Majorization in Quantum Adiabatic Algorithms

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We study the Majorization arrow in a big class of quantum adiabatic algorithms. In a quantum adiabatic algorithm, the ground state of the Hamiltonian is a guide state around which the actual state evolves. We prove that for any algorithm of this class, step-by-step majorization of the guide state holds perfectly. We also show that step-by-step majorization of the actual state appears if the running time becomes longer and longer. This supports the empirical viewpoint that step-by-step majorization seems to appear universally in quantum adiabatic algorithms. On the other hand, the performance of these algorithms discussed in this paper can all be estimated, which is exponential in the problem size. This can be looked as a strong evidence that step-by-step majorization is not a sufficient condition for efficiency.

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I. INTRODUCTION

In the past two decades, quantum computation has attracted a great deal of attention, because it was demonstrated that the performance of quantum computers exceeds that of classical computers for some computational tasks. Among quantum algorithms proposed so far Shor’s factorization algorithm [1] and Grover’s search algorithm [2] are two famous examples. However, it is believed that the design of quantum algorithms seems to be very difficult [3]. Thus studying the characters of quantum algorithms is a very important question. It has been observed that majorization theory seems to play an important role in the efficiency of quantum algorithms [2, 3, 4, 5]. For example, in many quantum algorithms the initial state of the system is an equal superposition state and the final state before measurement is some computational basis state. In the process of computation, the probability distribution associated to the state of the system in the computational basis is step-by-step majorized until it is maximally ordered. In [2] by carrying out a systematic analysis of a wide variety of quantum algorithms from the majorization theory point of view, the authors concluded that step-by-step majorization is found in the known instances of fast and efficient algorithms, such as quantum fourier transform, Grover’s algorithm, the hidden affine function problem. By contrary, in [3] the authors also offered a quantum algorithm that doesn’t show step-by-step majorization, which doesn’t provide any computational speed-up. These facts show that step-by-step majorization seems to be necessary for the efficiency of quantum algorithms.

In [2] and [3], the authors also studied majorization in quantum adiabatic algorithms. Quantum adiabatic computation is a novel paradigm for the design of quantum algorithms. In a quantum adiabatic algorithm, the evolution of the quantum register is governed by a Hamiltonian that varies continuously and slowly. If the initial state of the system is the ground state of the initial Hamiltonian, the state of the system at any moment in the whole process of computation will differ from the ground state of the Hamiltonian at that moment by a negligible amount. Thus, in a quantum adiabatic algorithm the ground state of the Hamiltonian is a guide, and the actual state of the system is always around this guide. (In this paper we call ground states in quantum adiabatic algorithms guide states.) In [4] and [5], through numerical simulations to several special cases the authors got an empirical conclusion that quantum algorithms based on adiabatic evolution naturally show step-by-step majorization provided that the Hamiltonians and the initial state are chosen with sufficient symmetry and the evolution is slow enough. In this paper, we will study the majorization arrow in a big class of quantum adiabatic algorithms. We prove that in any algorithm of this class step-by-step majorization of the guide state holds perfectly. For actual state, we show that step-by-step majorization appears as long as the running time of this algorithm becomes longer and longer. Based on these results, we analyze the relation between the efficiency of quantum algorithms and majorization.

The rest of the paper is organized as follows. In Sec. II we briefly review quantum adiabatic computation and majorization theory. In Sec. III we prove that step-by-step majorization of the guide state holds perfectly. For actual state, we show that step-by-step majorization appears as long as the running time of this algorithm becomes longer and longer. Finally, in Sec. V we discuss step-by-step majorization of the actual state.

II. QUANTUM ADIABATIC ALGORITHMS AND MAJORIZATION

For convenience of the readers, let us recall quantum adiabatic computation and majorization theory.

Quantum adiabatic computation, proposed by Farhi
is based on quantum adiabatic evolution. Suppose the state of a quantum system is $|\psi(t)\rangle (0 \leq t \leq T)$, which evolves according to the Schrödinger equation

$$i\frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle,$$

(1)

where $H(t)$ is the Hamiltonian of the system. Suppose $H_0 = H(0)$ and $H_1 = H(T)$ are the initial and the final Hamiltonians of the system. Then we let the Hamiltonian of the system vary from $H_0$ to $H_1$ slowly along some path. For example, an interpolation path is one choice,

$$H(t) = f(t)H_0 + g(t)H_1,$$

(2)

where $f(t)$ and $g(t)$ are continuous functions with $f(0) = g(T) = 1$ and $f(T) = g(0) = 0$ ($T$ is the running time of the evolution). Let $|E_0, t\rangle$ and $|E_1, t\rangle$ be the ground state and the first excited state of the Hamiltonian at time $t$, and let $E_0(t)$ and $E_1(t)$ be the corresponding eigenvalues. The quantum adiabatic theorem [5] shows that we have

$$\langle E_0, t|\psi(T)\rangle^2 \geq 1 - \varepsilon^2,$$

(3)

provided that

$$\frac{D_{\text{max}}}{g_{\text{min}}} \leq \varepsilon, \quad 0 < \varepsilon \ll 1,$$

(4)

where $g_{\text{min}}$ is the minimum gap between $E_0(t)$ and $E_1(t)$,

$$g_{\text{min}} = \min_{0 \leq t \leq T} |E_1(t) - E_0(t)|,$$

(5)

and $D_{\text{max}}$ is a measurement of the evolving rate of the Hamiltonian

$$D_{\text{max}} = \max_{0 \leq t \leq T} |\langle E_1, t| \frac{dH}{dt} |E_0, t\rangle|.$$  

(6)

Quantum adiabatic computation is a novel paradigm for the design of quantum algorithms. For example, Quantum search algorithm proposed by Grover [2] has been implemented by quantum adiabatic computation in [9]. Recently, the new paradigm for quantum computation has been used to try to solve some other interesting and important problems, such as Deutsch-Jozsa problem [10], [11], [12], hidden subgroup problem [13], 3SAT problem [7], [14], traveling salesman problem [15] and Hilbert’s tenth problem [10].

Let’s look a big class of typical quantum adiabatic algorithms, which is what we will discuss in this work. Suppose $f : \{0, 1\}^n \rightarrow R$ is a function that is bounded by a polynomial of $n$. Let $H_0$ and $H_1$ be the initial and the final hamiltonians of a quantum adiabatic evolution with a linear path $H(t)$. Concretely,

$$H_0 = I - |\alpha\rangle\langle \alpha|,$$

(7)

$$H_1 = \sum_{i=1}^{N} f(i)|i\rangle\langle i|,$$

(8)

$$H(s) = (1 - s)H_0 + sH_1,$$

(9)

where

$$|\alpha\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} |i\rangle, \quad N = 2^n,$$

(10)

and $s = s(t)$ a continuous increasing function with $s(0) = 0$ and $s(T) = 1$ ($T$ is the running time of the quantum adiabatic evolution). According to quantum adiabatic theorem, this class of algorithms can be used to minimize the function $f(i), i = 1, 2, ..., N$ (we suppose there is only one minimum). The quantum adiabatic algorithms for search problem in [9], hidden subgroup problem in [13], 3SAT problem in [14] and traveling salesman problem in [15] belong to this class.

Now let’s turn to the majorization theory. Majorization is an ordering on $N$-dimensional real vectors. Suppose $x = (x_1, x_2, ..., x_N)$ and $y = (y_1, y_2, ..., y_N)$ are two $N$-dimensional vectors. If $x$ is majorized by $y$, $y$ is more disordered than another. To be concrete, let $x^\prime$ mean $x$ re-ordered so the components are in decreasing order. We say $x$ is majorized by $y$, namely $x \prec y$, provided

$$\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i \text{ for } k = 1, 2, ..., N - 1 \text{ and } \sum_{i=1}^{N} x_i = \sum_{i=1}^{N} y_i.$$

(11)

It has been proven that majorization is at the heart of the solution of a large number of quantum information problems. For example, majorization characterizes when one quantum bipartite pure states can be transformed to another deterministically via local operations and classical communication [24]. More details about majorization see [21].

In [4] and [5], the authors related the majorization theory to quantum algorithms as follows: let $|\psi(m)\rangle$ be the state of the register of a quantum computers at an operating stage labeled by $m = 1, ..., M$, where $M$ is the total number of steps in the algorithm. Let $N$ be the dimension of the Hilbert space. Suppose $\{|i\rangle\}_{i=1}^{N}$ is the basis in which the final measurement is performed. Then suppose in this basis the state $|\psi(m)\rangle$ is

$$|\psi(m)\rangle = \sum_{i=1}^{N} a_i^{(m)} |i\rangle.$$

(11)

If we measure $|\psi(m)\rangle$ in the basis $\{|i\rangle\}_{i=1}^{N}$, the probability distribution associated to this state is

$$p_i^{(m)} = |\langle i| \psi(m)\rangle|^2,$$

(12)

where $i = 1, 2, ..., N$. If $p_i^{(m)} < p_{i+1}^{(m)}$ for every $m$, we say this algorithm majorizes step by step.

Especially, in [4] and [5] the authors applied majorization theory to study quantum adiabatic algorithms. By numerical simulation the authors observed that local quantum adiabatic search algorithm shows step-by-step majorization. For global adiabatic search algorithm, they...
observed that though step-by-step majorization doesn’t hold if the adiabatic inequality in Eq.(4) is tight, step-by-step majorization will appear as long as the evolution of the Hamiltonian becomes slower and slower. Note that these two quantum adiabatic search algorithms both belong to the big class of quantum adiabatic algorithms we have discussed above.

III. STEP-BY-STEP MAJORIZATION OF GUIDE STATES

In this work, we discuss majorization in this class of quantum adiabatic algorithms described by Eq.(7)-Eq.(10). Firstly let’s consider guide states of these algorithms. We prove that, for any quantum adiabatic algorithm of this big class, step-by-step majorization of the guide state holds perfectly.

Theorem 1 Suppose $H_0$ and $H_1$ given by Eq.(7) and Eq.(8) are the initial and the final Hamiltonians of a quantum adiabatic algorithm. Suppose this quantum adiabatic algorithm has a linear path given by Eq.(9). Then the guide state of this algorithm shows perfect step-by-step majorization.

Proof. Suppose the non-degenerate ground state of

$$H(s) = (1 - s)(I - |\alpha\rangle\langle\alpha|) + s \sum_{i=1}^{N} f(i)|i\rangle\langle i|, \quad (13)$$

is

$$|\psi(s)\rangle = (a_1, a_2, ..., a_N)^T, \quad (14)$$

and the corresponding eigenvalue is $\lambda(s)$. Suppose $\min_{1 \leq i \leq N} f(i) = 0$. Otherwise we can let

$$H(s) = H(s) - s \times I \times \min_{1 \leq i \leq N} f(i), \quad (15)$$

which doesn’t change the ground state of $H(s)$. Without loss of generality, we suppose $f(1) < f(2) \leq f(3) \leq ... \leq f(N)$. Because if we change $f(i)$ to $f'(i) = f(\pi(i))$, where $\pi$ is a permutation of $1, 2, ..., N$, the ground state of $H(s)$ for any $s \in [0, 1]$ will be changed to $(b_1, b_2, ..., b_N)^T$, where $b_i = a_{\pi(i)}$. This can be proved as follows. Let

$$H'(s) = (1 - s)H_0 + sH_1', \quad (16)$$

where

$$H_1' = \sum_{i=1}^{N} f'(i)|i\rangle\langle i|. \quad (17)$$

Note that $P_\pi H'(s)P_\pi = H(s)$, where $P_\pi$ is a permutation matrix such that $P_\pi |i\rangle = |\pi(i)\rangle$. So we have

$$P_\pi(b_1, b_2, ..., b_N)^T = (a_1, a_2, ..., a_N)^T. \quad (18)$$

That is to say, $b_i = a_{\pi(i)}$. We will find that a permutation of the components of ground states doesn’t affect our analysis later.

By the definitions of $|\psi(s)\rangle$ and $\lambda(s)$, we have

$$H(s)|\psi(s)\rangle = \lambda(s)|\psi(s)\rangle. \quad (19)$$

Substituting Eq.(13) and Eq.(14) into Eq.(19), we get

$$\frac{1 - s}{N} \sum_{i=1}^{N} a_i = t(s)a_i + sf(i)a_i, \quad (20)$$

where $t(s) = 1 - s - \lambda(s)$. Note that $t(s)$ is the biggest eigenvalue of

$$G(s) = (1 - s)I - H(s). \quad (21)$$

For every $s \in [0, 1)$ and $ds > 0$, an explicit calculation shows that

$$G(s + ds) - G(s) = -ds \times (|\alpha\rangle\langle\alpha| + \sum_{i=1}^{N} f(i)|i\rangle\langle i|). \quad (22)$$

Because $G(s) - G(s + ds)$ is a strictly positive matrix, $t(s)$ is a strictly decreasing function of $s$. It is easy to get $t(0) = 1$ and $t(1) = 0$. Then we have $0 < t(s) < 1$ for any $s \in (0, 1)$.

By Eq.(20) we can obtain

$$(t(s) + sf(i))a_i = (t(s) + sf(j))a_j, \quad i, j = 1, 2, ..., N. \quad (23)$$

Thus it can be proved that if we let $a_1 \geq 0$ (Note that, $c\lambda(s)$ is also the ground state of $H(s)$, where $|c| = 1$, we have $a_i > 0$ for any $s \in (0, 1)$ and any $i$. Otherwise if $a_i = 0$ for any $i$ we will have $\sum_{i=1}^{N} a_i^2 = 0$ according to Eq.(23), which is a contradiction. In this paper, we suppose the components of all ground states are real numbers. Substituting Eq.(23) into

$$\sum_{i=1}^{N} a_i^2 = 1, \quad (24)$$

gives

$$a_1^2 = \frac{1}{\sum_{i=1}^{N} \left(1 - \frac{1}{(t(s) + sf(i))^2}\right). \quad (25)$$

Note that $t(s)$ is a strictly decreasing function of $s$, which means $a_1$ is a strictly increasing function of $s$. However, for other $a_i$, the monotony is a little more complicated. It’s possible that they are not monotonous. However, we can prove that their increase and decrease are well-regulated. Namely, for $s > 0$ and $1 - s \geq ds > 0$, if $a_i \geq a_i'$, we have $a_j \geq a_j'$, where $(a_1', a_2', ..., a_N')^T$ is the ground state of $H(s + ds)$ and $i < j$.

Let’s prove this conclusion. From Eq.(23), we get

$$\frac{a_i}{a_j} = \frac{t(s) + sf(j)}{t(s) + sf(i)}. \quad (26)$$
and
\[ \frac{a'_i}{a'_j} = \frac{t(s + ds) + (s + ds)f(j)}{t(s + ds) + (s + ds)f(i)}. \]  
\( (27) \)

Because \( t(s) \) is a strictly decreasing function, it can be checked that
\[ \frac{t(s) + sf(j)}{t(s) + sf(i)} < \frac{t(s) + (s + ds)f(j)}{t(s) + (s + ds)f(i)}. \]  
\( (28) \)

So
\[ \frac{a_i}{a_j} < \frac{a'_i}{a'_j}. \]  
\( (29) \)

Thus if \( a'_i < a_i \), we have \( a'_j < a_j \).

According to the discussion above, we know that for every \( s \in (0,1) \) there is a special integer \( i_0(s) \). When \( i \leq i_0(s) \) we have \( a_i < a'_{i} \) and when \( i > i_0(s) \) we have \( a_i \geq a'_i \).

Now we are in a position to prove our first main conclusion. Namely,
\[ (a_1^2, a_2^2, ..., a_N^2)^T < (a'_1^2, a'_2^2, ..., a'_N^2)^T. \]  
\( (30) \)

Firstly, according to Eq.(23) it can be checked that the components of \((a_1^2, a_2^2, ..., a_N^2)^T\) and \((a'_1^2, a'_2^2, ..., a'_N^2)^T\) are in decreasing order. Secondly, for any \( s \in (0,1) \) and any \( k = 1, 2, ..., N \), if \( k \leq i_0(s) \), we have \( \sum_{i=1}^{k} a_i^2 \leq \sum_{i=1}^{k} a'_i^2 \) because \( a_i < a'_i \) for \( i \leq k \). If \( k > i_0(s) \), \( a_i > a'_i \) for \( i \geq k \), so we have \( \sum_{i=k+1}^{N} a_i^2 > \sum_{i=k+1}^{N} a'_i^2 \). Thus we also get \( \sum_{i=1}^{k} a_i^2 \leq \sum_{i=1}^{k} a'_i^2 \) because \( \sum_{i=1}^{N} a_i^2 = \sum_{i=1}^{N} a'_i^2 = 1 \). That completes the proof of Eq.(30), namely, step-by-step majorization holds if the adiabaticity is perfect.

Note that if the form of Eq.(7) doesn’t change, the ground state \( |\alpha\rangle \) in Eq.(7) can be replaced by any other vector of Hadamard basis. Because it can be proved if \( |\alpha\rangle \) is replaced by any other vector of Hadamard basis, for any \( s \) any component of the ground state of \( H(s) \) will not change up to the sign. Thus replacing the ground state in Eq.(7) doesn’t destroy majorization. Besides, the path along which the hamiltonian varies in Eq.(9) can be replaced by any interpolation path in Eq.(2) provided \( \frac{\Phi(t)}{f(t) + \Phi(t)} \) is a increasing function of \( t \), which doesn’t destroy step-by-step majorization either.

### IV. STEP-BY-STEP MAJORIZATION OF ACTUAL STATES

In the section above, we consider the guide state of a quantum adiabatic algorithm. In this section, we consider the actual state of the system. According to quantum adiabatic theorem, at any time of the whole computation process the actual state approaches the guide state. We show that step-by-step majorization of the actual state holds approximately \[3\], \[4\].

Suppose in actual adiabatic evolutions, the actual state of the system is
\[ |\psi'(s)\rangle = (b_1, b_2, ..., b_N)^T. \]  
\( (31) \)

Let
\[ B_k = \sum_{i=1}^{k} |b_i|^2, \quad k = 1, 2, ..., N. \]  
\( (32) \)

In \[3\] the authors studied \( s - B_1 \) curve (\( B_1 \) is the probability of finding the right solution) and \( s - B_2 \) curve of global quantum adiabatic evolution for search problem by numerical simulations. If step-by-step majorization holds perfectly, the curve should be monotonous. However, they observed that oscillation appears at the end of \( s - B_1 \) curve and \( s - B_2 \) curve, which destroys step-by-step majorization (See Figure.1). Furthermore, they also observed that the oscillation becomes weaker and weaker and step-by-step majorization appears as long as the running time becomes longer and longer.

![FIG: 1 A case that oscillation appears at the end of s - B1 curve (the solid curve). The dashed curve is s - B1 curve.](image)

Now, based on Theorem 1 we prove that for any quantum adiabatic evolution belonging to the big class discussed in this paper the oscillation at the end of \( s - B_1 \) curve \( \sum_{i=1}^{k} |b_i|^2 \leq \sum_{i=1}^{k} |b'_i|^2 \) curve, if any, will disappear if the running time becomes longer and longer.

We consider an arbitrary state of the system near the end of the quantum adiabatic evolution. Let
\[ A_k = \sum_{i=1}^{k} a_i^2, \quad k = 1, 2, ..., N, \]  
\( (33) \)

where \((a_1, a_2, ..., a_N)^T\) is the guide state of the quantum adiabatic algorithm as before. Then from Eq.(23) and
Eq. (25) we have
\[ A_k = \sum_{i=1}^{N} \left( \frac{1}{1 + s^i(f)} \right)^2, \quad k = 1, 2, ..., N, \]  
(34)
From Eq. (9) we get
\[ \frac{H(s)}{1 - s} = H_0 + \frac{s}{1 - s} H_1. \]  
(35)
It can be seen that \( \lambda(s) \), the ground state eigenvalue of \( \frac{H(s)}{1 - s} \), is a strictly decreasing function of \( s \) [21]. So
\[ \frac{d}{ds} \frac{\lambda(s)}{1 - s} > 0, \]  
(36)
which makes
\[ \frac{d}{ds} \left( t \frac{1 - s}{1 - s} \right) = \frac{d}{ds} \left( 1 - s - \frac{\lambda(s)}{1 - s} \right) < 0. \]  
(37)
A simple calculation shows that
\[ -\frac{dt}{ds} > \frac{t}{1 - s}, \]  
(38)
then
\[ \frac{d}{ds} (1 - s, 0) < s \left( \frac{1}{1 - s} + \frac{1}{1 - s} \right), \quad 0 < s < 1. \]  
(39)
Calculating the derivative of Eq. (34) we get
\[ \frac{dA_k}{ds} = 2 \sum_{i=1}^{N} \left( \frac{f(i) d(f)}{1 + s f(i)} - f(i) \frac{d(s)}{1 + s f(i)} \right) \]  
\[ + 2a_i^2 \sum_{i=k+1}^{N} a_i^2 \frac{f(i) d(f)}{1 + s f(i)}. \]  
(40)
Since
\[ f(j) \frac{d(f)}{1 + s f(j)} > f(i) \frac{d(f)}{1 + s f(i)} \]  
(41)
when \( N \geq j > i \geq 1 \), we have
\[ \frac{dA_k}{ds} > 2a_i^2 \sum_{i=k+1}^{N} a_i^2 \frac{f(i) d(f)}{1 + s f(i)}. \]  
(42)
Let
\[ m = \min_{2 \leq i \leq N} f(i). \]  
(43)
If \( m > 1 \),
\[ f(i) \frac{d(f)}{1 + s f(i)} > \frac{f(i) (1 + \frac{1}{s})}{1 + s f(i)} \]  
\[ > \frac{f(i)}{1 + s f(i)}, \]  
\[ > 1. \]  
(44)
Here, we use Eq. (39) and the fact \( t < 1 < f(i) \) and \( \frac{1}{1 - s} > 1 \). Similarly, if \( m < 1 \),
\[ f(i) \frac{d(f)}{1 + s f(i)} > \frac{f(i) (1 + \frac{1}{s})}{1 + s f(i)} \]  
\[ = m \frac{1}{m} \frac{f(i) + 1}{1 + \frac{1}{s} f(i)} > m. \]  
(45)
So, we have
\[ f(i) \frac{d(f)}{1 + s f(i)} > c, \]  
(46)
where \( c = \min \{ m, 1 \} \).
Substituting Eq. (46) into Eq. (42), we get
\[ \frac{dA_k}{ds} > 2c \cdot a_i^2 \sum_{i=k+1}^{N} a_i^2 \]  
(47)
Note that
\[ a_i^2 > \frac{\sum_{i=1}^{k} a_i^2}{k} = \frac{1}{k} A_k. \]  
(48)
We finally have
\[ \frac{dA_k}{ds} > \frac{2c}{k} A_k(1 - A_k). \]  
(49)
According to quantum adiabatic theorem, we know that for any positive \( \delta \) we have a finite running time \( T \) such that
\[ |\langle \psi'(s)|\psi(s)\rangle| \geq 1 - \delta^2/2 \]  
(50)
for any \( s \in (0, 1) \).
Since
\[ ||\psi| - |\psi'||^2 = 2 - 2|\langle \psi(s)|\psi'(s)\rangle| \]  
\[ < \delta^2, \]  
(51)
it can be seen that for any \( s \)
\[ \sum_{i=1}^{k} |a_i - b_i|^2 < \delta^2. \]  
(52)
Here, we choose the global phase of \( |\psi'(s)\rangle \) such that \( \langle \psi(s)|\psi'(s)\rangle \) is real. According to Cauchy’s inequality,
\[ \sum_{i=1}^{k} |a_i - b_i| < \sqrt{k} \delta. \]  
(53)
Because
\[ |a_i + b_i| < 2, \]  
(54)
we have
\[ \sum_{i=1}^{k} |a_i^2 - b_i^2| = \sum_{i=1}^{k} |a_i - b_i| \cdot |a_i + b_i| < 2\sqrt{k}\delta. \] (55)

Thus
\[ |\sum_{i=1}^{k} |a_i|^{2} - \sum_{i=1}^{k} |b_i|^{2}| \leq \sum_{i=1}^{k} |a_i^2 - b_i^2| < 2\sqrt{k}\delta. \] (56)

Now let us consider two points \((s_1, A'_k)\) and \((s_2, A_k)\) on \(s - A_k\) curve (about the guide state) and two points \((s_1, B'_k)\) and \((s_2, B_k)\) on \(s - B_k\) curve (about the actual state), where \(s_2 - s_1 = \Delta s, 0 < \Delta s \ll 1\). These four points are all near the end of the quantum adiabatic evolution. If step-by-step of the actual state holds, \(s - B_k\) curve should be a monotonically increasing curve. Suppose that Eq.(50) holds. According to Eq.(56) we have
\[ |A_k - B_k| < 2\sqrt{k}\delta, \quad |A'_k - B'_k| < 2\sqrt{k}\delta. \] (57)

On the other hand, according to the discussion above, we have
\[ A_k - A'_k > \frac{2c}{k} A_k(1 - A_k) \cdot \Delta s. \] (58)

Then if
\[ \delta < \frac{c}{2k\sqrt{k}} A_k(1 - A_k) \cdot \Delta s, \] (59)

we have \(B'_k < B_k\). Note that for arbitrary small \(\Delta s\) we can find corresponding \(\delta\) or running time \(T\) such that Eq.(59) holds. Thus, it can be judged that when the running time becomes longer and longer the oscillation at the end of \(s - B_k\) curve becomes weaker and weaker, and at the same time step-by-step majorization appears slowly. This explains the results of numerical simulations in \([4]\), which is a special case of our discussion above. This is consistent with our intuition. Because of quantum adiabatic theorem, we know that when the running time becomes longer and longer, the distance between the actual state of the system and the guide state becomes smaller and smaller. Since it has been shown that the guide state shows perfect step-by-step majorization, step-by-step majorization of the actual state appears slowly when the adiabaticity of the quantum adiabatic evolution becomes better and better is natural.

V. DISCUSSIONS AND CONCLUSIONS

The results of this paper can help us to understand the relation between the efficiency of quantum algorithms and step-by-step majorization \([4]\). Usually, except some simple cases, a decisive mathematical analysis of a quantum adiabatic algorithm is not possible, and frequently even the estimation of the running time is very difficult. However, the minimal running time of the class of quantum adiabatic algorithms discussed above can be estimated, which is exponential in \(n\), where \(n\) is the problem size \([17, 18, 19]\). In this connection, the efficiency of these algorithms is not very good. However, though this class of quantum adiabatic algorithms can’t offer exponential computational speed-up, they may be used to design quantum algorithms more powerful than corresponding classical ones. In \([1\, 2\, 3]\), the authors resolved quantum search problem via a quantum adiabatic algorithm of this class, which is optimal and gives rise to a quadratic speed-up. Similarly, In \([13\, 14]\) also by a quantum adiabatic algorithm of this class the author tried to solve the hidden subgroup problem. It can be shown that the latter is not optimal. So it is clear that the efficiency of some of this class of algorithms is different from that of others.

On the other hand, we have shown that if the running time of these algorithms becomes longer and longer they all show step-by-step majorization. Thus this result can be looked as a strong evidence that step-by-step majorization is not a sufficient condition for efficiency \([4]\).

In conclusion, we have shown that for any algorithm of a big class of quantum adiabatic algorithms, step-by-step majorization of the guide state holds perfectly. We also have shown that step-by-step majorization of the actual state holds approximately. The longer the running time is, the better majorization holds. This shows that majorization seems to appear universally in quantum adiabatic algorithms. If the initial and the final Hamiltonians are diagonalizable in Hadamard basis and computational basis respectively, does every quantum adiabatic algorithm show step-by-step majorization provided the running time is long enough? We have shown that majorization is not sufficient for the efficiency of quantum algorithms. A natural question is, is step-by-step majorization necessary for efficiency? In \([4]\) the authors showed that any known efficient quantum algorithm does obey step-by-step majorization. These questions need further research.

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