Defect Formation with Bulk Fields

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Abstract

It has recently been realized that brane-antibrane annihilation (a possible explanation for ending inflation) may result in defect formation, due to the dynamics of the tachyon field. Studies of this possibility have generally ignored the interaction of the brane fields with fields in the bulk; recently it has been argued [1] that interactions with bulk fields suppress or even eliminate defect formation.

To investigate the impact of bulk fields on brane defect formation, we construct a toy model that captures the essential features of the tachyon condensation with bulk fields. We study the structure of defects in this toy model, and simulate their formation and evolution on the lattice. We find that, while the energetics and interactions of defects are influenced by the size of the extra dimension and the bulk-brane coupling, the bulk-brane coupling does not prevent the formation of a defect network.
I. INTRODUCTION

The early universe is the most likely source for experimental input into physics at very high energies. Immediately after the Big Bang, energies as high as the Grand Unified (GUT) scale may have been explored. It is hoped that signatures of the processes that took place at that time will be visible in the universe today, for instance, in primordial perturbations or other relics of the very early universe.

Recently models for the early universe have become less ad-hoc and more inspired from String Theory. The brane world model is particularly attractive, since it contains the ingredients for a successful inflationary universe. The separation of a brane and an antibrane can act as an inflaton field, and the interaction of the pair allows one to calculate the corresponding inflaton potential. The collision and annihilation of the brane-antibrane pair provides a natural mechanism for ending inflation and reheating the universe.

One important prediction of a large class of brane inflation models is the formation of topological defects, typically networks of cosmic strings—it has been argued that brane-anti-brane annihilation in the very early universe leads to the formation of cosmic strings only. The formation of topological defects in brane-anti-brane annihilation has been well studied in String Theory and Field Theory, and represents a particularly testable potential signature for these models, since the defect network can have quite different properties from that of conventional cosmic string networks.

The discovery of explicit mechanisms for the stabilization of the compact extra dimensions makes this class of models even more appealing for phenomenology. These types of models have been mainly used to describe the inflationary universe, and the predictions these models make for the cosmological parameters can already be tested against the experimental data. At the same time, the mechanism that stabilizes the compactification poses additional challenges since it can make the topological defects unstable.

The formation of cosmic strings in brane-anti-brane annihilation has also been studied in the 4D low-energy effective theory, by analyzing the dynamics of the fields living in the world-volume of the brane. One usually considers only the complex tachyon field or can include the $U(1)$ gauge field that couples to the tachyon. Bulk fields are usually neglected. However, it has been argued that the inclusion of the bulk fields in the process of defect formation substantially changes the dynamics of defect formation, leading to a strong suppression of the defect formation rate. This is because the topological defects are themselves branes, so they couple to fields described by closed string modes, namely the dilaton and the Ramond-Ramond fields which propagate in the entire bulk space-time.

The energy of the gradients of the RR field along the compactified extra dimensions is large, so the formation of defects is strongly suppressed. More precisely, it was argued that deSitter quantum fluctuations cannot account for the formation of lower dimensional branes since the typical energy of the quantum fluctuations is of order the Hubble constant $H$ while the gradients of the fields spreading in the extra dimensions of size $R$ are of the order $1/R$. During brane inflation $H \ll 1/R$, so the formation of lower dimensional branes is suppressed.

In this paper we try to analyze the mechanism of defect formation when bulk fields are involved. We first construct a toy model which captures the essential features of the full 10-dimensional model that are responsible for the formation of the topological defects. Using this toy model we study the classical evolution of the brane and bulk fields, so the deSitter quantum fluctuations are responsible only for seeding perturbations which grow during the
subsequent evolution. The final configuration of the bulk field gradients is a result of this classical evolution. The energy of the system comes from tension of the brane-anti-brane pair, which can be modeled as the potential energy of the tachyon field sitting at the top of the potential. In our toy model this corresponds to the potential energy of a complex scalar living in the worldvolume of the brane-anti-brane pair.

We will present the SUGRA Lagrangian for the fields which live both inside the worldvolume of the brane-anti-brane system and the bulk, and we will construct a toy model with the same physical features. We will evolve the equations of motion of this toy model on a lattice and observe the formation of the defects. We will analyze the effect the coupling to the bulk fields has on the formation process as well as the characteristics of the defects once they are formed.

In section 2 we present the relevant supergravity Lagrangian and the toy model we use. In section 3 we analyze the equations of motion of the toy model and discuss the energetics for a defect-anti-defect pair. In section 4 we describe the results we obtained from lattice simulations. Finally the conclusions are presented in section 5. Details of the lattice implementation of our toy model are left to a technical appendix.

II. TACHYON CONDENSATION WITH BULK FIELDS

The usual approach to defect formation during tachyon condensation is to take into account only the fields that live in the worldvolume of the decaying non-BPS brane or brane-anti-brane pair. This approach seems to be motivated by the fact that the resulting lower-dimensional branes formed during the decay are localized inside the worldvolume of the parent brane. However, the final state defects are themselves D-branes, and therefore couple to bulk RR-fields. One should include the effects of these fields in the defect formation process.

When studying the formation of topological defects by the brane worldvolume fields only, one usually considers a single non-BPS brane with the action:

\[
S = -T_p \int d^{p+1}x \ e^{-\phi} V(T) \sqrt{\det (P [G_{ab} + B_{ab}] + 2\pi\alpha' [F_{ab} + \partial_a T \partial_b T])}, \tag{2.1}
\]

where \( P [G_{ab} + B_{ab}] \) represents the pull-back of the bulk metric and NS-NS two-form field on the brane and \( F_{ab} \) is the field strength of the Abelian world-volume gauge field \([13, 14, 15]\). This action was used extensively to study the evolution of the tachyon field in various special settings. The most common one is the spatially uniform field in a Friedmann-Robertson-Walker universe. In this case one usually sets the NS-NS two-form field and the brane gauge field to vanish everywhere, and study the time evolution of the tachyon field and scale factor of the universe. The uniform field does not lead to the formation of defects, it behaves like a pressureless fluid known as “tachyon matter” \([16]\).

When studying the formation of topological defects during tachyon condensation, one usually chooses a flat metric, sets the NS-NS field and the brane gauge field to vanish everywhere, but chooses a tachyon field that is both time and space dependent. The equation of motion for the tachyon field,

\[
\frac{\partial_a (\sqrt{-g} g^{\alpha\beta} \partial_\beta T)}{\sqrt{1 + \partial_a T \partial^a T}} - \frac{\sqrt{-g} g^{\alpha\beta} \partial_\beta T \partial_a (\partial_\mu T \partial^\mu T)}{2 (1 + \partial_a T \partial^a T)^{3/2}} - \frac{V'(T)}{V(T)} \frac{\sqrt{-g}}{\sqrt{1 + \partial_a T \partial^a T}} = 0, \tag{2.2}
\]
is non-linear, and one usually solves the equation around a point where \( T = 0 \). There
the space and time dependence can be approximated by a linear space profile with a time-
dependent slope, \( T (t, x) \simeq u(t) x \), and the resulting equation for \( u(t) \) can be solved. The
solution becomes singular in finite time, the occurrence of the singularity marking the forma-
tion of the topological defect. This result confirms the String Theory calculation in which a
linear tachyon profile \( T (x) = u x \) reproduces the correct tension of a codimension one brane
in the limit \( u \to \infty \).

The brane gauge field is usually included in the form of a uniform background electric
field \[17] or a constant gauge potential (Wilson line) \[18\]. The case where both the brane
gauge field and a worldvolume scalar field other than the tachyon are included was studied
in \[19\].

This action is highly non-linear and in order to perform a lattice regularization we prefer
an expression in which the square-root is expanded to quadratic order in the field strengths.
Also for a single non-BPS brane the tachyon is a real scalar which cannot be minimally
coupled to the world-volume gauge field.

It is therefore more convenient to consider the action for a brane-anti-brane pair. In the
case of a \( D9 - \overline{D9} \) pair the expanded action involving the complex tachyon coupled to the
gauge fields living inside each brane is given in Ref. \[20\]:

\[
S = 2T_{D9} \int d^{10} x e^{-\phi} e^{-2\pi \alpha' T F} \left[ 1 + 8\pi \alpha' \ln (2) D^\mu T D_\mu T + \frac{(2\pi \alpha')^2}{8} (F_\mu^+)^2 + \frac{(2\pi \alpha')^2}{8} (F_\mu^-)^2 + \frac{\beta \alpha'^2}{8} (F_\mu^+ - F_\mu^-)^2 \right].
\]

The two gauge fields live in the worldvolume of each brane and the tachyon field couples
only with one linear combination:

\[
D_\mu T = \partial_\mu T - (A^+_\mu - A^-_\mu) T. \tag{2.4}
\]

The brane fields couple with the Ramond-Ramond (RR) bulk fields through the Chern-
Simons coupling also given in Ref. \[20\]:

\[
S^{DD}_{RR} = T_{D9} \int C \wedge \text{Str} e^{2\pi \alpha' F}. \tag{2.5}
\]

One can expand the exponential above, and the leading order coupling between the brane
fields and the bulk RR field involves the same linear combination of brane gauge fields that
couples to the tachyon:

\[
S^{DD}_{RR} = 2\pi \alpha' T_{D9} \int C_{p-1} \wedge (F^+ - F^-) \tag{2.6}
\]

We see that the orthogonal linear combination, \( A^+_\mu + A^-_\mu \) does not couple to any other fields,
so we will drop it from the action. Including the kinetic terms for the RR field, the dilaton
and the metric, the 10-dimensional action describing the decay of the brane-anti-brane pair
is:

\[
S = \frac{1}{2\kappa_{10}^2} \int \sqrt{-G} d^{10} x \left[ e^{-2\phi} \left( R + 2 (\nabla \phi)^2 \right) - \frac{1}{2 (p)!} F_p^2 \right] - 2T_{D9} \int d^{10} x e^{-\phi} e^{-2\pi \alpha' T F} \left[ 1 + 8\pi \alpha' \ln (2) D^\mu T D_\mu T + \frac{(2\pi^2 + \beta) \alpha'^2}{8} (F_\mu^+ - F_\mu^-)^2 \right], \tag{2.7}
\]
where $F_p$ is the corresponding field strength for the potential $C_{p-1}$, $F_p = dC_{p-1}$. The $C_{p-1}$ will be the only field we consider here, we will not include the dilaton and the metric in the toy model we consider.

III. A TOY MODEL

We want to create the simplest toy model which captures the important features of the formation of defects with bulk fields included, and at the same time is amenable to a lattice regularization. This will allow us to follow the evolution to see whether defects form, and how the interactions with the bulk fields affect that formation. We want to include the minimal field content that will allow us to study the formation of the defects, so we will not include the metric and dilaton fields present in the full 10-dimensional model.

Since we want to study the formation of codimension 2 defects the brane must have at least 2 spatial dimensions, so we will choose a 2+1 dimensional brane. The defects that form are 0+1-dimensional and the corresponding bulk field that couples to their worldvolume must carry a single index. Therefore we will have a vector field living in the bulk. This field corresponds to the $C_{p-1}$ RR field in the 10-dimensional model. We choose the bulk to have the minimal space dimensionality, 1 space dimension more than the brane. Inside the brane we put the same field content as in the full 10-dimensional model, an Abelian gauge field corresponding to the linear combination $F^+_{\mu\nu} - F^-_{\mu\nu}$ and a complex scalar field corresponding to the complex tachyon $T$.

Therefore the model consists on a complex scalar $\phi$ charged under a gauge field $A_\mu$ living in a 2+1 dimensional “brane” (see Table I). As in the full String Theory case, the fields have a Chern-Simons coupling to the 3+1 dimensional bulk gauge field $C_\mu$. For our purposes it is sufficient to keep only the coupling of the brane gauge field to the bulk field and ignore the coupling of the scalar, which is higher order in the string coupling, as we discuss in Appendix A.

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<td>Bulk</td>
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TABLE I: Correspondence between the field content and dimensionalities for the two models.

We choose the complex scalar to be an Abelian Higgs field instead of a tachyon, since the tachyon field develops singularities in finite time and the simulation can only run until the first defect forms. The Higgs captures the essential features of the tachyon, but is well behaved after the defects have formed, so we can follow the evolution. The action of the
The model is:

$$\mathcal{L} = \int_{\mathcal{M}_3} dx^2 dt \left[ -\frac{1}{4g_{\text{brane}}^2} F^2 - D_\mu \phi D^\mu \phi^* - V(\phi) \right] - \frac{c_{\text{cs}}}{2} \int_{\mathcal{M}_3} F \wedge C + \int_{\mathcal{M}_4} dx^3 dt \left[ -\frac{1}{4g_{\text{bulk}}^2} H^2 \right].$$

(3.1)

We denote by $F = dA$ the field strength of the field $A$ and by $H = dC$ the field strength of the field $C$. The scalar covariant derivative is $D_\mu = \partial_\mu - iA_\mu$. One can easily check that the action is invariant under gauge transformations for each one of the fields $A$ and $C$. We take the standard symmetry breaking form for the scalar potential,

$$V(\phi) = \lambda (\phi^* \phi - v^2/2)^2.$$  

(3.2)

It is straightforward to derive the equations of motion which follow from this action. The simplest one to derive is the equation of motion for the scalar field since the scalar field $\phi$ does not couple directly with the bulk field:

$$\partial_\mu \partial^\mu \phi - 2iA^\mu \partial_\mu \phi - A_\mu A^\mu \phi - i\phi \partial^\mu A_\mu + \frac{\partial V}{\partial \phi^*} = 0.$$ 

(3.3)

One usually chooses the Lorentz gauge $\partial^\mu A_\mu = 0$ to further simplify the equation above. This can be convenient when solving for a static, isolated defect, but is not the most convenient one when performing lattice simulations. The equations of motion for the gauge fields are the usual ones with an extra term coming from the Chern-Simons interaction:

$$\partial_\mu F^{\mu\nu} + ig_{\text{brane}}^2 (\phi D^\nu \phi^* - \phi^* D^\nu \phi) + c_{\text{cs}} g_{\text{brane}}^2 \frac{\epsilon^{\alpha\beta\nu}}{2} H_{\alpha\beta} = 0,$$

(3.4)

$$\partial_\mu H^{\mu\nu} - c_{\text{cs}} g_{\text{bulk}}^2 \frac{\epsilon^{\alpha\beta\nu}}{2} F_{\alpha\beta} \delta(z) = 0.$$  

(3.5)

A. Vortex defects

At vanishing Chern-Simons interaction, $c_{\text{cs}} = 0$, these equations support a topological defect, the Nielsen-Olesen vortex [21]. In any nonsingular gauge, choosing cylindrical coordinates $(\rho, \theta, z)$ centered on the vortex, the phase of the scalar field $\phi$ winds by $2\pi n$, $n$ an integer, in going around the vortex. At large radii, the scalar rests in its broken phase minimum $|\phi|^2 = v^2/2$, and the gradient energy which the phase change would give rise to is exactly canceled by a gauge field, which in a particularly convenient gauge takes the value,

$$\sqrt{2} \phi(\vec{r}) = ve^{in\theta}, \quad A_\theta(\vec{r}) = \frac{n}{\rho}.$$  

(3.6)

Close to the core of the defect, the gauge field takes a smaller value and the scalar field leaves the vacuum manifold, with $\phi = 0$ at the exact vortex center. The defect core costs

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1 Here and throughout we use the $[-+++]$ metric convention and geometrical normalization for gauge fields, so covariant derivatives are of form $D_\mu = \partial_\mu - iA_\mu$. The scalar field charge is chosen to be 1. The mass dimensions of the parameters and fields appearing in the Lagrangian are, $[A] = [C] = 1$, $[\phi] = 1/2$, $[g_{\text{bulk}}^2] = 0$, $[g_{\text{brane}}^2] = [\lambda] = 1$, and $[c_{\text{cs}}^2] = 0$. 


finite energy and carries a net magnetic flux of $2\pi n$. Far from the vortex, the fields are locally in vacuum and all excitations are massive; so the mutual interactions of two vortices or a vortex-antivortex pair are exponentially suppressed with separation.

Examining Eq. (3.4) shows why these conclusions will change somewhat at nonzero values for the Chern-Simons coupling $c_{cs}$. Working in the same gauge, we see that only $F_{\rho \theta}$ is nonzero, so only $C^0$ is sourced by the Chern-Simons term. Its equation of motion is,

$$\nabla^2 C^0 - c_{cs} g_{\text{bulk}}^2 F_{12} \delta(z) = 0. \quad (3.7)$$

The component $F_{12} = F_{\rho \phi}$ acts as a surface charge density sourcing the bulk field.

Our goal is to try to understand the structure and energetics of an isolated defect, or a widely separated defect anti-defect pair, when this Chern-Simons interaction is taken into account. As a byproduct, we will determine the force between defects, which will no longer fall off exponentially. The force will be attractive between defects of opposite winding number, but repulsive between defects of the same winding number; our toy model does not include the dilaton and the graviton fields, which can provide compensating attractive interaction, so we cannot obtain the BPS states that correspond to two identical, parallel, branes.

One can perform a Kaluza-Klein reduction of the 3+1 dimensional bulk theory down to 2+1 dimensions. This will yield a theory containing a light $\phi$ and $A$ field and a massless $C$ field, together with a Kaluza-Klein tower of heavier $C$ field modes. It is possible to solve for the defect structure using this effective theory. Since the brane fields are directly the fields appearing in the 2+1D effective theory, their equations of motion change the least. The equation of motion for $\phi$ is unchanged, and the expression for $A^\mu$, Eq. (3.4), is changed only by the replacing $H$ with a sum over KK states of $H$ from each KK state. The change for the bulk field $C$ is slightly more complicated. The KK decomposition is a Fourier series expansion, so $\delta(z)$ in $z$ space becomes the constant $1/2\pi R$ in each KK mode. Therefore, the equation of motion for the $C^0$ field, Eq. (3.7), becomes,

$$\left(\nabla^2 - M_{kk}^2\right) C^0_m = \frac{c_{cs} g_{\text{bulk}}^2}{2\pi R} F_{12}, \quad M_{kk}^2 = \left(\frac{m}{R}\right)^2, \quad (3.8)$$

with $m$ the index of the KK field (the KK wavenumber).
We will not attempt to solve these equations in detail near the defect core, but will content ourselves with an understanding of the defect appearance at radii larger than the symmetry breaking scale, \( \rho \gg 1/g_{\text{brane}}v \), and larger than the radius of the extra dimension, \( \rho \gg R \). In this case, the appearance of \( M^2_{kk} \) in the equation for \( C^0_m \) leads to an exponential decay of \( C^0 \) away from the defect core for all KK states but the zero mode, which instead falls according to the 2D Coulomb’s law:

\[
C^0(\rho) = \frac{c_{cs}g^2_{\text{bulk}}}{2\pi R} \frac{\Phi}{2\pi} \ln(\rho/\rho_0),
\]

\[
\Phi \equiv \int d\theta \rho d\rho F_{12} = \oint \rho d\theta A_{\theta},
\]

where \( \Phi \) represents the total magnetic flux associated with the brane defect, and \( \rho_0 \) is a constant of integration. Physically, the charge distribution for the \( C \) field is a disk lying on the brane at the defect core, but the \( H \) electric field spreads out from this disk and becomes uniform across the extra dimension at a distance of a few \( R \), as illustrated by the cartoon in figure 1.

Unlike the case of vanishing \( c_{cs} \), the magnetic flux \( \Phi \) will not equal \( 2\pi \). There are two ways to see this; by considering the equations of motion, or by considering energetics. Looking first at the field equations, the requirement that the scalar field phase change by \( 2\pi n \) around the defect is topological and is not modified by the Chern-Simons term. However, the presence of the Chern-Simons term in Eq. (3.4) means that either the field strength or the Higgs field current must now be nonzero. The defect core is defined as the region where the gauge field strength is non-negligible; outside the core we expect \( F_{12} = 0 \), so only the Higgs field gradient term can balance the Chern-Simons term. This will happen if the gauge field is smaller than \( A_{\theta} = n/\rho \) (its value in the Nielsen-Olesen solution), corresponding to a flux which is smaller than \( \Phi = 2\pi n \), because in this case the derivative and gauge field parts of the covariant derivative \( D_{\theta} \) will not cancel when acting on \( \phi = e^{in\theta} v/\sqrt{2} \). In detail, the equation of motion becomes,

\[
\dot{g}^2_{\text{brane}} \left( \frac{1}{\rho} D_{\theta} \phi^* - \phi^* \frac{1}{\rho} D_{\theta} \phi \right) = c_{cs}g^2_{\text{brane}} \partial_\rho C^0,
\]

\[
v^2 \left( \frac{n - A_{\theta}/\rho}{\rho} \right) = c_{cs} \frac{g^2_{\text{bulk}}}{2\pi R} \frac{\Phi}{2\pi},
\]

where we used Eq. (3.9) in the second line. Solving for \( \Phi \), remembering that \( \Phi = 2\pi A_{\theta} \rho \), we find,

\[
v^2 \left( \frac{n - \Phi}{2\pi} \right) = \left( \frac{c_{cs}g^2_{\text{bulk}}}{2\pi R} \right) \frac{\Phi}{2\pi},
\]

\[
\Phi = 2\pi n \left[ 1 + \frac{c_{cs}g^2_{\text{bulk}}}{2\pi Rv^2} \right]^{-1}.
\]

**B. Defect Energetics**

Alternatively, we can understand this result by considering energetics. If the gauge field flux does not equal \( 2\pi n \), there will be a nonvanishing Higgs field gradient energy.
However, the larger the flux, the larger the \( C \) electric field. The vortex is the minimal energy configuration subject to the topological condition on the phase of \( \phi \). Therefore, the gauge field will take the value which minimizes the sum of these two energy densities;

\[
\frac{d}{dA_\theta} \left( D_\theta \phi^* D_\theta \phi + \frac{2\pi R}{2g_{\text{bulk}}} \partial_\rho C^0 \partial_\rho C^0 \right) = 0.
\]

(3.15)

In the first term, \( D_\theta \phi = (i\hbar/\rho - iA_\theta)\phi \), and \( \phi^* \phi = v^2/2 \) (since otherwise there would be an extensive potential energy). In the second term, \( C^0 \) is determined from Eq. (3.9). Also, \( A_\theta = \Phi/2\pi\rho \), as previously. The minimization therefore requires,

\[
\frac{d}{d\Phi} \left[ \frac{v^2}{2\rho^2} \left( n - \frac{\Phi}{2\pi} \right)^2 + \frac{2\pi R}{2g_{\text{bulk}}} \left( \frac{c_{\text{cs}} g_{\text{bulk}}^2 \Phi}{(2\pi)^2 R \rho} \right)^2 \right] = 0 \quad \Rightarrow \quad \Phi = 2\pi n \left[ 1 + \frac{c_{\text{cs}} g_{\text{bulk}}^2}{2\pi RV^2} \right]^{-1},
\]

(3.16)

the same answer we obtained directly from the equations of motion.

This calculation also allows us to find the energy density far from the brane core, which is,

\[
\epsilon = |D_\theta \phi|^2 + \frac{2\pi R}{2g_{\text{bulk}}} |\nabla C^0|^2,
\]

(3.17)

which using the above result for \( \Phi \) equals,

\[
\epsilon = \frac{v^2 n^2}{2\rho^2} \frac{a}{1 + a}, \quad a \equiv \frac{c_{\text{cs}} g_{\text{bulk}}^2}{2\pi RV^2}.
\]

(3.18)

Note several features of this energy density. First and most important, it falls off as \( 1/\rho^2 \) as one increases the distance \( \rho \) from the core of the defect. Second, as the Chern-Simons coupling \( c_{\text{cs}} \) is changed from 0 to \( \infty \), the energy density far from the core changes smoothly from 0, the value for a “local” (Abelian Higgs) topological defect, to \( v^2 n^2/2\rho^2 \), the value for a “global” topological defect (one for a complex, gauge singlet scalar field). The total energy of a defect, \( \int d\theta \rho d\rho \epsilon \), is logarithmically divergent;

\[
E = \int d\theta \rho d\rho \frac{v^2 n^2}{2\rho^2} \frac{a}{1 + a} = \pi v^2 n^2 \frac{a}{1 + a} \ln \rho_{\text{max}}/\rho_{\text{min}}.
\]

(3.19)

Here \( \rho_{\text{min}} \sim 1/g_{\text{brane}}v \) is a constant of integration which also absorbs the finite contribution from the defect core, and \( \rho_{\text{max}} \) is the distance where interactions with the fields of other defects become important. For instance, for a defect-antidefect pair of separation \( r \), the energy is \( E = 2\pi v^2 n^2 (a/[1+a]) \ln r/\rho_{\text{min}} \). This behavior shows that a defect antidefect pair will feel an attractive \( 1/r \) interaction, which is proportional to \( c_{\text{cs}}^2 \) when the Chern-Simons coupling is small but has a finite limit when it is large. The Chern-Simons term has replaced the exponentially weak long-range interaction with one more typical of Coulomb interactions in two dimensions.

One feature of Eq. (3.19) is that the energy associated with a defect is never larger than the energy of a global defect, that is, a defect without either the brane field \( A \) or the bulk field \( C \). We have shown this for the large distance contribution, but it is also true in the core; for while the bulk field \( C \) costs energy, it only does so if the gauge field \( A \) takes on a nonvanishing flux. The energetics are that the flux is present because it reduces the scalar field gradients, and it will always take a value which reduces the scalar gradients more than
it increases field strength energies. Therefore the energetics of defect formation are at least as favorable as in a global defect model. Also note that the larger the extra bulk dimension, the smaller the parameter $a$ and therefore the energy, as the $C^0$ field strength spreads out over a larger dimension and therefore carries a smaller field strength squared.

We can now understand why a network of defects is energetically permitted to form in the process of brane-antibrane annihilation, which is assumed to start with the tachyon field near zero. The analogous situation in the toy model is that the scalar field starts at zero, though with small quantum fluctuations away from zero. Consider an isolated defect, hypothetically formed after brane-anti-brane annihilation. There is a finite contribution to its energy from the defect core region, and a contribution arising from large $\rho$, which grows like $\log \rho$. However, inside the same radius, the potential energy initially available from the the scalar field grows like the area, $\rho^2$, and this always wins over the $\log \rho$ dependence. We can therefore conclude that for a large enough radius, $\rho$, there is always enough energy available for creating the defect. The reason is that the energy of the final defect is coming from the potential energy of the brane scalar field, or the tachyon in the String Theory case, and not from the quantum deSitter fluctuations.

This study has clarified the behavior of defects in the toy model and argued that their formation is not obstructed by energetic considerations. We now will verify these conclusions, and study the dynamics of the formation and evolution of the system of vortices which forms in brane-antibrane annihilation in this toy model, by studying it via nonperturbative lattice techniques.

**IV. LATTICE SIMULATION**

The previous section established a toy model and made preliminary investigations of the defect structure and energetics in this model. What we really want is to study the dynamics of this model and its defect formation in a nonperturbative way. Therefore we have implemented the toy model on the lattice. Excluding the Chern-Simons coupling, which will be presented in an appendix, the lattice regularization of the brane and bulk action is the usual one for an Abelian Higgs model in $2 + 1$ dimensions and a Abelian gauge field in $3 + 1$ dimensions. The fact that the fields are Abelian allows us to use the non-compact formulation of the lattice action.

The lattice equations of motion are most easily evolved in temporal gauge, $A^0 = 0$ and $C^0 = 0$. This choice is immaterial so long as we concentrate on gauge invariant observables. It makes it slightly harder to verify the field configuration for the vortex discussed in the previous section, since the two configurations are related by a time-dependent gauge transformation. However, we can still compare lattice studies to our results there using only gauge invariant quantities such as field strengths. In particular, defects are easily identified by the peak of magnetic field strength at their core, along with the peak in energy density.

Besides studying the static vortex solutions, the lattice implementation allows us to study the dynamical evolution of the fields, including the formation and annihilation of the vortices and the flow of energy from the brane into the bulk.

The goal of the simulations is to see whether defects form, and how the defect network subsequently evolves if they do form, when the fields are prepared in a state designed to mimic the initial state of the tachyon field right after a brane collision. Namely, we begin with vanishing gauge fields $A, B, \partial_t A, \partial_t B$, and with the scalar field $\phi$ at the unstable symmetric point $\phi = 0$. To simulate initial vacuum fluctuations, we add small fluctuations in the
scalar field. These will grow unstably, triggering the phase transition to the state where $\phi^* \phi \simeq v^2/2$ almost everywhere. We look for defects as localized bundles of $F_{12}$ field strength and $|D\phi|^2$ gradient energy, and explore their subsequent evolution. Our lattice simulations are generally performed using a lattice spacing $a/2 = 0.5$, with couplings $\lambda = g_{\text{brane}}^2 = v^2$ and $g_{\text{bulk}}^2 = 1$.

We find that defects do indeed form. At vanishing Chern-Simons number, the defects’ mutual attraction is exponentially weak, and they persist almost indefinitely in a large lattice volume. (We use square, periodic lattices. Because of the periodicity of the lattice, the net winding number of all defects, $\sum n$, is always zero, that is, there are always equal numbers of defects and anti-defects, so every defect always has a partner which it can annihilate off against.) At finite Chern-Simons number, there is long range attraction and the defects are observed to annihilate off much faster, though the larger the lattice volume, the longer it takes.

In order to confirm the two effects discussed earlier, we perform simulations in which we change each of two “control” parameters, the size of the extra dimension and the strength of the Chern-Simons coupling. This way we can see the effect of the size of the extra dimension on the energy of the bulk field and the effect of the strength of the Chern-Simons coupling on the magnetic flux through the core of the vortex.

The effect of $R_{\text{bulk}}$ is illustrated in Figure 2, which shows the evolution of the energy density on the brane and in the bulk for a configuration which was initially evolved with
FIG. 3: (color online) The magnetic field strength distribution of a defect-antidefect pair. The figure shows the field strength distribution for a series of defect-antidefect pairs in theories with progressively smaller Chern-Simons interactions. For the leftmost pair, the Chern-Simons term almost eliminates the field strength; for the rightmost pair, the Chern-Simons coupling has been turned off and the full field strength is obtained.

As the two defects approach each other there is an increase in the energy of the brane fields and a decrease of the one in the bulk field. The decrease in the bulk field energy is due to the fact that the pair of charges at small separation will generate a dipole field at large distances and this field has a lower total energy than two widely separated charges. The increase in the brane field energy can be thought of as the energy of motion of the two defects plus radiated brane field energy as they interact and annihilate.

Changing the strength of the Chern-Simons coupling tells another story, see Figure 3. Looking at the defects themselves, we can see the back-reaction of the bulk field on the brane fields. Increasing the strength of the coupling results in a reduction of the magnetic flux through the core of the defect, in agreement with the considerations of the previous section.

We also look at the energy transfer rate between the brane and the bulk. One of the complications of a symmetry breaking transition such as the one we are studying is, that the extra energy density associated with the initial field potential energy has to “go somewhere” during the simulation. Frequently people carry out such simulations using dissipative dynamics, meaning the extra energy is eaten up by friction. This is justified in our case if the Hubble expansion rate is large. Otherwise, for conservative evolution, the energy density typically finds its way into the plentiful short wavelength excitations, where it eventually becomes equipartitioned. Because there is a brane-bulk interaction in our simulations, the
FIG. 4: (color online) Energy transfer between the brane (solid red line) and the bulk (dotted green line) for different values of the Chern-Simons coupling. For large or small coupling the transfer is slow, so the bulk does not act as an efficient energy sink. A slow flow of energy can allow the brane scalar field to homogenize, reducing the final density of defects.

The bulk acts as an energy sink which can be much more efficient than Hubble expansion. If the energy of the brane is drained quickly, the fields will have less time to homogenize, increasing the density of topological defects. The question of energy leakage is also relevant when studying reheating.

When studying this effect we use a small lattice $32 \times 32 \times 8$ which allows us to evolve the system for a much longer time. As expected (see Figure 4), for a weak coupling the energy is transferred slowly to the bulk, so the brane fields have more time available to homogenize. However as the strength of the coupling is increased the energy transfer rate reaches a maximum, and then it decreases again. We can understand this effect if we take into account the fact that the magnetic flux of the brane gauge field acts as the electric charge for the bulk gauge field. We saw before that when increasing the strength of the coupling the charge increases. However, when the charge becomes too large, it becomes energetically expensive for brane magnetic fields to appear at all; therefore a very large Chern-Simons interaction simply prevents brane magnetic fields from taking large values, turning off the mechanism available for exchange of energy between brane and bulk.
V. DISCUSSION AND CONCLUSIONS

Previous studies of defect formation during tachyon condensation have neglected the role played by bulk fields. It was argued [1] that when bulk fields are included, the formation of topological defects is suppressed or prevented. The argument was that the energy of the deSitter fluctuations is of the order the Hubble constant during inflation, while the gradients of the fields along the compact dimensions have corresponding energies $1/R \gg H$; so there is not enough energy available for defect formation.

We have constructed a toy model that captures the essential features of the tachyon condensation process when both the brane worldvolume gauge field and the bulk RR field are included. In this model the gradients of the bulk field are formed as a result of classical field evolution and the energy is provided by the tension of the brane-anti-brane pair. Since this energy can also be described as the energy of the tachyon field placed at the top of the potential, in the toy model this corresponds to the potential energy of an Abelian Higgs field. We argued that inside a large enough area around a defect there is always enough energy in the form of potential energy of the scalar field such that there is no energetic constraint on the formation of topological defects.

In terms of energetics, the presence of the bulk with a Chern-Simons coupling increases the energy cost of a defect. The increase is greatest if the Chern-Simons coupling is large and if the extra dimension is small. However, the energy associated with a defect never exceeds the value in the global defect model, that is, the value it would take if there were no gauge field or if $g_{\text{brane}}^2 = 0$. The bulk field also changes the long range interactions between defects, so that their mutual attraction or repulsion shows power law, rather than exponential, dependence on separation. This speeds up the evolution of the defect network, so that defect-antidefect pairs find each other and annihilate more efficiently. Again, the defect network evolution is never faster than in the global defect model, which is known to support a defect network after a symmetry breaking transition.

Another effect we observe in the lattice simulations of the toy model is the decrease of the magnetic flux unit through the vortex core as a function of the Chern-Simons coupling constant. This flux also acts as the charge density for the bulk field and if the Chern-Simons coupling becomes too large this charge density becomes very small, reducing the energy leakage from the brane to the bulk.

One drawback of the toy model comes from using an Abelian Higgs field instead of a tachyon on the brane. We did this so that we could follow the evolution of the defects after they have formed while the tachyon develops singularities. At the same time the brane fields do not drop out of the dynamics once the defects have formed.

VI. ACKNOWLEDGEMENTS

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APPENDIX A: LEADING ORDER CHERN-SIMONS COUPLING

Here we expand the Chern-Simons coupling of the brane and bulk fields to second order in $\alpha'$. The full expression is

$$S_{RR}^{D9} = T_{D9} \int C \wedge \text{Str} \, e^{2\pi i \alpha' F},$$

(A1)

where

$$C = \sum_{p=\text{odd}} \frac{(-i)^{\frac{p-2}{2}}}{(p+1)!} C_{\mu_0...\mu_p} dx^{\mu_0} \wedge ... \wedge dx^{\mu_p}.$$  

(A2)

The supertrace of the matrix is defined as,

$$\text{Str} \, M = \text{Tr} \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) M,$$

(A3)

and $F$ is the curvature of the superconnection, given by

$$iF = \left( \begin{array}{cc} iF^+ - T \overline{T} & DT \\ DT & iF^- - T \overline{T} \end{array} \right).$$

(A4)

Here we are interested in the particular case of codimension 2 defects, so the relevant coupling will be with the $C_{p-1}$ RR-field that couples with the defects. We want to keep both the tachyon and the gauge fields non-zero, so we will expand the exponential inside the supertrace in powers of $\alpha'$:

$$\text{Str} \, e^{2\pi i \alpha' F} = \text{Tr} \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) + (2\pi \alpha') \text{Tr} \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \left( \begin{array}{cc} iF^+ - T \overline{T} & DT \\ DT & iF^- - T \overline{T} \end{array} \right)$$

$$+ \frac{(2\pi \alpha')^2}{2} \text{Tr} \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \left( \begin{array}{cc} iF^+ - T \overline{T} & DT \\ DT & iF^- - T \overline{T} \end{array} \right) \wedge \left( \begin{array}{cc} iF^+ - T \overline{T} & DT \\ DT & iF^- - T \overline{T} \end{array} \right)$$

$$+ ...$$

$$= (2\pi i \alpha') \left[ F^+ - F^- \right] + \left(2\pi \alpha' \right)^2 \left[ DT \wedge DT - iT \overline{T} \left( F^+ - F^- \right) - \frac{1}{2} (F^+ \wedge F^+ - F^- \wedge F^-) \right] + ....$$  

(A5)

The terms of the type $F \wedge F$ couple with $C_{p-3}$ and count only in the formation of codimension 4 defects. Therefore, in the case of a $D9 - \overline{D9}$ pair, the important couplings between the brane and the bulk fields are

$$T_{D9} \int -i C_8 \wedge \left\{ (2\pi i \alpha') \left[ F^+ - F^- \right] + \left(2\pi \alpha' \right)^2 \left[ DT \wedge DT - iT \overline{T} \left( F^+ - F^- \right) \right] \right\}.$$  

(A6)

Since we are interested in the simplest model which involves such a coupling, when we construct the toy model we keep only the interaction that corresponds to the leading term in $\alpha'$,

$$\left(2\pi \alpha' \right) T_{D9} \int C_8 \wedge \left[ F^+ - F^- \right].$$

(A7)
APPENDIX B: HIGHER DIMENSIONAL GENERALIZATIONS

We can extend the toy model we used to a higher dimensional one which will have fewer differences with respect to the full string-theory model. The most straightforward generalization would be to consider a 3 + 1-dimensional brane and a 9 + 1 dimensional bulk. The field content of the worldvolume theory would still be the Abelian Higgs model, but the bulk field will now be a rank-2 antisymmetric tensor field, which has the appropriate rank to couple to the 2-dimensional world-volume of a string. The Abelian Higgs model in 3 + 1 dimensions admits stable string-like defects which in our model will be charged under the bulk field. The Lagrangian of the model is:

\[
\mathcal{L} = \int_{M_4} d^3 x \, dt \left[ -\frac{1}{4 g_{\text{brane}}^2} F^2 - \partial_\mu \phi \partial^\mu \phi^* - V(\phi) \right] - \frac{c_{\text{cs}}}{2} \int_{M_4} F \wedge C + \int_{M_{10}} d^9 x \, dt \left[ -\frac{1}{12 g_{\text{bulk}}^2} H^2 \right],
\]

where as before we denote by \( F = dA \) the field strength of the field \( A \) and by \( H = dC \) the field strength of the field \( C \). The equations of motion now become:

\[
\partial_\mu F^{\mu\nu} + \frac{ig_{\text{brane}}^2}{2} (\phi D^\nu \phi^* - \phi^* D^\nu \phi) + c_{\text{cs}} g_{\text{brane}}^2 \epsilon^{\nu\alpha\beta\gamma} H_{\alpha\beta\gamma} = 0, \tag{B2}
\]

\[
\partial_\mu H^{\mu\nu\lambda} - c_{\text{cs}} g_{\text{bulk}}^2 \epsilon^{\alpha\beta\nu\lambda} F_{\alpha\beta} \delta (z) = 0. \tag{B3}
\]

If we now want to study how the solution for a Nielsen-Olesen string is modified by the presence of the bulk field, for the brane fields we make the usual ansatz for a string placed along the \( z \)-direction: far from the string the scalar and gauge fields are

\[
\sqrt{2} \phi(\vec{r}) = v e^{i \theta}, \quad A_\theta(\vec{r}) = z \frac{n}{\rho}, \tag{B4}
\]

with \( z \) a rescaling of the \( c_{\text{cs}} = 0 \) case which remains to be determined. We observe that for the bulk field only the \( C^{03} \) component is sourced by the Chern-Simons term. The entire analysis done for the lowest-dimensional toy model can be applied here with the only difference being the expression of the masses for the KK modes, which will be highly dependent on the details of the compactification. For the simplest case, a torus, the expression is:

\[
M_{kk}^2 = \sum_{i=1}^{6} \frac{m_i^2}{R_i^2}. \tag{B5}
\]

However, when studying the energetics of the defects, only the zero mode gives a potentially log-divergent contribution, and the details of the compactification are relevant only to an energy density associated with the core. Also, as in the toy model studied in the main text, the (global) defect with the \( \phi \) field varying by a \( 2\pi \) phase around the defect core but both the \( F \) and \( H \) field strengths vanishing gives an upper bound on the defect energetics. As in the lowest-dimensional toy model we have:

\[
C^{03}(\rho) = \frac{c_{\text{cs}} g_{\text{brane}}^2}{V} \Phi \frac{\ln(\rho/\rho_0)}{2\pi}, \tag{B6}
\]

\[
\Phi \equiv \int d\theta d\rho F_{12} = \oint \rho d\theta A_\theta, \tag{B7}
\]
with $V$ the (6-dimensional) volume of the compact manifold. This leads to the same result for the unit of magnetic flux through the vortex core:

$$\Phi = 2\pi n \left[ 1 + \frac{c_{\text{cs}} g_{\text{bulk}}^2}{V v^2} \right]^{-1},$$

(B8)

and for the energetics of the defect; the string tension involves a log, $\ln(\rho_{\text{max}}/\rho_{\text{min}})$, with $\rho_{\text{max}}$ the inter-string separation and $\rho_{\text{min}}$ the string core size:

$$T = \pi v^2 n^2 \frac{a}{1 + a} \ln \rho_{\text{max}}/\rho_{\text{min}}, \quad a \equiv \frac{c_{\text{cs}} g_{\text{bulk}}^2}{V v^2}.$$  (B9)

We could also have considered higher dimensional branes wrapped on cycles of the compact manifold, but again only the zero modes of the brane fields would participate in the formation of the defects.

APPENDIX C: 10 D GLOBAL DEFECTS IN THE ORIGINAL THEORY

The easiest setup to study the formation of defects in the original theory is to consider the action for a brane-anti-brane pair for only the tachyon field in flat space-time, with all the other fields turned off, since the lower-dimensional branes are in fact vortices of the complex tachyon field. The action is simply:

$$S = 2T_{D9} \int d^{10} X e^{-2\pi\alpha'T T} \left[ 1 + 8\pi\alpha' \ln (2) D^\mu T D_\mu T \right].$$  (C1)

One can also consider the action for lower-dimensional brane-anti-brane pairs. The equation of motion derived from the action above is:

$$\partial_{\mu} \partial^{\mu} T - 2\pi\alpha' T \partial_{\mu} T \partial^{\mu} T + \frac{T}{4 \ln 2} = 0.$$  (C2)

As in the case of the real tachyon field, one can approximate the profile of the field with a linear one as the vortex will form at the place where $T = 0$, and solve the resulting equation for the slope of the profile. The resulting defect formed in the decay of a $Dp - \bar{D}p$ pair is a $Dp - 2$ brane.

In order to understand this we have to go back to the calculation of the space-time action calculated on linear tachyon profiles and estimate the action in the limit of infinite slope. The calculation was done in Ref. [20] and we reproduce the important points here. In the case of a $D9 - \bar{D}9$ pair the space-time action for a linear tachyon profile is:

$$S (y^I) = 2T_{D9} \int dX^{10} e^{-2\pi\alpha'T T} \prod_{I=1}^{2} F (\pi\alpha' y^I).$$  (C3)

where $T^I = u^I X^I/\sqrt{\alpha'}$ and $y^I = (u^I)^2$. The function $F$ has the expression:

$$F (x) = \frac{4^x x^\Gamma (x)^2}{2 \Gamma (2x)}.$$  (C4)
and in the large argument limit it takes the form:

\[ F(x) \simeq \sqrt{\pi x}. \]  \hspace{1cm} (C5)

Calculating the action on the profile \( y^1 \to \infty \) and \( y^2 \to \infty \) the authors of Ref. \cite{20} obtain:

\[ S(y^I) = 2T_{D9} \int dX^{10} e^{-\frac{\pi}{2} \left[ y^1(X^1)^2 + y^2(X^2)^2 \right]} F(\pi \alpha' y^1) F(\pi \alpha' y^2) \]

\[ = 2T_{D9} \int dX^8 \sqrt{\frac{2}{y^1}} \sqrt{\frac{2}{y^2}} \sqrt{\pi^2 \alpha' y^1} \sqrt{\pi^2 \alpha' y^2} \to 4\pi^2 \alpha' T_{D9} \int dX^8. \]  \hspace{1cm} (C6)

The result gives the correct tension for a \( D7 \) brane, \( T_{D7} = (2\pi \sqrt{\alpha'})^2 T_{D9} \). Regarding the RR charge of the vortex, we can estimate it by using the result Eq. (A5) in the expression of the coupling between brane fields and the bulk RR fields,

\[ S_{RR} = T_{D9} \int C_8 \wedge e^{-2\pi \alpha' T} (2\pi \alpha')^2 dT \wedge dT = \frac{(2\pi \alpha')^2}{\alpha'} T_{D9} \int C_8, \]  \hspace{1cm} (C7)

which again reproduces the correct result for the \( D7 \) brane RR charge.

These results allow us to obtain an upper limit for the density of defects formed, based only on energetic considerations. The brane tension is equal to the mass per unit volume for the brane and the result \( T_{Dp-2} = (2\pi \sqrt{\alpha'})^2 T_{Dp} \) tells us that we can have at most one defect on each patch of area \( (2\pi \sqrt{\alpha'})^2 = (2\pi l_s)^2 \) where \( l_s \) is the string length. This is a very large density and it shows that constraints other than the energetic ones are more important in determining the final density of defects.

In a realistic model we expect that a very important role will be that of the other fields that we have neglected so far, namely the dilaton and the graviton. These two fields have universally attractive interactions and their presence allows for the existence of BPS states in which there is no interaction between identical, parallel, branes. We expect the presence of these fields to also change the evolution of the resulting network of defects, since our toy model allows for repulsive interactions between same-charge defects, while no repulsive interactions (except for very special situations, see \cite{3}) are possible in the full String Theory model.

**APPENDIX D: LATTICE IMPLEMENTATION**

The implementation of gauge fields on the lattice was first developed by Wilson \cite{23} and is by now standard. To render the number of degrees of freedom finite, the scalar field \( \phi \) is taken only to reside at a discrete set of points, a cubic (or square) lattice with spacing \( a \). Since we will want a finite-timestep update, it is also convenient to define the theory from the very beginning on a spacetime lattice, with temporal lattice spacing \( a_t \ll a \). The gauge fields are a connection (rule for parallel transport) and must be defined on the links (lines between nearest neighbor lattice sites). Therefore the lattice variable is

\[ A_\mu^\alpha(x) = \int_x^{x+a\hat{\mu}} A \cdot dl \sim a A_\mu^\alpha, \]  \hspace{1cm} (D1)

with \( \hat{\mu} \) the unit vector in the \( \mu \) direction. In a non-Abelian theory it is necessary to treat the exponent of this variable, rather than \( A_\mu^\alpha \) itself, as the natural variable, but in an Abelian
theory we are free to consider either \( A_\mu^L \) or \( \exp(-iA_\mu^L) \) as the native variable, and we will choose to use \( A_\mu^L \) (the noncompact formulation). Note that the variable \( A_\mu^L(x) \) is centered at \( x + a\hat{\mu}/2 \).

The terms in the action, Eq. (3.1), become,

\[
\int_{\mathcal{M}_3} D_i \phi D^i \phi^* \rightarrow a^2 \alpha_t \sum_{x \in \mathcal{M}_3} \sum_i \left| e^{-iA_\mu^L(x+a \hat{\mu})} - e^{-i\alpha^L} \right|^2, \tag{D2}
\]
\[
\int_{\mathcal{M}_3} D_0 \phi D_0 \phi^* \rightarrow a^2 \alpha_t \sum_{x \in \mathcal{M}_3} \sum_i \left| e^{-i\alpha^L} \phi(x+a \hat{\mu}) - \phi(x) \right|^2, \tag{D3}
\]
\[
\int_{\mathcal{M}_3} F_{ij} F^{ij} \rightarrow a^2 \alpha_t \sum_{x \in \mathcal{M}_3} \sum_{ij} \left[ A_{12}(x) + A_{ij}(x+a \hat{\mu}) - A_{12}(x+a \hat{\mu}) - A_{ij}(x) \right]^2. \tag{D4}
\]

and similarly for \( F_{0i} F^{0i} \) and the \( H^2 \) terms. These terms are all very standard in the lattice community. For the Chern-Simons term, the magnetic field is the sum of \( A \) fields going around a square,

\[
F_{ij}^{ij} = \frac{1}{a^2} \left( A_{12}^L(x) - A_{12}^L(x+a \hat{\mu}) - A_{ij}^L(x) + A_{ij}^L(x+a \hat{\mu}) \right) = 0, \tag{D5}
\]

which, note, is centered at \( x + (i+j) a \hat{\mu}/2 \). Therefore the \( C \) field must be averaged over the four sites, \( x, x+a \hat{\mu}, x+a \hat{\mu}, \) and \( x + a(i+j) \). Further, since \( C^0 \) lives at the half time-step, it must be averaged over \( C(t) \) and \( C(t - \alpha_t/2) \);

\[
\int_{\mathcal{M}_3} F^{12} C^0 \rightarrow a^2 \alpha_t \sum_{x \in \mathcal{M}_3} F^{12} L \times \frac{1}{8 \alpha_t} \sum_{x \in \mathcal{M}_3} C^0 L = 0. \tag{D6}
\]

The sums for \( F^{01} C^2 \) and \( F^{20} C^1 \) work similarly.

Alternatively, one can re-arrange this sum in terms of what \( A \) fields couple to each \( C \) field;

\[
\sum_{x \in \mathcal{M}_3} F^{12} C^0 = \sum_{x \in \mathcal{M}_3} C^0 L(x) \times \frac{1}{8} \sum_{x \in \mathcal{M}_3} F^{12} L = 0. \tag{D7}
\]

Note that \( F^{12} \) involves a signed sum, see Eq. (D5), so the “middle” lines in the field strength squares above cancel off, the contribution associated with one \( C^0 \) field “link” is,

\[
F^{12} C^0 \rightarrow 0. \tag{D8}
\]

In the continuum, time evolution requires fixing initial field values \( \phi, A, C \) and their time derivatives \( D_0 \phi, F_{0i}, H_{0i} \). On the spatiotemporal lattice, we fix the links on two initial time slices, \( t = 0 \) and \( t = a_t \). The choice is constrained by Gauss’ law, which is the extremization of the lattice action with respect to a temporal link \( A^0(x) \) or \( C^0(x) \). Extremization of the action with respect to the \( \phi \) and \( A \) variables on the \( t = a_t \) layer fixes the values on the \( t = 2a_t \) layer once we choose values for the temporal links (choose the time dependent gauge), which
is most conveniently done by setting $A^0_L = C^0_L = 0$. In the $c_{cs} = 0$ theory, there is a 1-1 correspondence between equations of motion and $t = 2a_t$ variables; variation of $A^i_L(x, a_t)$ determines $A^i_L(x, 2a_t)$ explicitly. At finite Chern-Simons term, the relations between the $t = a_t$ equations of motion and the $t = 2a_t$ fields are implicit and must be solved iteratively by perturbing in $c_{cs}$. This works because the Chern-Simons term has one derivative, while the $F_{0i}^2$ term has two; so the iteration converges in powers of $a_t/a$.

We should also verify that the action used is gauge invariant. Changing the gauge at point $x$ means shifting $\phi(x) \to e^{-i\theta} \phi(x)$ and making a change in the gauge fields which will keep the $\phi$ field derivative term unchanged. Examining Eq. (D2), we see that this requires,

$$\phi(x) \to e^{-i\theta} \phi(x) \quad (D9)$$
$$A^i_L(x) \to A^i_L(x) + \theta \quad (D10)$$
$$A^i_L(x-a^i) \to A^i_L(x-a^i) - \theta \quad (D11)$$

so each link starting at $x$ is shifted by $\theta$ and each link ending at $x$ is shifted by $-\theta$. This clearly cancels in the field strength, $F^{ij}$; each corner of the square in Eq. (D5) has one link entering and one link leaving. This ensures the gauge invariance of the $F_{ij}^2$ term in the action, and the gauge invariance to changes in the $A$ field gauge of the $F \wedge C$ term. What about $C$ field gauge changes in the Chern-Simons term?

To see that the Chern-Simons term is invariant to $C$ field gauge change, first note what such a gauge change adds to the action. Each $C$ link entering or leaving the site $x$ is shifted by $\pm \theta$ (+ if it leaves the site). This contributes $\theta$ times a signed sum of $F_{\mu\nu}^L$ “squares” to the action. But when we look at the signed sum in detail, see Figure 5, we find that it is

[Figure 5: Magnetic fields contributing to the variation of the action with respect to a gauge change in the $C$ field at the center of the cube. This sum of $A$ field links exactly cancels—a lattice implementation of a “boundary of a boundary.”]

Specifically, the Chern-Simons term means, for instance, that the variation of the action with respect to $A^i_L(x, a_t)$ depends on $A^i_L(x, 2a_t)$, which is to be determined. One starts by guessing that $A^i_L(x, 2a_t) = A^i_L(x, a_t)$ to evaluate the equation of motion at $x, a_t$. This gives a first guess for $A^i_L(x, 2a_t)$ (and simultaneously, all other links at time $2a_t$). This guess is used to compute the Chern-Simons term, redetermining the equation of motion at time $a_t$ and a better guess for the variables at $2a_t$. The process is iterated until numerical convergence, typically of order 4-7 iterations.
the surface of a box, such that each link appears in two $F_{\mu\nu}$, with opposite orientation; the sum cancels.


