UNITARITY CONSTRAINTS ON HELICITY FLIP AMPLITUDES

IN MESON-NUCLEON SCATTERING

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A B S T R A C T

From direct channel unitarity we derive bounds on the s channel helicity flip amplitude in terms of the non-flip amplitude. The conventional Regge pole parametrizations lead to values of the helicity flip amplitudes, which are rather close to these bounds.

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1. - INTRODUCTION

Some years ago, MacDowell and Martin ¹⁾ investigated constraints on scattering amplitudes which follow from direct channel unitarity and some well-established experimental data like total cross-sections and total elastic cross-sections. Especially they found a lower bound for the slope of the absorptive part of the scattering amplitude at $t = 0$, which reads for sufficiently high energies :

$$a = \left. \frac{\frac{d}{dt} \text{Im} A(s,t)}{\text{Im} A(s,t)} \right|_{t=0} \geq \frac{1}{9} \frac{\sigma_{\text{tot}}^2}{4\pi \sigma_{\text{el im}}} = a_0 \quad (1)$$

Here $\sigma_{\text{el im}}$ is that part of the total elastic cross-section which comes from the absorptive part of the scattering amplitude alone. Feeding in the experimental data for σ_{tot} and σ_{el} on the right-hand side, one gets for a_0 a value of about 3.5 to 4 GeV^{-2} which is rather close to the value of a one would expect from the slope of the diffraction peak. Recently, Roy and Singh ²⁾ have extended this result to values of $t \neq 0$.

In this paper, we want to consider constraints of the helicity flip amplitude which follow in a similar way from direct channel unitarity, the total cross-section, the total elastic cross-section and some additional assumptions on the non-flip amplitude. Such constraints may be interesting for Regge pole models where non-flip and flip residues are parametrized independently. If we neglect contributions of the real parts of the flip and non-flip amplitudes, we also get a lower bound for the slope of the diffraction peak

$$\left. \frac{d}{dt} \ln \frac{d\sigma}{dt} \right|_{t=0} \geq 2 a_0 \quad (2)$$

This bound follows immediately from the MacDowell-Martin formula, if spin flip contributions are neglected too. It is shown in Section 2 that the inequality (2) is true even if the absorptive parts of the spin flip contributions are taken into account.

In Sections 3 and 4 we derive a bound for the absorptive part of the helicity flip amplitude for the charge exchange process $\pi^- p \rightarrow \pi^0 n$, which implies a bound on the helicity flip residue of the φ Regge pole.

2. - A BOUND ON THE HELICITY FLIP AMPLITUDES IN THE ELASTIC MESON-NUCLEON SCATTERING

For the derivation of this bound, we have to solve the following maximizing problem.

The imaginary part of the non-flip amplitude and its first derivative with respect to t at $t = 0$

$$\begin{aligned} A_0 &= 2q \operatorname{Im} \varphi\left(\frac{1}{2}, \frac{1}{2}\right) \Big|_{t=0} = \sum_{\vartheta} (2\vartheta+1) (a_+^{\vartheta} + a_-^{\vartheta}) \\ A_1 &= 8q^3 \frac{d}{dt} \operatorname{Im} \varphi\left(\frac{1}{2}, \frac{1}{2}\right) \Big|_{t=0} = \sum_{\vartheta} (2\vartheta+1) (a_+^{\vartheta} + a_-^{\vartheta}) \vartheta(\vartheta+1) \end{aligned} \quad (3)$$

as well as that part of the total elastic cross-section which arises from the imaginary parts of the flip and non-flip amplitudes

$$\Sigma = \frac{q^2}{2\pi} \sigma_{el \text{ im}} = \sum_{\vartheta} (2\vartheta+1) (a_+^{\vartheta^2} + a_-^{\vartheta^2}) \quad (4)$$

are assumed to be given. Then we look for an upper bound on

$$B = \sqrt{8} q (1-z_s)^{-\frac{1}{2}} \operatorname{Im} \varphi\left(-\frac{1}{2}, \frac{1}{2}\right) \Big|_{t=0} = \sum_{\vartheta} (2\vartheta+1) (a_+^{\vartheta} - a_-^{\vartheta}) \left(\vartheta + \frac{1}{2}\right) \quad (5)$$

which is compatible with Eqs. (3), (4) and the unitary inequalities

$$0 \leq a_{\pm}^{\vartheta} \leq 1 \quad (6)$$

The problem is solved by means of the method with Lagrangian multipliers ³⁾. One easily realizes that the following distribution of partial wave amplitudes leads to an upper bound on B :

$$a_{\pm}^J = \begin{cases} \frac{1}{2\sigma} \left[\pm \left(J + \frac{1}{2} \right) - \alpha_0 - \alpha_1 J (J+1) \right] & \text{for } J < J_{\pm} \\ 0 & \text{for } J > J_{\pm} \end{cases} \quad (7)$$

where $\sigma \geq 0$, $\alpha_0 \leq 0$, $\alpha_1 \geq 0$.

The parameters α_0 , α_1 and σ are just the Lagrangian multipliers which correspond to A_0 , A_1 and Σ . They are determined by inserting the distribution (7) into Eqs. (3), (4). $J_{\pm}$ is the largest value of J which is compatible with the positivity condition of the partial wave amplitudes. This means, for sufficiently large t ,

$$\pm \left(J_{\pm} + \frac{1}{2} \right) - \alpha_0 - \alpha_1 J_{\pm} (J_{\pm} + 1) = 0 \quad (8)$$

We are left with the problem of calculating the Lagrangian multipliers α_0 , α_1 and σ . Substituting the sums over J by integrals, we are led to the following equations :

$$\begin{aligned} A_0 &= \frac{1}{12} \frac{x_+^3}{\sigma(1-\xi)} P_0(\xi) \\ A_1 &= \frac{1}{30} \frac{x_+^5}{\sigma(1-\xi)} P_1(\xi) \\ \Sigma &= \frac{1}{120} \frac{x_+^4}{\sigma^2(1-\xi)^2} P_2(\xi) \end{aligned} \quad (9)$$

where $x_+ = J_+ + \frac{1}{2}$ and $\xi = (J_- + \frac{1}{2}) / (J_+ + \frac{1}{2})$ are related to the Lagrangian multipliers α_0 and α_1 via Eq. (8). The functions $P_i(\xi)$ $i = 0, 1, 2$ are polynomials in ξ :

$$P_0(\xi) = 1 + 2\xi + 2\xi^3 + \xi^4$$

$$P_1(\xi) = 1 + \frac{3}{2}\xi + \frac{3}{2}\xi^5 + \xi^6$$

(10)

$$P_2(\xi) = 1 + 4\xi + 5\xi^2 + 5\xi^4 + 4\xi^5 + \xi^6$$

In terms of the quantities x_+ , ξ and σ the upper bound on B reads :

$$B \leq \frac{1}{20} \frac{x_+^4}{\sigma(1-\xi)} P_3(\xi) \quad (11)$$

where

$$P_3(\xi) = 1 + \frac{5}{3}\xi - \frac{5}{3}\xi^4 - \xi^5 \quad (12)$$

It is quite easy to eliminate σ and x_+ such that the bound (11) only depends on the parameter ξ :

$$b = \frac{1}{\sqrt{-t}} \frac{\text{Im} \varphi(-\frac{1}{2}, \frac{1}{2})}{\text{Im} \varphi(\frac{1}{2}, \frac{1}{2})} \Big|_{t=0} = \frac{1}{2q} \frac{B}{A_0} \leq \frac{0.6 \sqrt{2}}{18\pi} \frac{\sigma_{tot}}{\sqrt{\sigma_{el} \text{ in}}} \cdot \frac{P_3(\xi) \sqrt{P_2(\xi)}}{P_0(\xi)^2} \quad (13)$$

The remaining parameter ξ is determined from

$$a = \frac{\frac{d}{dt} \text{Im} \varphi(\frac{1}{2}, \frac{1}{2})}{\text{Im} \varphi(\frac{1}{2}, \frac{1}{2})} \Big|_{t=0} = \frac{1}{4q^2} \frac{A_1}{A_0} = \frac{0.48}{8\pi} \frac{\sigma_{tot}^2}{\sigma_{el} \text{ in}} \frac{P_1(\xi) P_2(\xi)}{P_0(\xi)^3} \quad (14)$$

The positivity condition of the partial wave amplitudes (7) is guaranteed by Eq. (8). In order to ensure the upper bound on the partial wave amplitudes [cf. Eq. (6)] we have to take into account the following condition :

$$\frac{10}{8} \frac{\sigma_{el, \text{flip}}}{\sigma_{\text{tot}}} \frac{P_0(\xi)}{P_1(\xi)} (1+\xi)^2 \leq 1 \quad (15)$$

In Figure 1, we have plotted b versus a for the case of π^-p scattering at $p_{\text{lab}} = 18$ GeV, where $\sigma_{\text{tot}}(\pi^-) = 25.5$ mb and $\sigma_{\text{el}}(\pi^-) = 3.8$ mb. We get this value of $\sigma_{\text{el}}(\pi^-)$ if we take for $d\sigma/dt$ in the interval $0 \leq |t| \leq 0.2$ the data of Foley et al. ⁴⁾ and in the interval $|t| > 0.2$ GeV² the data of Harting et al. ⁵⁾. Notice that the value $\sigma_{\text{el}}(\pi^-) = 4.1$ mb, which one can find for instance in the data compilation of Giacomelli ⁶⁾, has been calculated without taking into account the more precise data of Foley et al. ⁴⁾. The dotted line in Fig. 1 represents the bound on b for the value $\sigma_{\text{el}}(\pi^-) = 4.1$ mb.

The bound on b increases with a which simply means that the flip amplitude can increase more rapidly if the non-flip amplitude decreases faster. Notice that the bound on b becomes zero at $a = a_0 = 3.95$. This is just the lowest value of a which is allowed according to the MacDowell-Martin bound [cf. Eq. (1)]. The bound on b implies a bound on the slope of the diffraction peak, if we neglect the real parts of the helicity flip and non-flip amplitudes

$$\frac{d}{dt} \ln \frac{d\sigma}{dt} = 2a - b^2 \geq 2a - b_{\text{max}}^2 \quad (16)$$

Due to the parabolic structure of the bound on b , the right-hand side is always larger than $2a_0$. Thus it turns out that the MacDowell-Martin bound (2) is true even if the imaginary part of the helicity flip amplitude is taken into account. We have also calculated the values for a and b from some Regge pole fits. Point 1 represents solution I, point 2 solution III of the fits of Rarita et al. ⁷⁾. Point 2 shows indeed a considerable violation of the bound. It is interesting to see

that this solution is also in disagreement with the finite energy sum rules ⁸⁾. Finally point 3 corresponds to a parametrization where the Pomeron trajectories contribute to the invariant amplitude A , i.e., the t channel helicity non-flip amplitude alone ⁹⁾.

Figure 2 shows the corresponding bound for π^+p scattering at $p_{\text{lab}} = 18$ GeV, where $\sigma_{\text{tot}}(\pi^+) = 23.8$ mb and $\sigma_{\text{el}}(\pi^+) = 3.6$ mb. Summing up the bounds for the π^-p and the π^+p cases, one can isolate a bound on the $I_t = 0$ - i.e., the P, P' - helicity flip contributions. A similar bound on the Pomeron residues in NN scattering has been derived already in Ref. 10).

3. - A BOUND ON THE HELICITY FLIP CONTRIBUTIONS FOR THE CHARGE EXCHANGE PROCESS $\pi^-p \rightarrow \pi^0n$

The derivation of a bound on inelastic amplitudes is more complicated, since their partial wave amplitudes do not have positivity properties. However, the partial wave amplitudes for the charge exchange processes $\pi^-p \rightarrow \pi^0n$ and for the elastic processes $\pi^\pm p \rightarrow \pi^\pm p$ are simply connected by isospin invariance. This means that we have to incorporate the information we have about the elastic processes in our maximizing program.

The total cross-section $\sigma_{\text{tot}}(\pi^\pm)$, the total elastic cross-sections $\sigma_{\text{el im}}(\pi^\pm)$ and the total cross-section for the charge exchange process $\sigma_{\text{im}}(\pi^0)$ are assumed to be given. In order to get a convergent bound, we also need the first derivative of the $I_t = 0$ non-flip amplitude at $t = 0$, i.e., $A_1(\pi^+) + A_1(\pi^-)$ [cf. Eq. (3)]. Then we look for an upper bound on

$$2 B^{(-)} = (B(\pi^+) - B(\pi^-)) \quad (17)$$

$$= \sum_{\mathfrak{F}} (2\mathfrak{F}+1) (a_+^{\mathfrak{F}}(\pi^+) - a_-^{\mathfrak{F}}(\pi^+) - a_+^{\mathfrak{F}}(\pi^-) + a_-^{\mathfrak{F}}(\pi^-)) (\mathfrak{F} + \frac{1}{2})$$

which is compatible with the unitarity inequalities

$$0 \leq a_{\pm}^{\pm}(\pi^{\pm}) \leq 1 \quad (18)$$

The auxiliary function for $2B^{(-)}$ reads in this case

$$\begin{aligned} L(a_{\pm}^{\pm}(\pi^{\pm})) = & 2B^{(-)} + \sum_{Q=\pm} \alpha_0(\pi^Q) \left(A_0(\pi^Q) - \sum_{\pm} (2\pm+1) (a_{+}^{\pm}(\pi^Q) + a_{-}^{\pm}(\pi^Q)) \right) \\ & + \alpha_1 \sum_{Q=\pm} \left(A_1(\pi^Q) - \sum_{\pm} (2\pm+1) (a_{+}^{\pm}(\pi^Q) + a_{-}^{\pm}(\pi^Q)) \right) \pm (\pm+1) \\ & + \sum_{Q=\pm} \sigma(\pi^Q) \left(\sum_{\pm} (\pi^Q) - \sum_{\pm} (2\pm+1) (a_{+}^{\pm}(\pi^Q)^2 + a_{-}^{\pm}(\pi^Q)^2) \right) \\ & + \sigma(\pi^0) \left(\sum_{\pm} (\pi^0) - \sum_{\pm} (2\pm+1) [(a_{+}^{\pm}(\pi^+) - a_{+}^{\pm}(\pi^-))^2 + (a_{-}^{\pm}(\pi^+) - a_{-}^{\pm}(\pi^-))^2] \right) \end{aligned} \quad (19)$$

A maximum of $B^{(-)}$ occurs if

$$\frac{\partial L}{\partial a_s^{\pm}(\pi^Q)} = 0 \quad s = \pm, \quad Q = \pm \quad (20)$$

and if the 2×2 matrix of the second order derivatives

$$\begin{pmatrix} \frac{\partial^2 L}{\partial a_{+}^{\pm}(\pi^+)^2} & \frac{\partial^2 L}{\partial a_{+}^{\pm}(\pi^+) \partial a_{+}^{\pm}(\pi^-)} \\ \frac{\partial^2 L}{\partial a_{-}^{\pm}(\pi^+) \partial a_{-}^{\pm}(\pi^-)} & \frac{\partial^2 L}{\partial a_{-}^{\pm}(\pi^-)^2} \end{pmatrix} \quad (21)$$

has non-negative eigenvalues. The last condition imposes the following constraints on the Lagrangian multipliers $\sigma(\pi^{\pm})$, $\sigma(\pi^0)$

$$\sigma(\pi^{\pm}) + \sigma(\pi^0) \geq 0 \quad (22)$$

$$d = (\sigma(\pi^+) + \sigma(\pi^0))(\sigma(\pi^-) + \sigma(\pi^0)) - \sigma(\pi^0)^2 \geq 0 \quad (23)$$

From Eq. (20) one easily derives the following distribution of partial wave amplitudes which leads to a maximum of $B^{(-)}$:

a) $S = Q = \pm$

$$a_s^{\pm}(\pi^Q) = \begin{cases} \frac{\sigma(\pi^Q)}{2d} (\mathfrak{J} + \frac{1}{2} - \bar{\alpha}^{\pm}(\pi^Q)) & \text{for } \mathfrak{J} < \mathfrak{J}_s^Q \\ \frac{\mathfrak{J} + \frac{1}{2} - \alpha^{\pm}(\pi^Q)}{2(\sigma(\pi^Q) + \sigma(\pi^0))} & \text{for } \mathfrak{J}_s^Q < \mathfrak{J} < \mathfrak{J}_s^Q \\ 0 & \text{for } \mathfrak{J} > \mathfrak{J}_s^Q \end{cases} \quad (24)$$

b) $S = -Q = \pm$

$$a_s^{\pm}(\pi^Q) = \begin{cases} \frac{\sigma(\pi^Q)}{2d} (-\mathfrak{J} - \frac{1}{2} - \bar{\alpha}^{\pm}(\pi^Q)) & \text{for } \mathfrak{J} < \mathfrak{J}_s^Q \\ 0 & \text{for } \mathfrak{J} > \mathfrak{J}_s^Q \end{cases} \quad (25)$$

Here we have used the following notations :

$$\begin{aligned} \alpha^{\pm}(\pi^Q) &= \alpha_0(\pi^Q) + \alpha_1 \mathfrak{J}(\mathfrak{J}+1) \\ \bar{\alpha}^{\pm}(\pi^Q) &= \alpha(\pi^Q) + (\alpha(\pi^+) + \alpha(\pi^-)) \frac{\sigma(\pi^0)}{\sigma(\pi^{\pm})} \end{aligned} \quad (26)$$

The four J values J_S^Q , $S = \pm$, $Q = \pm$ are determined in such a way that all the partial wave amplitudes obey the positivity condition (18) :

$$\begin{aligned} \mathfrak{J}_s^Q + \frac{1}{2} - \alpha^{\pm}(\pi^Q) &= 0 \quad \text{for } S = Q \\ -(\mathfrak{J}_s^Q + \frac{1}{2}) - \bar{\alpha}^{\pm}(\pi^Q) &= 0 \quad \text{for } S = -Q \end{aligned} \quad (27)$$

4. - EVALUATION OF THE BOUND

If we insert the distribution (24), (25) into the partial wave expansions for $A_0(\pi^\pm)$, $A_1(\pi^+) + A_1(\pi^-)$, $\Sigma(\pi^\pm)$ and $\Sigma(\pi^0)$, we are led to a system of six non-linear equations for the six Lagrangian multipliers $\alpha_0(\pi^\pm)$, α_1 , $\sigma(\pi^\pm)$ and $\sigma(\pi^0)$, which in general has to be solved numerically. For a study of the structure of the bound, we consider first the simpler case, where

$$\alpha_0(\pi^+) = \alpha_0(\pi^-) = \alpha_0 \quad \sigma(\pi^+) = \sigma(\pi^-) = \sigma \quad (28)$$

That means we calculate a bound on $B^{(-)}$, where only the sums

$$A_i^{(+)} = \frac{1}{2} (A_i(\pi^+) + A_i(\pi^-)) \quad i = 0, 1$$

$$\Sigma^{(+)} = \frac{1}{2} (\Sigma(\pi^+) + \Sigma(\pi^-))$$

and $\Sigma(\pi^0)$ are assumed to be given. The specialization (28) leads to a distribution of partial wave amplitudes, where

$$\begin{aligned} a_+^{\tilde{+}}(\pi^+) &= a_-^{\tilde{-}}(\pi^-) \\ a_-^{\tilde{-}}(\pi^+) &= a_+^{\tilde{+}}(\pi^-) \end{aligned} \quad (29)$$

which provides

$$A_0(\pi^+) = A_0(\pi^-) \quad (30)$$

$$\Sigma(\pi^+) = \Sigma(\pi^-) \quad (31)$$

$$B(\pi^+) = -B(\pi^-) \quad (32)$$

Since the experimental values for

$$\frac{\sigma_{tot}(\pi^+) - \sigma_{tot}(\pi^-)}{\sigma_{tot}(\pi^+) + \sigma_{tot}(\pi^-)} \quad (33)$$

and

$$\frac{\sigma_{el}(\pi^+) - \sigma_{el}(\pi^-)}{\sigma_{el}(\pi^+) + \sigma_{el}(\pi^-)} \quad (34)$$

are indeed quite small, we expect that the restriction (28) does not worsen essentially the bound on $B^{(-)}$. Notice that this bound on $B^{(-)}$ can be saturated only if $B^{(+)} = \frac{1}{2}(B(\pi^+) + B(\pi^-)) = 0$ [cf. Eq. (32)]^{*}. The system of equations for the four Lagrangian multipliers reads :

$$\begin{aligned} A_0^{(+)} &= \frac{x_+^3}{\sigma} \frac{1}{1-\xi^2} \frac{1}{1+s} \frac{1}{6} P_0(\xi, s) \\ A_1^{(+)} &= \frac{x_+^5}{\sigma} \frac{1}{1-\xi^2} \frac{1}{1+s} \frac{1}{30} P_1(\xi, s) \\ \Sigma^{(+)} &= \frac{x_+^4}{\sigma^2} \frac{1}{(1-\xi^2)^2} \frac{1}{(1+s)^2} \frac{1}{60} P_2(\xi, s) \\ \Sigma(\pi^0) &= \frac{x_+^4}{\sigma^2} \frac{1}{(1-\xi^2)^2} \left(\frac{1}{(1+s)^2} \frac{1}{60} P_2(\xi, s) - \frac{1}{12} P_3(\xi, s) \right) \end{aligned} \quad (35)$$

where

$$\begin{aligned} x_+ &= \beta_+^+ + \frac{1}{2} = \beta_-^- + \frac{1}{2} \\ x_- &= \beta_+^- + \frac{1}{2} = \beta_-^+ + \frac{1}{2} \end{aligned} \quad (36)$$

^{*}) This means in the Regge pole model that the Pomeron trajectories P and P' obey conservation of s channel helicities.

are related to α_0 and α_1 via Eq. (27), i.e.,

$$\alpha_0 = - \frac{x_+ x_-}{x_+^2 - x_-^2} (x_- + S x_+) \quad (37)$$

$$\alpha_1 = \frac{x_+ + S x_-}{x_+^2 - x_-^2}$$

Furthermore, we have used the abbreviations

$$\xi = \frac{x_-}{x_+} \quad (38)$$

$$S = \frac{\sigma}{\sigma + 2\sigma(\pi^0)} \quad (39)$$

The P 's are polynomials in ξ and S ; they are listed in the Table.

In terms of the parameters x_+ , S , σ and ξ , the bound on B can be written as

$$B^{(+)} \leq \frac{x_+^4}{\sigma} \frac{S}{1+S} \frac{1}{1-\xi^2} \frac{1}{3\sigma} P_4(\xi, S) \quad (40)$$

It is quite easy to eliminate x_+ and σ . The remaining parameters ξ and S are determined from

$$\frac{\Sigma(\pi^0)}{\Sigma^{(+)}} = R = 1 - S(1+S)^2 \frac{P_3(\xi, S)}{P_2(\xi, S)} \quad (41)$$

$$a^{(+)} = \frac{\frac{d}{dt} \Im \varphi^{(+)}(\frac{1}{2}, \frac{1}{2})}{\Im \varphi^{(+)}(\frac{1}{2}, \frac{1}{2})} \Big|_{t=0} = \frac{1}{4q^2} \frac{A_1^{(+)}}{A_0^{(+)}}$$

$$= \frac{0.12}{8\pi} \frac{\sigma_{tot}^{(+)^2}}{\sigma_{el inu}^{(+)}} \frac{P_1(\xi, S) P_2(\xi, S)}{P_0(\xi, S)^3} \quad (42)$$

$$5(1+S) \frac{\sigma_{el}^{(+)} \text{ inu}}{\sigma_{tot}^{(+)}} \frac{P_0(\xi, S)}{P_2(\xi, S)} \left(\frac{S^2}{4} \frac{(1-\xi)^2}{1+S\xi} + \xi(\xi+S) \right) \leq 1 \quad (43)$$

$$5S \frac{\sigma_{el}^{(+)} \text{ inu}}{\sigma_{tot}^{(+)}} \frac{P_0(\xi, S)}{P_2(\xi, S)} \left(\frac{1}{4} \frac{(1-\xi)^2}{1+S\xi} + \xi(\xi+S) \right) \leq 1 \quad (44)$$

The inequalities (43) and (44) have to be satisfied in order to ensure the upper bounds [cf. Eq. (18)] on the partial wave amplitudes (24) and (25). Finally, we get for the bound on the $I_t = 1$ helicity flip amplitude

$$b^{(-)} = \frac{1}{\sqrt{-t}} \frac{\Im m \varphi^{(-)}(-\frac{1}{2}, \frac{1}{2})}{\Im m \varphi^{(+)}(\frac{1}{2}, \frac{1}{2})} \Big|_{t=0} = \frac{1}{2q} \frac{B^{(-)}}{A_0^{(+)}} \quad (45)$$

$$\leq \frac{0.2 \sqrt{0.6}}{\sqrt{8\pi}} \frac{\sigma_{tot}^{(+)}}{\sqrt{\sigma_{el}^{(+)} \text{ inu}}} \frac{P_+(\xi, S) / \sqrt{P_2(\xi, S)}}{P_0(\xi, S)^2}$$

where

$$\sigma_{tot}^{(+)} \text{ el inu} = \frac{1}{2} \left(\sigma_{tot}^{(+)}(\pi^+) \text{ el inu} + \sigma_{tot}^{(+)}(\pi^-) \text{ el inu} \right) \quad (46)$$

Figure 3 shows the bound on $b^{(-)}$ versus $a^{(+)}$ for $p_{lab} = 18$ GeV where

$$\sigma_{tot}^{(+)} = 24.75 \text{ mb} \quad \sigma_{el}^{(+)} = 3.75 \text{ mb} \quad \sigma_{el}(\pi^0) = 24 \mu b \quad (47)$$

The bound on $b^{(-)}$ increases with $a^{(+)}$; it vanishes at the MacDowell-Martin point $a^{(+)} = a_0$ [cf. Eq. (1)]. The quantity $b^{(-)}$ is related to the ratio b_-/b_+ of the t channel ϱ - residues¹¹⁾ via

$$b^{(-)} = \frac{\sigma^{(-)}}{\sigma^{(+)}} \frac{1}{2M} \left(\frac{b_-}{b_+} \alpha_\varrho(0) - 1 \right) \quad (48)$$

The single ϱ Regge pole model¹⁰⁾ of the charge exchange processes $\pi^- p \rightarrow \pi^0 n$ provides a rather large value for the ratio b_-/b_+ of about 20. Taking for $(\sigma^{(-)}/\sigma^{(+)}) = 0.035$ the experimental value at $p_{\text{lab}} = 18$ GeV and $\alpha_\varrho(0) = 0.57$ one finds for $b^{(-)}$ a value of about 0.19. The straight line $b^{(-)} = 0.19$ in Fig. 3 indicates that this value is compatible with the unitarity bound on $b^{(-)}$ only if $a^{(+)}$ is larger than 3.93. Indeed, if we take the single ϱ Regge pole model seriously, the bound on $b^{(-)}$ can be improved since in this case

$$\sigma_{\text{el inel}}(\pi^0) = \int_{-t_0}^0 dt \frac{d\sigma}{dt} \cos^2 \pi \frac{\alpha_\varrho(t)}{2} \approx 14 \mu b \quad (49)$$

where $\alpha_\varrho(t) = 0.57 + t$, $t_0 = 1.5 (\text{GeV})^2$.

The bound which corresponds to this value of $\sigma_{\text{el inel}}(\pi^0)$ is represented in Fig. 3 by the dotted line. In this case $a^{(+)}$ must be larger than 4.04.

We have also calculated the bound for the general case where the individual cross-sections $\sigma_{\text{tot}}(\pi^\pm)$ and $\sigma_{\text{el}}(\pi^\pm)$ are really taken into account. However, we could not find any essential improvement of the bound. In order to understand this feature, let us concentrate on the case

$$\alpha_0(\pi^+) \neq \alpha_0(\pi^-)$$

but

$$\sigma(\pi^+) = \sigma(\pi^-)$$

If we consider the quantities

$$F = (A_0^{(+)} , A_1^{(+)} , Z^{(+)} , Z(\pi^0) , B) \quad (50)$$

and

$$G = A_0(\pi^+) - A_0(\pi^-) \quad (51)$$

as functions of the Lagrangian multiplier

$$\Delta\alpha = \alpha_0(\pi^+) - \alpha_0(\pi^-) \quad (52)$$

it turns out that

$$F(\Delta\alpha) = F(\Delta\alpha=0) + O(\Delta\alpha^2) \quad (53)$$

$$G(\Delta\alpha) = O(\Delta\alpha) \quad (54)$$

Since the difference (51) of the total cross-sections is quite small, we expect only a small value of $\Delta\alpha$ [cf. Eq. (54)]. In a first order approximation, however, where only linear terms in $\Delta\alpha$ are taken into account, the quantities (50) do not change. Therefore, we cannot expect an improvement of the bound which is verified by a numerical calculation.

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$$P_0(\xi, s) = 3\xi^4 + s(1 + 2\xi^2 + 2\xi^3 + \xi^5) + 3s^2\xi$$

$$P_1(\xi, s) = 5\xi^6 + s(2 + 3\xi^2 + 3\xi^5 + 2\xi^7) + 5s^2\xi$$

$$\begin{aligned} P_2(\xi, s) = & 10\xi^6 + 10s(3\xi^5 - \xi^7) \\ & + s^2(2 + 8\xi^2 + 25\xi^4 - 32\xi^5 - 20\xi^6 + 32\xi^7 - 5\xi^8) \\ & + 2s^3(6\xi + 14\xi^3 + 5\xi^4 - 15\xi^5 + 4\xi^6 + 5\xi^7 + \xi^8) \\ & + 5s^4(4\xi - 3\xi^4 + \xi^8) \end{aligned}$$

$$P_3(\xi, s) = 2\xi^6 + 2s(3\xi^5 - \xi^7) + s^2(3\xi^4 - 16\xi^8)$$

$$P_4(\xi, s) = 3 + 5\xi^2 - 8\xi^5 + s(8\xi - 5\xi^4 - 3\xi^6)$$

TABLE : Definition of the polynomials $P_i(\xi, s)$
in Eqs. (35) and (40).

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FIGURE CAPTIONS

Figure_1 The bound on the helicity flip amplitude $b(\pi^-)$ [cf. Eq. (13)] as a function of the first derivative of the non-flip amplitude $a(\pi^-)$ at $t = 0$ [cf. Eq. (14)] for two different values of $\sigma_{el}(\pi^-)$:
dotted line $\sigma_{el}(\pi^-) = 4.1$ mb ;
solid line $\sigma_{el}(\pi^-) = 3.8$ mb.
The points 1 and 2 correspond to solution I and III of Ref. 7). The point 3 comes from a fit where the Pomeron contributes only to the invariant amplitude A [cf. Ref. 9)].

Figure_2 The bound on the helicity flip amplitude $b(\pi^+)$ [cf. Eq. (13)].

Figure_3 The bound on the $I_t = 1$ helicity flip amplitude $b^{(-)}$ [cf. Eq. (45)] as a function of the first derivative of the $I_t = 0$ non-flip amplitude $a^{(+)}$ at $t = 0$ [cf. Eq. (42)].

FIG.1

FIG. 2

FIG. 3