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ASYMPTOTIC EQUALITY OF CROSS-SECTIONS
FOR LINE-REVERSED REACTIONS

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A B S T R A C T

We show that if the ratio $\frac{d\sigma}{dt}(AB \rightarrow CD) / \frac{d\sigma}{dt}(\bar{C}\bar{B} \rightarrow \bar{A}\bar{D})$ has a generalized limit, for energies tending to infinity at fixed momentum transfer, this generalized limit is unity, if, in addition, the phases of the two amplitudes grow separately less fast than the logarithm of the energy. It is shown by explicit counter-examples that it is not sufficient to impose such a condition on the relative phase of the two amplitudes. Generalizations to non-fixed momentum transfer (case of a shrinking forward peak) are discussed.

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1. - INTRODUCTION

This paper is the first of a series of papers on generalization of the Pomeranchuk theorem to differential and integrated cross-sections for line reversed processes. What we want to discuss is the limit

$$\frac{d\sigma}{dt}(AB \rightarrow CD) / \frac{d\sigma}{dt}(\bar{C}B \rightarrow \bar{A}D)$$

for fixed momentum transfer (or possibly a momentum transfer going to zero very slowly as $s \rightarrow \infty$) and s (square of centre-of-mass energy) going to infinity. This question has been discussed first on models. For instance, Van Hove ¹⁾, assuming that the amplitude for the direct reaction behaves like $sf^P(t)$ proves that $f^P(t) = -f^{A*}(t)$ (P for particles, A for anti-particles) and that therefore $d\sigma^P/dt = d\sigma^A/dt$. This was generalized by Logunov et al. ²⁾ to amplitudes of the type

$$s^{\alpha(t)} (\log s)^{\beta(t)} (\log \log s)^{\gamma(t)}$$

with α , β and γ real.

Later one began to realize what was the relevant feature of this model : it was in fact that the phase of the reaction amplitude has a limit at infinite energy. One of us ³⁾ showed that if, above a certain energy, the phase of both reaction amplitudes stays between $n\pi$ and $(n+1)\pi$ and $n'\pi$ and $(n'+1)\pi$, the limit of $(d\sigma^P/dt)/(d\sigma^A/dt)$, if it exists in the generalized sense (i.e., including possibly zero and infinity), has to be unity. On the other hand, Meiman ⁴⁾ proposed a weaker condition to ensure $(d\sigma^P/dt)/(d\sigma^A/dt) \rightarrow 1$: he defines reduced amplitudes

$$\hat{F}_P(s, t, u) = F(s, t, u) \frac{\prod_i (s - s_i^*)}{\prod_i (s - s_i)}$$

where the s_i are the zeros of F in the upper half plane and demands that the phase of $\hat{F}_P/\hat{F}_A$ be growing less fast than $\log s$ for $s \rightarrow \infty$. Of course, if F has no zeros or a finite number of zeros this is reduced to a condition on the phase of F_P/F_A whereas if this number is not finite we have to take into account the phases coming from the zeros. Last year

we have found that a manner to overcome this difficulty is to require similar conditions separately on the phases of the two amplitudes. This result was quoted on different occasions ⁵⁾. In a recent paper ⁶⁾ Medvedev and Lomsadze claim that the condition $\text{phase}(F_P/F_A)/\log s \rightarrow 0$ together with $\log|F_P/F_A|/\log s \rightarrow 0$ is sufficient to ensure $(d\sigma^P/dt)/(d\sigma^A/dt) \rightarrow 1$ at least for a sequence of energies. We do not understand this statement, for which an explicit counter-example had already been built by Gribov (quoted by Meiman ⁴⁾). Here we shall prove that the generalized limit of $(d\sigma^P/dt)/(d\sigma^A/dt)$ if it exists has to be unity if separately $(\text{phase of } F^P)/\log s$ and $(\text{phase of } F^A)/\log s$ go to zero as s goes to infinity. These conditions cannot be weakened. A counter-example has been already proposed by Białyas, Siciak and Wit ⁷⁾ for $\text{phase}(F^P/F^A) \sim \log s$. We also construct counter-examples where $\text{phase } F^P + \text{phase } F^A \sim \log s$.

In the limit case $|\text{phase } F^P| \leq \log s$, $|\text{phase } F^A| \leq \log s$ we can still manage to prove something, namely that $\lim(d\sigma^P/dt)/(d\sigma^A/dt)$ cannot be zero or infinity.

In the last section, we discuss the possible generalizations of these results to the case where t , instead of being fixed, tends slowly to zero like, for instance, $-(\log s)^{-\alpha}$. This is a non-trivial complication, contrary to the general belief.

To end this Introduction, let us insist on the fact that the present work is only a building block inside a larger construction, because the condition on the phases must be justified from outside. In the case of elastic amplitudes, positivity and analyticity help supplying the necessary information. This will be the object of a future publication and has been already partly summarized at the 1971 Ettore Majorana International School of Subnuclear Physics ⁵⁾.

2. - THE ZEROS OF THE AMPLITUDE

We remind the reader that

$$F_P(s, t) = F(s + i\epsilon, u, t) \quad s > 0 \quad (1)$$

with

$$s + t + u = \sum M_i^2$$

$$F_A(s, t) = F(u, s + i\epsilon, t) \quad (2)$$

and if one assumes T invariance

$$F_P(\sum M_i^2 - t - s + i\epsilon, t) = F_A^*(s, t)$$

It is convenient, for fixed real t , to introduce a variable

$$E = \frac{s - u}{2} = s - \sum \frac{M_i^2}{2} + \frac{t}{2} \quad (3)$$

then

$$F_P(-E + i\epsilon) = F_A^*(E + i\epsilon) \quad (4)$$

$F_P(E)$ is analytic in the upper half plane apart possibly a finite region ⁸⁾, and furthermore for $t \leq 0$ it is polynomially bounded ⁹⁾. We call $\phi(E)$ the phase of F for $E = \text{real} + i\epsilon$. Hence we have $\phi(E) = \phi_P(E)$ for $E > 0$ where ϕ_P is the phase of the amplitude $A + B \rightarrow C + D$ and $\phi(-E) = -\phi_A(E) + n\pi$ where ϕ_A is the phase of the amplitude $\bar{C} + B \rightarrow \bar{A} + D$. Of course to make the phase unique we have to assume that $|F_P(E)|$ and $|F_A(E)|$ have no zeros above a certain energy and that $F_P(E)$ and $F_A(E)$ are continuous. This assumption, incidentally can be checked experimentally for $t \leq 0$ since the vanishing of the modulus of F corresponds to the vanishing of the differential cross-section.

Let us now define

$$G(E) = \exp \frac{E}{\pi} \int_{-\infty}^{+\infty} \frac{\phi(E') dE'}{E'(E'-E)} \quad (5)$$

The integral appearing in (5) is meaningful because all through this paper we assume $|\phi(E)| \leq \text{const } \log E$. $G(E)$, by construction, has the same phase as $F(E)$ and further has no zeros and no poles in the cut complex E plane. Hence $F(E)/(F(0)G(E))$ is a real entire function which has the same zeros as $F(E)$.

With the condition $|\phi(E)| < \text{const } \log|E|$ one can easily get a bound on $G(E)$ in complex directions :

$$|G(E)| > \exp \left[-\text{Const. } \log|E| \left(\log|E| + \log \frac{|E|}{\text{Im } E} \right) \right] \quad (6)$$

From the fact that $|F(E)|$ is bounded by a polynomial it is not very difficult to conclude that the entire function $F(E)/F(0)G(E)$ is of order zero and can therefore be written as a convergent product over the zeros of F . The only difficulty lies in the fact that the bound (6) is singular for $\text{Im } E = 0$. However, this difficulty is easily got rid of by noticing that $\log|F(E)/G(E)|$ is subharmonic and therefore less or equal to its mean value along a circle centered at the point under consideration. Since this integral is convergent, there is no problem. Hence

$$\begin{aligned} F(E)/(F(0) \cdot G(E)) &= H(E) \\ H(E) &= \prod_i \left(1 - \frac{E}{E_i}\right) \prod_j \left(1 - \frac{E}{E_j}\right) \prod_j \left(1 - \frac{E}{E_j^*}\right) \\ &\equiv H_R(E) \cdot H_+(E) \cdot H_+^*(E^*) \end{aligned} \quad (7)$$

The E_i 's are the real zeros, which, by assumption, are confined in a finite region. The E_j 's are the complex zeros, with $\text{Im } E_j > 0$.

From the polynomial bound of $F(E)$ and the estimate (6), a more precise estimate can be made about the zeros of the entire function $H(E)$.
Using the notations of Boas ¹⁰⁾

$n(t)$ = number of zeros inside the circle $|E| < t$

$$N(t) = \int_0^t \frac{n(t')}{t'} dt' \quad (8)$$

we have, according to Jensen's theorem

$$N(|E|) \leq \overline{\log. (H(|E|e^{i\theta}))} \leq \text{Const.} (\log |E|)^2 \quad (9)$$

and

$$n(|E|) \leq \frac{N(|E|^2)}{\int_{|E|}^{|E|^2} \frac{dt}{t}} \leq \text{const.} \log |E|. \quad (10)$$

Let us now see what happens if, in addition, one assumes

$$\frac{|\phi_P(E) + \phi_A(E)|}{\log. E} \rightarrow 0 \quad (11)$$

or equivalently

$$\frac{|\phi(E) - \phi(-E)|}{\log. E} \rightarrow 0 \quad (12)$$

Then, from (5), one finds ^{*}), instead of (6)

$$|\log G(E)| < \sigma (\log |E|)^2 + O \log \left(\frac{|E|}{|\text{Im} E|} \right) \quad (13)$$

^{*}) We recall ¹⁰⁾ that $f(x) = o(g(x))$ means that $f(x)/g(x)$ approaches zero.

and hence

$$\begin{aligned} N(|E|) &< \sigma (\log |E|)^2 \\ n(|E|) &< \sigma (\log |E|) \end{aligned} \quad (14)$$

With these estimates one can find a bound on the phase of the product $\prod (1 - E/E_j)$ for $\text{Im } E \leq 0$. We have

$$\begin{aligned} \left| \text{Arg } \prod \left(1 - \frac{E}{E_j} \right) \right| &\leq \left| \text{Arg } \prod_{|E_j| < 2|E|} \left(1 - \frac{E}{E_j} \right) \right| + \left| \text{Arg } \prod_{|E_j| \geq 2|E|} \left(1 - \frac{E}{E_j} \right) \right| \\ &\leq n(2|E|) \cdot \pi + 2|E| \int_{2|E|}^{\infty} \frac{dn(t)}{t} \\ &\leq (\pi + 2) (n(2|E|)) + 2|E| \int_{2|E|}^{\infty} \frac{n(t) dt}{t^2} \end{aligned} \quad (15)$$

In summary :

(i) if we only assume

$$|\phi_P| \text{ and } |\phi_A| \leq \text{const. } \log E \quad (16)$$

we get

$$\left| \text{Arg } \prod_j \left(1 - \frac{E}{E_j} \right) \right| \leq \text{const. } \log E \quad (17)$$

(ii) if we make the stronger assumption (11) together with (16), we get

$$\left| \text{Arg. } \prod_j \left(1 - \frac{E}{E_j} \right) \right| = o \log E \quad (18)$$

Finally we would like to notice that the result (18) holds with weaker assumptions than (11)-(16). In fact, as it is easily verified, $|\rho^P + \rho^A| = o((\log E)^2)$ and $|\rho^P - \rho^A| = o(\log E)$ are sufficient assumptions in order to get, from (5), almost the same bound as (13) for $|\log G|$ and the bound (18) for the product of the zeros.

3. - THE RATIO $(d\sigma^P/dt)/(d\sigma^A/dt)$

The easiest way of getting the result we want is to combine the results of Section 2 with the work of Meiman ⁴⁾ who introduces the amplitude

$$\hat{F}_P(E) = F_P(E) \frac{\prod (1 - \frac{E}{E_j^*})}{\prod (1 - \frac{E}{E_j})} \quad (19)$$

(with $\text{Im } E_j > 0$). Meiman shows that if the phase of $\hat{F}_P/\hat{F}_A$ grows less fast than $\log E$, then the generalized limit of $|\hat{F}_P/\hat{F}_A| = |F_P/F_A|$ has to be unity. Now

$$\begin{aligned} & \phi \left(\hat{F}_P / \hat{F}_A \right) \\ &= \phi_P - \phi_A - 2 \text{Arg} \prod \left(1 - \frac{E}{E_j} \right) + 2 \text{Arg} \prod \left(1 + \frac{E}{E_j^*} \right) \end{aligned} \quad (20)$$

Therefore if :

(i)

$$\frac{|\phi_P - \phi_A|}{\log E} \rightarrow 0$$

(ii)

$$\frac{|\phi_P + \phi_A|}{\log E} \rightarrow 0$$

we find, using (18), that $\phi(\hat{F}_P/\hat{F}_A)/\log E \rightarrow 0$ and the only possible generalized limit for $(d\sigma^P/dt)/(d\sigma^A/dt)$ is unity.

However, we want to give here an independent proof which will have the advantage of studying the limit case $\phi_P - \phi_A \sim \log E$ at the same time. We consider the function

$$\frac{1}{E} \log \frac{G(E)}{G(-E)} = g(z) \quad (21)$$

$$z = E^2$$

along the cut we have

$$\text{Im} \left(\frac{1}{E} \log \frac{G(E)}{G(-E)} \right) = \frac{1}{E} [\phi_P(E) - \phi_A(E)] \quad (22)$$

since by construction the phase of G is the same as the phase of F (this is the main difference of our treatment with that of Meiman who tries to replace F by a function with the same modulus while we try to preserve the phase).

With the new variable z we can easily evaluate $g(z)$ for $z < 0$, i.e., E pure imaginary :

$$g(-x) = \frac{1}{\pi} \int_0^\infty \frac{\text{Im} g(y) dy}{y+x} \quad (23)$$

Due to the results of Section 2, the representation (23) holds. We find :

(i) if

$$|\phi_P(E) - \phi_A(E)| \leq C \log E \quad (24)$$

$$g(-x) < \frac{1}{\sqrt{|x|}} \log x$$

(ii) if $|\phi_P(E) - \phi_A(E)| = \sigma(\log E)$

$$g(-x) = \frac{1}{\sqrt{|x|}} \sigma(\log x) \quad (25)$$

On the other hand, one can also evaluate $g(-x)$ by using "inverse" dispersion relations

$$\begin{aligned} g(-x) &= \frac{\sqrt{x}}{\pi} \int_0^\infty \frac{d\gamma}{\sqrt{\gamma}} \frac{\operatorname{Re} g(\gamma)}{\gamma + x} \\ &= \frac{2\sqrt{x}}{\pi} \int_0^\infty \frac{dE}{E(x+E^2)} \log \left| \frac{G(E)}{G(-E)} \right| \end{aligned} \quad (26)$$

(notice that the singularity at $E = 0$ is fictitious because the logarithm vanishes). It is extremely easy to see that :

(i) if $\lim \left| \frac{G(E)}{G(-E)} \right| \neq 1$

we get

$$g(-x) \sim \frac{\log x}{\sqrt{|x|}}$$

(ii) if $\left| \frac{G(E)}{G(-E)} \right| \rightarrow +\infty$

or if
$$\left| \frac{G(E)}{G(-E)} \right| \rightarrow 0$$

$$|g(-x) \sqrt{|x|} / \log |x| \rightarrow \infty$$

We can draw the following conclusions :

(i) if
$$|\phi_P - \phi_A| \leq \text{const.} \log E$$

we cannot have

$$\left| \frac{G(E)}{G(-E)} \right| \rightarrow \infty$$

or

$$\left| \frac{G(E)}{G(-E)} \right| \rightarrow 0$$

(ii) if
$$|\phi_P - \phi_A| < \text{const.} o(\log E)$$

and if
$$\left| \frac{G(E)}{G(-E)} \right|$$

is continuous, which we assume all through this paper, then unity must belong to the set of limiting values of $|G(E)/G(-E)|$ and, in particular if the limit exists it must be unity.

It is clear that if F has no zeros then $F = GF(0)$ and the discussion stops here. In this case it is true that $|\phi_P - \phi_A|$ alone determines the asymptotic behaviour of $|F^P(E)/F^A(E)|$. The same conclusion holds for a finite number of zeros.

What remains to be done is to show that the conclusions obtained for G also apply to $F = HG$.

For this purpose we write

$$g(-x) + h(-x) = \frac{2\sqrt{x}}{\pi} \int_0^{\infty} \frac{dE}{E(E^2+x)} \log \left| \frac{F(E) H_R(-E)}{F(E) H_R(E)} \right| \quad (27)$$

where

$$h(-x) = \frac{2\sqrt{x}}{\pi} \int_0^{\infty} \frac{dE}{E(E^2+x)} \log \left| \frac{H(E) H_R(-E)}{H_R(E) H(-E)} \right|$$

and $H(E)/H_R(E) = H_+(E) H_+^*(E^*)$ has only complex zeros, those of H_+ being in $\text{Im } E > 0$. (H_R corresponds to the finite number of real zeros.) We know the behaviours of the right-hand side of (27) corresponding to $|F(E)/F(-E)| \rightarrow \infty, 0$, or a constant different of unity. We know also the behaviours of $g(-x)$ corresponding to $|\phi_P - \phi_A| < \text{const } \log E$ or $= o(\log E)$. All we have to do is to evaluate $h(-x)$, discuss compatible behaviours of both the right-hand side and the left-hand side of (27), and in this way the constraints about $|\phi_P + \phi_A|$ will appear. For real E we remark that the modulus of both $H_+(E) H_+^*(E^*)/H_+(-E) H_+^*(-E^*)$ and $(H_+^*(E^*)/H_+(-E))^2$ are the same but this last function has no zeros in $\text{Im } E > 0$. In order to evaluate $h(-x)$ we can deform the path of integration in the upper half E plane and replace with $[-\infty, +\infty]$. The pole at $E = +ix$ gives a contribution

$$h(-x) = \frac{2}{i\sqrt{x}} \log \left(\frac{H_+^*(-i\sqrt{x})}{H_+(-i\sqrt{x})} \right)^2 = \frac{8}{\sqrt{x}} \sum_j \text{Arg} \left(1 + i \frac{\sqrt{x}}{E_j} \right)$$

This term can be estimated from (17) or (18) according to what is assumed for $(\phi_P + \phi_A)$.

The conclusions are the following :

- (i) if both $(\phi_P + \phi_A)$ and $(\phi_P - \phi_A)$ are $o(\log E)$, unity belongs to the set of limiting values of $|F_P(E)/F_A(E)|$ for $E \rightarrow \infty$. If the generalized limit of $|F_P/F_A|$ exists it is of course unity ;

(ii) if $(\phi_P + \phi_A)$ and $(\phi_P - \phi_A)$ are both bounded by $\text{const log } E$ we cannot have

$$|F_P(E)/F_A(E)| \rightarrow \infty \text{ or } 0$$

so the generalized limit of

$$|F_P(E)/F_A(E)|$$

if it exists is necessarily finite.

We shall see in the next section that in the intermediate cases

$$|\phi_P - \phi_A| = o(\log E)$$

with

$$\phi_P + \phi_A = O(\log E)$$

or

$$|\phi_P - \phi_A| = O(\log E)$$

with

$$\phi_P + \phi_A = o(\log E)$$

one cannot draw a better conclusion on the limit of $|F_P/F_A|$.

Finally we want to mention another general result linking the asymptotic ratios of particle and antiparticle differential cross-sections corresponding to two different fixed transfer values t_1 and t_2 . Let us assume that for t_1 and t_2 , $|\phi^{P,A}| < \text{const log } E$ and call δ^P and δ^A the differences of the phases for these two values. A similar analysis as above can be done and we obtain, if

$$\delta^P = o(\log E), \quad \delta^A = o(\log E)$$

then

$$\left((d\sigma^P/dt) / (d\sigma^A/dt) \right)_{t=t_1} / \left((d\sigma^P/dt) / (d\sigma^A/dt) \right)_{t=t_2} \rightarrow 1$$

if this ratio has a generalized limit. This implies that if at $t = t_1$, $|F^P(E)/F^A(E)| \rightarrow \delta \neq 1$, then at $t = t_2$ we have also $|F^P(E)/F^A(E)| \rightarrow \delta$.

4. - COUNTER-EXAMPLES

Long ago, Białas, Siciak and Wit⁷⁾ have proposed the following very simple counter-example :

$$F(E) \simeq E^{\alpha_1 + i\alpha_2} \quad (29)$$

Then

$$\left| \frac{F(E+i\epsilon)}{F(-E+i\epsilon)} \right| \rightarrow \exp(\pi\alpha_2) \quad (30)$$

If we examine this example more carefully, we see that

$$\phi_P = -\phi_A = \alpha_2 \log E \quad (31)$$

Hence $\phi_P + \phi_A$ remains bounded while $\phi_P - \phi_A$ behaves like $\log E$. This fits with the absence of zeros of (29) for $|E| \rightarrow \infty$ since according to the results of Section 2, it is $\phi_P + \phi_A$ which controls the number of zeros. In fact let us make in passing the following general remark : if the amplitude has no zeros (or a finite number of zeros) and if on the real axis the amplitude is polynomially bounded above and below $E^{-n} < |F(E)| < E^m$, then $\phi_P + \phi_A$ cannot tend to $+\infty$ or $-\infty$ for $E \rightarrow \infty$. Indeed one can write a dispersion relation

$$\log \left| \frac{F(E) F(-E)}{|F(0)|^2} \right| = \frac{E^2}{\pi} \int_0^\infty \frac{dE' (\phi_P(E') + \phi_A(E'))}{E' (E'^2 - E^2)} \quad (32)$$

Hence if $\phi_P + \phi_A$ is, say, negative we have for $E = iy$

$$\log \left| \frac{F(iy) F(-iy)}{|F(0)|^2} \right| > \frac{1}{2} \int_0^Y \frac{dE' |\phi_P(E') + \phi_A(E')|}{E'} \quad (33)$$

and hence, clearly if $\phi_P + \phi_A$ increases to infinity we exceed the polynomial bound. If on the contrary $\phi_P + \phi_A$ is negative we have the same inequality as in Eq. (33), the left-hand side being now

$$\log \left| \frac{F(0)^2}{F(iy) F(-iy)} \right|$$

We want now to turn to other counter-examples in which the set of limiting values of $|F_P/F_A|$ does not contain unity in spite of the fact that $|\phi_P - \phi_A| = o(\log E)$. Such an example has already been supplied by Gribov to Meiman⁴⁾. It is :

$$F = A \exp \sqrt{E_0 - E} + B \exp \sqrt{E_0 + E} \quad (34)$$

We have for E large enough such that the real term in (34) is negligible

$$\phi_P(E) \simeq \phi_A(E) \simeq \sqrt{E} + O\left(\frac{1}{\sqrt{E}}\right)$$

Nevertheless

$$\frac{F_P}{F_A} \rightarrow \frac{A}{B}$$

The only objection to this example is that $\phi_P + \phi_A \sim \sqrt{E}$. We would like to have instead $\phi_P + \phi_A \sim \log E$. This can be done, however.

Define

$$\psi(z) = \frac{\sin[\lambda \log(1 + \sqrt{-z})]}{\exp\left(\frac{\sqrt{-z}}{\pi} \int_0^\infty \frac{\log|\sin[\lambda \log(1 + i\sqrt{x})]| dx}{\sqrt{x}(x-z)}\right)} \quad (35)$$

such a function has, by construction, a modulus unity along the positive real axis. Because asymptotically

$$\log|\sin \lambda \log(1 + i\sqrt{x})| \simeq \frac{1}{2} \log \left| \sinh^2\left(\frac{\lambda\pi}{2}\right) + \sin^2(\lambda \log \sqrt{x}) \right|$$

it is easy to see that for $z < 0$ large enough one has

$$|\psi(z)| < 1 / \sinh\left(\frac{\lambda\pi}{2}\right) \quad (36)$$

So if we take $\sinh(\lambda\pi/2) > 1000$ we have $|\psi(z)| < 1/1000$ for $z < -z_0$. On the other hand the phase of $\psi(z)$ along the cut $z > 0$ is twice the phase of the numerator [the "old timers" of potential scattering will recognize an analogy with the expression of the S matrix in terms of the Jost function $S(E) = f(k)/f(-k)$ where, by construction, $f(-k)$ has no zeros in the upper half k plane in the absence of bound states, and the phase of S is twice the phase of f].

The phase of the numerator of ψ is easily seen to grow like $(-\lambda/2)\log z$: the numerator follows an ellipse centred at the origin in the Argand diagram.

Now we are ready to present our counter-example, analogous to the one of Gribov :

$$F(E) = \psi(E) + \frac{1}{i0} \psi(-E) \quad (36)$$

with $\sinh(\lambda\pi/2) = 1000$. For $E \rightarrow +\infty$ the first term in (36) dominates, whereas for $E \rightarrow -s$, the second term dominates. Hence

$$10 \left(1 + \frac{1}{100}\right)^{-1} \left(1 - \frac{1}{10,000}\right) < \left| \frac{F_P(E)}{F_A(E)} \right| < 10 \left(1 - \frac{1}{100}\right)^{-1} \left(1 + \frac{1}{10,000}\right) \quad (37)$$

$|\phi_P - \phi_A|$ remains bounded and $|\phi_P + \phi_A|$ behaves like $2\lambda \log E$.

4. - GENERALIZATIONS TO VARIABLE t

In all the previous sections, we have worked with the fixed t reaction amplitude. However, for several purposes it is useful to work with variable t , for instance, to investigate the equality of cross-sections for particles and antiparticles inside a shrinking diffraction peak. It is generally assumed that this creates no problem, that, instead of working with $F(s, t)$, one can use $F(s, t(s))$ where $t(s)$ is an analytic function of s . This belief is incorrect. Of course, as long as $t(s)$ is an analytic function of $1/s$ around $s = \infty$, there is indeed no problem because one can manage to have $t(s) < 0$, i.e., physical, both for $s \rightarrow +\infty$ and $-\infty$. However, the most interesting case is $t(s) \sim -(\log s)^{-\alpha}$, from a physical point of view, and, then, one cannot have $t(s)$ real both for $s \rightarrow +\infty$ and $s \rightarrow -\infty$. Here we want to see what can be done in this situation. Of course we want to preserve above all the analyticity of the amplitude in a cut plane. So we take

$$t(s) = \frac{\sqrt{-s}}{\pi} \int_0^\infty \frac{(-\Delta(s')) ds'}{\sqrt{s'} (s' - s)} \quad (38)$$

where $\Delta(s) > 0$, $\Delta(s) \searrow 0$ and there exists $n > 0$ such that

$$\Delta(s) \left(\log \left(\frac{s}{s_0} \right) \right)^n \nearrow \quad (39)$$

For instance if we consider, like Bessis¹¹⁾, reasonable physical assumptions, then $n = 2 + \epsilon$ is sufficient. $t(s)$ is real and negative for $s < 0$ so that the cut $u > 0$ is mapped on the negative real axis in the s variable. For $s > 0$ we have

$$\text{Im } t(s) = -\frac{\sqrt{-s}}{\pi} \int_0^\infty \frac{(\Delta(s) - \Delta(s'))}{\sqrt{s'} (s' - s)} ds' \quad (40)$$

Therefore from (39), $\text{Im } t(s)$ is positive. One can also estimate $\text{Im } t(s)$ using

$$\left| \Delta(s) - \Delta(s') \right| < \Delta(s) \left| 1 - \frac{(\log s)^n}{(\log s')^n} \right|$$

One finds

$$|\text{Im } t(s)| < C(n) \frac{\Delta(s)}{\log s} \quad (41)$$

It is also important to compare $\text{Re } t(s)$ for $s > 0$ and $s < 0$

$$\text{Re } t(-s) - \text{Re } t(s) = -\frac{\sqrt{s}}{\pi} \int_0^\infty \frac{(\Delta(s') - \Delta(s))}{\sqrt{s'} (s' + s)} ds' \quad (42)$$

Here the inequalities (39) allow only to get

$$|\text{Re } t(-s) - \text{Re } t(s)| < C(n) \frac{\Delta(s)}{\log s} \quad (43)$$

even though a smoother behaviour of $\Delta(s)$ is expected to lead to

$$\text{Re } t(-s) - \text{Re } t(s) \sim \frac{\Delta(s)}{(\log s)^2}$$

Of course, one can consider also slower shrinking, for instance $\Delta(s)(\log \log s)^n$ leads to

$$|\operatorname{Im} t(s)| < \frac{\Delta(s)}{\log s \log(\log s)} \quad (44)$$

Anyway with these functions, one can repeat the considerations of the previous sections. However, some caution must be used to get the physical implications : what one would like is a statement on

$$\lim_{s \rightarrow \infty} \left| \frac{F_P(s, -\Delta(s) = t)}{F_A(s, -\Delta(s) = t)} \right|$$

However, what one gets is a statement on

$$\lim_{s \rightarrow \infty} \left| \frac{F_P(s, t = -\Delta(s) + i \frac{\Delta(s)}{\log s})}{F_A(s, t = -\Delta(s) (1 + O(\frac{1}{\log s})))} \right|$$

one can reach a conclusion on the ratio of the physical cross-section only after a careful study of the amplitude is made in the complex t plane for physical energies. A small shift of t in the physical region is relatively harmless but a shift in which t gets a small imaginary part may be dangerous. This will be discussed in forthcoming papers.

5. - CONCLUDING REMARKS

We have found two sets of conditions on the phases of reaction amplitudes in which we can get some information on the behaviour of

$$\frac{d\sigma}{dt}(A+B \rightarrow C+D) / \frac{d\sigma}{dt}(\bar{C}+B \rightarrow \bar{A}+D)$$

at infinite energy. If the fixed t phases $|\delta_P - \delta_A|$ and $|\delta_P + \delta_A|$ are bounded by $\log E$ we can only say that this ratio cannot tend to zero or tend to infinity. The interest of this assumption about the phases is that, for elastic amplitudes, it holds quite generally, provided the modulus of the amplitude does not vanish in the physical region (an assumption which can be tested experimentally). A more precise result is obtained if one knows that the phases satisfy $|\delta_P + \delta_A| = o(\log E)$, $|\delta_P - \delta_A| = o(\log E)$. Then it can be shown that unity belongs to the set of limiting values of $(d\sigma_P/dt)/(d\sigma_A/dt)$ so that if the generalized limit exists, it is one. In the general case of line reversed reactions, we do not know any way to justify this assumption. However, we shall show in forthcoming publications that this can be established under some conditions involving only cross-sections and hence measurable quantities for the case of elastic scattering.

We see no way of weakening these conditions on the phases. In particular, the counter-example we have built shows the importance of the condition $|\delta_A + \delta_P| = o(\log E)$.

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