Spin polarization and chiral symmetry breaking at finite density

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Abstract

We investigate the possibility of the spin polarization of quark matter at zero temperature and moderate baryon density. The Nambu-Jona-Lasinio (NJL) model including interactions in vector and axial-vector channel (coupling constant $G_2$) as well as scalar and pseudoscalar channel (coupling constant $G_1$) is used, and the self consistent equations for the spin polarization and chiral condensate are solved numerically in the framework of the Hartree approximation. Numerical calculations show that in the one flavor model the spin polarization is possible at finite density if the ratio $G_2/G_1$ is larger than $O(1)$. We find that the interplay between the spin polarization and the chiral symmetry plays an important role. If the chemical potential of quarks reaches a certain finite value, the spin polarization begins to occur because the quark still has relatively large dynamical mass, which is generated by spontaneous chiral symmetry breaking. On the other hand, when the spin polarization occurs, it operates to lower the dynamical quark mass a little. If one increases the chemical potential furthermore, the spin polarization becomes weaker and finally disappears because the dynamical quark mass is reduced by restoring gradually the chiral symmetry.

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1 Introduction

The density of matter in cores of neutron stars is of order $O(10^{15}\text{g/cm}^3)$, and the neutron star will give us useful information on how the matter behaves with very high density. In neutron stars, it is supposed that nuclear matter or quark matter exists. One of the distinctive features of the neutron star is that it has a strong magnetic field $O(10^{12}\text{Gauss})$. In order to explain the origin of the strong magnetic field in neutron stars, the possibility of the spin polarization in nuclear matter has been studied by many authors [1]; however, no definite conclusion has been obtained. Recently, instead of nuclear matter, several authors investigate the possibility of the spin polarization in quark matter [2, 3], which is expressed by axial vector $\langle \bar{\psi}\gamma_5\gamma\psi \rangle = \langle \bar{\psi}\Sigma\psi \rangle$ with $\Sigma$ being the spin operator of the quark in the relativistic quantum field theory [4]. This article is one of such investigations. We consider the possibility of the spin polarization of quark matter at zero temperature and moderate baryon density at which asymptotic freedom of quarks does not hold. In such a density region, the interplay between the spin polarization and spontaneous chiral symmetry breaking of QCD will be important.

QCD describes the dynamics of quarks and gluons. For the study of neutron stars with high baryon density, one needs to understand the properties of QCD at finite density, which have been investigated vigorously by use of effective theories of QCD [5]. At low temperature and low density, quarks and gluons are confined and chiral symmetry is broken spontaneously, while at low temperature and high density, deconfinement occurs and chiral symmetry will be restored, furthermore a new phase called color superconductivity may be realized [6, 7]. In Ref. [3], a coexistent phase of spin polarization and color superconductivity in high-density QCD is investigated. The authors of Ref. [3] use the one-gluon-exchange interaction as an effective interaction between quarks, which implies that the density is taken to be high so that one can treat the coupling constant $\alpha_s = g^2/4\pi$ very small. The quark mass is treated as the constant parameter not depending on the density, and the numerical calculations are carried out for several different values of the quark mass. It is shown that the spin polarization remarkably depends on the value of the quark mass. Especially, the spin polarization does not occur in the limit of zero quark mass, which is shown analytically by the use of the self consistent equation in the framework of the mean field approximation [3].

In this paper, we investigate the possibility of the spin polarization of quark matter at zero temperature and moderate baryon density, at which spontaneous symmetry breaking of chiral symmetry plays an important role. When the quark number density increases, the quark mass can not be regarded as a constant, because the chiral symmetry is gradually restored and the dynamical quark mass decreases [8, 9]. How the spin polarization is influenced by the dynamical quark mass which decreases due to restoration of chiral symmetry? Conversely, how the dynamical quark mass is influenced by occurrence of the spin polarization? The point we concentrate is the interplay between the spin polarization and chiral symmetry. Although the relation between the chiral symmetry breaking phase and the color superconducting phase has been discussed [10],
we do not consider the color superconductivity in the present paper and focus on the relation between chiral symmetry and the spin polarization.

The method of our approach is as follows. Since the color gauge coupling constant is not small at the moderate density, one cannot use the one-gluon-exchange interaction as an effective interaction between quarks. Instead, effective theories of QCD such as the Nambu-Jona-Lasinio (NJL) model [11] are often used at that moderate density region. The NJL model realizes spontaneous symmetry breaking of chiral symmetry while it does not confine quarks; nevertheless the properties of mesons can be well described by the NJL model [9]. The model we use is the NJL type model including interactions in vector and axial-vector channel as well as scalar and pseudoscalar channel, which model is used to deal with (axial) vector meson modes [12, 13, 9].

The paper is organized as follows. The model we use is described in the next section and the chemical potential $\mu$ of quarks is introduced so as to deal with the system with finite density. Using the Hartree approximation (the mean field approximation), we can obtain the propagator of the quark in the presence of the mean fields of chiral condensate, quark number density, and axial-vector field. In section 3, the self consistent equations for chiral condensate, quark number density, and axial-vector field related to the spin polarization are derived respectively in the framework of the mean field approximation. These simultaneous self consistent equations are solved numerically in section 4. The results of the numerical calculations tell us how the spin polarization arises or how the dynamical quark mass changes as the chemical potential of quarks is varied. Section 5 is devoted to conclusion.

2 Formulation and the Hartree approximation

In this section, the model we use is introduced. Assuming the presence of the mean fields of chiral condensate, $\langle \bar{\psi}\psi \rangle$, quark number density, $\langle \bar{\psi}\gamma^0\psi \rangle$, and axial-vector, $\langle \bar{\psi}\gamma^5\gamma^3\psi \rangle$, at finite density, we calculate the propagator of the quark in the Hartree approximation.

The following NJL type model [9] with the number of colors $N_c = 3$ and the number of flavors $N_f = 1$ is used,

$$L = L_0 + L_S + L_V,$$

$$L_0 = \bar{\psi}(i\partial - m)\psi,$$

$$L_S = \frac{G_1}{2} \left[ (\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\psi)^2 \right],$$

$$L_V = -\frac{G_2}{2} \left[ (\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma^\mu\psi) + (\bar{\psi}\gamma^\mu\gamma_5\psi)(\bar{\psi}\gamma^\mu\gamma_5\psi) \right].$$

The vector and axial-vector interaction term $L_V$ is needed for treatment of vector meson or axial-vector meson in the NJL type model. This lagrangian is chiral invariant if the current quark mass $m$ is zero. We make use of this lagrangian in order to investigate
the interplay between the spin polarization and chiral symmetry, and regard $m$ and the ratio $G_2/G_1$ as free parameters.

The chemical potential $\mu$ of quarks is introduced to treat the system with finite density, $\mathcal{L} + \mu \bar{\psi} \gamma^0 \psi$. Now, we suppose that the following mean fields exist at finite density,

$$\langle \bar{\psi} \psi \rangle, \quad \langle \bar{\psi} \gamma^0 \psi \rangle, \quad \langle \bar{\psi} \gamma_5 \gamma^3 \psi \rangle,$$

where $\langle \bar{\psi} \gamma_5 \gamma^3 \psi \rangle = \langle \tilde{\psi} \Sigma_2 \psi \rangle$ is an expectation value of the $z$-component of the quark’s spin $\Sigma$; one can choose the $z$-component as the direction of the spin polarization without loss of generality. In the vacuum state ($\mu = 0$), chiral symmetry is broken spontaneously, $\langle \bar{\psi} \psi \rangle \neq 0$, and no spin polarization occurs, $\langle \bar{\psi} \gamma_5 \gamma^3 \psi \rangle = 0$. As noted in the previous section, color superconductivity is not considered in this paper. Using the Hartree approximation (mean field approximation), we obtain the lagrangian with bilinear form,

$$\mathcal{L}_{MF} = \bar{\psi} \left( i \slashed{\partial} - m \right) \psi + G_1 \langle \bar{\psi} \psi \rangle \bar{\psi} \psi - G_2 \langle \bar{\psi} \gamma^0 \psi \rangle \bar{\psi} \gamma^0 \psi + G_2 \langle \bar{\psi} \gamma_5 \gamma^3 \psi \rangle \bar{\psi} \gamma_5 \gamma^3 \psi$$

$$+ \mu \bar{\psi} \gamma^0 \psi$$

$$\equiv \bar{\psi}_\alpha^c (G^{-1}_A)_{\alpha \beta} \gamma^0 \gamma^c \beta^\prime,$$

where $\alpha, \beta = 1, 2, 3, 4$ are the spinor indices, and $c, c' = 1, 2, 3$ are the color indices. The $G_A^{-1}$ is rewritten more compact form as

$$(G^{-1}_A)_{\alpha \beta} = \left[ i \slashed{\partial} + G_1 \langle \bar{\psi} \psi \rangle + \left\{ \mu - G_2 \langle \bar{\psi} \gamma^0 \psi \rangle \right\} \gamma^0 + G_2 \langle \bar{\psi} \gamma_5 \gamma^3 \psi \rangle \gamma_5 \gamma^3 \right]_{\alpha \beta}$$

$$\equiv \left[ i \slashed{\partial} - M + \mu_\tau \gamma^0 + U_A \gamma_5 \gamma^3 \right]_{\alpha \beta},$$

with

$$M \equiv m - G_1 \langle \bar{\psi} \psi \rangle,$$

$$\mu_\tau \equiv \mu - G_2 \langle \bar{\psi} \gamma^0 \psi \rangle,$$

$$U_A \equiv G_2 \langle \bar{\psi} \gamma_5 \gamma^3 \psi \rangle.$$

The chemical potential $\mu$ is shifted to $\mu - G_2 \langle \bar{\psi} \gamma^0 \psi \rangle$ in $G_A^{-1}$. In momentum space,

$$G^{-1}_A = \left[ \slashed{\partial} + \mu_\tau \gamma^0 - M + U_A \gamma_5 \gamma^3 \right]$$

$$\equiv \left[ \slashed{p} - M + U_A \gamma_5 \gamma^3 \right],$$

with

$$p^\mu = (p_0, \mathbf{p}) \equiv (q_0 + \mu_\tau, \mathbf{q}).$$

The quark propagator $G_A$ ($G_AG_A^{-1} = 1$) is obtained as (see Appendix A)

$$G_A = (1) \left[ \slashed{p} - M + U_A \gamma_5 \gamma^3 \right]^{-1}.$$
\[
\begin{align*}
= (1) \frac{1}{(p^2 - M^2 - U_A^2)^2 - 4 U_A^2 (M^2 + p_z^2)} \\
\times \left[ (p^2 - M^2 + U_A^2) M + (p^2 - M^2 - U_A^2) \dot{p} - 2 U_A^2 p_z (\gamma^3) \\
+ 2 U_A p_z p_\mu (\gamma_5 \gamma^\mu) - U_A (p^2 + M^2 - U_A^2) (\gamma_5 \gamma^3) \\
+ 2 U_A M \left( p_0 \sigma^{12} - p_x \sigma^{02} + p_y \sigma^{01} \right) \right].
\end{align*}
\]

The propagator \( G_A \) has four energy poles \( p_0 = \epsilon_n \) \((n = 1, 2, 3, 4)\),

\[
\begin{align*}
\epsilon_1 &= \epsilon_- = \sqrt{p_\perp^2 + \left( \sqrt{M^2 + p_z^2} - U_A \right)^2}, \\
\epsilon_2 &= \epsilon_+ = \sqrt{p_\perp^2 + \left( \sqrt{M^2 + p_z^2} + U_A \right)^2}, \\
\epsilon_3 &= -\epsilon_1, \\
\epsilon_4 &= -\epsilon_2,
\end{align*}
\]

where \( p_\perp^2 = p_x^2 + p_y^2 \) and \( \epsilon_{1,2} \) represents positive energy while \( \epsilon_{3,4} \) represents negative energy. Peculiarities of the Fermi surface of \( \epsilon_n \) are described in detail in Ref.\[3\]

### 3 Self consistent equation

In this section, we derive self consistent equations in the framework of the mean field approximation, each of which can be calculated analytically in the zero temperature limit. These self consistent equations will be solved numerically in the next section. The self consistent equations for \( \langle \bar{\psi} \psi \rangle \), \( \langle \bar{\psi} \gamma^0 \psi \rangle \), and \( \langle \bar{\psi} \gamma_5 \gamma^3 \psi \rangle \) are expressed with the quark propagator \( G_A \), respectively,

\[
\begin{align*}
\langle \bar{\psi} \psi \rangle &= -i \text{Tr} \{ G_A \}, \\
\langle \bar{\psi} \gamma^0 \psi \rangle &= -i \text{Tr} \{ G_A \gamma^0 \}, \\
\langle \bar{\psi} \gamma_5 \gamma^3 \psi \rangle &= -i \text{Tr} \{ G_A \gamma_5 \gamma^3 \}.
\end{align*}
\]

They are rewritten in terms of \( M, \mu, \) and \( U_A \) that are introduced in Eq.\( (5) \),

\[
\begin{align*}
M &= m + i G_1 \text{Tr} \{ G_A \}, \\
\mu_r &= \mu + i G_2 \text{Tr} \{ G_A \gamma^0 \} = \mu - G_2 \rho, \\
U_A &= -i G_2 \text{Tr} \{ G_A \gamma_5 \gamma^3 \},
\end{align*}
\]

where we have introduced the quark number density, \( \rho = \langle \bar{\psi} \gamma^0 \psi \rangle = -i \text{Tr} \{ G_A \gamma^0 \} \). These three equations are simultaneous equations for three unknowns, namely the dynamical quark mass \( M \), the "shifted" quark chemical potential \( \mu_r \), and the spin polarization \( U_A \).
3.1 Self consistent equation for $M$.

When the system with finite temperature $T$ is concerned, one can employ the imaginary time formulation [14, 15], according to which the $q_0$ in the propagator Eq.(5) is replaced by the Matsubara frequency, $i\omega_j = i(2j + 1)\pi T, (j \in \mathbb{Z})$. Let us begin with the calculation of $\text{Tr}\{G_A\}$ in the self consistent equation for $M$, Eq.(13). In the imaginary time formulation, one needs to perform the Matsubara sum over $q_0 = i\omega_j$,

\[
\text{Tr}\{G_A\} = \int \frac{d^3q}{(2\pi)^3} \left( i T \sum_{j=-\infty}^{\infty} \right) \text{tr}\{G_A\}
\]

\[
= \int \frac{d^3p}{(2\pi)^3} \left( i T \sum_{j=-\infty}^{\infty} \right) N_c \left\{ \frac{4 M(p^2 - M^2 + U_A^2)}{(p^2 - M^2 - U_A^2)^2 - 4 U_A^2 (M^2 + p_z^2)} \right\},
\]

(16)

where $p_0 = q_0 + \mu_t = i \omega_j + \mu_t$. The result of the sum is (see Appendix B)

\[
\text{Tr}\{G_A\} = i N_c \int \frac{d^3p}{(2\pi)^3} \left[ \sum_{n=1,2,3,4} \frac{M \left\{ \sqrt{M^2 + p_z^2 + (-1)^n U_A} \right\}}{\epsilon_n \sqrt{M^2 + p_z^2}} \tilde{f}(\epsilon_n - \mu_t) \right],
\]

(17)

where $\tilde{f}(\epsilon_n - \mu_t) \equiv 1/\{\epsilon^\beta(\epsilon_n - \mu_t) + 1\}$ is the Fermi-distribution ($\beta = 1/T$). In the zero temperature limit $T \to 0$, the self consistent equation for $M$ then becomes

\[
M = m - G_1 N_c \int \frac{d^3p}{(2\pi)^3} \left[ \sum_{n=1,2} \frac{M \left\{ \sqrt{M^2 + p_z^2 + (-1)^n U_A} \right\}}{\epsilon_n \sqrt{M^2 + p_z^2}} \theta (\mu_t - \epsilon_n) \right.

\hspace{1cm} + \left. \sum_{n=3,4} \frac{M \left\{ \sqrt{M^2 + p_z^2 + (-1)^n U_A} \right\}}{\epsilon_n \sqrt{M^2 + p_z^2}} \theta (\sqrt{\Lambda^2 + M^2} - |\epsilon_n|) \right].
\]

(18)

Since the terms coming from the negative energy ($\epsilon_{3,4} < 0$) contribution in Eq.(17) diverge in the zero temperature limit, we regularize these terms by $\theta (\sqrt{\Lambda^2 + M^2} - |\epsilon_n|)$ in Eq.(18) with cutoff $\Lambda$.

3.2 Self consistent equations for $\mu_t$, $U_A$.

First, the self consistent equation for $\mu_t$, Eq.(14), is calculated. As the $\text{Tr}\{G_A\}$ in Eq.(13), $\text{Tr}\{G_A\gamma^0\}$ is calculated as follows,

\[
\text{Tr}\{G_A\gamma^0\} = i N_c \int \frac{d^3p}{(2\pi)^3} \left[ \sum_{n=1,2,3,4} \tilde{f}(\epsilon_n - \mu_t) - 2 \right].
\]

(19)

The self consistent equation for $\mu_t$ is then

\[
\mu_t = \mu - G_2 N_c \int \frac{d^3p}{(2\pi)^3} \sum_{n=1,2} \theta (\mu_t - \epsilon_n),
\]

(20)
in the zero temperature limit $T \to 0$.

Secondly, the self consistent equation for $U_A$, Eq.(15), is considered. $\text{Tr}\{G_A \gamma^5 \gamma^3\}$ is evaluated as

$$\text{Tr}\{G_A \gamma^5 \gamma^3\} = -i N_c \int \frac{d^3 p}{(2\pi)^3} \left[ \sum_{n=1,2,3,4} \frac{\{U_A + (-1)^n \sqrt{M^2 + p_z^2}\}}{\epsilon_n} \theta(\mu_r - \epsilon_n) \right],$$

and in the limit $T \to 0$, it becomes

$$\text{Tr}\{G_A \gamma^5 \gamma^3\} = -i N_c \int \frac{d^3 p}{(2\pi)^3} \left[ \sum_{n=1,2} \frac{\{U_A + (-1)^n \sqrt{M^2 + p_z^2}\}}{\epsilon_n} \theta(\mu_r - \epsilon_n) + \sum_{n=3,4} \frac{\{U_A + (-1)^n \sqrt{M^2 + p_z^2}\}}{\epsilon_n} \right],$$

where the second term in the right-hand side represents the contribution of the Dirac sea ($\epsilon_{3,4} < 0$). Here, we define the spin polarization of the vacuum state $\mu = 0$ ($\mu_r = 0$) to be zero by subtracting as follows,

$$U_A = -i G_2 \left[ \text{Tr}\{G_A \gamma^5 \gamma^3\} - \text{Tr}\{G_A \gamma^5 \gamma^3\}|_{\mu_r=0} \right],$$

the definition implies that the contribution of the Dirac sea to the spin polarization is neglected. The self consistent equation for $U_A$ eventually takes the form

$$U_A = -G_2 N_c \int \frac{d^3 p}{(2\pi)^3} \sum_{n=1,2} \frac{\{U_A + (-1)^n \sqrt{M^2 + p_z^2}\}}{\epsilon_n} \theta(\mu_r - \epsilon_n),$$

in the zero temperature limit.

We finally obtained the self consistent equations for the dynamical quark mass $M$, Eq.(18), the ”shifted” quark chemical potential $\mu_r$, Eq.(20), and the spin polarization $U_A$, Eq.(24), in the zero temperature limit. Each of three equations can be calculated analytically if $0 \leq U_A < M$, and the results are shown in Appendix C.

4 Numerical calculations

In this chapter, the numerical solutions for the three self consistent equations Eq.(18), (20), and (24) are obtained, by which we can know how the quark spin polarization $U_A$ and the dynamical quark mass $M$ behaves as the quark chemical potential $\mu$ varies. Giving the five input parameters ($G_1$, $G_2$, $m$, $\Lambda$, $\mu$), we solve these simultaneous equations of three unknowns ($M$, $\mu_r$, $U_A$).
At the beginning, let us discuss the input parameters. In the NJL model analysis, the ratio $G_2/G_1$ is taken to be $G_2/G_1 \sim 0.7$ in the vacuum state when the number of quark flavor is three\cite{13}. We shall, however, regard $G_2/G_1$ and $m$ as free parameters in our one flavor model since we are interested in studying the relationship between spin polarization and spontaneous symmetry breaking of chiral symmetry at finite density by the use of this effective theory. In other words, the pion decay constant or the vector meson mass calculated by our model with these input parameters does not necessarily fit the experimental data. Now, before going into the case of $G_2/G_1 \neq 0$, let us consider the case of $G_2/G_1 = 0$, which enables us to concentrate on the issue of chiral symmetry.

4.1 Numerical results with $G_2/G_1 = 0$

When $G_2 = 0$, input parameters are $(G_1, \ m, \ \Lambda, \ \mu)$, and an unknown is $(M)$ because of $U_A = 0$ and $\mu_r = \mu$. The value of the effective quark mass (dynamical mass) $M_0$ in the vacuum state $\mu = 0$ is determined by the current quark mass $m$, the coupling constant $G_1$, and the cutoff $\Lambda$. We put $G_1$ and $\Lambda$ as follows,

$$G_1 = 0.95 \times 10^{-5} \text{ MeV}^{-2}, \quad \Lambda = 900 \text{ MeV}.$$ (25)

For several choices of the current mass $m = 0, 5 \text{ MeV, } 130 \text{ MeV}$, the self consistent equation Eq.(18) gives the value $M_0 = 295 \text{ MeV, } 316 \text{ MeV, } 601 \text{ MeV}$, respectively. In Fig.1, the numerical results of the $M$’s self consistent equation Eq.(18) are shown for

Figure 1: Dynamical quark mass $M$(MeV) as a function of the chemical potential $\mu$ (MeV) when the ratio is $G_2/G_1 = 0$. The current quark mass $m$ is taken to be 0 MeV (solid), 5 MeV (short-dashed), and 130 MeV (long-dashed), respectively.
\( m = 0, 5 \text{ MeV}, 130 \text{ MeV} \), respectively. These graphs show that, if the current mass \( m \) becomes heavier, the value of the chemical potential \( \mu \) at which chiral symmetry is restored becomes larger. The reason of such a behavior is as follows. From the self consistent equation Eq. (18) (see also Eq. (C.1)), the dynamical mass \( M \) takes the constant value \( M_0 \) when \( \mu_r \) is in the range \( 0 \leq \mu_r \leq M_0 \). If \( \mu_r \) exceeds the value \( M_0 \), the dynamical mass \( M \) decreases and the quark number density \( \rho = N_c (\mu^2 - M^2)^{3/2} \theta (\mu - M) / 3\pi^2 \) begins to take a positive value.

4.2 Numerical results with \( G_2/G_1 > 0 \)

All the numerical calculations are carried out with the input parameters of \( G_1 \) and \( \Lambda \) given in Eq. (25). As mentioned before, \( m \) and \( G_2/G_1 \) are regarded as free parameters. At first, we put \( m = 0 \) and find the value of \( G_2/G_1 \) by numerical calculations so that the spin polarization occurs (\( U_A > 0 \)) in some chemical potential region (\( \mu < \Lambda \)). The same procedures are performed for other values, \( m = 5 \text{ MeV}, 130 \text{ MeV} \), respectively.

First, we start with the chiral limit case, \( m = 0 \). By numerical calculations for the three self consistent equations Eq. (18), Eq. (20), and Eq. (24), we find that it is necessary to satisfy \( G_2/G_1 \approx 7.37 \) if \( m = 0 \) in order to occur the spin polarization. When \( G_2/G_1 = 7.37 \), the spin polarization \( U_A \) has the nontrivial solution in a chemical potential range of about 370 MeV to 470 MeV, and the numerical results are shown in Fig.2 and Fig.3. As seen from Fig.2, chiral symmetry is restored completely at \( \mu \approx 630 \text{ MeV} \), which value is considerably larger than the value \( \mu \approx 340 \text{ MeV} \) at which chiral symmetry is restored completely in the case of \( G_2/G_1 = 0 \) and \( m = 0 \) in Fig.1. This is because the lagrangian Eq. (1) contains the vector type interaction and thereby the chemical potential is shifted by an amount \( G_2 \rho \mu = \mu_r + G_2 \rho \).\[8, 16\]. This graph Fig.2 has the peculiarity; the line of the graph is lowered in the range \( 370 \text{ MeV} \lesssim \mu \lesssim 470 \text{ MeV} \). This peculiarity will be discussed later. In Fig.3, the nontrivial solution of \( U_A \) is figured, and one finds that the quark spin polarization occurs in the chemical potential range \( 370 \text{ MeV} \lesssim \mu \lesssim 470 \text{ MeV} \). It is notable that the spin polarization disappears when the chemical potential becomes too large, \( \mu \gtrsim 470 \text{ MeV} \), and this point will be also discussed in the later subsection.

Next, when \( m = 5 \text{ MeV} \), we find by numerical calculations that it is necessary to satisfy \( G_2/G_1 \approx 6.3 \) in order to occur the spin polarization \( U_A > 0 \). One can see that the necessary value \( G_2/G_1 \approx 6.3 \) in the case of \( m = 5 \text{ MeV} \) is smaller than the value \( G_2/G_1 \approx 7.37 \) in the chiral limit case \( m = 0 \). Although our NJL model of one flavor at finite density differs from the NJL model of three flavor in the vacuum state, the above value \( G_2/G_1 \approx 6.3 \) is fairly larger than the value \( G_2/G_1 \approx 0.7 \) which value is expected in the NJL model of three flavor in the vacuum state \[13\]. We then take another large value of \( m \) and find the value \( G_2/G_1 \) necessary for obtaining spin polarization \( U_A > 0 \).

Finally, when the current mass is taken to be \( m = 130 \text{ MeV} \), we find that it is necessary to satisfy \( G_2/G_1 \approx 1.89 \) so as to occur the spin polarization \( U_A > 0 \).
Figure 2: Dynamical quark mass \( M \) (MeV) as a function of the chemical potential \( \mu \) (MeV) when the ratio is \( G_2/G_1 = 7.37 \) and the current quark mass is \( m = 0 \). The \( M \) in Fig.2 and the nontrivial solution \( U_A \) in Fig.3 are the solution of the simultaneous equations Eq.(18), Eq.(20), and Eq.(24).

Figure 3: Spin polarization \( U_A \) (MeV) as a function of the chemical potential \( \mu \) (MeV) when the ratio is \( G_2/G_1 = 7.37 \) and the current quark mass is \( m = 0 \). The \( M \) in Fig.2 and the nontrivial solution \( U_A \) in Fig.3 are the solution of the simultaneous equations Eq.(18), Eq.(20), and Eq.(24).
Figure 4: Dynamical quark mass $M$(MeV) as a function of the chemical potential $\mu$ (MeV) when the ratio is $G_2/G_1 = 1.89$ and the current quark mass is $m = 130$ MeV. The $M$ in Fig.4 and the nontrivial solution $U_A$ in Fig.5 are the solution of the simultaneous equations Eq.(18), Eq.(20), and Eq.(24).

Figure 5: Spin polarization $U_A$(MeV) as a function of the chemical potential $\mu$ (MeV) when the ratio is $G_2/G_1 = 1.89$ and the current quark mass is $m = 130$ MeV. The $M$ in Fig.4 and the nontrivial solution $U_A$ in Fig.5 are the solution of the simultaneous equations Eq.(18), Eq.(20), and Eq.(24).
While $G_2/G_1 \gtrsim 1.89$ is still larger than the value 0.7, we understand that the necessary value $G_2/G_1$ for obtaining spin polarization becomes smaller if the current quark mass $m$ becomes larger. The numerical results for $m = 130$ MeV and $G_2/G_1 = 1.89$ are represented in Fig.4 and Fig.5. The value of $\mu$ at which the dynamical mass $M$ begins to decrease in Fig.4 is larger than that value of $\mu$ in Fig.2, because the current mass $m$ has larger value 130 MeV in Fig.4 than in Fig.2. When the current mass $m$ is 130 MeV (constituent mass $M_0 = 601$ MeV), the spin polarization occurs in the chemical potential range $730$ MeV $\lesssim \mu \lesssim 860$ MeV as seen from Fig.5. When, on the other hand, $m$ is 0 (constituent mass $M_0 = 295$ MeV), the spin polarization occurs in the range $370$ MeV $\lesssim \mu \lesssim 470$ MeV as seen from Fig.3. The difference between these two ranges of $\mu$ in Fig.5 and Fig.3 comes from the fact that the chemical potential $\mu$ must exceed the value of the constituent mass $M_0$ so that the quark number density $\rho$ has a positive value.

### 4.3 Effect of the spin polarization $U_A$ on the dynamical quark mass $M$, and vice versa.

We have solved numerically the simultaneous self consistent equations for $M$, $\mu_r$, and $U_A$ when the current mass $m = 0$, 5 MeV, 130 MeV, respectively. Since these equations are the simultaneous ones, $U_A$ has an effect on $M$, and vice versa. In this subsection, we study the effect of $U_A$ on $M$ and the effect of $M$ on $U_A$ through the case of $m = 0$. The behavior of the dynamical mass $M$ in Fig.2 is influenced by the spin polarization $U_A$, and the behavior of $U_A$ in Fig.3 is influenced by the dynamical mass $M$.

First, let us consider the effect of the spin polarization $U_A$ on the dynamical mass $M$. As mentioned before, the behavior of $M$ in Fig.2 has the peculiarity that the line of the graph is lowered in the range $370$ MeV $\lesssim \mu \lesssim 470$ MeV. This peculiarity will be caused by the influence of the spin polarization $U_A$ being non-zero in that range of $\mu$ as seen in Fig.3. In order to confirm this, we solve numerically the self consistent equation for $M$ and that for $\mu_r$ with setting $U_A = 0$ in the whole range of $\mu$, $0 \leq \mu \leq \Lambda$. The numerical result is shown with the dotted line in Fig.6, and the graph of Fig.2 is also shown in Fig.6 with the solid line for comparison. When the solid line is compared with the dotted line in Fig.6, we find that if the quark spin polarization occurs $U_A > 0$, it operates to lower the dynamical quark mass $M$ which is generated by spontaneous symmetry breakdown of chiral symmetry.

Next, let us consider the effect of the dynamical mass $M$ on the spin polarization $U_A$. As seen in Fig.2, $M$ is a monotone decreasing function of $\mu$, which passes through a point $(\mu, M) = (452$ MeV, 219 MeV). As mentioned before, the behavior of $U_A$ in Fig.3 has the peculiarity that the spin polarization $U_A$ disappears when the chemical potential becomes too large, $\mu \gtrsim 470$ MeV. This peculiarity may be caused by the fact that the dynamical mass $M$ is the monotone decreasing function of $\mu$. In order to confirm this, we examine how the $U_A$ behaves when the $M$ is taken to be a constant. Specifically
Figure 6: Dynamical quark mass $M$(MeV) as a function of the chemical potential $\mu$ (MeV) when the ratio is $G_2/G_1 = 7.37$ and the current quark mass is $m = 0$. The dotted line represents the dynamical mass when there is no spin polarization ($U_A = 0$). The solid line represents the dynamical mass with the nontrivial solution $U_A$ (the same result shown in Fig.2).

Figure 7: Spin polarization $U_A$(MeV) as a function of the chemical potential $\mu$ (MeV) when the ratio is $G_2/G_1 = 7.37$. The solid line is a part ($\mu \geq 452$ MeV) of the nontrivial solution shown in Fig.3. The dotted line is obtained when we set the dynamical mass $M = \text{constant} = 219$ MeV.
speaking, we solve numerically the self consistent equation for $U_A$ and that for $\mu_r$ with setting $M = \text{constant} = 219 \text{ MeV}$ \(^2\) in the range of $\mu \geq 452 \text{ MeV}$. The numerical result is shown with the dotted line in Fig.7, which shows that the spin polarization $U_A$ becomes larger as $\mu$ increases even in the range of large $\mu \approx 470 \text{ MeV}$ if the $M$ is constant (219 MeV). In Fig.7, a part of the graph of Fig.3 ($\mu \geq 452 \text{ MeV}$) is also shown with the solid line. By comparing this solid line with the dotted line in Fig.7, we find that, when the chiral symmetry is gradually restored as the chemical potential $\mu$ increases ($\mu \geq 452 \text{ MeV}$), the dynamical quark mass $M$ reduces ($M \leq 219 \text{ MeV}$, see Fig.2) and this reduction of the dynamical quark mass $M$ operates to suppress the spin polarization $U_A$ (see the solid line in Fig.7).

So far we have argued the special case of $m = 0$, yet the qualitative properties of the effect of $U_A$ on $M$ and the effect of $M$ on $U_A$ are the same with the cases of $m = 5 \text{ MeV}$ and 130 MeV.

5 Conclusion

We studied the relation between the quark spin polarization and spontaneous chiral symmetry breaking at finite density by use of the NJL type model with the number of colors $N_c = 3$ and the number of flavors $N_f = 1$, which includes interactions in vector and axial-vector channel. The self consistent equations for chiral condensate $\langle \bar{\psi} \psi \rangle$, spin polarization $\langle \bar{\psi} \Sigma \psi \rangle$, and quark number density $\langle \bar{\psi} \gamma^0 \psi \rangle$ are derived respectively in the framework of the Hartree approximation (mean field approximation). These simultaneous equations are solved numerically with regarding the quark current mass $m$ and the coupling constant ratio $G_2/G_1$ as free parameters. Consequently, we find that in the flavor $N_f = 1$ model the spin polarization is possible at finite density if the ratio $G_2/G_1$ is larger than $O(1)$. When the current quark mass $m$ becomes heavier, the necessary value $G_2/G_1$ for obtaining the spin polarization becomes smaller. For example, if $m = 130 \text{ MeV}$, the necessary value is $G_2/G_1 \approx 1.89$.

The spin polarization occurs in the following way. The spin polarization begins to occur at a certain value of the chemical potential, at which the dynamical quark mass is still a few hundred MeV. The reason why the spin polarization occurs are that the quark number density is finite and that the dynamical quark mass remains relatively large value about a few hundred MeV. If the effective mass $M$ of the quark is taken to be a small value such as 5 MeV, the spin polarization does not take place. When one increases the chemical potential furthermore, the spin polarization becomes weaker and finally disappears, where the chiral symmetry is gradually restored and the dynamical mass $M$ is reduced from its value in the vacuum state. The reason why the spin polarization disappears at the large chemical potential is that the dynamical quark mass is reduced by restoring gradually the chiral symmetry. The effects of the dynamical mass $M$ on the spin polarization $U_A$ are discussed above; conversely, what is

\(^2\) The $M = \text{constant} = 219 \text{ MeV}$ is not a solution of the self consistent equation for $M$. 

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the effect of the spin polarization on the dynamical mass? The numerical calculations show that, if the spin polarization occurs $U_A > 0$, it operates to lower the dynamical quark mass $M$ which is generated by spontaneous chiral symmetry breaking.

In studying the quark spin polarization with the moderate density at which chiral symmetry is partially broken spontaneously, it is necessary to consider the relation between the spin polarization and spontaneous chiral symmetry breaking [3]; the present paper attempts to investigate such issue. However, there remains several problems not discussed here, and we show below;

- It is necessary to make sure that the nontrivial solution $U_A$ of the self consistent equations gives the minimum value of the thermodynamical potential.

- The effect of quark confinement is not considered, since the NJL model we use is an effective theory that does not confine quarks. We therefore treat the system at finite density described by the NJL model as quark matter in the present paper. Although the lattice QCD simulation demonstrates quark confinement, it is difficult to apply it to the system with finite density. The development of this field is anticipated.

- We need to study the case of $SU(2)$ or $SU(3)$ flavors in which the condition of charge neutrality is imposed. According to the NJL model analysis in the vacuum state, the ratio $G_2/G_1$ is taken to be $G_2/G_1 \sim 0.7$ when the number of flavor is three. In our one flavor model at finite density, it is necessary to satisfy $G_2/G_1 \sim 1.89$ in order to occur spin polarization if the current quark mass is $m = 130$ MeV. We therefore need to study the spin polarization around $G_2/G_1 \sim O(1)$ in the case of three flavors.

- If the color superconductivity is considered, one needs to investigate the relationship among the spin polarization, spontaneous chiral symmetry breaking, and the color superconductivity.

- We have done the numerical calculations with $T = 0$. It would be interesting to study numerically the finite temperature case $T > 0$.

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Appendix

A The calculation of the propagator $G_A$.

The $G_A$, Eq. (8), is calculated as follows. Following the notation used in the textbook of Bjorken and Drell [17], we write

$$\left[ \not{p} - M + U_A \gamma_5 \gamma^3 \right] = \begin{pmatrix} p_0 - U_A \sigma_3 - M & -p \cdot \sigma \\ p \cdot \sigma & -p_0 + U_A \sigma_3 - M \end{pmatrix}.$$  \hspace{1cm} (A.1)

By the way, the inverse of the matrix

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$  \hspace{1cm} (A.2)

can be obtained by

$$X^{-1} = \begin{pmatrix} -C^{-1}DU & Y \\ U & -B^{-1}AY \end{pmatrix},$$  \hspace{1cm} (A.3)

where

$$U \equiv (B - AC^{-1}D)^{-1},$$
$$Y \equiv (C - DB^{-1}A)^{-1},$$  \hspace{1cm} (A.4)

provided $B^{-1}$ and $C^{-1}$ exist. Using this formula, we get Eq.(8).

B Summation over the Matsubara frequency

Let us use the standard technique [14, 15] to calculate the sum $\sum_{j=-\infty}^{\infty} g(p_0 - i\omega_j + \mu_r)$ over the Matsubara frequency $\omega_j = (2j + 1)\pi T$. Let $g(p^0)$ be a function of $p^0$ which decreases faster than $1/p^0$ for $|p^0| \to \infty$. Putting $p_0 = i\omega_j + \mu_r$, we obtain the following equation by the contour integration [15],

$$\frac{1}{\beta} \sum_{j=-\infty}^{\infty} g(p_0 = i\omega_j + \mu_r) = \sum_n \frac{\text{Res} g(p_0)}{e^{\beta(p_0 - \mu_r)} + 1}.$$  \hspace{1cm} (B.1)

Now, in Eq. (16), we take as follows

$$g(p_0) = \frac{4M(p^2 - M^2 + U_A^2)}{(p^2 - M^2 - U_A^2)^2 - 4U_A^2(M^2 + p^2)}$$
$$= \frac{4M(p^2 - M^2 + U_A^2)}{2(\epsilon_2 - \epsilon_1)} \sum_{n=1}^{4} \frac{(-1)^n}{\epsilon_n} \frac{1}{(p_0 - \epsilon_n)},$$  \hspace{1cm} (B.2)

then Eq.(17) is derived by the formula Eq.(B.1).
C  Explicit expressions for the self consistent equations

The self consistent equations Eq. (18), Eq. (20), and Eq. (24) with $T = 0$ can be calculated analytically when $0 \leq U_A < M$. The results are

$$M = m - G_1 \frac{N_c}{(2\pi)^2} M \left[ \left\{ (\mu_r - 3U_A) \sqrt{(\mu_r - U_A)^2 - M^2} ight. ight.$$
$$+(2U_A(\mu_r - U_A) - M^2) \log \frac{\mu_r - U_A + \sqrt{(\mu_r - U_A)^2 - M^2}}{M} \right] \theta(\mu_r - (M + U_A))$$
$$+ \left\{ (\mu_r + 3U_A) \sqrt{(\mu_r + U_A)^2 - M^2} ight. ight.$$
$$+(-2U_A(\mu_r + U_A) - M^2) \log \frac{\mu_r + U_A + \sqrt{(\mu_r + U_A)^2 - M^2}}{M} \right] \theta(\mu_r - (M - U_A)) \right]$$
$$+ G_1 \frac{N_c}{(2\pi)^2} M \left[ \mu_r \rightarrow \sqrt{\Lambda^2 + M^2} \right], \quad \text{(C.1)}$$

$$\mu_r = \mu - G_2 \frac{N_c}{(2\pi)^2} \left[ \left\{ \frac{1}{3} \left( -2M^2 + (\mu_r - U_A)(2\mu_r + U_A) \right) \sqrt{(\mu_r - U_A)^2 - M^2} ight. ight.$$
$$-U_A M^2 \log \frac{\mu_r - U_A + \sqrt{(\mu_r - U_A)^2 - M^2}}{M} \right] \theta(\mu_r - (M + U_A)) \right]$$
$$- G_2 \frac{N_c}{(2\pi)^2} \left[ U_A \rightarrow -U_A \right], \quad \text{(C.2)}$$

$$U_A = G_2 \frac{N_c}{(2\pi)^2} \left[ \left\{ \frac{1}{3} \left( 4M^2 - (\mu_r - U_A)(\mu_r + 2U_A) \right) \sqrt{(\mu_r - U_A)^2 - M^2} ight. ight.$$
$$-(\mu_r - 2U_A) M^2 \log \frac{\mu_r - U_A + \sqrt{(\mu_r - U_A)^2 - M^2}}{M} \right] \theta(\mu_r - (M + U_A)) \right]$$
$$- G_2 \frac{N_c}{(2\pi)^2} \left[ U_A \rightarrow -U_A \right]. \quad \text{(C.3)}$$
References


