Extended 2d generalized dilaton gravity theories

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We show that an anomaly-free description of matter in (1+1) dimensions requires a correction of
the 2d relativity principle, which is connected to a noncommutativity of 2d Minkowski space that
introduces a 2d Planck length. Then, in order to describe dynamically this noncommutative structure
of the tangent space, we propose to extend the usual 2d generalized dilaton gravity models by a non-
standard Maxwell component, which acts as a quantum correction affecting the topology of space–
time. In addition, we prove that the extended dilaton theories can be formulated as Poisson–Sigma
models based on a nonlinear deformation of the extended Poincaré algebra.

Keywords: quantum gravity, dilaton gravity, two-dimensional models, noncommutativity, nonlinear
symmetries, Sigma models

I. INTRODUCTION

Since the discovery of the accelerated and flat universe, dilaton theories became more likely to describe
the effective theory of gravity in four dimensions than the Einstein–Hilbert one. On the other hand, su-
perstring theory is still afflicted with several theoretical problems and it is too complicated to work with.
Nowadays, 2d dilaton gravity theories are widely regarded as invaluable tools for researching several aspects
of quantum gravity. Indeed, they provide a unitary quantum theory of gravity in a background independent
formulation [1, 2, 3], with a lot of application in black hole physics [4, 5, 6].

In this paper, we show that an anomaly-free description of the elementary relativistic systems in (1+1)
dimensions requires a correction of the 2d relativity principle, which turns out to be connected to a noncom-
mutativity of 2d Minkowski space that introduces a two-dimensional Planck length. Therefore, we propose
a specific class of non-standard dilaton–Maxwell theories, which extends the usual 2d generalized dilaton
gravity models, so that this noncommutative structure of the tangent space can be properly described. The
additional U(1) gauge sector acts as a quantum correction that affects the topology of space–time. Finally,
we prove that this class can be formulated as a family of nonlinear gauge theories [7] based on a deformation
of the extended Poincaré algebra.

This paper is organized as follows. In Sec. II, we will review briefly the quantum anomaly-free 2d
relativistic particle, which will be employed, in Sec. III, to formulate a correction of the 2d relativity prin-
ciple and to probe the noncommutative structure of 2d Minkowski space. In Sec. IV, we will introduce the
extended 2d generalized dilaton gravity theories and show that they can be formulated as Poisson–Sigma
models. Finally, in Sec. V, we will summarize and comment our results, besides pointing out other future
developments.

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II. THE ANOMALY-FREE 2D RELATIVISTIC PARTICLE

Specifically in 1+1 dimensions, a relativistic particle in the absence of Yang–Mills potentials and space–time curvature is still under a certain interaction. We have already proven, in Ref. [8], that this situation is associated with an anomaly that is generated by a non-trivial two cocycle in the second cohomology group of the two-dimensional Poincaré group \( \mathcal{P} \). In the next section, it will be shown that an anomaly-free description of elementary systems like this requires a correction of the 2d relativity principle, according to which the physical laws should be covariant under the centrally extended Poincaré group \( \overline{\mathcal{P}} \).

The extended Poincaré algebra \( \mathfrak{h}_3 \) is given by

\[
[P_a, J] = \sqrt{-h} \epsilon_a^b P_b, \quad [P_a, P_b] = B \epsilon_{ab} I, \quad \text{and} \quad [P_a, I] = [J, I] = 0, \tag{1}
\]

where \( \epsilon^{01} = -\epsilon_{01} = 1 \) and the indices \( a \) and \( b \) are raised and lowered by the metric \( h_{ab} = \text{diag}(1, -1) \), with dimension \( \text{dim}(h_{ab}) = L^{-2} \) and \( h := \det h_{ab} = -1 \). We shall adopt units where only the speed of light is set to \( c = 1 \). The generators of translations are \( P_a := T_a \), and their dimensions are \( L^{-1} \). The generator of Lorentz transformations is \( J := T_3 \), which is dimensionless. The dimension of the central generator \( I := \bar{T}_3 \) is inverse of action \( \text{dim}[h]^{-1} \) and the central charge \( B \) has dimension \( \text{dim}[B] = L^{-2} \times [h] \). The group law \( g^b(\theta^m, \alpha^m, \beta^m) = g(\theta^m, \alpha^m, \beta^m) g(\theta^m, \alpha^m, \beta^m) \) determined by Eq. (1) is given by \( \theta^m_\beta = \theta^m + \Lambda(\alpha^m)_b \theta^m \), \( \alpha^m = \alpha^m + \alpha^m \), and \( \beta^m = \beta^m + (B/2) \theta^m \epsilon_{ab} \Lambda(\alpha^m)^b \theta^m \), where \( \Lambda(\alpha^m)^b \theta^m = \delta^a_b \cosh \alpha + \sqrt{-h} \epsilon^a_c \sinh \alpha \).

For the Jackiw–Teitelboim model and the conformally related string inspired dilaton gravity [9, 10, 11], the dimension specific interaction has already received a gauge theoretic formulation, according to which it would have a geometrical origin, stemming from the volume two form of space–time. We agree with these authors in that this interaction should receive a gravitational interpretation. However, in Ref. [8], our algebraic analysis has shown in contrast that it is caused by a Wess–Zumino (WZ) potential, which is a quantum effect associated with the aforementioned anomaly.

Moreover, when we formulate this interaction for the extended generalized dilaton models, in Sec. IV, it will become clear that the WZ potential is independent of the geometry, being rather connected to the topology of space–time (cf. the comment following Eq. (3)). We are showing below that the anomaly cancellation, for matter in the absence of gravity, induces a noncommutative structure on 2d Minkowski space. Although we consider only the case of a 2d relativistic particle for simplicity, similar arguments should apply to matter fields.

The anomaly-free 2d relativistic particle can be described in Hamiltonian form by

\[
S[q^a, \chi, p_b, \pi, u^m, v^m] = \int_{\tau_1}^{\tau_2} d\tau \left( p_a \dot{q}^a + \pi \dot{\chi} - u^m \phi_m - v^m C_m \right), \tag{2}
\]

where \( \chi \) is an internal gauge degree of freedom with dimension of action and transforming as \( \chi' = \chi + \beta + (B/2) \theta^a \epsilon_{ab} \Lambda^b \phi^c \), which is introduced to neutralize the WZ term, so that the action is rendered invariant under \( \overline{\mathcal{P}} \). This addition produces another primary first class constraint, \( \phi_1 = \pi + 1 \), besides \( \phi_2 = [p_a - (B/2) \pi \epsilon_a q^b] \) (note the minimal coupling to the WZ potential). As a result, the reduced phase–space becomes parametrized only by the position coordinates \( q^a \), independently of any gauge fixation [3].

Therefore, in order to reach the reduced phase–space, we fix the scale of \( \tau \) by imposing the usual canonical gauge condition \( C_2 = \bar{p}(\tau) - \bar{p}(\tau_0) \) \( - B(\tau - \tau_0) \), where \( \bar{p}(\tau) := p_1 - (B/2) \pi q^0 \) is the kinematical momentum. This is a sensible choice, inasmuch as the particle is accelerated by a constant physical force equal to \( B \). In this formulation, the usual condition \( q^0 = \tau \) appears (assuming \( q^0(\tau_0) = \tau_0 \)) as a Hamilton equation, together with \( q^1(\tau) = q^1(\tau_0) + B \Delta \tau / \overline{B} \), where \( \Delta \tau = \sqrt{m^2 + \bar{p}(\tau)^2} - \sqrt{m^2 + \bar{p}(\tau_0)^2} \) is the variation of relativistic energy \( E(\tau) := -[p_0 + (B/2) q^1] \).

On the other hand, the gauge degree of freedom \( \chi \) can be naturally interpreted as the phase of the particle’s wave function, by imposing the canonical gauge condition \( C_1 = \chi - S(q^1, \tau) \), where \( S(q^1, \tau) = S(E(q^1), \tau) \) is the action function of the anomalous sector of the theory, which can be expressed in terms
of the variable $E = Bq^1$ by

$$S(E, \tau) = \frac{(\hat{p}(\tau_0) - 2E + \Delta E)\Delta E}{2B} + \frac{m\Delta t'}{2} + \left( E - \frac{\Delta E}{2} \right)\Delta \tau,$$

(3)

with the proper time interval being $\Delta t' = (m/B)(\text{arsinh}[\hat{p}(\tau)/m] - \text{arsinh}[\hat{p}(\tau_0)/m])$ and $\Delta \tau = \tau - \tau_0$. The Eq. 3 is a solution for the relativistic Hamilton-Jacobi equation associated with the Hamiltonian $H(q^0, q^1, \tau) = Bq^1 - \hat{E}(\tau)q^0$. Note that, in this gauge, the Hamiltonian does not coincide with the total energy, which is given by $E_T = \hat{E}(\tau) + \mathcal{E}_{\text{pot}}$, where $\mathcal{E}_{\text{pot}} = -E/2$ is the potential energy.

The quantum description of the system [3] can be given in terms of the base kets $\{|E\rangle\}$. If the initial state is $|\alpha\rangle = \int_{-\infty}^{+\infty} dE c_E(\tau_0)|E\rangle$, where $c_E(\tau_0)$ is some known complex function of $E$ satisfying $\int_{-\infty}^{+\infty} dE |c_E(\tau_0)|^2 = 1$, then for $\tau > \tau_0$ the state ket will be $|\alpha, \tau; \tau\rangle = \int_{-\infty}^{+\infty} dE c_E(\tau)e^{(iE/h)\Delta \tau}|E\rangle$, where

$$c_E(\tau) = \exp\left[\frac{i(S(E, \tau) - E\Delta \tau)}{h}\right]c_{E-\Delta E}(\tau_0).$$

(4)

III. NONCOMMUTATIVE 2D MINKOWSKI SPACE

The Hamiltonian analysis of the anomaly-free relativistic particle, sketched in the previous section, brings out the unusual situation in which the reduced phase–space is spanned by the position coordinates $q^a$ alone. Exploiting this isomorphism, we discover that the quantum particle lives in a noncommutative 2d Minkowski space. Indeed, calculating the fundamental Dirac brackets associated with the constraints $\{\phi_m, C_n\}$ and applying Dirac’s quantum condition, we can see that the operators corresponding to the components of the position two vector do not commute with each other:

$$[\widehat{q}^a, \widehat{q}^b] = -\frac{ihe^{ab}(\sqrt{-h})^2}{B}.$$  

(5)

This implies a new kind of uncertainty relation, $\langle (\Delta \hat{q}^a)^2 \rangle \langle (\Delta \hat{q}^b)^2 \rangle \geq \hbar^2/(4B^2)$ — with the dispersions of the coordinates given by $\langle (\Delta \hat{q}^a)^2 \rangle = \langle (\hat{q}^a)^2 \rangle - \langle \hat{q}^a \rangle^2$, — which precludes a simultaneous localization of the particle in time and space. In order to probe this quantum space–time, we can examine the evolution of a Gaussian wave packet with initial state specified by $c_E(\tau_0) = \left[1/(\pi^{1/4}\sqrt{B})\right]\exp[-E^2/(2d^2)]$, assuming $\langle \hat{q}^1 \rangle(\tau_0) = \langle \hat{q}^0 \rangle(\tau_0) = 0$. The first remarkable fact that we learn, using Eq. 3, is that the width of the wave packet does not increase with time, since the dispersions $\langle (\Delta \hat{q}^1)^2 \rangle(\tau) = d^2/(2B^2)$ and $\langle (\Delta \hat{q}^0)^2 \rangle(\tau) = h^2/(2d^2) + d^2/(2B^2)$ are constant.

Not less surprising is the existence of a minimum temporal dispersion Gaussian wave packet, with $\langle (\Delta \hat{q}^0)^2 \rangle(\tau) = h/|B|$ and $\langle (\Delta \hat{q}^1)^2 \rangle(\tau) = h/(2|B|)$. Although this is not an exact coherent state, since its uncertainty product satisfies only approximately the condition $\langle (\Delta \hat{q}^1)^2 \rangle \langle (\Delta \hat{q}^0)^2 \rangle \approx \hbar^2/(4B^2)$, it is the Gaussian wave packet that more closely resembles a classical particle or a space–time point. Consequently, we can define a 2d Planck length in terms of the shortest observable time interval:

$$l_P = \sqrt{\frac{\hbar}{|B|}}.$$  

(6)

It is worth mentioning that this noncommutative space–time does not present any discrete structure in the absence of curvature, since the position coordinates $q^a$ have been quantized by unbounded operators with continuous spectra [8]. Moreover, although the time coordinate $q^0$ is involved in the commutation relations of Eq. 5, the particle theory defined by Eq. 2 is perfectly unitary in the sense of Ref. [13], due
to Eq. (4). Nevertheless, causality violations are expected at Planck scale, since the quantum description of the system employs a time operator.

Another important point concerns Poincaré invariance. Canonical noncommutativity relations like Eq. (5) usually indicate Lorentz symmetry breaking. However, we learned in Sec. III that the Poincaré transformation rules between inertial observers are rather deformed in the sector of translations (see Eq. (1)). In fact, the dispersion relation $E_T = -p_0$ is obviously extended Poincaré invariant and tends to $E_T = \sqrt{m^2 + p^2(n)}$ in the limit when $B \rightarrow 0$ (at subPlanckian length scales), so that there is no preferred observer. Hence, this dimension specific type of deformation contrasts with the four-dimensional case, inasmuch as it affects all energy scales below the Planck scale. On the other hand, the fact that the laws relating extended Poincaré frames involve the observer-independent length scale $\tilde{l}_P$, given by Eq. (6), is familiar from doubly-special relativity theories [14], thus providing a correction to the relativity principle in (1+1) dimensions.

IV. EXTENDED DILATON GRAVITY THEORIES

The corrected relativity principle formulated above expresses the fact that the relevant dynamical group in two dimensions is the extended Poincaré group $\bar{\mathcal{P}}$. Hence, the 2d generalized dilaton gravity theories are expected to be extended by an additional $U(1)$ gauge sector, which determines the noncommutative structure of the tangent space and provides some fundamental gravitational interpretation for the central charge. To accomplish such an extension, we propose the following action, which can be written in first order employing Cartan variables:

$$S^{(\text{EF OG})} = \frac{1}{G} \int_M \left[ \eta_a D e^a + \eta_2 d\omega + G \eta_3 da + \epsilon \left( V(\eta^a \eta_a, \eta_2) + \frac{BG \eta_3}{W(\eta_2)} \right) \right],$$

where $M$ is the space–time manifold, $G$ is the 2d gravitational constant of dimension $[G] = [\hbar]^{-1}$, $\epsilon = -(\epsilon^{a b} a^b)/2$ is the volume two form and $D e^a = d e^a + \sqrt{-\hbar} e^a \omega b$. The zweibeine $e^a$ and the dilaton $\eta_2$ are dimensionless, the spin connection $\omega$ has dimension $[\omega] = L^{-1}$, and the Lagrange multipliers $\eta_a$ (related to the torsion of space–time) have dimension $[\eta_a] = L^{-1}$.

The extension introduces the WZ potential $a$ of dimension $[a] = L^{-1} \times [\hbar]$ — which is responsible for the dimension specific interaction described previously — and a fourth dimensionless Lagrange multiplier $\eta_3$, so that the scalar field $BG \eta_3$ describes a dynamical cosmological constant. Besides, it brings a new fundamental physical constant $B$ of dimension $[B] = L^{-2} \times [\hbar]$, which corresponds to the conserved $U(1)$ charge and shall be interpreted according to Eq. (6). Provided there are no matter couplings to $a$, the field equation resulting from its variation implies $\eta_3 = 1$, since the central charge $B$ has been absorbed in the WZ potential. Hence, the cosmological constant is given by $\lambda_0 = BG$ and it has dimension $[\lambda_0] = L^{-2}$.

The dynamical content of the theory is determined by the Lorentz invariant functions $W(\eta_2)$ and $V(\eta^a \eta_a, \eta_2)$. The former allows a non-minimal coupling of the additional gauge sector to the geometric one — with the minimal coupling corresponding to the choice $W(\eta_2) = 1$ — and the latter has the form $V(\eta^a \eta_a, \eta_2) = U(\eta_2)[(\sqrt{-h} \eta^2 \eta_a)/2] + V(\eta_2)$. They allow the integration of two Casimir functions, which parametrize all the solutions for some specific model. The first one has the meaning of energy and is given by $C_1 = [(\sqrt{-h} \eta^2 \eta_a)/2] \epsilon^{Q(\eta_2)} = Z(\eta_2, \eta_3)$, where $Q(\eta_2) = \int_{\eta_2} U(y) dy, Z(\eta_2, \eta_3) = \int_{\eta_2} \epsilon^{Q(y)} V(y, \eta_3) dy$, and $V(\eta_2, \eta_3) = V(\eta_2) + [BG \eta_3]/W(\eta_2)$. The other is $C_2 = \eta_3$, which has already been interpreted above.

We emphasize that the extended generalized theories (EF OG’s) are not trivially equivalent to the ordinary dilaton gravity models; they are legitimate extensions of the latter. In fact, in the absence of matter and using $\eta_3 = 1$ it is straightforward to show that the field equations derived from Eq. (7) encompass those of the unextended theories with $V(\eta^a \eta_a, \eta_2; \lambda_0) = U(\eta_2)[(\sqrt{-h} \eta^2 \eta_a)/2] + V(\eta_2, 1)$, together with the topological equation

$$da + \frac{B \epsilon}{W(\eta_2)} = 0.$$  

(8)
Concerning the field equations derived from Eq. (7), it is important to note that, in general, the WZ potential \( a \) will not be redundantly expressed in terms of the geometric variables \((e^a, \omega)\) \[^{10}\], but its dual field strength \( *da = -B/W(\eta_2) \) will be determined by the dilaton. Moreover, although Eq. (6) can always be integrated locally, it will generally require an atlas containing more than one chart for the space–time manifold \( M \), due to the singularities that do occur in the \( a \) field \[^{11}\]. Indeed, assuming that \( M \) is contracted to a bounded manifold with the WZ potential vanishing on the boundary, it is possible to keep the dilaton everywhere non-singular, so that the integral over \( M \) of the second term on the left side is made a well-defined finite non-vanishing quantity, while the first term integrates to zero, if \( a \) is non-singular.

It has already been realized that non-standard 2d dilaton–Maxwell theories constitute a powerful generalization of the usual dilaton models, as a recent application to the exact string black hole exemplifies \[^{15}\]. As for the EFOG’s, they include some particular dilaton-Maxwell gravity models that have been explored previously, such as the conformally transformed string inspired model \[^{2, 10, 11, 12}\], the extended de Sitter gravity \[^{10}\], and the Kaluza–Klein reduction of Chern–Simons gravity (featuring kink solutions) \[^{16, 17}\].

More importantly, they provide systems that have not been addressed in the literature yet. A particularly interesting new class of models represents an extension of the \( a - b \) family \[^{18}\], presenting black hole solutions. It consists of EFOG’s with potentials of the form \( U(\eta_2) = -a/\eta_2 \) and \( V(\eta_2, \eta_3) = -[BG\eta_3/(\eta_2)^{a+b}]/2 \), where the global scale factor \( E \) has been identified with \( E = |B|G \) and thus acquires a fundamental role in this context. The extensions of the most important models are included in this case: the extended spherically reduced gravity \[^{19, 20, 21}\], the extended Callan–Giddings–Harvey–Strominger model \[^{22}\], and the extended Jackiw–Teitelboim one \[^{23, 24}\]. In addition, the EFOG’s allow the description of extended models with torsion, like the extended Katanov–Volvich one \[^{25, 26}\], for which \( V(\eta_2, \eta_3) = \beta(\eta_2)^2/2 - (BG\eta_3)/2 \), where \( \alpha \) and \( \beta \) are coupling constants.

A parenthetical remark concerns the aforementioned identification, involving the global scale factor \( E \), and illuminates the physical interpretation of the central charge. For the 2d dilaton model which results from the spherical reduction of Einstein’s theory of gravity in four dimensions, we have the relation \( G = EG_N \), where \( G_N \) is the 4d Newton constant. But, since \( E = |B|G \), we deduce somewhat surprisingly that \( |B| = G_N^{-1} \). Substituting the last equation in Eq. (6), we can see that the 2d Planck length \( l_P \) coincides with the four-dimensional one (in units with \( c = 1 \)).

The theories described by Eq. (7) are particular expressions of the more general Poisson–Sigma models \[^{18, 27}\], which are nonlinear generalizations of the BF theories that have recently found application in superstring theory. In fact, performing the identifications \( X^I \equiv (\eta^a, \eta_2, G\eta_3) \) and \( A_I \equiv (e_a, \omega, a) \) (with \( I \in \{1, 2, 3, 4\} \)), we can see that Eq. (7) can be written as

\[
S^{(PSM)} = \frac{1}{G} \int_M \left( A_I \wedge dX^I + \frac{1}{2} P^{IJ}(X)A_I \wedge A_J \right),
\]

in which \( P^{IJ}(X) = \{X^I, X^J\} \) is the Poisson tensor associated — by the correspondence \( X^I \leftrightarrow (P_a, J, I) \) — to the nonlinear deformation of the extended Poincaré algebra given by

\[
[P_a, J] = \sqrt{-h} e_a^b P_b, \quad [P_a, P_b] = \epsilon_{ab} \left( V(P^a P_a, J) + \frac{BI}{W(J)} \right), \quad \text{and} \quad [P_a, I] = [J, I] = 0,
\]

where \( V(\eta^a\eta_2, \eta_2) + [BG\eta_3]/W(\eta_2) \) is the potential that determines a model in Eq. (7).

The generalized Jacobi identity \( P^{IJ}_K P^{KL} + \text{cycl}(I, J, L) = 0 \) (where the subscript “, \( K \)” stands for the partial derivative \( \partial/\partial X^K \)) can be straightforwardly checked for Eq. (10). Moreover, for most physically interesting models \( P^{IJ}(X) \) will be a polynomial, so that the nonlinear algebra of Eq. (10) will be a finite \( W \)-algebra. Nevertheless, for each case it should be tested whether Eq. (10) has the structure of a reduced Kirillov–Poisson algebra \[^{28}\]. Note that, for \( W(\eta_2) = 1 \) and in the limit when \( V \rightarrow 0 \), the algebra of Eq. (10) tends to \( i_2 \) (see Eq. (11)) just as Eq. (9) becomes the BF theory describing the zero curvature limit of the extended de Sitter gravity \[^{10, 12}\].
V. CONCLUDING REMARKS

We have seen that the interaction between a 2d relativistic particle and the WZ potential is a quantum effect tied to the non-triviality of the second cohomology group of the two-dimensional Poincaré group \( P \). Moreover, an anomaly-free description of elementary systems like this requires a correction of the relativity principle in two dimensions, which is connected to a noncommutative structure of 2d Minkowski space that introduces an observer-independent length scale \( l_P \); see Eqs. (5), (6).

It is important to emphasize that this noncommutativity comes out as a result in our formulation, rather than being assumed a priori. Further, although we have only considered point particles in order to establish our argument, the same feature is also expected to show up in the case of matter fields, inasmuch as it stems from a general property of the dynamical symmetries: the connection between the quasi-invariance of a Lagrangian and the second cohomology group of its dynamical group.

These facts motivated us to add a non-standard Maxwell component to the usual dilaton gravity models, so that they could be formulated as nonlinear gauge theories based on a deformation of the extended Poincaré algebra; see Eqs. (7), (9), (10). In our formulation, a 2d Planck length \( l_P \) appears related to the cosmological constant \( \lambda_0 = BG \), through a conserved U(1) charge \( B \), and the WZ potential \( a \) is connected to the topology of space–time by Eq. (8).

It remains to examine the coupling of matter fields to the extended generalized dilaton gravity theories and to probe the global structure of the solutions to the field equations derived from the latter. We are particularly concerned with the extension of the noncommutativity to the space–time manifold and its topological consequences. A Planck length is an unusual feature in two dimensions, which introduces a natural ultraviolet cut-off that may lead to space and/or time discretization under certain conditions still to be researched. Our results shall be published soon elsewhere.

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