Towards understanding Regge trajectories in holographic QCD

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Abstract

We reassess a work done by Migdal on the spectrum of low-energy vector mesons in QCD in the light of the AdS-QCD correspondence. Recently, a tantalizing parallelism was suggested between Migdal’s work and a family of holographic duals of QCD. Despite the intriguing similarities, both approaches face a major drawback: the spectrum is in conflict with well-tested Regge scaling. However, while holographic duals can be modified to accommodate Regge behavior, so far it was not clear how this could be achieved in Migdal’s approach. In this paper we claim that Regge behavior cannot be achieved in the latter unless quark-hadron duality breakdown is taken into account. We also suggest that this duality breakdown in QCD is dual to the dilaton’s infrared profile in Regge-like holographic models.

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1 Introduction

The recent years have seen an emergence of 5-dimensional modelling in an effort to make contact with QCD, the main motivation under most of the models relying upon the gauge/gravity duality conjectured by Maldacena almost a decade ago\(^2\).

Even though the original Maldacena conjecture related a slice of superstring theory on anti-de Sitter (AdS) background with \(N = 4\) SYM, both the belief that QCD is rooted in string theory and, especially, the high-energy conformal symmetry of QCD yield room for hoping that the conjecture may be extended beyond its actual formulation to provide new insights into the strong interactions. Indeed, rather simple models capturing some of the features of QCD like chiral symmetry breaking and confinement are in remarkable agreement with phenomenology.

The common setting of these models is a 5-dimensional AdS background. A 4-dimensional boundary brane is introduced, upon which Standard Model fields are bound to propagate. Confinement is modelled by cutting the bulk space down at another boundary brane, thus introducing an infrared scale that mimics \(\Lambda_{QCD}\). Spontaneous chiral symmetry breaking \(^3\), quark masses \(^3\) and OPE condensates \(^7\) have also been implemented over the last years. However, the manifestly wrong scaling of the predicted spectrum, namely \(m_n^2 \sim n^2\), has been a burden to all these models, not only from the phenomenological side but also because their eventual connection to string theory was doubtful.

Only recently was it realized \(^8\) that the wrong mass-scaling was not an essential drawback of holographic models of QCD, but due to the choice of boundary conditions. One can draw an analogy between the 5-dimensional Kaluza-Klein modes in the bulk of the aforementioned models and the quantum mechanical excitations in a square well potential. By changing the profile of the boundaries one can dial the energy spectrum at will and should be able to get to the desired mass-scaling. In the language of holographic QCD, this amounts to a dynamical dilaton, scaling (asymptotically) as \(\Phi(z) \sim z^2\), whereas the metric is kept (asymptotically) purely AdS. Therefore, hard-wall models, \(i.e.,\) those with a sharp infrared brane, yield \(m_n^2 \sim n^2\), as opposed to infrared-improved models, where the asymptotic form of the dilaton ensures \(m_n^2 \sim n\).

It turns out, as first pointed out in \(^9\), that hard-wall holographic models bear a tantalizing resemblance with a work done by Migdal \(^10\) almost 30 years ago on characterizing the spectrum of vector mesons in QCD. This very ambitious project relied on linking short distance QCD to the meson spectrum through quark-hadron duality. The information on short distances was used to make a Padé approximant to the whole correlator and thereby information on the spectrum was extracted. This procedure was designed to be a systematic step by step approximation, in the sense that adding more information on short distances one should get closer to the true QCD spectrum. Thus, at least formally, the programme was improvable. Therefore, when the Padé for the leading order perturbation theory yielded a spectrum with poles at zeros of Bessel functions, meaning that the tower of vector mesons in QCD should scale as \(m_n^2 \sim n^2\), the apparent contradiction with experimental evidence was seen as an artifact of the truncation. Should more orders in the OPE be added, the spectrum would smoothly reshuffle to display Regge trajectories.

The similarity between Migdal’s result and that of hard-wall holographic models is, in a sense, not so surprising. In the simplest hard-wall models the 5-dimensional background is filled with a pure AdS metric, meaning that the correlator looks like that of a free quark

\(^2\)See, for instance, \(^1\)\(^2\) for comprehensive reviews.
theory. Both approaches therefore start from the same conformally invariant correlator. The interesting thing is that the analogy seems to go further and one might even argue \[11\] that holographic models can be interpreted as the limit of a series of Padé approximants. A qualitative understanding of this can be found in the deconstructed version of holographic QCD models \[12\], where an open moose model of hidden local symmetries \[13\] was shown to be interpretable as the latticized version of a fifth dimension. A \([N,N]\) Padé approximant would then be the dual picture to an open moose with \(N\) hidden symmetries.

In \[11\], Migdal’s procedure was intended to shed some light on AdS-QCD. Here we reverse the point: we reassess Migdal’s work with the benefit of using the results of AdS-QCD. A natural question then arises: if Regge scaling was achieved in holographic models through a non-trivial dilaton background, what is the analog in Migdal’s approach? We will see that one is forced to include non-perturbative effects beyond the OPE, generically referred to as quark-hadron duality violations in the QCD literature \[14\]. As a bonus, we will be able to grasp the meaning of the dilaton’s quadratic scaling in holography, an interesting question partially answered in the context of string theory \[9\]. Still, one would like to have some physical intuition in the context of QCD.

The outline of the paper will be as follows: in section 2 we will introduce a toy model with explicit Regge scaling, whose Padé approximant will be discussed in section 3. We will work in the large-\(N_c\) limit of QCD. There, resonances are narrow and therefore mass-poles are well-defined. The aim is to use our toy model, where results can be worked out analytically, as a probe to reassess the accuracy of Migdal’s programme. Section 4 will be devoted to the relationship between quark-hadron duality violations and the meson spectrum, whereas contact with holographic models is discussed in the last section.

2 A toy model

Following \[10\], we consider the following two-point vector correlator

\[
\Pi^{\mu\nu}(q) = i \int d^4 x e^{iq\cdot x} \langle 0 | T\{ V^\mu_{\bar{u}d}(x)(V^\nu_{\bar{u}d})^\dagger(0) \} |0 \rangle , \tag{1}
\]

with currents defined as

\[
V^\mu_{\bar{u}d}(x) = \bar{u}(x)\gamma^\mu d(x) . \tag{2}
\]

In the chiral limit, Lorentz symmetry implies that

\[
\Pi^{\mu\nu}(q) = (q^\mu q^\nu - q^2 g^{\mu\nu})\Pi_V(q^2) , \tag{3}
\]

namely that there is only one form factor, which satisfies the following dispersion relation

\[
\Pi_V(q^2) = q^2 \int_0^\infty \frac{dt}{t(t-q^2-i\varepsilon)} \frac{1}{\pi} \text{Im} \Pi_V(t) + \Pi_V(0) , \tag{4}
\]

where \(\Pi_V(0)\) is a subtraction constant. In the strict large-\(N_c\) limit, a two-point Green’s function consists of single particle exchange of an infinite number of stable mesons \[15\]. Therefore, the spectral function reads

\[
\frac{1}{\pi} \text{Im} \Pi_V(t) = \sum_{n=0}^\infty E_n^2 \delta(t-M_V^2(n)) . \tag{5}
\]
Unfortunately, this is the furthest one can go in full generality. Decay constants and masses in large-$N_c$ QCD are unknown. However, phenomenology in $N_c = 3$ QCD seems to indicate that Regge behavior holds and, in the vectorial channel, already sets in from the very first resonance. Therefore, we will choose as an educated ansatz \[ F^2_n = F^2, \quad M^2_V(n) = m^2_V + an. \] Inserting (6) back into (11) we end up with the following expression for the correlator

\[
\Pi_V(q^2) = \frac{F^2}{a} \left[ \psi \left( \frac{m^2_V}{a} \right) - \psi \left( \frac{-q^2 + m^2_V}{a} \right) \right],
\]

where $\psi(\xi)$ is the digamma function, defined as\[ \psi(\xi) = \frac{d}{d\xi} \log \Gamma(\xi). \]

The free parameters $F$ and $a$ can now be fixed upon matching with the first terms in the OPE for $\Pi_V(Q^2 \equiv -q^2)$. Indeed, in the Euclidean half-plane, the digamma function admits the following integral representation

\[
\psi(\xi) = \int_0^\infty dt \left( \frac{e^{-t}}{t} - \frac{e^{-\xi t}}{1 - e^{-t}} \right),
\]

from which the large-$\xi$ behavior can be extracted

\[
\psi(\xi) = \log \xi - \frac{1}{2\xi} - \sum_{n=1}^{\infty} \frac{B_{2n}}{2n \xi^{2n}}.
\]

Thus, the OPE for our model reads

\[
\Pi_V(Q^2) \simeq \Pi^{OPE}_V(Q^2) = -\frac{F^2}{a} \log \frac{Q^2}{\mu^2} + \sum_{k=1}^{\infty} \frac{c_{2k}}{Q^{2k}},
\]

where the condensates are given by (8)

\[
c_{2k} = (-1)^k \frac{F^2 a^{2k-2}}{k} B_k \left( \frac{m^2_V}{a^2} \right),
\]

and $B_k(\xi)$ stand for the Bernoulli polynomials. On the other hand, in real QCD, to leading order in $\alpha_s$,

\[
\Pi^{OPE}_V(Q^2) = -\frac{4}{3} \frac{N_c}{(4\pi)^2} \log \frac{Q^2}{\mu^2} + ...
\]

Thus, we immediately get

\[
\frac{4}{3} \frac{N_c}{(4\pi)^2} = \frac{F^2}{a}.
\]

\[3\text{Notice the connection between this model and the Veneziano model (Beta function) for fixed spin, i.e.,}

\[ B(n, s) = \frac{\Gamma(n) \Gamma(s)}{\Gamma(n + s)}. \]

In both cases, the mass-poles are dictated by the poles of the Gamma function.
As it stands, the model only fulfills one short-distance constraint. In principle, one could refine the model in order to match the first OPE condensates. This can be easily achieved by relaxing our initial ansatz for the decay constants and letting an arbitrary number of them as free parameters, which can then match as many OPE terms as wanted. For such extensions of our model, we refer the reader to [20]. However, in this paper we will not be interested in the phenomenological implications of the model. For all our considerations we can safely stick to the simpler version.

3 Building the Padé approximant

With our model for the two-point vector current correlator of (7), we can now attempt to build its Padé approximant. With the advantage of knowing the final answer, we will try to assess the validity of the procedure followed in [10]. We begin with some notation and definitions.

A \([N,M]\) Padé approximant to a function is defined as the quotient of two polynomials \(P_M(q^2)\) and \(Q_N(q^2)\),

\[
\Pi_V(q^2) \simeq \frac{P_M(q^2)}{Q_N(q^2)},
\]

such that the first \(N + M + 1\) derivatives of the function at a point match those of the approximant. With full generality, we can set \(Q_N(0) = 1\). Notice that one should be able to define (at least formally) a Taylor expansion of the function around a point, even if the expansion is divergent. The next issue to address is that of convergence. One would wish that the \([N,M]\) rational should converge to the function as \(N,M \to \infty\). This is true for a class of functions known as Stieltjes functions. Stieltjes functions admit the following power expansion about the origin

\[
f(z) = \sum_{n=0}^{\infty} f_n z^n,
\]

with the coefficients (also called moments) given by the following integrals

\[
f_n = (-1)^n \int_0^R u^n \nu(u) \, du,
\]

where \(R\) is the radius of convergence of (17) and \(\nu(u)\) is some positive definite integration measure. Therefore, (17) admits the following representation as a Stieltjes integral

\[
f(z) = \int_0^R \frac{du}{1 + uz} \nu(u).
\]

With the change of variable \(u = \frac{1}{t}\) one obtains

\[
f(z) = \int_R^{\infty} \frac{dt}{t(t + z)} \nu\left(\frac{1}{t}\right),
\]

which matches (11) with the following identifications\(^5\)

\[
z = -q^2, \quad \nu\left(\frac{1}{t}\right) = \frac{1}{\pi} \text{Im} \Pi_V(t).
\]

\(^4\)Padé approximants have been a useful tool for widespread applications in condensed matter and particle physics. See, for instance, the very interesting examples of [16].

\(^5\)Notice that our ansatz for the spectrum assumes a mass gap with the first resonance sitting at \(q^2 = m_V^2\). Therefore, \(R = m_V^2\) in (20). The spectral integral beginning at \(t = 0\) or \(t = m_V^2\) thus yields the same result. We will hereafter use the former notation for simplicity.

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Therefore, as recently pointed out in [17], the vector vacuum polarization admits a representation as a Stieltjes integral. This is most fortunate, and a series of interesting consequences ensue\(^6\). First and foremost, it ensures convergence of the Padé approximant of (16) for the special subset \([\{M+J\}, M]\), \(J \geq -1\). The convergence region covers the whole complex \(q^2\) plane, save for the Minkowski axis, where the poles sit. Furthermore, all the poles of the approximant are simple and located on the negative real axis, i.e., on the physical axis, and residues are positive.

Therefore, from a physical point of view, the Padé approximant to (7) can be built, at least formally, out of the infinite number of terms of the chiral expansion. Padé poles can then be interpreted as meson mass-poles whereas Padé residues play the rôle of the meson decay constants. Convergence then means that our Padé approximant has to match asymptotically the real QCD spectrum.

The strategy followed in [10] was to construct the symmetric \([N,N]\) approximant around an expansion point \(q^2 = -\mu^2\) on the far Euclidean axis. The Padé equations then read [10, 11]

\[
\frac{d^n}{dq^2} \left[ \Pi_V(q^2)Q_N(q^2) - P_N(q^2) \right]_{q^2 = -\mu^2} = 0, \quad n = 0, \ldots, 2N. \tag{22}
\]

It is worth noting at this point that the convergence theorems above no longer apply if one builds the Padé away from the origin. Interesting exceptions are meromorphic functions. There it can be proved that convergence is guaranteed as long as the Padé is constructed on any non-singular point of the real \(q^2\) axis\(^7\). In the strict large-\(N_c\) limit Green’s functions are known to be meromorphic, which means, in particular, that with the model of section 2 the Padé constructed around an arbitrary \(q^2 = -\mu^2\) converges. However, for finite \(N_c\) nothing compels our Padé to approach the original function. It is true that often Padé approximants converge even when there is no theorem to guarantee their convergence, but this pragmatic approach is far from being rigorous.

Here we are just interested in the mass spectrum, which is encoded in \(Q_N(q^2)\). We will therefore keep only the upper half of the previous set of equations, where the contribution of \(P_N(q^2)\) vanishes, namely

\[
\frac{d^n}{dq^2} \left[ \Pi_V(q^2)Q_N(q^2) \right]_{q^2 = -\mu^2} = 0, \quad n = N + 1, \ldots, 2N. \tag{23}
\]

Use of Cauchy’s integral formula leads to

\[
\frac{d^n}{dq^2} \left[ \Pi_V(q^2)Q_N(q^2) \right]_{q^2 = -\mu^2} = \frac{n!}{2\pi i} \oint_{\gamma} \frac{dq^2}{(q^2 + \mu^2)^{n+1}} \left[ \Pi_V(q^2)Q_N(q^2) \right], \tag{24}
\]

where \(\gamma\) is the contour depicted in figure 1. We can relate the previous expression to the spectral function by using the conventional strategy in sum rules of splitting the contour in two paths (see, for instance, [19]). In \(N_c = 3\) QCD one can readily show that the integral over the circle \(|q^2| = s_0\) vanishes when the contour is sent to infinity, and only the integral over the physical axis contributes. This is related to the fact that the asymptotic behavior of the spectral function goes to a constant and so\(^8\)

\[
\lim_{t \to \infty} \frac{1}{t^N} \text{Im} \Pi_V(t) Q_N(t) \sim t^N. \tag{25}
\]

\(^6\)See, for instance, [18].

\(^7\)I thank G. López-Lagomasino for helping me clarify this issue.

\(^8\)In the strict large-\(N_c\) limit, equation (25) is ill-defined because of the discrete nature of the spectral function. However, one can show that (25) still holds. See the Appendix for a detailed proof.
Figure 1: Contour integration chosen in (24). The singularities on the right half-plane are the poles of \( \Pi_V(q^2) \), sitting at \( q^2 = m^2 + ak \) for natural \( k \). The pole on the left half-plane is the \((n+1)\)-th multiple pole explicitly shown in the denominator of (24). The circle is defined by \(|q^2| = s_0\), which will be eventually sent to infinity.

Therefore, \( N + 1 \) derivatives are sufficient to get a Källén-Lehmann representation for the Padé equations, namely

\[
\frac{d^n}{d(q^2)^n} \left[ \Pi_V(q^2) Q_N(q^2) \right]_{q^2=-\mu^2} = n! \int_0^\infty \frac{dt}{(t+\mu^2)^{n+1}} \left[ \frac{1}{\pi} \text{Im} \Pi_V(t) Q_N(t) \right],
\]

where we made use of Schwarz reflection formula and the reality of \( Q_N(q^2) \). Following Migdal’s strategy, at that point we will rely on (local) quark-hadron duality and assume that the spectral function can be inferred from the OPE of (12). Then,

\[
\frac{1}{\pi} \text{Im} \Pi_V(t) \simeq \frac{4}{3} \frac{N_c}{(4\pi)^2} \theta(t) + \sum_{k=0}^{\infty} c_{(2k+2)} \delta^{(k)}(t),
\]

where \( \delta^{(k)}(t) \) stands for the \( k \)-th derivative of the Dirac delta and \( \delta^{(0)}(t) \equiv \delta(t) \). If we just keep the constant term in (27) – i.e., the logarithmic piece in (12) –, then the Padé equations (26) reduce to

\[
\int_0^\infty \frac{dt}{(t+\mu^2)^{n+1}} Q_N(t) = 0, \quad n = N + 1, \ldots, 2N.
\]

By reorganizing the powers of \((t+\mu^2)\) in (28), Migdal realized that \( Q_N(q^2) \) are proportional to Jacobi polynomials. Actually, it seems that the first derivation of the Padé approximant to the logarithm was performed by Jacobi, Rouché and Gauss\(^9\). Their result for the poles was

\[
Q_N(q^2) = 2F_1 \left( -N, -N; 1; -\frac{q^2}{\mu^2} \right) = (q^2 + \mu^2)^N P_N^{(0,0)} \left( \frac{\mu^2 - q^2}{\mu^2 + q^2} \right),
\]

where \( 2F_1(a, b; c; d) \) is Gauss’ hypergeometric function and \( P_N^{(0,0)} \) stands for the \((0,0)\)-Jacobi polynomial, i.e., the Legendre polynomial. Up to an overall constant out front, this last expression is the one found in the recent analysis of (11).

\(^9\)See [22] for details.
At low energies \((q^2 \ll \mu^2)\) the argument of the Jacobi polynomial above can be Taylor-expanded to yield
\[
\lim_{q^2 \to 0} \frac{\mu^2 - q^2}{\mu^2 + q^2} = \left(1 - 2\frac{q^2}{\mu^2}\right).
\]
Then, using the asymptotic formula [21]
\[
\lim_{N \to \infty} P_N^{(0,b)} \left(1 - \frac{\xi^2}{2N^2}\right) = J_0(\xi),
\]
one ends up with
\[
Q_N(q^2) = J_0 \left(2N \sqrt{\frac{q^2}{\mu^2}}\right),
\]
As anticipated, conformal invariance in the correlator generates a Padé made of Bessel functions. In order to bring the Padé approximant to the actual mass-scaling we have departed from, the natural step would be to refine the model so that it matches the first \(k\) OPE condensates, namely the right outmost terms in \([24]\). This can be easily achieved, \(e.g.\), by relaxing our ansatz \([6]\) so that the first \(k\) decay constants \(F_n\) are left as free parameters. As an academic exercise, let us imagine that all OPE condensates can be accounted for this way. Padé equations would then be brought to the form
\[
\int_0^{\infty} \frac{dt}{(t + \mu^2)^{n+1}} Q_N(t) = -12\frac{\pi^2}{N_c} \sum_{k=0}^{\infty} \frac{c_{(2k+2)}}{\mu^{2(n+1)}} M_N(k), \quad n = N + 1, \ldots, 2N,
\]
where the right-hand side is just a constant and \(M_N(k)\) is the rather cumbersome expression
\[
M_N(k) = \sum_{j=0}^{k} (-1)^{k+j} \frac{(n+j)!}{n!} \binom{k}{j} \frac{Q_N^{(k-j)}(0)}{\mu^{2j}},
\]
where \(Q_N^{(i)}\) stands for the \(i\)-th derivative of \(Q_N(q^2)\).

The terms in the right-hand side of \([33]\) shift the location of the Padé poles away from the Bessel roots of \([22]\). This is what one would naturally expect. However, notice that our model, which clearly is not QCD, can potentially fulfill all QCD short distance constraints (once the decay constants are left as free parameters). Therefore, Migdal’s Padé construction is now unable to distinguish our toy model from real QCD. This casts serious doubts on the reliability of the approach outlined above. In the next section we will see that the assumption of quark-hadron duality in the context of Migdal’s programme is unjustified.

### 4 The rôle of quark-hadron duality violations

It is a well-known fact that the OPE is not an allowed expansion in the whole \(q^2\) complex plane. At least on the physical axis the expansion breaks down. The difference between the spectral function coming from the OPE \([27]\) and the spectral function of our toy model \([5]\) basically reflects this fact: the OPE is unable to reproduce the singular structure of mass-poles we started from. The analytical properties of an equally-spaced array of simple poles has turned to a constant and an overlap of multiple poles at the origin. Despite this clear mismatch, the substitution of Green’s functions for their OPE, known as the assumption of quark-hadron duality, has played a prominent rôle, for instance, in the development of
QCD sum rules. Therefore, it is worth reassessing this assumption in the context of Migdal’s programme.

Far in the Euclidean half-plane, the asymptotic behavior of Green’s functions is given by their OPE, whose analytic continuation into the Minkowski region is (up to exponentially suppressed terms) given by \[23, 24\]

\[\Pi_V(q^2) \simeq \Pi_V^{OPE}(q^2) + \Delta(q^2), \quad (35)\]

where the extra term above has to be of exponential type\(^{10}\) except on the physical axis, where the OPE is no longer an allowed expansion. Quite generally \[20, 23\],

\[\Delta(q^2) \simeq e^{-|q^2|F(\phi,N_c)}H(q^2). \quad (36)\]

The presence of this extra term can be understood on the basis that the OPE, being a regular expansion, loses track of the precise details of the spectrum, which are then encoded in the \(\Delta(q^2)\) function. At this point, one should ask why terms like (36) can be expected to restore our initial spectrum. After all, such duality breakdown is expected to vanish exponentially, as (36) seems to indicate\(^{11}\).

Again, because of the ill-defined nature of the large-\(N_c\) limit due care has to be exercised. In \(N_c = 3\) QCD, and even at large but finite \(N_c\), resonances in the spectrum develop a width, expected to be proportional to

\[\Gamma_V(n) \sim \frac{\sqrt{n}}{N_c}, \quad (37)\]

\(n\) being the excitation number. Therefore, increasing widths of resonances makes them overlap and eventually the spectrum is smeared out into a continuum: local duality sets in. This means that at sufficiently large values of momentum \(t\) on Minkowski axis the duality picture has to be very accurate. Actually, based on a model, the authors of \[24\] found

\[\text{Im } \Delta(t) \sim \exp\left(-\frac{\alpha t}{N_c}\right)\cos \beta t \quad (38)\]

as a particular realization of (36), \(\alpha\) and \(\beta\) being constants, such that duality pieces turn out to be unimportant compared to the OPE from \(n \sim N_c\) onwards.

However, in the strict large-\(N_c\) limit \(N_c\) is sent to infinity from the very beginning. Resonances therefore stay infinitely narrow no matter the excitation number and duality never sets in \((n \sim N_c \to \infty)\). This means that \(F(\phi,N_c \to \infty)\) has to kill the exponential damping of (36) on the Minkowski axis \((\phi = 0, 2\pi)\).

In our specific toy model, (36) can be computed analytically. One needs to recall the reflection property of the digamma function,

\[\psi(\xi) = \psi(-\xi) - \pi \cot \pi \xi - \frac{1}{\xi}, \quad (39)\]

in order to be able to analytically-continue the OPE to the Minkowski region. Matching to (35), one gets

\[\Delta(q^2) = \frac{\pi F^2}{a} \cot \left[\frac{\pi - q^2 + m_V^2}{a}\right]. \quad (40)\]

\(^{10}\)This is related to the fact that the OPE is a non-convergent expansion.

\(^{11}\)It is worth stressing that we are not considering the instanton-induced duality violations, which, as Migdal noted, are suppressed at large-\(N_c\). Instead, we are referring here to the duality breakdown due to resonances, which is a leading order effect in the \(1/N_c\) expansion.
Recall that indeed the cotangent encodes information about the poles of the digamma function, something that the OPE fails to reproduce. Poles are equidistantly located on the real axis and are given by the expression

\[ \pi \cot \pi \xi = \frac{1}{\xi} + 2\xi \sum_{n=1}^{\infty} \frac{1}{\xi^2 - n^2}. \]  

In short, duality pieces become unimportant as long as one can no longer trace the position of the mass-poles back in the spectrum. However, the large-\(N_c\) and low-\(q^2\) limits taken by Migdal were intended to avoid the continuum region of QCD, precisely where the assumption of duality is allowed. At low energies resonance poles are well-defined only because duality violations are sizeable. As \(N_c\) is increased, the onset of duality occurs at higher and higher energies and eventually, in the strict large-\(N_c\) limit, duality violations remain unsuppressed.

Therefore, as a matter of principle, Migdal’s approach would have never reached the spectrum of QCD. One should add the duality violating piece, the analog of (40) in QCD, in the spectral function (27). Unfortunately, very little is known about the form of duality violations in real QCD. The best one can do is to resort to models in the framework of large-\(N_c\). In that context Migdal’s approach turns out to be rather inconvenient, because at large-\(N_c\) there is no need to discretize the spectral function.

5 Connection with holographic QCD models

In this section we will pursue the relationship between QCD and holographic QCD duals suggested in [11]. If Migdal’s original work is related to hard-wall holographic models, there should be a similar relationship between our toy model and holographic duals with built-in Regge behavior. This relationship was recently found in [8].

The 5-dimensional model they started from can be written as

\[ S = -\int d^4 x \, dz \, e^{-\Phi(z)} \sqrt{\frac{g_{\hat{\mu} \hat{\nu}}}{4g_5^2}} \text{Tr} \left[ (F_{\hat{\mu} \hat{\nu}} F_{\hat{\mu} \hat{\nu}})_L + (F_{\hat{\mu} \hat{\nu}} F_{\hat{\mu} \hat{\nu}})_R \right], \]

where the hatted indices \(\hat{\mu}, \hat{\nu} = 0, 1, 2, 3, 4\), the metric is parametrised as

\[ g_{\hat{\mu} \hat{\nu}} \, dx^{\hat{\mu}} \, dx^{\hat{\nu}} = e^{2A(z)} (\eta_{\mu\nu} \, dx^\mu \, dx^\nu + dz^2), \quad \eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1), \]

and \(\Phi(z)\) stands for the dilaton field. For our purposes, namely the spectrum of vector mesons, additional bulk fields in the action (42) are irrelevant.

The eigenvalue equation for the vectorial modes can be expressed as a Sturm-Liouville differential equation

\[ \frac{d}{dz} \left( e^{-B(z)} \frac{d}{dz} V_n \right) + m_n^2 e^{-B(z)} V_n = 0, \]

where \(B(z) = \Phi(z) - A(z)\). After a change of variable, the previous equation can be recast in the Schrödinger-like form

\[ \left\{ -\frac{d^2}{dz^2} + V(z) \right\} \psi_n(z) = m_n^2 \psi_n(z), \quad V(z) = \left( \frac{B'}{2} \right)^2 - \frac{B''}{2}. \]

\(^{12}\text{See [14, 20, 24] for some recent studies.}\)
In the so-called hard-wall models, \( A = - \log z \), such that the metric is AdS, and the dilaton is taken to be constant, \( \Phi(z) = \phi \), \( \epsilon \leq z \leq \Lambda \), where \( z = \epsilon \) and \( z = \Lambda \) are the four-dimensional boundaries. Then the potential is given by
\[
V(z) = \frac{3}{4 z^2}, \tag{46}
\]
and the Schrödinger equation has Bessel functions as solutions, yielding \( m_n^2 \sim n^2 \).

The authors of \[8\] realized that in order to get to the right scaling one has to modify the infrared boundary conditions. The requirement of Regge behavior therefore leads to a (asymptotically) unique choice of the dilaton background, namely \( \Phi(z) \sim z^2 \). In particular, the simplest potential incorporating the right leading order QCD short distances and Regge behavior takes the form
\[
V(z) = z^2 + \frac{3}{4 z^2}, \tag{47}
\]
which corresponds to the three-dimensional harmonic oscillator in quantum mechanics. The harmonic term is the dilaton contribution, whereas the centrifugal barrier term is due to the AdS metric. The Schrödinger equation is solved through Laguerre polynomials with the spectrum scaling as
\[
m_n^2 = 4n + 4. \tag{48}
\]
Furthermore, the decay constants can be computed to give a constant,
\[
F_n^2 = \frac{2}{g_5^2} \equiv F^2. \tag{49}
\]
The metric therefore plays no rôle in fixing the infrared boundary of the 5-dimensional background, which turns out to be the cornerstone to have a Regge-like spectrum\(^\text{13}\).

As noted by \[8\], comparison of (48) and (49) with (6) shows that the 5-dimensional model with linear confinement is dual to the toy model of section 2 with \( a = 4 \) and \( m_V^2 = 4 \).

The analogy one can draw now is straightforward: on the one hand hard-wall models in holographic QCD can be brought to the right mass-scaling through infrared effects carried by the dilaton; on the other hand, Migdal’s result can only show Regge scaling through infrared effects collected under the name of quark-hadron duality breakdown. We therefore suggest that the introduction of the precise dilaton profile \( \Phi(z) \sim z^2 \) in \[8\] is the dual version of taking into account the specific quark-hadron duality breakdown of \[10\].

6 Conclusions

Using a model of large-\( N_c \) QCD with built-in Regge behavior we have shown that out of the OPE one cannot recover the spectral function, as conveyed in a work by Migdal. Following his same strategy, we have used Padé approximants to try to link short distances to resonance masses. Starting from the parton model logarithm the inferred spectrum would look like
\[
\frac{J_{\nu-1}(\xi)}{J_\nu(\xi)} = \frac{2\nu}{\xi} + 2\xi \sum_{n=1}^{\infty} \frac{1}{\xi^2 - j_{\nu,n}^2}, \quad (\nu = 0), \tag{50}
\]
\(^\text{13}\)To illustrate the point even further, one could get rid of the centrifugal barrier in (47) at the price of changing the metric from AdS to flat space-time. The spectrum would then be dictated by Hermite polynomials, but its scaling would still be \( m_n^2 \sim n \). Obviously, the interplay of the metric and the dilaton will give rise to a different background and this will affect, e.g., the determination of decay constants and the slope of the spectrum (the \( a \) parameter in our toy model) but, importantly, will not change its scaling properties.
namely, masses would sit at Bessel zeros instead of matching the Gamma function poles of our model,

$$\pi \cot \pi \xi = \frac{1}{\xi} + 2\xi \sum_{n=1}^{\infty} \frac{1}{\xi^2 - n^2}. \quad (51)$$

Inclusion of condensates and higher order terms in the perturbative expansion might shift the position of the poles in (50), but these ultraviolet corrections will not drive the poles to (51), because short distances bear no information on the details of the spectrum. This is related to the well-known fact that different functions can lead to the same asymptotic behavior. Therefore, from perturbative QCD and, most generally, the OPE, there is no hope of trying to reconstruct the spectrum. That same statement was already made in a slightly different context in [25].

The fact that resonances can be distinctly singled out in the spectrum signals strong quark-hadron duality breakdown. In the strict large-$N_c$ limit, where resonances are infinitely narrow, duality violations do persist at all energies. For large but finite $N_c$, on the other hand, they turn out to be exponentially suppressed. Whenever distinction between resonances is no longer possible, because their widths overlap, duality-violating terms are accordingly small; the spectrum then appears as a continuum which is well described by the OPE. The concept of duality is thus essentially linked to the distinguishability of resonances in the spectrum. Therefore, if we attempt to reconstruct the spectrum of QCD, the assumption of duality in Migdal’s work is inherently inconsistent.

We would like to stress that, admittedly, there are interesting similarities between Migdal’s work and certain holographic models. Recently it was claimed [27] that the extension of Migdal’s programme to three point functions carried out in [26] has its counterpart in the holographic light-front wavefunctions of [28]. However, when it comes to real QCD the virtues of Migdal’s approach should not be overestimated. For one thing, we have seen that it fails as a method of getting the spectrum of QCD. In addition, the case of three-point functions is not free from subtleties: Källén-Lehmann representations no longer apply and a number of assumptions have to be made [26]. Therefore, in our opinion the similarities found in the literature are suggestive and can even be insightful, but should be considered with circumspection, for their eventual connection with real QCD is more than dubious.

In this work we have also suggested a connection between quark-hadron duality breakdown and the infrared dependence of the dilaton in holographic QCD models. In the simplest of the hard-wall holographic models, the anti-de Sitter metric extends all along the fifth dimension between the boundary branes and the predicted spectrum is that of (50). In order to get to (51), the picture advocated in [8] is that a non-trivial dilaton in the infrared is needed. In a qualitative way, therefore, the quadratic dilaton in holography is playing the rôle of quark-hadron duality breakdown in QCD. This is really plausible and appealing, since both are infrared effects.

The assumption that the whole holographic duality was leading in an essential way to non-Regge behavior was based on the belief that the metric in the ultraviolet was determining the spectrum. This is essentially the same philosophy lying behind the programme carried out by Migdal. Presently, there is increasing evidence that breaking of conformal symmetry induced by the metric in the ultraviolet is tantamount to adding more terms in the OPE\(^{14}\), while the spectrum remains basically unscathed. It is the introduction of non-perturbative infrared effects, namely the specific profile of the dilaton (which is capturing in an effective

\(^{14}\)See, e.g., the strategy followed in [7] to model the QCD condensates. See also, e.g., [29, 30] for the effects of modifying the metric in the infrared.
way the form of quark-hadron duality breakdown in QCD) that fixes the mass scaling to the
sought Regge spectrum.

This brings up the interesting question of whether there is hope in the AdS-QCD models
to build the holographic dual of QCD. Unfortunately, to achieve this goal successfully it seems
that a better knowledge of non-perturbative QCD is mandatory.

Acknowledgements

I am grateful to Maurizio Piai, D. T. Son and Mithat Unsal for useful conversations and
to Chris Herzog, Santi Peris and Matt Strassler for their critical reading of the manuscript
and the suggestions that followed. I also would like to acknowledge very fruitful correspon-
dence with S. J. Brodsky, G. F. de T´eramond, J. Hirn, G. L´opez-Lagomasino, I. Low and
E. J. Weniger. My thanks to the Department of Physics of the University of Wash-
ington at Seattle, where this work has been carried out, for their hospitality. The author is supported
by the Fulbright Program and the Spanish Ministry of Education and Science under grant
no. FU2005-0791.

Appendix

As stated in the main text, the vanishing of contour integrals in two-point functions can be
inferred from the asymptotic behavior of the spectral function, leading in our case to

$$
\lim_{t \to \infty} \frac{1}{\pi} \Im \Pi_V(t) \mathcal{Q}_N(t) \sim t^N.
$$

(52)

However, in the large-$N_c$ limit the above equation is strictly speaking ill-defined. Therefore,
one has to adopt a slightly different strategy. We start from

$$
\frac{d^n}{d(q^2)^n} \left[ \Pi_V(q^2) \mathcal{Q}(q^2) \right]_{q^2=\mu^2} = \frac{n!}{2\pi i} \oint_{\gamma} \frac{dq^2}{(q^2 + \mu^2)^{n+1}} \left[ \Pi_V(q^2) \mathcal{Q}(q^2) \right] =
$$

$$
= \frac{n!}{2\pi i} \oint_{|q^2|=s_0} \frac{dq^2}{(q^2 + \mu^2)^{n+1}} \left[ \Pi_V(q^2) \mathcal{Q}(q^2) \right] + \frac{n!}{\pi} \int_0^{s_0} dt \frac{1}{(t + \mu^2)^{n+1}} \left[ \frac{1}{\pi} \Im \Pi_V(t) \mathcal{Q}_N(t) \right].
$$

(53)

where the limit $s_0 \to \infty$ is to be understood henceforth. To get to (26) in the main text we
have to show that the integral in the second line above identically vanishes. Even though we
will carry all explicit calculations with the model of section 1, we will find that this statement
is a general result of large-$N_c$ QCD itself.

Our model reads

$$
\Pi_V(q^2) = \frac{F^2}{a} \left[ \psi \left( \frac{m_V^2}{a} \right) - \psi \left( \frac{-q^2 + m_V^2}{a} \right) \right],
$$

$$
\frac{1}{\pi} \Im \Pi_V(t) = F^2 \sum_{n=0}^{\infty} \delta(t - m_V^2 - an).
$$

(54)

Recall that the digamma function $\psi(\xi)$ is analytic everywhere in the complex $\xi$ plane except
for simple poles at negative integer values of the argument, i.e., $\xi = 0, -1, -2, \ldots$. Moreover,
it satisfies

$$
\text{Res} \left[ \psi(\xi); \xi \right] = -1, \quad \xi = 0, -1, -2, \ldots
$$

(55)
The second line in (53) can then be straightforwardly evaluated by reiterative application of Cauchy’s residue theorem. The integrand has a pole of order \( n+1 \) in the Euclidean axis sitting at \( q^2 = -\mu^2 \) along with an infinite number of equidistant (simple) poles in the Minkowski axis, at

\[
q^2 = m_V^2 + ak, \quad k = 0, 1, \ldots
\]  

(56)

Application of the residue theorem then yields

\[
\frac{n!}{2\pi i} \oint_{|q^2|=s_0} \frac{dq^2}{(q^2 + \mu^2)^{n+1}} \left[ \Pi_V(q^2) Q_N(q^2) \right] = \lim_{q^2\to-\mu^2} \frac{d^n}{d(q^2)^n} \left[ \Pi_V(q^2) Q_N(q^2) \right] + \\
+ \lim_{q^2\to m_V^2+ak} n! \sum_{k=0}^{\infty} \left[ \frac{q^2 - (m_V^2 + ak)}{(q^2 + \mu^2)^{n+1}} \Pi_V(q^2) Q_N(q^2) \right].
\]  

(57)

Obviously, the first term is the left-hand side we started from in (53). Therefore, the second line above has to cancel the integral over the spectral function in (53). Indeed,

\[
\int_0^{s_0} \frac{dt}{(t + \mu^2)^{n+1}} \left[ \frac{1}{\pi} \lim_{t\to q^2} \Pi_V(t) Q_N(t) \right] = F^2 \sum_{k=0}^{\infty} \frac{Q_N(m_V^2 + ak)}{(m_V^2 + ak + \mu^2)^{n+1}},
\]  

(58)

whereas

\[
\lim_{q^2\to m_V^2+ak} \sum_{k=0}^{\infty} \left[ \frac{q^2 - (m_V^2 + ak)}{(q^2 + \mu^2)^{n+1}} \Pi_V(q^2) Q_N(q^2) \right] = \\
= \lim_{q^2\to m_V^2+ak} F^2 \sum_{k=0}^{\infty} \left[ \left( \frac{q^2 - m_V^2}{a} + k \right) \psi \left( \frac{q^2 - m_V^2}{a} \right) \right. \\
\left. \frac{Q_N(q^2)}{(q^2 + \mu^2)^{n+1}} \right],
\]  

(59)

which equals (58) but for a change in sign\(^{15}\).

Now we only need to show that the first line in (57) cancels (59), or equivalently, as we have just checked, equals (58). In other words, the contribution from the infinite poles on the Minkowski axis counterplays that of the pole at the Euclidean axis. Applying the binomial formula to the first line of (57), one finds

\[
\frac{d^n}{d(q^2)^n} \left[ \frac{F^2}{a} \psi \left( \frac{q^2 - m_V^2}{a} \right) Q_N(q^2) \right] \bigg|_{q^2=-\mu^2} = \\
= -\sum_{j=1}^{n} \binom{n}{j} \left( \frac{-1}{a} \right) \psi_j \left( \frac{q^2 - m_V^2}{a} \right) Q_N^{(n-j)}(q^2) \bigg|_{q^2=-\mu^2} = \\
= F^2 \sum_{k=0}^{\infty} \sum_{j=1}^{n} \frac{n! \cdot Q_N^{(n-j)}(-\mu^2)}{(n-j)! \cdot (m_V^2 + ak + \mu^2)^{j+1}},
\]  

(60)

where in the last line we have used the definition of the polygamma function, namely

\[
\psi_j(\xi) = \frac{d^j}{d\xi^j} \psi(\xi) = (-1)^{j+1} j! \sum_{k=0}^{\infty} \frac{1}{(\xi + k)^{j+1}}.
\]  

(61)

\(^{15}\)This becomes obvious if one deforms the path \( \gamma \) such it does not include the pole in the Euclidean. Then Cauchy’s integral theorem asserts that the integral over the circle equals that over the physical axis.
The last line in (60) can be recast in the more useful way

\[
\sum_{k=0}^{\infty} \left( \frac{1}{(m_V^2 + ak + \mu)^{n+1}} \right) \sum_{j=1}^{n} \frac{n!}{(n-j)!} Q_N^{(n-j)}(-\mu^2)(m_V^2 + ak + \mu^2)^{n-j}.
\]  

Therefore, in order to match (58), it follows that

\[
Q_N(m_V^2 + ak) = \sum_{j=1}^{n} \frac{1}{(n-j)!} Q_N^{(n-j)}(-\mu^2)(m_V^2 + ak + \mu^2)^{n-j},
\]

but the sum over \(j\) is just the Taylor expansion of \(Q_N(q^2)\) around \(q^2 = -\mu^2\), evaluated at \(q^2 = m_V^2 + an\), and truncated after \(n+1\) terms. However, being \(Q_N(q^2)\) a polynomial of degree \(N\), the previous relation is exact if \(n > N\), which is one of our starting points. This completes the proof.

Recall that the proof, as anticipated, relies only upon the analytic properties of two-point Green’s functions in large-\(N_c\), namely the existence of no cuts and just simple poles. Therefore, the result is model-independent and valid for large-\(N_c\) QCD.

References


