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MODEL INDEPENDENT CALCULATION OF K_p DISPERSION RELATIONS

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A B S T R A C T

Using an extrapolation procedure proposed by one of us, which is based on rigorous properties of Padé approximants, we compute forward K_p dispersion relations. Among other results, we get averages of the absorptive part of the amplitude in the unphysical cut, and the absolute bounds $12.39 \leq \bar{G}^2/4\pi \leq 16.45$, and statistical estimate and errors $\bar{G}^2/4\pi = 14.15 \pm 1.17$, for the combination $\bar{G}^2 = G_{p\Lambda K}^2 + 0.92 G_{p\Sigma K}^2$. This is to be compared with the result of the polynomial extrapolation of Cutkosky and Deo, $\bar{G}^2/4\pi = 15^{+6}_{-4}$, the lower bound of Rogers, $\bar{G}^2/4\pi \geq 10$, and a host of model-dependent calculations for which $\bar{G}^2/4\pi$ ranges between 4 and 20.

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Kaon-nucleon dispersion relations, in particular in the forward direction (KNFDR), present the problem of a short but strong unphysical cut which prevents the straightforward calculation of the KNFDR to obtain, for example, the coupling constants $G_{\Lambda} \equiv G_{\Lambda pK}$ and $G_{\Sigma} \equiv G_{\Sigma pK}$. The attempts to solve KN dispersion relations can be classified into two main groups.

Authors in the first group use model-dependent parametrizations for Kp partial waves and then adjust the parameters to fit experimental data [see Refs. 1), 2) ; the second reviews earlier work]. Of these parametrizations, two kinds (so-called constant scattering length and K matrix with zero effective range) give values of $G_{\Lambda}^2/4\pi$ around 5, in violent disagreement with SU(3) predictions. On the contrary, the K matrix approach with variable range yields $G_{\Lambda}^2/4\pi \sim 13$, which is about the SU(3) value ^{*}). Still other authors ³⁾ obtain $G_{\Lambda}^2/4\pi$ around 8. Needless to say, all error estimates are highly unreliable and the calculations strongly model dependent. This has been demonstrated very strikingly by consistency tests ^{4), 5)} which no model could pass.

Authors in the second group try to get model independent results. If we denote by $A_{\pm}(E)$ to the imaginary part of the $K^{\pm}p$ forward scattering amplitude, with E the kaon lab. energy, all models give $A_{\pm}(E) \geq 0$ over the unphysical cut [associated with the reaction $K^{\pm}p \rightarrow \pi(\Lambda \text{ or } \Sigma) \rightarrow K^{\pm}p$]. This is used by Rogers ⁴⁾ to apply a set of inequalities devised by Tiktopoulos and Treiman ⁶⁾ to constrain the values of the coupling constant. This method [not totally model independent, as models were used to smooth out the real part of the $K^{\pm}p$ amplitude, $D_{\pm}(E)$, used as input] gave hopelessly big upper bounds and the lower bound $\bar{G}^2/4\pi \geq 10$. $\bar{G}^2$ is the combination of coupling constants that appears in the KNFDR,

$$\bar{G}^2 = G_{\Lambda}^2 + \xi(E) G_{\Sigma}^2, \quad (1a)$$

where

$$\xi(E) = \frac{X_{\Sigma} (\mu^2 - E_{\Lambda}^2)(E + E_{\Lambda})}{X_{\Lambda} (\mu^2 - E_{\Sigma}^2)(E + E_{\Sigma})}, \quad X_Y = \frac{(M_Y - M)^2 - \mu^2}{4M^2}, \quad E_Y = \frac{M_Y^2 - M^2 - \mu^2}{2M} \quad (1b)$$

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^{*}) This value of G_{Λ} is not only important from an SU(3) point of view ; for example ³⁾, if $G_{\Lambda}^2/4\pi \sim 14$, then a certain Kp amplitude must possess a fixed Regge pole, which would be absent if $G_{\Lambda}^2/4\pi \sim 5$.

and M_Y is the mass of hyperon Y , $Y = \Lambda$ or Σ ; M, μ are the masses of p, K . $\xi(E)$ is a slowly varying function of E . According to all indications, G_Σ^2 is small compared to G_Λ^2 , so that

$$\bar{G}^2 \approx G_\Lambda^2.$$

Another model independent method has been put forward by Cutkosky and Deo ⁷⁾: they subtract the contribution to the KNFDR of the known terms, viz., physical dispersive integrals over the $K^\pm p$ total cross-sections, $\sigma_\pm(E)$, and the subtraction terms (scattering lengths), to be left with a function possessing only the unphysical cut. This cut is mapped over the contour of an ellipse in a new variable, z , and the cut plane goes over the interior of the ellipse. Thus, the resulting function $\varphi(z)$ is analytic inside the ellipse except for poles at z_Λ and z_Σ . These can be disposed of by considering $(z - z_\Lambda)(z - z_\Sigma)\varphi(z)$ which therefore can be convergently expanded in Legendre or other similar polynomials. The coefficients of this (truncated) expansion can be fitted to experimental values of $\varphi(z)$ and this extrapolates the function to the points z_Λ, z_Σ where it gives G_Λ^2, G_Σ^2 . The value, and statistical errors obtained, are $\bar{G}^2/4\pi = 15^{+6}_{-4}$. All papers using this method, or variants of it ^{7),8)} give $\bar{G}^2/4\pi$ centred around 15. It should be emphasized, however, that the errors given in these papers do not take into account the errors of the extrapolation.

What we want to do here is to use an extrapolation method recently proposed by one of us [Ref. 9), Section 8 and especially Ref. 10] based on techniques of Padé approximants and the moment problem. Before describing the method, as applied to our case, we discuss our input.

Letting $\sigma_\pm$ be the $K^\pm p$ total cross-sections, and k the kaon lab. momentum, we define the functions $f, \bar{f}$:

$$\begin{aligned} f(E) &\equiv \frac{\pi}{E^2 - \mu^2} \left[D_+(E) + \frac{1}{2} D_+(\mu) \left(1 - \frac{E}{\mu}\right) - \frac{1}{2} D_-(\mu) \left(1 + \frac{E}{\mu}\right) \right] \\ &\quad - \frac{1}{2\pi} \int_0^\infty dk' \frac{\sigma_+(k')}{E'(E' - E)} - \frac{1}{2\pi} \int_{k_1}^\infty dk' \frac{\sigma_-(k')}{E'(E' + E)} \\ &= \frac{\chi_\Lambda}{(E_\Lambda^2 - \mu^2)(E + E_\Lambda)} \bar{G}^2 + \bar{f}(E), \end{aligned} \quad (2a)$$

$$\bar{f}(E) = \int_{E_0}^{E_1} dE' \frac{\rho(E')}{E' + E}, \quad \rho(E') \equiv \frac{A_-(E')}{\mu^2 - E'^2}. \quad (2b)$$

E_0 is the beginning of the unphysical cut, $E_0 = [(m_\pi + M_\Lambda)^2 - M^2 - \mu^2]/2M$, m_π the pion mass and we have taken $E_1 = 0.49016$ GeV slightly below the physical K^-p threshold to avoid the singularity caused by the subtraction at $E' = -\mu$. This only requires knowledge of $A_-(\mu)$, which can be taken from experiment. A detailed account of this evaluation, as well as of others to come, shall be published elsewhere ¹¹⁾. Now, $D(E)$ and hence $f(E)$ can be obtained at certain values E_n of E from forward elastic differential cross-sections or Coulomb interference measurements. The errors in $f(E_n)$ will be obtained composing errors on $D(E)$ with the contributions of errors on $\sigma_\pm(E')$, $D_\pm(\mu)$, $A_-(\mu)$; the values of $f(E_n)$, together with the errors can be found in Fig. 1.

The values of $D_\pm(\mu)$, $A_-(\mu)$ are given in terms of the scattering lengths, $\alpha_\ell^I = a_\ell^I + i b_\ell^I$. We have taken ^{12), 2)}:

$$\begin{aligned} a_0^0 &= -1.67 \pm 0.03 F; & a_0^1 &= -0.095 \pm 0.030 F \\ b_0^0 &= 0.69 \pm 0.02 F; & b_0^1 &= 0.59 \pm 0.05 F; \end{aligned} \quad (3a)$$

low and medium energy cross-sections were taken from Ref. 13); at high energy, the data of Serpukhov ¹⁴⁾ were used and, above 55 GeV, we took:

$$\sigma_-(k) = 19.64 + (18.68) k^{-0.814}, \quad \sigma_+(k) = 18.2 \pm 1.5. \quad (3b)$$

The errors in (3b) are irrelevant, as the over-all contribution of the high energy region is, at most, of 6%. Also the errors coming from medium and low energy $\sigma_\pm$ are seen to be small although the fact that the dispersion integrals are singular makes error estimates difficult. In fact, most of the errors in $f(E_n)$ come from errors in $D_\pm(E_n)$.

Now, by using unitarity, one can show ¹⁵⁾ that $\rho(E')$ can be written as $\rho(E') = \rho_{\text{even}}(E') - \rho_{\text{odd}}(E')$, where $\rho_{\text{even/odd}}$ are the contributions of the partial waves with even/odd orbital angular momentum, and are positive. In this situation it can be shown ^{9),10)} that what essentially amounts to Padé approximants can be constructed out of the values $f(E_n)$ and converge towards $f(E)$ and also, although not only in average sense, towards $\rho(E')$. Furthermore, it is possible to compute explicitly the way in which errors in $f(E_n)$ propagate to errors in the approximations of $f(E)$ or to averages of $\rho(E')$. If we want to be totally model-independent this is as far as we can go. However, all models predict

$\rho_{\text{odd}} \ll \rho_{\text{even}}$ (this is rigorously true at both end-points of the unphysical cut, due to threshold factors). Even straining the parameters in the models, all it is possible to get is $\rho(E')$ negative around the $\Sigma(1385)$ resonance and right below the opening of the $\pi\Sigma$ channel ; and here the absolute value of ρ is extremely small compared with the value of ρ elsewhere, so that even if ρ was negative at some points, the negative contribution would be negligible. If, according to this, we assume $\rho \geq 0$, then we can go much further. We will describe separately both possible situations.

1. - No hypothesis on ρ

We approximate $\bar{f}(E)$ by second kind Padé approximants ¹⁶⁾ $\bar{f}_\nu(E)$. To reproduce correctly the asymptotic behaviour of $\bar{f}$ we take only approximants of the form $[1,0]$. The coefficients in the Padé approximants are fixed by requiring that $\bar{f}_\nu(E_n) = \bar{f}(E_n)$; thus, the errors in the $\bar{f}(E_n)$ induce errors in the $\bar{f}_\nu(E_n)$ thereby limiting the effective degree ν that can be used. Due to this, it is impossible to go beyond $\nu = 2$, if we assume nothing about ρ . Hence, we fit $f(E) = (\text{pole}) + (a_0/E + b_1)$ to 73 experimental points and get the following results :

$$a_0 = 53.9 \pm 2.7 ; b_1 = 0.45 \pm 0.01 ; \bar{G}^2 = 14.3 \pm 3.1 ; \chi^2 = 54.2. \quad (4)$$

Details on the stability of the fit, etc., will appear in Ref. 11). Calculated values of f , $D_\pm$ may be found in Figs. 1, 2.

The excellent value of χ^2 is to be noted as well as the economy of parameters : 2+1 coupling constant, vs., 4+1 in Ref. 8) and 5+1 in Ref. 17). The extrapolated values of $D_\pm(E)$ are shown in Fig. 2. We now turn to the calculation of ρ .

It has been shown ^{9),10)} that the averages of $\rho(E')$ around the point

$$\begin{aligned}\tilde{E}_j^{(N)} &= \frac{1}{M_j^{(N)}} \sum_{n=0}^N \lambda_n(N, j) \left\{ (E_1 - E_0) - E_n \log \frac{E_n + E_1}{E_n + E_0} \right\} \in [E_0, E_1]; \\ M_j^{(N)} &= \sum_{n=0}^N \lambda_n(N, j) \log \frac{E_n + E_1}{E_n + E_0}, \\ \lambda_n(N, j) &= (-1)^j (E_n + E_0)^j (E_1 + E_n)^{N-j} / \prod_{\substack{v=0 \\ v \neq n}}^N (E_v - E_n),\end{aligned}\quad (5)$$

are given by the quantities

$$\langle \rho(\tilde{E}_j^{(N)}) \rangle = \frac{1}{M_j^{(N)}} \sum_{n=0}^N \lambda_n(N, j) \bar{f}(E_n). \quad (6)$$

The absolute errors in the averages are given by

$$\Delta \rho(\tilde{E}_j^{(N)}) = \frac{1}{M_j^{(N)}} \sum_{n=0}^N |\lambda_n(N, j)| \Delta \bar{f}(E_n). \quad (7)$$

Equation (7) [and the χ^2 in (4)] shows clearly why we cannot go to higher approximants: the errors become enormous. Thus here we can only obtain two averages of ρ in the unphysical cut. If we want to be totally model-independent this is as far as we can get.

2. - ρ positive

If we assume $\rho \geq 0$ or, what is still weaker, that ρ has its first averages positive: $\langle \rho(\tilde{E}_j^{(N)}) \rangle \geq 0$, $N = 0, 1, 2, 3, 4$, then we can prove ¹⁰⁾ that Eqs. (2b) imply a set of inequalities on $\bar{f}$, namely,

$$\sum_{n=0}^N \lambda_n(N, j) \bar{f}(E_n) \geq 0, \quad j = 0, \dots, N. \quad (8)$$

These inequalities, which extend those of Ref. 6), permit us to filter the errors on $\bar{f}(\bar{E}_n)$, as it is obvious that not all of the error bars [computed from the experimental errors in $f(E)$ and the error in $\bar{G}^2$ obtained above] will be compatible with each other through inequalities (8). Since it is impossible to filter all values of f , we have selected $f(E_n)$ at eight different energies : the two points at which Coulomb interference measurements are available ¹⁸⁾ and, at low energy, three points from K^+p and another three from K^-p selected among the most reliable ones.

The higher the order of the constraints used [i.e., the larger the value of N in (8)] the more the error bands are reduced. Thus, although more and more errors enter into Eq. (7), the over-all error $\Delta\rho$ does not increase as fast as when ρ is oscillating, because the $\Delta\bar{f}$ also becomes smaller. This will permit us to go to Padé approximants of high orders ($\nu = 3, 4$) [higher order approximants begin to develop unstabilities, and will be discussed in detail in Ref. 11]. Thus, we shall be able to give averages of ρ around three or four different points. The results are presented in Fig. 3. Also this permits us to calculate absolute error bands for $\bar{G}^2$, which will be the extreme values of $\bar{G}^2$ compatible with the values of $\bar{f}$ being inside their filtered error bands. The results are

$$12.39 \leq \bar{G}^2/4\pi \leq 16.66 \quad \text{if using (8) with } N=2 \quad (9a)$$

$$12.39 \leq \bar{G}^2/4\pi \leq 16.45 \quad \text{if using (8) with } N=3 \quad (9b)$$

The corresponding statistical values and errors are

$$\bar{G}^2/4\pi = 14.18 \pm 1.38, \quad (8) \text{ with } N=2 \quad (10a)$$

$$\bar{G}^2/4\pi = 14.15 \pm 1.17, \quad (8) \text{ with } N=3 \quad (10b)$$

These figures demonstrate clearly the stability and accuracy of the extrapolation. The improvement of our best results (9b), (10b) over the bounds of Rogers ⁴⁾ and of Cutkosky and Deo ⁷⁾ is obvious and is due to our use of a more complete set of inequalities, to the procedure of error filtration, and to the superiority of the extrapolation method.

In conclusion, we think this work has demonstrated the possibility of obtaining model-independent estimates of KN dispersion relations, with reasonably small errors. We hope to have exemplified the usefulness of Padé approximants to perform extrapolations. Our results for $\bar{G}^2$, three averages of $A_-(E')$ on the unphysical cut and $D_{\pm}(E)$, constitute, we believe, reliable estimates of such quantities.

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FIGURE CAPTIONS

- Figure_1   Experimental values of $f(E)$ and calculated values (solid line) using $f(E) = (\text{pole}) + a_0/(E+b_1)$, a_0 , b_1 , pole as in Eqs. (4).
- Figure_2   Experimental and calculated values of $D_{\pm}(E)$.
a) $D_+(E)$;
b) $D_-(E)$.
- Figure_3   Averages of ρ around different points with absolute error bars. ($\rho \geq 0$).

FIG 1

FIG 2a.

FIG 2b.

FIG 3