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CURRENTS AND GREEN'S FUNCTIONS FOR DUAL MODELS - I

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A B S T R A C T

In dual models with unit intercept, we show that it is possible to explicitly construct currents and/or Green's functions in a way consistent with factorization in the original Hilbert space of physical particle states. In contrast with previous attempts, amplitudes involving more than one current or off-shell particle satisfy that consistency and have only correctly located singularities. We prove in a few particular cases that the currents and Green's functions thus defined satisfy the conservation rules expected by analogy with similar problems of quantum field theories. The full conformal invariance of dual models with unit intercept is particularly useful for our approach.

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I. - INTRODUCTION

The problem of defining currents in a model of strong interactions is of obvious interest : many dynamical properties of hadrons are tested via electromagnetic or weak currents (e.g., chiral symmetry, current algebra, scaling properties). Despite many attempts ¹⁾⁻³⁾, one cannot claim that a consistent definition of currents has been achieved for dual models. By consistent, we mean that dual currents should be formulated within the original Hilbert space of physical particle states, which is defined from the analysis of purely hadronic dual amplitudes involving only on-shell external states (enlarging the original spectrum ²⁾ clearly leads into the trouble of redefining the hadronic amplitudes for the newly introduced states). Another related consistency requirement is the location of singularities (poles) in various channels of multicurrent amplitudes, which should also be compatible with the particle interpretation of these singularities (most of the proposals ³⁾ fail to satisfy that point ^{*)}).

Another long standing and closely related problem is the definition of off-shell dual amplitudes (or Green's functions). For example, these are required for a correct computation and renormalization of dual loop diagrams.

Both problems are connected with the realization of a conventional Lagrangian formulation of dual models. Such a realization still seems rather far away, but it would be of great help in the understanding of other properties of dual models, for example of the construction of loops and of their renormalization, which are in remarkably close parallel with similar properties of field theory, particularly in the Virasoro case, where there are enough subsidiary conditions to cancel all the ghosts of the theory.

In this paper, we propose explicit rules for the construction of dual off-shell amplitudes and/or currents satisfying the above-mentioned consistency requirements. These rules are closely related to the rules for the construction of dual multiloop amplitudes, and use the same powerful mathematical tools. In the case of vector currents, for example, our proposal

*) We are grateful to N. Surlas for pointing out to us that, contrary to a wide-spread belief, multicurrent amplitudes in the model of Kikkawa and Sato ³⁾ also have those unphysical singularities : this can be checked by explicit computation of an amplitude with two adjacent currents.

realizes a generalized vector dominance model. It turns out that all the formalism and many physical properties of those currents (e.g., conservation), are particularly beautiful and simple when the dual model exhibits the full conformal invariance which allows for the (presumably complete) cancellation of ghosts.

Since the formalism and mathematical manipulations are rather complicated for multicurrent amplitudes, we shall only describe here their prominent features ; we leave the detail of their construction for a subsequent publication, where we also hope to study more closely the algebra of those currents and their light-cone properties. In the present paper, we only assume that the reader is somewhat familiar with one-loop dual amplitudes. We also restrict our treatment to the conventional model. The basic ideas could easily be extended to the other conformal invariant dual models presently known, except, perhaps, for the fully symmetric model (or generalized Virasoro model).

Whether our amplitudes will turn out to be more useful as off-shell extrapolations or as currents is not clear at the present time. We shall then use both words for the same objects.

II . - OPERATORS FOR OFF-SHELL STATES

In this section we write down operators describing the emission of off-shell states, and motivate our choice.

In the usual N point function for on-shell particles, the external particles are emitted from points in a cyclic order along the Koba-Nielsen ⁴⁾ circle. It has been suggested by Nambu ⁵⁾ and Nielsen ⁶⁾ that a natural extension of the dual model to off-shell amplitudes or currents would correspond to emitting the off-shell states at some points inside the circle (together with their electrostatic image with respect to that circle). However, a precise formulation of that suggestion in the operator formalism ⁷⁾ has encountered several difficulties : one is the apparent impossibility of introducing isospin quantum numbers for the current, because quantum numbers in dual models are usually thought of as propagating only along the Koba-Nielsen circle, and not wandering inside the disc ; a related problem is the construction of vector currents which also seemed impossible ⁸⁾. Another, more serious difficulty of the first attempts ³⁾ to implement that idea, was the badly located singularities of multicurrent amplitudes : singularities (poles) in channels with more than one current would always depend on the q^2 of those currents, and not only on the total momentum of that channel, a feature clearly incompatible with factorization ; this led these authors to abandon that scheme ⁹⁾.

In this paper, we show by explicit construction that a model can be built, which is free of all these difficulties.

The starting point of our construction is the following remark. There exists in the dual resonance model instances of particles emerging from the inside of the Koba-Nielsen disc : one looks at orientable non-planar one-loop graphs ^{10),11)} ; such graphs exhibit the so-called "Pomeranchuk singularity" in the limit where the radius of the annulus ¹¹⁾ goes to zero. In that limit, one obviously obtains a candidate for the emission of particles from a point inside the Koba-Nielsen circle by making a kind of multipole expansion of the graph. This means making a factorization of the Pomeron singularity à la Fubini-Veneziano : one would write the amplitude of Fig. 1 in the following way :

$$A = \sum_n R_n(p_i, q_j) P_n(p^2)$$

where $P_n(p^2)$ would include the information on the actual nature (pole, cut) of the Pomeron, and R_n would be a finite sum of products of Lorentz tensors :

$$R_n(p_i, q_j) = \sum_{m=1}^{d(n)} T_m^{(n)}(p_i) T_m^{(n)}(q_j)$$

The quantities $T_m^{(n)}(p_i)$ would be the candidates for the scattering of on-shell ground states with momenta p_i and one current with momentum $p = -\sum_i p_i$. The Lorentz tensor structure of $T_m^{(n)}$ would tell about the spin content of this current.

Another reason for considering the Pomeronchuk graph is that it is the only one-loop singularity which is mediated by resonances. All the other one-loop graphs can be written as tadpoles.

Let us show how this idea works in the simplest case : we find for the lowest lying Pomeron singularity :

$$\langle q_1 \dots q_M | \Sigma_T(p) | p_1 \dots p_N \rangle \simeq \mathcal{F}_M(q_1 \dots q_M, -p) P_0(p^2) \mathcal{F}_N(p_1 \dots p_N, p)$$

where

$$\mathcal{F}_N(p_1 \dots p_N, p) = g^{N-1} \int_0^\pi d\theta_N \int_0^{\theta_N} d\theta_{N-1} \dots \int_0^{\theta_3} d\theta_2 \prod_{1 \leq i < j \leq N} [\sin(\theta_j - \theta_i)]^{-p_i \cdot p_j}$$

and

$$P_0(p^2) = g^2 \left(\frac{\pi}{2}\right)^{p^2} \int_0^1 dw \int_w^1 \frac{dx}{x} \mu(w) q^{-\frac{1}{4} - \frac{p^2}{4}}$$

$$q = \exp \frac{2\pi^2}{\ln w}$$

This result has been obtained straightforwardly from expanding the integrand of the non-planar diagram in power of the integration variable q and keeping the lowest order only. It has been assumed that the intercept is one, otherwise the integrals do not factorize¹²⁾. Since the integration measure $\mu(w)$ contains terms of the form $\ln q$, $P_0(p^2)$ instead of having poles in p^2 has a right-hand cut starting at $p^2 = -1$; the next term in the expansion in powers of q ; $P_1(p^2)$ would have a priori a cut starting at $p^2 = 3$, and so on. We note that this procedure of expanding in power of q allows us to define the form factors $\mathcal{F}_N$ uniquely because the integrals factorize by themselves (up to normalization factors which can eventually depend on p^2 ; here these factors have been chosen so that the amplitude is meromorphic in p^2).

The form factors can now be considered for themselves and one can forget about the Pomeron from which they have been heuristically derived. They have factorization properties :

$$\mathcal{F}_N(p_1, \dots, p_N, p) = \langle 0 | t(p) D(p) | p_1 \dots p_N \rangle \quad (1)$$

$$t(p) = \pi^{-\alpha(p^2)} \exp \left[\frac{1}{2} (\alpha | \tilde{\Delta} | \alpha) + (p | \tilde{F}_T | \alpha) \right] \quad (2)$$

$$\tilde{\Delta} = \sum_{n \neq 0} \left(\frac{1}{n} \right) M_+ \left(-\frac{1}{n} \right)$$

$$\tilde{F}_T = - \sum_{m \neq 0} \left(\frac{1}{m} \right)$$

The operator $t(p)$ has a very close relation with the self-energy non-planar operator $\sum_T(p)$ from which it has obtained by letting w go to 1 and keeping only the operator part acting on the oscillators on one side. The key formula to check Eq. (1) uses the expansion of the sine function as an infinite product, namely :

$$\begin{aligned}
 (p_1 \dots p_N | \tilde{\Delta} | p_1 \dots p_N) &= \sum_{i,j=1}^N p_i \cdot p_j (p_j | \tilde{\Delta} | p_i) \\
 &= - \sum_{i,j=1}^N p_i \cdot p_j \sum_{n \neq 0} \ln \left[\frac{1 - (p_j - p_i)/m}{(1 - p_j/m)(1 - p_i/m)} \right] \\
 &= \sum_{i,j=1}^N p_i \cdot p_j \ln \left[\frac{\pi(p_j - p_i) \sin \pi p_i \sin \pi p_j}{\sin \pi(p_j - p_i) \pi p_i \pi p_j} \right]_{(3)}
 \end{aligned}$$

the proof of Eq. (1) is then completed by the change of variables $\theta_i = \pi \cdot p_i$ *).

One would now take $\langle 0 | t(p) \rangle$ as describing the emission of an off-shell scalar **). One immediate advantage of such a choice is the following. Because of their close relationship with multiloop amplitudes, one expects amplitudes with many off-shell particles which would be constructed with $\langle 0 | t(p) \rangle$ to exhibit only properly located poles in multicurrent channels. However, as it stands, such a choice is unacceptable for two reasons : the first one can be seen already at the simplest level $\mathcal{F}_N(p_1, \dots, p_N, p)$ has the property of being cyclic symmetric in the variables $p_1, \dots, p_N$ keeping p fixed. This is a natural consequence of the properties of the self-energy operator from which $\mathcal{F}_N$ has been derived and is not bad in itself ; however, it is clearly incompatible with the introduction of isospin quantum numbers for the off-shell state with momentum p . (We introduce isospin via the usual Chan-Paton factors ¹⁴⁾, ***)). The second, more serious, reason to reject that choice can be seen at the level of two currents : because of the duality property of

*) One must also apply that treatment to the term $(p | \tilde{F}_T | a)$. The terms of the form $p_i p_j \ln(p_i - p_j)$ cancel with the same expressions coming from the rest of the tree vector $|p_1, \dots, p_N\rangle$.

**) The amplitudes $\mathcal{F}_N(p_1, \dots, p_N; p)$ for one off-shell scalar are then exactly those of Ref. 13).

***) It also turns out that the duality properties of $\mathcal{F}_N$ are incompatible with the definition of off-shell vector states, which is related to the fact that the Pomeron singularity has only even angular momenta ¹²⁾.

Fig. 2, one could, in a channel with two off-shell scalars, obtain again the Pomeron singularity ^{*}). We now propose a modification which solves both problems.

We first discuss the modification in the simple case of the function $\mathcal{F}_3(p_1, p_2, p_3; p)$ (Fig. 3)

$$\begin{aligned} \mathcal{F}_3(p_1, p_2, p_3; p) &= g^2 \int_0^\pi d\theta_3 \int_0^{\theta_3} d\theta_2 [\sin(\theta_3 - \theta_2)]^{-p_3 p_2} [\sin \theta_3]^{-p_3 p_1} [\sin \theta_2]^{-p_2 p_1} \\ &= g^2 \frac{\sqrt{\pi}}{2} \frac{\Gamma(-\frac{1}{2}\alpha(p^2)) \Gamma(-\frac{1}{2}\alpha_s) \Gamma(-\frac{1}{2}\alpha_t) \Gamma(-\frac{1}{2}\alpha_u)}{\Gamma(-\frac{1}{2}\alpha_s - \frac{1}{2}\alpha_t) \Gamma(-\frac{1}{2}\alpha_t - \frac{1}{2}\alpha_u) \Gamma(-\frac{1}{2}\alpha_u - \frac{1}{2}\alpha_s)} \end{aligned} \quad (4)$$

with

$$\begin{aligned} \alpha_s &= 1 + \frac{1}{2} (p_1 + p_2)^2 & \alpha_t &= 1 + \frac{1}{2} (p_2 + p_3)^2 \\ \alpha_u &= 1 + \frac{1}{2} (p_1 + p_3)^2 \end{aligned}$$

On Eq. (4) the duality properties of $\mathcal{F}_3$ are manifest : it is completely symmetric in the momenta p_1, p_2, p_3 . It is only a sum of poles in the variable p^2 , without background or exponential increase ; in fact, one has, in the cut p^2 plane

$$\mathcal{F}_3 \underset{p^2 \rightarrow -\infty}{\sim} g^2 \frac{\sqrt{\pi}}{2} \left[-\frac{1}{2} \alpha(p^2) \right]^{-\frac{1}{2}} B\left(-\frac{1}{2} \alpha_s ; -\frac{1}{2} \alpha_t\right)$$

α_s, α_t fixed.

^{*}) This is because a two-loop graph has three partition functions which can diverge simultaneously, as shown in Ref. 15).

To obtain an amplitude which would have only planar duality, one should divide the integration region of the integral defining $\mathcal{F}_3$. Usually, such procedures lead to unacceptable asymptotic behaviours. However, it turns out that it is not so in our case : we define

$$\mathcal{F}_3' (P_1, P_2, P_3, P) = g^2 \int_0^{\frac{\pi}{2}} d\theta_3 \int_0^{\theta_3} d\theta_2 [\sin(\theta_3 - \theta_2)]^{-P_3 \cdot P_2} [\sin \theta_3]^{-P_3 \cdot P_1} [\sin \theta_2]^{-P_1 \cdot P_2} \quad (5)$$

This new amplitude has poles only in the variables α_s, α_t and p^2 (planar duality). It cannot be explicitated in a form as simple as that of Eq. (4), but only as the convergent series ¹⁶⁾ :

$$\begin{aligned} \mathcal{F}_3' &= g^2 B(-\alpha_s, -\alpha_t) \sum_{n=0}^{\infty} \frac{\Gamma(-\frac{1}{2}\alpha_t + n) \Gamma(-\frac{1}{2}\alpha_s + n)}{\Gamma(-\frac{1}{2}\alpha_t) \Gamma(-\frac{1}{2}\alpha_s)} \\ &\times \frac{\Gamma(\frac{1}{2} - \frac{1}{2}\alpha_s - \frac{1}{2}\alpha_t)}{\Gamma(n + \frac{1}{2} - \frac{1}{2}\alpha_s - \frac{1}{2}\alpha_t) n!} \frac{\Gamma(-\frac{1}{4}p^2 + n - \frac{1}{2}) \Gamma(1/2)}{\Gamma(-\frac{1}{4}p^2 + n)} \end{aligned}$$

In this form it is manifest that it is only a sum of poles in p^2 [at fixed $\alpha(s)$ and $\alpha(t)$]. The asymptotic behaviour at fixed $\alpha(s)$ and $\alpha(t)$ in the cut p^2 plane is

$$\mathcal{F}_3' \underset{p^2 \rightarrow -\infty}{\sim} \frac{g^2}{2} \sqrt{\frac{\pi}{-p^2}} B(-\frac{1}{2}\alpha(s); -\frac{1}{2}\alpha(t))$$

The behaviour is at fixed $p^2, \alpha(t); \alpha(s) \rightarrow -\infty$ is :

$$\alpha(t) \leq -1 \quad \mathcal{F}_3' \sim -g^2 [-\alpha(s)]^{\frac{\alpha(t)-1}{2}} \Gamma(-\frac{\alpha(t)-1}{2}) \frac{2^{-\frac{1}{2}(1+\alpha(t))}}{1+\alpha(t)}$$

$$\alpha(t) \geq -1 \quad \mathcal{F}_3' \sim g^2 \frac{1}{2} [-\alpha(s)]^{\alpha(t)} B(-\frac{1}{2}\alpha(p^2); \frac{1+\alpha(t)}{2})$$

it is thus Regge-behaved, however, this introduces a new trajectory

$\tilde{\alpha}(t) = (\alpha(t)-1)/2$. The contribution of this new trajectory is negligible whenever $\alpha(p^2) \simeq 2N$ where one recovers the usual Regge behaviour in $(-\alpha(s))\alpha(t)$. This new trajectory does not introduce any new states in the model since it materializes only when the usual Regge trajectory $\alpha(t) \simeq 2N+1$. Near such a point the factorization tells us immediately that we do not introduce any new state. Such a situation is not without analogy with the photo-production fits, where, to obtain a reasonable fit, fixed poles are needed besides the usual Regge poles.

Finally, the factorization properties of $\mathcal{F}'_3$ are exhibited by the following remark : in Eq. (1), one obtains $\mathcal{F}'_3$ by replacing $t(p)$ by

$$t'(p) = \left(\frac{\pi}{2}\right)^{-\alpha(p^2)} \exp \left[\frac{1}{2} (a | \tilde{\Delta}' | a) + (p | \tilde{F}'_\tau | a) \right]$$

$$\tilde{\Delta}' = \sum_{n \neq 0} \left(\frac{1}{2n}\right) M_+ \left(-\frac{1}{2n}\right)$$

$$\tilde{F}'_\tau = -\sum_{m \neq 0} \left(\frac{1}{2m}\right) \quad (6)$$

This then defines $\mathcal{F}'_N(p_1, \dots, p_N; p)$:

$$\mathcal{F}'_N(p_1, \dots, p_N, p) = g^{N-1} \int_0^{\frac{\pi}{2}} d\theta_N \dots \int_0^{\theta_3} d\theta_2 \prod_{1 \leq i < j \leq N} [\sin(\theta_j - \theta_i)]^{-P \cdot Q} \quad (7)$$

On this form, it is clear that $\mathcal{F}'_N$ has only the poles of the planar channels of Fig. 4. It can also be shown (see Section IV) that it is only a sum of poles in the variable p^2 , at fixed values of the other invariants of the planar channels of Fig. 3. We shall also comment in Section IV about the uniqueness of the choice of $\pi/2$ instead of π as the upper bound for the integration variable θ_N .

As was explained before, multicurrent amplitudes computed with the operator of Eq. (2) contain the unphysical Pomeron singularity in a two-current channel. This can be seen in the following way : the tadpole $t(p)$ of Eq. (2) is (for $p = 0$) just the limit of the planar tadpole when the generator P of its group of projective transformations becomes parabolic. When dealing with a two-current amplitude, one must introduce the group generated by the parabolic transformations P_1 and P_2 of the two currents. It has been shown in Ref. 15) that of all the elements of this group, only those of the form $(P_1 P_2)^n$ can themselves be parabolic. When they do become parabolic, the amplitude exhibits the third Pomeron singularity which, by duality (Fig. 2), one expects to occur in a two-Pomeron channel. Whereas such a property is natural for currents with vacuum quantum numbers, it would be undesirable for currents with isospin to exhibit this singularity.

We now remark that the modification which takes Eq. (2) into Eq. (6) amounts to replace the generator P of the group of projective transformations of the planar tadpole by P^2 . Hence, when computing multicurrent amplitudes with Eq. (6), instead of the group generated by P_1 and P_2 , one deals with the group generated by P_1^2 and P_2^2 . In this new group, there can never be any new parabolic transformation. This generalizes to multicurrent amplitudes, where we can now expect that no unphysical singularity will occur.

It is amusing to remark that in this scheme the detailed structure of the operator $t'(p)$ is important to give the right singularities but is almost irrelevant to prove the duality properties of the general N current amplitude. Indeed one can show that $t(p)$ and $t'(p)$ satisfy :

$$\langle 0 | t'(p) x^R \Omega = \langle 0 | t(p) x^R$$

and therefore

$$\langle 0 | t(p) D(p) V(p_N) \dots V(p_1) | 0 \rangle = \langle 0 | t(p) \mathcal{D}(p) | p_1 \dots p_N \rangle$$

where $\mathcal{D}(p)$ is the usual projected twisted propagator. Therefore a multicurrent amplitude will be obtained by putting the states $\langle 0 | t(p_i) \mathcal{D}(p_i)$ on the external lines of an N Reggeon vertex ; since these states are projected under $L_0 - L_+ - 1$ the duality properties of the multicurrent graph will come only from those of the N Reggeon vertex.

To sum up the findings of this section, we have constructed two kinds of scalar currents, one with vacuum quantum numbers $[\text{operator } t(p)]$ which can couple to the Pommeranchuk singularity, and one to which isospin Chan-Paton factors can be attached and which is decoupled from the Pommeranchuk singularity $[\text{operator } t'(p)]$.

III. - VECTOR CURRENTS

To construct the operator an off-shell vector state, we go back to the idea of a multipole expansion of the non-planar graph : the scalar off-shell state was represented by a charge at the centre of the Koba-Nielsen circle. By analogy with what occurs for on-shell states ¹⁷⁾, we expect an off-shell vector particle to contain also a dipole contribution. We also expect that there will be poles in p^2 starting at $p^2 = 0$, mass of the first vector particle of the theory. Hence, we are led to the rather natural choice

$$V'_\mu(p) = -\frac{1}{4} \left(\frac{\pi}{2}\right)^{-\alpha(p^2)} \langle 0 | t'(p) (a_\mu | s) \frac{1}{L_0 - 1} \quad (8)$$

where the vector $|s\rangle$ has components :

$$\begin{aligned} S_n &= \pi^n \frac{\sqrt{n}}{n!} \left(\frac{\partial}{\partial \theta}\right)^n \sin \theta \Big|_{\theta=0} = \pi^n \frac{\sqrt{n}}{n!} (-1)^{\frac{1}{2}(n-1)} \quad (n \text{ odd}) \\ &= 0 \quad (n \text{ even}) \end{aligned}$$

One should of course use an expression which has negative charge conjugation ¹⁸⁾. So that the full operator is

$$V_\mu(p) = V'_\mu(p)(1 - \Omega)$$

Ω being the twisting operator ¹⁹⁾.

Applying this operator to a state $V(p_2)|0\rangle$ we obtain the form factor :

$$\begin{aligned} A_2^\mu &= -\frac{g}{4} (p_2^\mu - p_1^\mu) \int_0^{\frac{\pi}{2}} d\theta \sin \theta^{-1-\alpha(p^2)} \sin 2\theta \\ &= g \frac{(p_2^\mu - p_1^\mu)}{p^2} \end{aligned}$$

This form factor comes out as being entirely ρ dominated, the ρ mass coinciding in our case with the photon mass (this coincidence makes the investigation of the algebra of currents rather difficult : our model for vector currents is similar to a field-algebra model, and field algebra is singular when the ρ mass goes to zero). This shows that the electromagnetic form factor needs not necessarily have all the poles which could possibly couple to the current. However, the form factors for resonance production off a ground state have in general higher order poles.

Applying this operator V_μ to the tree of Fig. 3, one obtains the amplitude :

$$\begin{aligned}
 A_3^\mu = & -\frac{g^2}{4} \int_0^{\frac{\pi}{2}} d\theta_3 \int_0^{\theta_3} d\theta_2 [\sin(\theta_3 - \theta_2)]^{-P_3 P_2} [\sin \theta_3]^{-P_3 P_1} [\sin \theta_2]^{-P_1 P_2} \\
 & \times [P_2^\mu \sin 2\theta_2 + P_3^\mu \sin 2\theta_3] \\
 & + \frac{g^2}{4} \int_0^{\frac{\pi}{2}} d\theta_1 \int_0^{\theta_1} d\theta_2 [\sin(\theta_2 - \theta_1)]^{-P_1 P_2} [\sin \theta_2]^{-P_2 P_3} [\sin \theta_1]^{-P_1 P_3} \\
 & \times [P_2^\mu \sin 2\theta_2 + P_3^\mu \sin 2\theta_1]
 \end{aligned}$$

By the same arguments as in the preceding section, one can show that this amplitude has an acceptable asymptotic behaviour in p^2 (no exponential increase in the cut p^2 plane). We thus realize a kind of generalized vector dominance model.

The first problem to study is the transversality properties of V_μ . At this stage, the simplest method²⁰⁾ is to use an infinitesimal change of variables in Eq. (7). Specifically, we make the change

$$\theta \longrightarrow \theta + 2\epsilon \sin^2 \theta$$

we expand in powers of ϵ , and write that the coefficient of ϵ is zero :

$$0 = \int_0^{\frac{\pi}{2}} d\theta_N \dots \int_0^{\theta_3} d\theta_2 \prod_{1 \leq i < j \leq N} [\sin(\theta_j - \theta_i)]^{-P_i \cdot P_j} \left\{ - \sum_{1 \leq i < j \leq N} P_i \cdot P_j (\sin 2\theta_i + \sin 2\theta_j) \right. \\ \left. + 2 \sum_{i=2}^N \sin 2\theta_i \right\} \\ + \int_0^{\frac{\pi}{2}} d\theta_{N-1} \int_0^{\theta_{N-1}} d\theta_{N-2} \dots \int_0^{\theta_3} d\theta_2 \prod [\sin(\theta_j - \theta_i)]^{-P_i \cdot P_j}$$

After some kinematics (using the mass-shell condition $p_1^2 = -2$) :

$$\int_0^{\frac{\pi}{2}} d\theta_N \dots \int_0^{\theta_3} d\theta_2 \prod [\sin(\theta_j - \theta_i)]^{-P_i \cdot P_j} \left[\sum_{i=1}^N P \cdot P_i \sin 2\theta_i \right] \\ = \int_0^{\frac{\pi}{2}} d\theta_{N-1} \dots \int_0^{\theta_3} d\theta_2 \prod [\sin(\theta_j - \theta_i)]^{-P_i \cdot P_j} \quad (10)$$

On the left-hand side of Eq. (10), one recognizes the action of $p^\mu V'_\mu$ on a tree. The right-hand side is even under twisting of the off-shell line. Hence, on an arbitrary tree :

$$p^\mu V_\mu |tree\rangle = p^\mu V'_\mu (1 - \Omega) |tree\rangle \\ = 0 \quad (11)$$

This proves the transversality of the amplitude $V_\mu(p)|tree\rangle$ except at $p^2 = 0$ (V^μ could have the form $v^\mu - p \cdot v p^\mu / p^2$). At $p^2 = 0$, one notices that $V^\mu(p)|tree\rangle$ has a pole (the ρ meson pole) whose residue is the amplitude for the process : $\langle tree | \rightarrow \rho$ meson on-shell. It is well known²¹⁾ that such an amplitude is transverse. This completes the proof of the transversality of $V^\mu(p)|tree\rangle$.

Equation (11) tells us that $p^\mu V_\mu$ is a spurious state (i.e., it has zero scalar product with an arbitrary tree). Hence it should be possible to express this property in terms of the Virasoro operators L_n 's²⁰⁾, which implement the infinitesimal change of variables of Eq. (9). In the Appendix, we show that the following operator identities hold :

$$\langle 0 | t'(p) (1 - \Omega) = 0$$

and

$$p_\mu V^\mu(p) = \langle 0 | t'(p) \left[\sum_{m=0}^{\infty} \frac{\pi^{2m+1} (-1)^m}{2 (2m+2)!} L_{2m+1} \right] \frac{1}{L_0 - 1} (1 - \Omega)$$

These two identities clearly mean that $p_\mu V^\mu(p)$ annihilate an arbitrary tree. We shall give yet another much simpler proof of this fact in the next section.

Notice that the existence of the full set of Ward identities (which is the full conformal invariance of the model) plays a crucial role in the proof of the transversality of the off-shell vector states.

IV. - REFORMULATION OF AMPLITUDES WITH ONE OFF-SHELL STATE

From Eq. (7), it is clear that the main part of the action of $\langle 0 | t'(p)$ on a tree is to make a change of variables :

$$p_i - p_j \rightarrow \sin \frac{\pi}{2} (p_i - p_j)$$

This fact suggests that, by using more thoroughly the full conformal invariance of dual models with unit intercept (which allows for a wide variety of changes of variables), one could find a much simpler form for the operators of Section II, which would give the same result when applied on a tree. This is what we are going to show, and it will also elucidate the connection with previous works ^{13), 22)}.

We start from Eq. (7) on which we make the following change of variables

$$x_i = \frac{\sin \theta_{i+1} \sin(\theta_N - \theta_{i+2})}{\sin \theta_{i+2} \sin(\theta_N - \theta_{i+1})} \quad i = 1, 2, \dots, N-3 \quad (12)$$

This leads to

$$\begin{aligned} \mathcal{G}'_N &= g^{N-1} \int_0^{\frac{\pi}{2}} d\theta_N \int_0^{\theta_N} d\theta_{N-1} \prod_{i=1}^{N-3} dx_i x_i^{-\alpha_{i,i+1}-1} \\ &\cdot \prod_{1 \leq i < j \leq N-3} (1 - x_i x_{i+1} \dots x_j)^{-p_{i+1} p_{j+2}} \\ &\cdot \frac{[\sin \theta_N]^{-\frac{1}{2} p^2}}{\sin \theta_N \sin \theta_{N-1} \sin(\theta_N - \theta_{N-1})} \left[\frac{\sin(\theta_N - \theta_{N-1})}{\sin \theta_{N-1}} \right]^{-\frac{1}{2} p^2 - p \sum_{i=1}^{N-1} p_i} \\ &\cdot \prod_{i=2}^{N-2} \left[\frac{\sin(\theta_N - \theta_i)}{\sin \theta_N} \right]^{p p_i} \end{aligned}$$

with

$$\alpha_{i,j} = 1 + \frac{1}{2} (p_i + p_{i+1} + \dots + p_j)^2$$

We introduce the variable

$$r = \frac{\sin \theta_{N-1}}{\sin(\theta_N - \theta_{N-1})}$$

and the notation

$$p_i = x_i x_{i+1} \dots x_{N-3}$$

to obtain

$$\begin{aligned} \mathcal{F}'_N &= g^{N-1} \int \prod_{i=1}^{N-3} dx_i x_i^{-\alpha_{i,i+1}-1} \prod_{1 \leq i < j \leq N-3} (1 - p_i/p_{j+1})^{-p_{i+1} \cdot p_{j+2}} \\ &\times \int_0^{\pi/2} d\theta_N (\sin \theta_N)^{-1-\alpha(p^2)} \int_0^\infty dr r^{-1-\alpha_{1,N-1}} \prod_{i=2}^{N-1} (1 + 2r p_{i-1} \cos \theta_N + r^2 p_{i-1}^2)^{-\frac{p_i \cdot p_j}{2}} \end{aligned}$$

On this form, one recognizes that $\mathcal{F}'_N$ can be factorized in the configuration of Fig. 5 :

$$\mathcal{F}'_N = \int_0^{\pi/2} d\theta (\sin \theta)^{-1-\alpha(p^2)} \int_0^\infty dr r^{-1-\alpha_{1,N-1}} \langle 0, p_N | \exp(p | W(\theta) | a) (-r)^H | \text{tree} \rangle$$

where

$$| \text{tree} \rangle = V(p_{N-1}) D V(p_{N-2}) \dots V(p_2) | 0 \rangle$$

and the diagonal matrix W is

$$W(\theta)_{mm} = \frac{1}{\sqrt{m}} \cos m \theta \delta_{mm}$$

We must now factorize the propagator with variable r ; for this, we use the identity :

$$\Omega^+ = \int_0^\infty dr r^{-1-\alpha_{1,N-1}} (-r)^H = \int_0^1 dx x^{-1-\alpha(p^2)+H} (1-x)^{-A}$$

and, after some simple manipulations on $\langle 0, p_N | \exp(p|W(\theta)|a) \Omega^+$, we arrive at the following form corresponding to the factorization in the configuration of Fig. 6 :

$$\mathcal{F}'_N = \int_0^{\frac{\pi}{2}} d\theta \sin \theta^{-1-\alpha(p^2)} \langle 0, p | e^{(p|\phi|a)} V(p_N) D V(p_{N-1}) \dots V(p_2) | 0 \rangle \quad (13)$$

where ϕ is a diagonal matrix with elements

$$\phi_{n,m} = \frac{1}{2} \left[(1 + e^{i\theta})^n + (1 - e^{-i\theta})^m \right] \delta_{n,m}$$

We thus have refactorized the amplitude for one off-shell scalar with a linear exponential, instead of the quadratic exponential of Eq. (6). On Eq. (13), it is also clear that $\frac{1}{2}\pi$ instead of π is the only modification on the θ integration which can possibly retain the structure of $\mathcal{F}'_N$ as a sum of poles in p^2 . Any other choice would introduce an exponential behaviour for $p^2 \rightarrow \infty$ in some direction.

Equation (13) can also be written as ($z = 1 - e^{-i\theta}$)

$$\mathcal{F}'_N = -i \int_{(c)} \frac{dz}{1-z} \left| \frac{z-z^*}{2} \right|^{-1-\alpha(p^2)} \left[\left(\frac{p}{2} z + \left(\frac{p}{2} z^* \right) \right] V(p_N) \dots V(p_2) | 0 \rangle \quad (14)$$

(c) being the integration contour of Fig. 7.

On this last form, the interpretation in terms of Koba-Nielsen variables, and the connection with Ref. 13) are manifest : the charge is emitted at the conjugate points z and z^* in Fig. 7.

Before discussing the implications of Eqs. (7) and (13), we give the formula which can be obtained by the same manipulations for the off-shell vector states :

$$V_\mu(p) |tree\rangle = \int_0^{\pi/2} d\theta \sin \theta^{-\alpha(p^2)} (0,p|T_\mu(z,z^*) e^{(p|\phi|a)} (1-\Omega) \times V(p_N) \mathcal{D} V(p_{N-1}) \dots V(p_2) |0\rangle \quad (15)$$

with

$$T^\mu(z,z^*) = \sum_{m=1}^{\infty} \sqrt{m} (z^{m-1} + z^{*m-1}) a_m^\mu$$

On this form, one recognizes that the residue of the first pole in p^2 is precisely the one-current operator of Refs. 22) and 23).

In Eq. (15), the transversality of V_μ is also transparent :
from

$$p^\mu T_\mu = 2 \left(\frac{d}{dz} + \frac{d}{dz^*} \right) (p|\phi(z,z^*)|a) \quad (16)$$

and

$$\langle 0| e^{(p|\phi|a)} (1-\Omega) = 0$$

one obtains trivially :

$$p^\mu V_\mu(p) |tree\rangle = 0$$

a proof which is simpler than the one of the Appendix !

We now discuss the implications of Eqs. (7) and (13). These two equations give the same amplitudes for one off-shell particle, but differ completely when used to build amplitudes with more than one off-shell state.

If one uses the linear exponential [Eq. (13)], one gets results similar to those of Refs. 8) and 13) : these results have the expected interpretation in terms of the analogue model : the electrostatic energy of a system of charges in a plane. However, one gets unphysical singularities in channels with more than one off-shell state : the location of the poles in such channels depend on the p^2 of the off-shell states, a result which is incompatible with the particle interpretation of these singularities.

If one uses the quadratic exponential [Eq. (6)], one does not run into these difficulties : as we shall show in a subsequent publication, one retains the interpretation of the analogue model, but the region of integration comes out restricted in such a way that only physical singularities occur (this is easily understood by the close relationship of quadratic exponentials with multiloop amplitudes). Also notice that the measure of integration is trivial in the case of linear exponentials, whereas it will depend on the number of dimensions in a highly non-trivial way in the case of quadratic exponentials.

Of course, one can argue that since Eqs. (6) and (13) give the same result when applied to an on-shell tree, they differ only by general gauge transformations of the type considered in Ref. 24). Hence, if one knew the fully projected propagator (i.e., projected with respect to all the L_n 's), using Eqs. (6) or (13) with this propagator would give the same result for many off-shell states. Considering all what this fully projected propagator is supposed to do : undo the change of variables of Eq. (12), then do another one, generalizing to the multiloop case the Jacobi transformation of the single loop case, restrict the integration region to yield only physical singularities, introduce a generalized partition function in the measure of integration, one concludes that such a propagator must be highly non-trivial ! Not having this propagator at our disposal, we explain in the next section how to investigate the main features of multicurrent amplitudes.

V. - CONSTRUCTION OF MULTICURRENT AMPLITUDES

In this section, we describe the program which we shall follow from the construction of multicurrent amplitudes and some of the results obtained. We do not have at our disposal a fully projected propagator, but only ²⁵⁾ the propagator $\mathcal{D}$ projected for L_1 . In this case, as was just explained, one should not use the linear exponentials [Eq. (13)] to compute multicurrent amplitudes. Instead, one should use the quadratic exponentials [Eq. (6)]. In this way because of their close relation with the self-energy operators, one expects only properly located singularities.

One will start by writing the operators $t'(p)$ or $t(p)$ in a way which makes their relations with projective transformations obvious

$$t(p) = \pi^{-\alpha(p^2)} \langle 0 | \exp \left[\frac{1}{2} (a | \Gamma(\mathcal{P}-I) | a) + p (a | \Gamma(\mathcal{P}-I) | \infty) \right] \rangle_{(17)}$$

where $\Gamma = M_+ M_-$

$$\mathcal{P} = \sum_{m=-\infty}^{+\infty} P^m \quad P = M_-[-1] \quad P(z) = (z+1) - (1)$$

as we see the operator $t(p)$ is associated with a homographic transformation of the parabolic type. Therefore when we will do the sewing of these operators we will get a group of transformations generated by parabolic transformations. The algebraic manipulations involved will be a particular case of those effected by Cremmer ²⁶⁾ to compute two-loop graphs starting with one-loop tadpole operators. When these algebraic manipulations are effected, one obtains an integral representation for, let us say, the two-current N particle amplitude which has the following features.

It contains multiplicatively in the integrand the following parts.

- A generalized partition function $\prod_q [f(w_q)]^{-4}$ coming from the infinite dimensional determinant one has to evaluate. This function blows up exponentially at one point only in the case of vacuum quantum number currents ($t(p)$) and remains finite when currents with quantum numbers ($t'(p)$) are considered.

- A part depending on products of different momenta : $\exp - \sum_{i \neq j} (p_i p_j / 2) N(z_i, z_j)$
 $N(z_i, z_j)$ being the Neumann function of the surface we consider. However, because we consider a group generated by parabolic transformations, this means [see Refs. 27] that the surface we consider is a sphere with handles which have vanished, therefore merely a sphere. So, the Neumann function will be a cross-ratio. However, it will not be a cross-ratio in the variable z_i, z_j , but in new variables Z_i, Z_j . The transformation which casts the z_i into the Z_i is called the Jacobi transformation. An example of such transformation is shown for the one-current case by Eq. (3) where the variables ρ_i have been replaced by $Z_i = \exp 2i\pi \rho_i$. When this transformation has been done, this part takes the form described in the original version of Drummond's paper

$$\prod_{i \neq j} |Z_j - Z_i| |Z_j - \tilde{Z}_i|^{-\frac{p_i \cdot p_j}{4}}$$

Z_i being real if p_i is on-shell and complex if p_i is off-shell.

- A part depending on the square of the off-shell momenta

$$\prod_i |Z_i - \tilde{Z}_i|^{-p_i^2/2}$$

When Z_i and Z_j approach each other on the real axis, these factors contribute to the normal resonance singularities in $(p_i + p_j)^2$.

- Additional factors depending also on the square of the off-shell momenta :

$$\prod_i a_i^{-\frac{p_i^2}{8}}$$

these factors look like generalized partition functions. They are a generalization of the quantity $\exp -(4\pi^2/(1-w))$ met in the one-loop diagram.

The normal resonance singularities in Drummond's model occur when $Z_i \rightarrow \tilde{Z}_j$, i.e., when the points Z_i and Z_j touch each other on the real axis. Then we get the normal poles in the trajectory $\alpha((p_i + p_j)^2)$.

There are also abnormal singularities when two points touch each other off the real axis : $Z_i \rightarrow Z_j$. This is the cause of the failure of the original version of Drummond's model because the integrand behaves like $|Z_i - Z_j|^{-(p_i p_j / 4)}$ and gives rise to a singularity in the variable $p_i p_j$, which ruins the factorization. In our model, this problem is solved in the following way.

Consider for instance a two-current amplitude. When we start to compute it we have two parabolic transformations :

$$\frac{1}{P_i(z) - \eta_i} = \frac{1}{z - \eta_i} + \frac{1}{c_i} \quad i = 1, 2$$

which generate the group of transformations which define the Neumann functions : a projective invariant quantity is $\rho_{12} = [(\eta_1 - \eta_2)^2 / c_1 c_2]$. This is the only projective invariant quantity which is left to define the group. Several expressions in the integrand depend only on this parameter :

- the generalized partition function $\prod_{\alpha \neq 1, 2} f(w_\alpha)^{-4}$;
- the factors a_1, a_2 ;
- the cross-ratio

$$R_{12} = \frac{(z_1 - z_2)(\tilde{z}_1 - \tilde{z}_2)}{(z_1 - \tilde{z}_2)(\tilde{z}_1 - z_2)}$$

One can show that $\rho_{12} \geq 4$. ρ_{12} is also related to the multiplier of the transformation $P_1 P_2 (w_3)$ and $\rho_{12} \geq 4$ implies that $0 \leq w_3 \leq 1$. When $\rho_{12} \rightarrow 4$ then $w_3 \rightarrow 1$ ("third divergence" phenomenon) and the partition function blows up exponentially. But also, because of their relation with ρ_{12} , the cross-ratio behaves like :

$$R_{12} \sim \exp - \frac{4\pi^2}{1 - w_3}$$

thus $z_1 = z_2$ is obtained off the real axis when $w_3 \rightarrow 1$. This is an interesting interpretation of Drummond's singularity. One has also :

$$a_1 \sim a_2 \sim R_{12}$$

Therefore the integrand behaves like :

$$\exp - \frac{2\pi^2}{1 - w_3} \left[-\frac{1}{3} - \frac{1}{4} (p_1 + p_2)^2 \right] \sim q_3^{-\alpha_2 (p_1 + p_2)^2}$$

and we get the Pomeranchuk singularity in the variable $(p_1 + p_2)^2$.

This was to be expected : because of the duality transformation of Fig. 2, we know that two currents with vacuum quantum numbers could couple to the Pomeranchuk trajectory.

In the case of currents with quantum numbers, the cross-ratio R_{12} turns out to be bounded below : $R_{12} \geq R_0 \neq 0$ so that the Pomeranchuk singularity does not occur.

We write below for the sake of completeness the expression for two currents (p_1, p_2) and N particles on-mass shell $(p_3, \dots, p_{N+2})$

$$\int \prod_{i=3}^{N+2} \frac{d\tilde{z}_i}{dV_{abc}} \frac{d\eta_1 d\zeta_1 d\eta_2 d\zeta_2}{C_1^2 C_2^2} \left[\prod_{\alpha \neq 1,2} f(w_\alpha) \right]^{-4} \prod_{1 \leq i < j \leq N+2} |(z_i - \tilde{z}_j)(\tilde{z}_i - z_j)|^{-\frac{p_i p_j}{2}} \\ \times \left(\frac{z_1 - \tilde{z}_1}{2i} \right)^{-\frac{p_1^2}{2}} \left(\frac{z_2 - \tilde{z}_2}{2i} \right)^{-\frac{p_2^2}{2}} \left[\frac{(z_1 - z_2)(\tilde{z}_1 - \tilde{z}_2)}{(z_1 - \tilde{z}_2)(\tilde{z}_1 - z_2)} \right]^{-\frac{p_1 p_2}{4}} a_1^{-\frac{p_1^2}{8}} a_2^{-\frac{p_2^2}{8}} \quad (18)$$

z_1, z_2 are complex variables while $z_3, z_4, \dots, z_{N+2}$ are real. (η_1, c_1) , (η_2, c_2) are the variables associated with each current. They are related by the Jacobi transformation to the variables z_1, z_2 as well as to the w_α 's and a_1, a_2 . We will explicit these relations in a subsequent paper. This expression is easily generalized to a N current amplitude.

Of course, this result is not absolutely complete since our current still propagates spurious states. It is, however, a wide-spread belief, based on the experience of the elimination of the spurious states in the one-loop case ²⁸⁾ that this removal will mainly change the power of the generalized partition function from -4 to -3 , while the momentum-dependent part will keep its interpretation in term of Abelian integrals.

VI. - CONCLUSION

The main advantage of the approach to the off-shell extension of dual models which we have presented here is its internal consistency. Its main drawback is the rather involved manipulations necessary to prove that self-consistency, although the main features of multicurrent amplitudes have rather simple interpretations.

Because of the complication of multicurrent amplitudes, and of the unphysical characteristics of the low energy part of the spectrum of presently known dual models, fine details, like current algebras and low energy theorems are particularly hard to test. (Properties more easily accessible are the high energy behaviours of multicurrent amplitudes.)

Another interesting application would be the investigation of the internal symmetries of dual models. The degeneracy of the spectrum of dual models is much higher than expected from usual Regge-behaved models. One may expect, in analogy with the hydrogen atom, or the non-relativistic harmonic potential, that this degeneracy has its origin in some internal symmetry of dynamical nature. This symmetry might be revealed by a study of the commutation relations of currents corresponding to higher order expansions of the Pomeron singularity, and thus to lower daughter trajectories

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APPENDIX

We show here the relationship between the divergence of the vector current and the Virasoro gauges. Let us consider the operator :

$$J_{\mu} = \langle 0 | t'(p) (a_{\mu} | s)$$

We remark that $\tilde{\Delta}'_{m+n}$ and $(\tilde{F}'_T)_{m+n}$ are related by

$$\tilde{\Delta}'_{m,n} = (-1)^{n-1} \sqrt{\frac{n}{m}} \binom{m+n-1}{n} (\tilde{F}'_T)_{m+n} \quad (A.1)$$

while

$$(\tilde{F}'_T)_{2n} = (-1)^n \frac{\pi^{2n}}{2n!} B_{2n} (\tilde{F}'_T)_{2n+1} = 0 \quad (A.2)$$

The coefficients B_{2n} are the Bernoulli numbers defined by the recursion relation :

$$\sum_{k=0}^n \binom{n}{k} B_k = B_n \quad B_0 = 1 \quad (A.3)$$

We consider now the divergence of the vector current :

$$D = p^{\mu} J_{\mu} = \langle 0 | \left[\sum_{n=0}^{\infty} p \cdot a_{2n+1} \pi^{2n+1} \frac{\sqrt{2n+1}}{(2n+1)!} (-1)^n \right] t'(p)$$

And introduce the Virasoro operators :

$$L_N = L_N(-p) = \sqrt{N} p a_N - \frac{1}{2} \sum_{n=1}^{N-1} \sqrt{n(N-n)} a_n a_{N-n} - \sum_{n=1}^{\infty} \sqrt{n(n+N)} a_n^+ a_{n+N} \quad (A.4)$$

Their commutation properties with respect to the operator $t'(p)$ is expressed through the identity :

$$t'(p) L_N t'(p)^{-1} = L_N - \sum_{M=N+2}^{\infty} (-1)^{\frac{M-N}{2}} \frac{\pi^{M-N}}{(M-N)!} B_{M-N} J_M^{(N)} \quad (A.5)$$

$J_M^{(N)}$ has the following property :

$$\langle 0 | J_M^{(N)} \left(L_0 - M + \frac{p^2}{2} \right) = 0 \quad (A.6)$$

and its expression is :

$$J_M^{(N)} = -\sqrt{M} p a_M + \sum_{n=N+1}^{M-1} \sqrt{\frac{n}{M-n}} (n-N) \binom{M-N-1}{n-N} (-1)^{n-N} a_n a_{M-n} \quad (A.7)$$

Let us now consider the divergence D :

$$D = \langle 0 | S_1 t'(p) = \langle 0 | (\bar{L}_1 + R_3) t'(p)$$

where S_1 has been split into an eigenvector of R with eigenvalue 1 ($\langle 0 | \bar{L}_1$) and a sum of eigenvectors of R with eigenvalues greater or equal to 3 ($\langle 0 | R_3$). One recognizes that $\langle 0 | \bar{L}_1 = \pi \langle 0 | L_1$. Therefore, using (A.5), we get :

$$D = \langle 0 | [S_3 t'(p) + t'(p) \pi L_1]$$

where $\langle 0 | S_3$ has eigenvalues of R greater or equal to 3. One splits again S_3 : $\langle 0 | S_3 = \langle 0 | (\bar{L}_3 + R_5)$. $\bar{L}_3$ is found equal to

$$\langle 0 | \bar{L}_3 = -\frac{\pi^3}{13} \langle 0 | L_3(-p)$$

which we can again commute with $t'(p)$ using (A.5).

One can then show by recursion that :

$$D = \langle 0 | \left(\pi^{2n+1} \frac{2}{(2n+2)!} (-1)^n L_{2n+1} + R_{2n+3} \right) t'(p) + \langle 0 | t'(p) \sum_{m=0}^{n-1} 2 \pi^{2m+1} \frac{(-1)^m}{(2m+2)!} L_{2m+1}$$

where $\langle 0 | R_{2n+3}$ is a sum of eigenvectors of R with eigenvalues greater or equal to $2n+3$. The proof of this recursion relation is too lengthy to be presented here, but involves only (A.5) and the property of the Bernoulli numbers (A.3).

Letting n go to infinity, we get :

$$D = \langle 0 | t'(p) \left[\frac{4}{\pi} (L_0 - 1) + \sum_{m=0}^{\infty} 2 \pi^{2m+1} \frac{(-1)^m}{(2m+2)!} W_{2m+1} \right] \quad (A.8)$$

$$W_{2m+1} = L_{2m+1} - L_0 + 1$$

the second term will annihilate any multiperipheral tree ending with a propagator $1/(L_0 - 1)$, while the second will not contribute since

$$\langle 0 | t'(p) (L_0 - 1) \frac{1}{L_0 - 1} = \langle 0 | t'(p) (1 - \Omega) = 0$$

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FIG.1

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FIG.3

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FIG.5

FIG.6

FIG.7