian of the form $^{31}$:

$$H = \int \left( N\mathcal{H} + X^a D_a + Y^k G_k + Z C \right) \mathrm{d}^3 x.$$  \hspace{1cm} (22)$$

The detailed construction of the constraints $\mathcal{H}, D_a, G_k$ and $C$ and the proof of their first class nature are given in $^{31}$.

The quantization of gravity in the canonical variables $(A^i_a, E^a_i; \phi, \pi)$ follows formally as

$$\hat{\mathcal{H}} \Psi = 0 \hspace{1cm} (23)$$

$$\hat{D}_a \Psi = 0 \hspace{1cm} (24)$$

$$\hat{G}_k \Psi = 0 \hspace{1cm} (25)$$

$$\hat{C} \Psi = 0 \hspace{1cm} (26)$$

where $\Psi = \Psi[A^i_a, \phi]$. Here all the constraints $\mathcal{H}, D_a, G_k$ and $C$ are treated on an equal basis. If we eliminate the conformal constraint $C$ at the classical level by freezing $\phi$ to be a constant, say $\phi = \beta^{1/4}$, then the phase space reduces immediately to that of the standard spin-gauge formulation described in section $^V$. Though the quantization in this reduced
phase space using (12)–(14) has less quantum constraints, the price to pay is a $\beta$-dependent quantum ambiguity. We see that the Barbero-Immirzi ambiguity originates from gauge fixation, as discussed in section III. In contrast, the quantization using (23)–(26) is free from this ambiguity. This provides strong evidence that the true phase space in which quantum gravity occurs has the canonical coordinates $(A^i_a, E^a_i; \phi, \pi)$ and that $\mathcal{H}, \mathcal{D}_a, \mathcal{G}_k$ and $\mathcal{C}$ constitute the complete set of gravitational constraints.

VII. CONCLUDING REMARKS AND FUTURE VISION

A discussion has been given of recent developments in unifying the two great theories of modern physics – general relativity and quantum mechanics. Inspired by promising progress on loop quantum gravity, we have reviewed its underlying spin-gauge structure which enables the powerful loop and indeed the spin network quantization techniques to conquer the ‘unquantizable’. However, the quantum ambiguity due to a free (Barbero-Immirzi) parameter existing in the present loop quantum gravity suggests that the theory is not final yet. After all, a motivating rationale for loop quantum gravity is the very parameter free approach to quantum gravity. We take the presence of the Barbero-Immirzi ambiguity as the indication that the gauge symmetries for canonical gravity must be further enlarged to incorporate conformal symmetry, in addition to spin-gauge symmetry. The need for conformal symmetry originates from the fact that the Barbero-Immirzi parameter is a scaling parameter. By locally ‘gauging’ this scaling invariance we obtain an extended phase space of general relativity with conformal symmetry. Indeed, starting from this extended phase space we see that the Barbero-Immirzi ambiguity arises if the conformal symmetry is gauge fixed prior to quantization, which is equivalent to the present loop quantum gravity. Therefore, our vision for the new face of quantum gravity is a fresh new phase space with conformal and spin-gauge symmetries as the unambiguous basis for unifying general relativity and quantum mechanics.

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