Why the cosmological constant is small and positive

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Within conventional big bang cosmology, it has proven to be very difficult to understand why today’s cosmological constant is so small. In this paper, we show that a cyclic model of the universe can naturally incorporate a dynamical mechanism that automatically relaxes the value of the cosmological constant, taking account of contributions to the vacuum density at all energy scales. Because the relaxation time grows exponentially as the vacuum density decreases, nearly every volume of space spends an overwhelming majority of the time at the stage when the cosmological constant is small and positive, as observed today.

1 Introduction

One of the greatest challenges in physics today is to explain the small positive value of the cosmological constant or, equivalently, the energy density of the vacuum. The observed value, $7 \times 10^{-30}$ g/cm$^3$, is over one hundred twenty orders of magnitude smaller than the Planck density, $10^{93}$ g/cm$^3$, as the universe emerges from the big bang, yet its value is thought to be set at that time. Even more puzzling, the vacuum density receives a
series of contributions from lower energy physical effects, including the electroweak and quantum chromodynamics (QCD) transitions, that only become significant at a later stage. Explaining today’s tiny value requires a mechanism capable of canceling many very different contributions with near-perfect precision.

One long-standing hope had been to find a symmetry (1) or quantum gravity effect (2, 3) that forces the vacuum density to be zero. Another hope had been to find a relaxation mechanism driving it to zero in the hot early universe, as the universe expands. These hopes have been hard to reconcile with cosmic inflation and, in any case, have been dashed by recent observations indicating that the vacuum density is small, positive and very nearly constant (4, 5). Now it is apparent that one does not want a complete cancellation of the cosmological constant. And, in order for a relaxation mechanism to operate within the standard inflationary picture, the relaxation time must at first be much longer than the Hubble time, so inflation can take place; then much shorter than the Hubble time so that nucleosynthesis and structure formation can occur; and then, after that, much longer than the Hubble time again so that the vacuum density is nearly constant today, as observed. Despite many attempts, no simple and compelling mechanism has been found. The frustration has been enough to drive many physicists to consider anthropic explanations (6, 7), in which one assumes that the vacuum density takes on all possible values in different regions of space, but that life is only possible in one of the rare regions where the vacuum density is exponentially small.

In this paper, we point out that a cyclic model of the type described in Refs. (8, 9) re-opens the possibility of solving the cosmological constant problem with a natural, monotonic relaxation mechanism. In these models, each cycle consists of a hot big bang followed by a nearly vacuous period of dark energy domination, ending with a crunch which initiates the next bang. The duration of a cycle is typically of the order of a trillion
years. There is no known limit to the number of cycles that have occurred in the past, so the universe today can plausibly be exponentially older than today’s Hubble time, and still form galaxies and stars as observed today. Within this cyclic framework, it is reasonable to consider mechanisms for relaxing the cosmological constant whose timescale is always far greater than today’s Hubble time. The cosmological constant is exponentially smaller than one might have guessed based on the big bang picture precisely because the universe is exponentially older than the big bang estimate, so the cosmological constant has had a very long time to reduce in value from the Planck scale to the miniscule value observed today. Furthermore, we will show that it is natural to have mechanisms in which the relaxation time increases exponentially as the vacuum density approaches zero from above, resulting in a universe in which nearly every volume of space spends an exponentially longer time in a state with small, positive cosmological constant than in any other state. This is in stark contrast to anthropic explanations according to which the only regions of space ever capable of producing galaxies, stars, planets and life are exponentially rare.

2 Dynamical Relaxation: A Worked Example

As a specific example of a dynamical relaxation mechanism, we adapt an idea first discussed by Abbott (10) in the context of standard big bang cosmology (see also Ref. (19)). In Abbott’s model, the vacuum energy density of a scalar field gradually decays through a sequence of exponentially slow quantum tunneling events, relaxing an initially large positive cosmological constant to a small value. In spite of some appealing features, Abbott found that the mechanism failed, as we will explain, within the context of a big bang universe, essentially because the relaxation occurs far too slowly compared to a Hubble time. In this paper, however, we show that the mechanism becomes viable within the
cyclic universe picture.

Abbott’s proposal introduces an axion-like scalar field $\phi$ coupled to the hidden non-abelian gauge fields through a pseudoscalar coupling $(\phi/f)F^*F$, with $f$ some high energy mass scale. The theory is assumed to have a classical symmetry

$$\phi \rightarrow \phi + \text{constant},$$

which is softly broken at low energies by various effects. Integrating out the gauge fields induces a potential $-M^4 \cos(\phi/f)$, where $M$ is the scale where the gauge coupling becomes strong. (Fields of this type are commonly invoked to suppress CP-violation in the strong interactions \cite{11,12,13} and are also ubiquitous in string theory.)

It is natural for $M$ to be very small, as a consequence of the slow (logarithmic) running of the coupling in a nonabelian gauge theory. For example, in QCD with six flavors, $\Lambda_{QCD} = M_{Pl}\exp(-2\pi/(7\alpha_{QCD}(M_{Pl})) \sim 100 \text{MeV}$ if the coupling strength at the Planck scale $\alpha_{QCD}(M_{Pl}) \sim 1/50$. (Here and below, $M_{Pl} = (8\pi G)^{-\frac{1}{2}}$). In Abbott’s model for the hidden axion field, $M$ replaces $\Lambda_{QCD}$ and is similarly expressed in terms of the relevant coupling to hidden gauge fields. For example, if the hidden sector were exactly like QCD, taking $\alpha(M_{Pl}) \sim 1/75$ would give $M \sim 10^{-3} \text{eV}$, a viable value for our model. (Our choices are less extreme than those in Abbott’s paper; in the 1980’s, his goal was to obtain a very small vacuum density, whereas ours is to explain the observed value.)

The cosine potential breaks the symmetry \cite{14} down to a discrete subgroup, $\phi \rightarrow \phi + 2\pi N$. The discrete symmetry is also assumed to be softly broken, by a term producing a ‘washboard’ effective potential:

$$V(\phi) = -M^4 \cos \left( \frac{\phi}{f} \right) + \epsilon \frac{\phi^2}{2\pi f} + V_{\text{other}},$$

where $V_{\text{other}}$ includes all other contributions to the vacuum density. (The linearity of the second, soft breaking term is inessential: any potential will do as long as it is very gently
Figure 1: The effective cosmological constant $\Lambda_{total}$ for the washboard potential defined in Eq. 2 can take discrete values depending on which minimum $\phi$ occupies. In the scenario presented here, the time spent in the lowest positive minimum is exponentially greater than the entire time spent in all other minima.

sloping in the region of interest, around $V = 0$.) Provided $\epsilon < M^4$, (2) has a set of equally spaced minima $V_N$, with effective cosmological constant $\Lambda_{total}$ spaced by $V_N - V_{N-1} = \epsilon$ (Fig. 1). No matter what $V_{other}$ is, there is a minimum with $\Lambda_{total} = V_0$ in the range $0 \leq V_0 < \epsilon$. Although $\epsilon$ must be chosen to be very small in order to account for today’s tiny vacuum density, this choice is technically natural within the model since all quantum corrections to $\epsilon$ are proportional to $\epsilon$. Hence, Abbott’s model is a self-consistent low-energy effective theory capable of cancelling contributions to the vacuum density coming from any other source.

In Abbott’s scheme, the smallness of the cosmological constant today is related through the relaxation mechanism to the smallness of the parameters $M$ and $\epsilon$ in the potential $V(\phi)$. Effectively, the intractable problem of naturally obtaining an exponentially small cosmological constant is transmuted into a tractable problem of naturally obtaining small axion interaction parameters.

Abbott assumed the universe emerges from the big bang with some large positive value
of $\phi$ and quickly settles into a minimum with large positive $V_N$, driving a period of de Sitter expansion which dilutes away any matter and radiation. Over time the field $\phi$ then works its way slowly but inexorably downhill. In flat space-time, the tunneling events would occur at a constant rate, independent of $N$. However, once the effects of gravity are included, the tunneling rate becomes slower and slower as $V_N$ decreases. As we shall see, the universe remains in the last positive minimum for a relative eternity compared to the time spent in reaching it. This is the basis for our claim that the most probable value for the vacuum density in the model is that of the last positive minimum.

Assuming the field starts high up the potential, $V_N >> M^2 M_{Pl}^2$, de Sitter fluctuations overwhelm the energy barriers and the field makes its way quickly downhill. But as $V_N$ falls below $M^2 M_{Pl}^2$, the barriers become increasingly significant and the field progresses downwards by quantum tunneling via bubble nucleation (14). Upward tunneling is also allowed, but hugely suppressed in the parameter range of interest (17).

For simplicity, we shall focus on the parameter range $f^2/M_{Pl}^2 << \beta < 1$, where $\beta \equiv \epsilon/M^4$. In the semiclassical approximation, the rate for nucleating bubbles of vacuum energy density $V_{N-1}$ beginning from the $V_N$ phase is $\Gamma(N) \propto \exp(-B(N))$ where $B(N)$ is the Euclidean action for the tunneling solution. In order to describe the scaling of $B(N)$ with $N$, we shall neglect unimportant numerical coefficients and approximate $V_N \approx \beta N M^4$.

As $\phi$ tunnels towards minima with decreasing $N$, the nucleation rate decreases monotonically through three scaling regimes which match smoothly onto one another:

- For $N \geq M_{Pl}^2/(f^2 \beta) \equiv N_{NM}$, the de Sitter radius is smaller than the bubble wall thickness $\sim fM^{-2}$ and the relevant instanton is the Hawking-Moss solution (14). In this regime, $B(N) \propto N^{-2}$.
- For $N_{CD} < N < N_{NM}$, where $N_{CD} \equiv M_{Pl}^2 \beta/f^2$, the relevant instantons are of the
Coleman-De Luccia type \( (14) \) and the thin wall approximation becomes increasingly accurate. The bubbles are in the scaling regime described by Parke \( (15) \), where the bubble radius is controlled by gravitational effects. In this regime, \( B(N) \propto N^{-3/2} \).

- As \( N \) falls below \( N_{CD} \), the bubble radius becomes much smaller than the de Sitter radius and the instantons are well-approximated by the flat spacetime bubble solution. Although gravitational effects increase the action by only a small factor in this regime, the correction is very important because \( B_0 \) is so large. The leading gravitational correction is given by

\[
B(N) = B_0 \left( 1 - \frac{3}{2} \frac{(V_N + V_{N-1})T^2}{M_P^2 \epsilon^2} \right),
\]

where the flat spacetime bubble action \( B_0 = \frac{27}{8} \pi^2 T^4 / \epsilon^3 \), with \( T \) the wall tension. In a cosine potential this is \( 8M^2 f \). \( B_0 \) is an enormous number \( \sim 10^{110} \) for plausible parameters \( f \sim 10^{14} \) GeV, \( \beta \sim 0.1, M \sim 10^{-3} \) eV. The gravitational correction causes the bubble action to decrease linearly with \( N \) in this final regime. Thus, as \( N \) approaches zero from above, the time spent at vacuum density \( V_N \) scales parametrically as \( \exp \left( -B_0 (N/N_{CD}) \right) \) where \( N_{CD} \) is given above. For example, with our chosen parameters the time spent at the last positive value of the vacuum energy density is more than \( 10^{10^{110}} \) times longer than the entire time spent before it.

The whole process ends when the field \( \phi \) tunnels through to negative potential energy. Then, the negative potential causes the space within the bubble to collapse in a time of order one Hubble time. (For this reason, it makes no difference if the field could have tunneled further downhill or not since the region will collapse before it tunnels further downhill.) Space outside the bubble continues to expand from cycle to cycle, so there always remain regions with positive cosmological constant. Hence, the relaxation process we have described naturally leads to a universe which is overwhelmingly likely to possess a small positive cosmological constant, in agreement with observation.
Despite its attractive features, the proposal proves to be fatally flawed in a standard big bang cosmology setting, as Abbott himself pointed out, due to the ‘empty universe problem.’ Each time the universe is caught in a minimum, it undergoes a period of inflation that empties out all matter and radiation. When a bubble is nucleated, its interior is nearly empty, too. At most, it contains an energy density $\epsilon$ and even if this is turned entirely into matter and radiation it is far too low to make planets, stars or galaxies. In fact, whatever density does lie within the bubble is rapidly diluted away by the next bout of de Sitter expansion. The process continues; new bubbles are formed within the old but at each stage, the energy density is far too small to explain the observed universe. In effect, the problem is that the relaxation process is too slow for standard big bang cosmology, so that the universe is empty by the time the cosmological constant reaches the requisite value.

3 Cyclic Model with Dynamical Relaxation

With this thought in mind, we now turn to the cyclic model of the universe (8,9). According to the cyclic picture, the big bang is collision between orbifold planes (branes) along an extra dimension of space, as might occur in heterotic M-theory (18). A weak, spring-like force draws the branes together at regular intervals, resulting in periodic collisions that fill the universe with new matter and radiation. After each collision, the branes separate and start to re-expand, causing the matter and radiation to cool and spread out. Eventually, the matter and radiation become so dilute that the potential energy associated with the inter-brane force takes over.

In the low energy four-dimensional effective theory, the inter-brane distance can be described by a modulus field $\psi$ which moves back and forth along its effective potential. When the branes are far apart, the potential energy density is positive and acts as dark
energy, causing the branes to expand at an accelerating rate and diluting away the matter and radiation created at the bang. At the same time, the force draws the branes together, causing the potential energy density to decrease from positive to negative. As the branes accelerate towards one another, their expansion slows.

Ripples in the branes caused by quantum fluctuations are amplified by the inter-brane force as the branes approach one another into a scale-invariant spectrum of growing energy density perturbations. The branes remain stretched out, though, and any matter and radiation within them remains dilute. So, after a period of a trillion years or so, the nearly empty branes collide, creating new matter and radiation and initiating a new cycle of cosmic evolution. In dealing directly with the big bang singularity, the cyclic scenario poses new challenges to fundamental theory, and some aspects are still being actively debated (20, 21, 22, 23, 24). Here, we shall assume the cyclic picture is valid.

Now let us suppose we add to this story the axion-like field $\phi$ and the associated hidden gauge sector, as entities on one of the two branes. Surprisingly, although it was not invented for this purpose, the cyclic model has just the right properties to make Abbott’s mechanism viable, leading to the prediction we have emphasized: a small, positive cosmological constant. Four features inherent to the cyclic model play a key role in rendering the combined model viable:

First and foremost, the cyclic model regularly replenishes the supply of matter and radiation, instantly solving the ‘empty universe problem.’ Brane collisions occur every trillion years or so, an infinitesimal time compared to the eons it takes the universe to tunnel from one minimum to the next. So, between each step down the washboard potential, the universe undergoes exponentially many cycles. Each bubble that is nucleated fills with matter and radiation at the cyclic reheat temperature $T_{\text{reheat}} \sim 10^8 \text{ GeV}$ or so (21), at each new brane collision. The value of $\epsilon$ is far smaller than the energy scale associated
with the collision so the washboard potential has little effect on reheating. Instead, it controls the low energy density, de Sitter like phase of each cycle, ensuring the cycling solution is a stable attractor \[ (7) \]. For \( f >> T_{\text{reheat}} \), \( \phi \) is only weakly coupled to the matter and radiation, and the reheating process does not significantly affect the evolution of \( \phi \).

A second essential element of the cyclic model is the orbifold (brane) structure. If \( \phi \) were coupled to the usual 4d Einstein metric, its kinetic energy would be strongly blueshifted during the periods of Einstein-frame contraction. Instead of proceeding in an orderly manner down the washboard potential, it would be excited by the contraction and jump out of the minimum, accelerating off to infinity as the crunch approached. In the cyclic model the behavior is quite different because \( \phi \) couples to the induced metric on the brane, not the 4d effective Einstein-frame metric. The brane expands exponentially from cycle to cycle and never contracts to zero; only the extra dimension that separates the branes does that. Consequently, the kinetic energy of \( \phi \) is red shifted and diluted during every cycle, even during the phases when the extra dimension (and the Einstein-frame 4d effective scale factor) contracts to zero. Thus \( \phi \) remains trapped in its potential minimum for exponentially long periods until the next bubble nucleation occurs.

The reheating of the universe at the beginning of each cycle also does not excite \( \phi \) because it is so weakly coupled. In fact, by causing the expansion to decelerate and hence suppressing the de Sitter fluctuations in \( \phi \), the matter and radiation actually decrease the nucleation rate. The majority of tunneling events occur during vacuum energy domination, which is the longest phase of each cycle.

A third advantageous feature of the cyclic model is that, because the homogeneity and isotropy of the universe and the generation of density perturbations are produced by very low-energy physics, there is no inflation and, hence, no need to tune the relaxation to be slow and then fast.
A fourth critical aspect of the cyclic model is that dark energy acts as a stabilizer. By diluting the density of matter and radiation and any random excess kinetic energy of the branes produced at the previous bounce, the dark energy ensures that the cycling solution is a stable attractor \( [9] \). When we add the washboard potential, the dark energy density depends on \( V(\phi) \). The value decreases by \( \epsilon \) each time a bubble is nucleated. As long as the dark energy density is positive, the cyclic solution remains a stable attractor. Once the sum becomes negative, the periodic cycling comes to an end. Most likely, the interior of the negative potential energy bubble collapses into a black hole, detaching itself from the universe outside it and ending cycling in that small patch of space; but the rest of the universe continues to cycle stably.

Putting these ideas together, the cyclic model and Abbott’s mechanism are merged into a new scenario that significantly modifies both. In the combined picture, there are two fundamental timescales that govern the long-term evolution of any patch of universe: the cycling time \( \tau_{\text{cycle}} \) and the time it takes to nucleate bubbles, \( \tau_N \). The latter increases exponentially as the universe tunnels from large \( N \) towards \( N = 0 \), and during each of the stages we have described, \( \tau_N \) is exponentially greater than \( \tau_{\text{cycle}} \). So, for each jump in \( \phi \) the universe undergoes many cycles and many big bangs. When \( V_N \) is large, the vacuum energy density dominates the universe at an earlier point in the cycle, before matter has a chance to cool and form stars, planets or life. But nothing happens to disrupt the evolution. The universe simply continues cycling as \( \phi \) continues to hop down the potential, each step taking exponentially longer than the one before. Finally, \( V_N \) becomes small enough that structure begins to form. How big \( N \) is before this occurs depends on \( \epsilon \); for our example above, galaxy formation occurs during the last few hundred steps or so. However, exponentially more time and more cycles are spent at \( V = V_0 \) than at any other value.
4 Discussion

We have focused on Abbott’s particular mechanism, but we can extract from this case the conditions that are generally required: (1) a relaxation time much greater than today’s Hubble time; and, (2) a dynamics which collapses or recycles any regions with negative cosmological constant on a much shorter time scale. In our example, the relaxation time increases as the cosmological constant approaches zero, so that the system spends most of the time at the lowest positive value. However, it is also interesting to consider other parameter ranges or other forms for $V(\phi)$, including the pure linear potential invoked in the anthropic model of Ref. (6), which has no local minimum to be fixed. Here the relaxation time decreases as the cosmological constant approaches zero from above. By introducing cycling and restricting attention to the past light cone of any observer, we find that most galaxies are produced when the vacuum density is smaller, but not much smaller than the matter density.

In either example, our result is a universe in which the cosmological constant $\Lambda(t)$ is an ultra-slowly varying function of time $t$ and in which virtually every patch of space proceeds through stages of evolution that include ones in which $\Lambda(t)$ is small enough to be habitable for life. It is interesting to contrast this situation with the anthropic picture, especially versions based on inflationary cosmology, for which the fraction of habitable space is infinitesimally small. All other things being equal, a theory that predicts that life can exist almost everywhere is overwhelmingly preferred by Bayesian analysis (or common sense) over a theory that predicts it can exists almost nowhere.

Although the relaxation time scale is far too slow to be detectable, the general picture we have suggested here can be falsified. First, since it relies on the cyclic model, it inherits the cyclic prediction for primordial gravitational waves (25). Second, one might
for other implications of having an exponentially long time for fields or couplings to evolve. For example, axions in QCD and string theory with $f \gg 10^{12}$ GeV are well-motivated theoretically, but ruled out in conventional inflationary theory because de Sitter fluctuations typically excite the field to a value where its energy density overdominates the universe today (26). Some propose resolving this dilemma, also, using the anthropic principle (27), but, then, the same reasoning suggests that axions should contribute all or most of the dark matter density today (28). In the alternative picture we have presented, though, there is no inflation and axions are never excited. So, finding axions with large $f$ and negligible density would be an embarrassment for the inflationary picture but would fit naturally in the picture outlined here. Similar considerations apply to other solutions to the strong CP problem (30) where a very long relaxation time may be useful.

References and Notes


26. For a recent discussion, see P. Fox, A. Pierce, and S. Thomas, hep-th/0409049.


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