EDDY CURRENTS IN PULSED MAGNETS

(Example of the Booster Bending Magnet)

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1. Eddy current effects due to the vertical field component.

2. Approximate method for the computation of eddy-current effects due to vertical field component.

3. Approximate method for the computation of eddy-current effects due to horizontal field component.

4. Power dissipation.

5. Overall conclusions.
In the design of pulsed magnets, such as synchrotron and beam extraction or transport elements, the question of the influence of the eddy currents on the magnetic field as a function of time has often to be considered when the bending power must be known with great accuracy.

In considering the literature \(^1\) to \(^7\), we have found that the solution is often either approximate or not directly applicable to practical cases.

In this report we reconsider the problem and give solutions for a synchrotron magnet formed by a stack of thin laminations limited by thick end plates (example of the Booster Bending Magnet).

The influence of the eddy currents associated with the vertical and the horizontal field components are considered separately. The former, although with the limitation of certain simplifying assumptions, is treated rigourously (chapter 1) and in a simplified form (chapter 2). A good agreement is obtained between the two results. Of course, eddy currents are considered to exist only in the thick end plates. The latter, namely the influence of the eddy currents associated with the horizontal field component, is treated approximately in chapter 3. In this case, the thickness of the laminations does not affect the results.

In chapter 4, the power dissipated by the eddy currents due to both field components is computed.

Overall conclusions are drawn in chapter 5.

* Eddy-current effects in the thin laminations (1-2 mm) have been computed and found to be about two orders of magnitude less important than in the thick end plates (20-30 mm).
1. Eddy current effects due to the vertical field component

1.1. Derivation of the partial equation

In a charge-free medium the Maxwell equations (MKS system) are

\[ \text{curl} \vec{H} = \sigma \text{curl} \vec{E}, \quad \text{curl} \vec{E} = \mu \frac{\partial \vec{H}}{\partial t} \]
\[ \text{div} \vec{H} = 0, \quad \text{div} \vec{E} = 0 \]

Since

\[ \text{curl} \text{curl} \vec{H} = \text{grad} \text{div} \vec{H} - \Delta \vec{H} = \sigma \text{curl} \vec{E} = -\sigma \mu \frac{\partial \vec{H}}{\partial t}, \]

it follows that

\[ \Delta \vec{H} = \sigma \mu \frac{\partial \vec{H}}{\partial t}, \quad (1) \]

and in our discussion in Cartesian coordinates

\[ \frac{\partial^2 \vec{H}}{\partial x^2} = \sigma \mu \frac{\partial \vec{H}}{\partial t}, \quad H = H(x,t). \quad (2) \]

1.2. Derivation of the boundary conditions

Consider one lamination of the form shown in the figure, excited by a winding having \( N \) turns and a resistance \( R \). If no current flows in the winding prior to \( t = 0 \),

\[ H(x,0) = 0. \quad (3) \]

Because of symmetry in the choice of the point \( x = 0 \)

\[ \frac{\partial}{\partial x} H(x,t) \bigg|_{x=0} = 0. \quad (4) \]

Neglecting the leakage flux in the surrounding air,

\[ V(t) = N \frac{\partial \phi}{\partial t} + R I(t), \quad (5) \]
1.3. Transformed boundary-value problem

The boundary-value problem (2), (3), (4), (9) is of a form suitable for the use of an integral transformation or of the Heaviside operational calculus. Consider the particular case of the Laplace transformation $f(s) = \mathcal{L}\{f(t)\} = \int_{0}^{\infty} e^{-st} f(t) \, dt$. (10)

Applying (10) to (2) and taking account of (3) yields

$$\frac{d^2 H}{dx^2} - \gamma_0^2 s \, H = 0, \quad H = H(x, s), \quad \gamma_0^2 = \frac{\mu_R \mu_o}{\rho} = \sigma \mu_R \mu_o. \quad (11)$$

The transformed boundary-value problem is therefore (4), (9) and (11). The general solution of (11) is

$$H(x, s) = A(s) \, \text{ch} \, \gamma, \quad B(s) \, \text{sh} \, \gamma, \quad \gamma = \gamma_0 \sqrt{s}, \quad (12)$$

where $A(s)$ and $B(s)$ are unknown functions.

The symmetry condition (4) requires $B(s) = 0$. Hence

$$H(x, s) = A(s) \, \text{ch} \, \gamma, \quad L\{H(t)\} = \frac{1}{a} \int_{a}^{\infty} H(x, s) \, dx = \frac{A(s)}{a \gamma} \, \text{sh} \, \gamma, \quad (12a)$$

and

$$V(s)/N = A(s) \left[ \frac{k}{\gamma a} \, \text{sh} \, \gamma + k_1 \, \text{ch} \, \gamma + \frac{k_2 s}{a \gamma} \, \text{sh} \, \gamma \right]. \quad (13)$$

Equation (13) fixes the unknown function $A(s)$, and thus the solution

$$H(x, t) = \frac{1}{2 \pi i} \int_{c-i\infty}^{c+i\infty} e^{st} V(s) \, \frac{\text{ch} \gamma \sqrt{s}}{s} \, ds. \quad (14)$$
\[D_m = (k_2 + k_1 \gamma_0 a^2 + \frac{k_2}{y_m^2}) \frac{k_1}{k_2 y_m^2 - k} + k_2 - \frac{k_2}{y_m^2},\]

\[H(t) = \frac{E_0}{N(k + k_1)} - \frac{2E_0}{N} \sum_{m=1}^{\infty} \frac{-y_m^2 t}{D_m y_m^2} \frac{\tan \gamma_0 a y_m}{\gamma_0 a y_m}\]

\[+ \frac{E_1}{N(k + k_1)} \left( t + \frac{1}{6} a^2 \gamma_0^2 - \frac{k_2}{k_1 + k} \frac{k_1 + k}{5} \gamma_0^2 a^2 \right)\]

\[+ \frac{2E_1}{N} \sum_{m=1}^{\infty} \frac{-y_m^2 t}{D_m y_m^4} \frac{\tan \gamma_0 a y_m}{\gamma_0 a y_m}.\]

1.4. Magnet constituted by a stack of thin laminations terminated at both end by thick plates

This type of construction can be considered typical of proton-synchrotrons for which a welded structure is retained for the magnet blocks.

The thin laminations have thickness of 1 - 2 mm, whereas the thick end plates show in general thicknesses of several tens mm (20 - 50), determined essentially by mechanical considerations. From the point of view of eddy currents one can assume that the ones in the thin laminations have practically no influence on the magnetic field of the gap, while one can fear an influence from the thick end plates. To study the latter, one would have to consider the pattern of the flux lines in the lamination in the region of the gap.
by an eddy current loop. In particular, it is valid for two lines, one
in a thin lamination since we have assumed that there is no eddy-current
there, and the other at the surface of a thick lamination. For the latter,
it is further assumed that \( H_a \) in the gap does not depend on \( a \), in other
words that the \( \int H \, dh \) in the gap is sufficiently well represented also for
the particular line chosen by the average field \( H_a \) times \( h \). This assump-
tion would be correct if \( 2a < \frac{h}{2} \) which in general is not so well verified
(ex. Booster \( 2a = 23 \text{ mm}; \frac{h}{2} = 35 \text{ mm} \)), but it is maintained here for
simplicity.

One has then:

\[
NI(t) = H_{am}(t)h + \overline{H}_m(t)\ell
\]

\[
NI(t) = H_a(t)h + H(a,t)\ell.
\]  

From these equations it follows that

\[
H_{am}(t) = \mu_R \overline{H}_m(t), \quad H_a(t) = \mu_R \overline{H}(t)
\]

\[
H_{am}(t) = \frac{\mu_R h}{h + \frac{\ell}{\mu_R}} \overline{H}(t) + \frac{\ell}{h + \frac{\ell}{\mu_R}} H(a,t).
\]

Hence the total flux is

\[
\dot{\Phi}_m(t) + \dot{\Phi}(t) = \mu_0 H_{am}(t)(L - 4a)W + \mu_R \mu_0 \overline{H}(t) 4aW
\]

\[
= \mu_0 (L - 4a)W \left[ \frac{\mu_R h}{h + \frac{\ell}{\mu_R}} \overline{H}(t) + \frac{\ell}{h + \frac{\ell}{\mu_R}} H(a,t) \right] + 4aW \mu_R \mu_0 \overline{H}(t).
\]  

Substituting (21) and the second equation of (20) into (5) yields
\[
\text{tgh} \ a \sqrt{s} = - \frac{\gamma_0 a \sqrt{s} (k_4 + k_2 s)}{k_3 + k_1 s}, \quad (25)
\]

and they are again purely imaginary.

Letting as before \( \sqrt{s} = i y_m \), \( y_m \) purely real, \( x_m = \gamma_0 a y_m \), the algebraic equation to be solved on the digital computer becomes

\[
tg x_m = \frac{k_4 - \frac{k_2}{\gamma_0 a^2} x_m^2}{k_3 - \frac{k_1}{\gamma_0 a^2} x_m^2}, \quad m = 1, 2, \ldots \quad (26)
\]

For \( V(t) = E_o + E_1 t \), \( V(s) = \frac{E_o}{s} + \frac{E_1}{s^2} \) the solution (24a) is

\[
H(x, t) = \frac{E_o}{N(k_3 + k_4)} - \frac{2E_o}{N} \sum_{m=1}^{\infty} \frac{-y_m^2 t}{y_m^2 D_m} \frac{\cos \gamma_0 y_m x}{\cos \gamma_0 y_m a}
\]

\[
+ \frac{E_1}{N(k_3 + k_4)} \left( t + \frac{1}{2} \gamma_0 a x^2 - \frac{k_1 + k_2 + \left( \frac{k_3}{6} + \frac{k_4}{2} \right) \gamma_0 a^2}{k_3 + k_4} \right)
\]

\[
+ \frac{2E_1}{N} \sum_{m=1}^{\infty} \frac{-y_m^2 t}{y_m^4 D_m} \frac{\cos \gamma_0 y_m x}{\cos \gamma_0 y_m a},
\]

where

\[
D_m = \left( k_1 - k_2 y_m^2 \gamma_0 a^2 + \frac{k_3}{y_m^2} + k_4 \gamma_0 a^2 \right) \frac{k_4 - k_2 y_m^2}{k_1 y_m^2 - k_3} + k_1 + 2k_2 - \frac{k_3}{y_m^2}.
\]
Table 1. Eddy current effects in Booster Bending Magnet

\[ \mu = \text{constant (in both thin and thick laminations)} \]

<table>
<thead>
<tr>
<th>Time [sec]</th>
<th>( \frac{H_{am}(t) - H(t)}{H_{am}(t)} \cdot 100 = \frac{\Delta H(t)}{H_{am}} \cdot 100 )</th>
<th>( \frac{\Delta L_{eq}}{L_{eq}} \cdot 1000 )</th>
</tr>
</thead>
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<tr>
<td></td>
<td>( a=10 \text{ mm} ) ( a=12.5 \text{ mm} ) ( a=15 \text{ mm} )</td>
<td>( a=10 \text{ mm} ) ( a=12.5 \text{ mm} ) ( a=15 \text{ mm} )</td>
</tr>
<tr>
<td>0.02</td>
<td>( \mu=1000 ) ( \mu=3000 )</td>
<td>( \mu=1000 ) ( \mu=3000 )</td>
</tr>
<tr>
<td>0.04</td>
<td>7.40</td>
<td>4.61</td>
</tr>
<tr>
<td>0.06</td>
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<td>3.21</td>
</tr>
<tr>
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</tr>
<tr>
<td>0.10</td>
<td>3.35</td>
<td>2.20</td>
</tr>
<tr>
<td>0.20</td>
<td>2.91</td>
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</tr>
<tr>
<td>0.30</td>
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</tr>
<tr>
<td>0.40</td>
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</tr>
<tr>
<td>0.50</td>
<td>1.00</td>
<td>0.81</td>
</tr>
<tr>
<td>0.60</td>
<td>0.81</td>
<td>0.69</td>
</tr>
<tr>
<td>1.00</td>
<td>0.68</td>
<td>0.60</td>
</tr>
</tbody>
</table>

where:

\[ H_{am}(t) = (\text{average}) \text{ field in the gap under thin laminations} \]

\[ H_a(t) = \text{average """""""" thick """"""""} \]

\[ \frac{\Delta L_{eq}}{L_{eq}} \cdot 100 = \frac{4a}{L} \frac{H_{am}(t) - H_a(t)}{H_{am}(t)} \cdot 100. \]

\[ L = 1.66 \text{ m}. \]
At large $t$, one has instead:

$$\left[ \frac{\Delta H}{H_{am}}(t) \right]_{a=a_2} \approx \frac{a_2^2}{a_1^2} \left[ \frac{\Delta H}{H_{am}}(t) \right]_{a=a_1} \quad \text{for large } t.$$

For what concerns ii), the fraction of the total length $L$ occupied by the end plates is obviously proportional to $a$ so that for $\frac{\Delta L_{eq}}{L_{eq}}$ the above relations become:

$$\left[ \frac{\Delta L_{eq}}{L_{eq}} \right]_{a=a_2} = f(t, a_2, a_1) \left( \frac{a_2}{a_1} \right)^3 \left[ \frac{\Delta L_{eq}}{L_{eq}} \right]_{a=a_1} \quad \text{General}$$

$$\left[ \frac{\Delta L_{eq}}{L_{eq}} \right]_{a=a_2} \approx \left( \frac{a_2}{a_1} \right)^2 \left[ \frac{\Delta L_{eq}}{L_{eq}} \right]_{a=a_1} \quad \text{for } t \approx 0$$

$$\left[ \frac{\Delta L_{eq}}{L_{eq}} \right]_{a=a_2} \approx \left( \frac{a_2}{a_1} \right)^3 \left[ \frac{\Delta L_{eq}}{L_{eq}} \right]_{a=a_1} \quad \text{for large } t.$$

1.6.2 Influence of the permeability $\mu_R$

As to be expected an increase of permeability makes the eddy-current influence smaller as it is illustrated by the case $\mu_R = 3000$ for $a = 10$ mm in Table 1.
The approximate method for computing the influence of the eddy currents due to the vertical field component on the equivalent length of a laminated bending magnet having thick end plates is based on the following physical assumptions:

Let us for the moment consider one magnet end plate in accordance with Fig. 1 and assume that no fringing field exists. The end plate has an excitation winding the ohmic resistance of which we neglect. Under this assumption and with no eddy current effects, a linear current \( I(t) \) will flow if a step voltage \( E_0 \delta(t) \) is applied. The ohmic resistance of the entire bending magnet will also be neglected. As will be shown, very close agreement with the method exposed in the first part of this report is obtained even at neglected ohmic resistance, provided a step voltage is assumed. In part I an \( E_0 \delta \left(1 + \frac{t}{T}\right) \)-type voltage had to be applied in order to get a linearly rising current, the \( E_0 \frac{t}{T} \) voltage component serving mainly to compensate the voltage drop of that current in the excitation winding resistance.

Applying thus a step voltage to the assumed end plate winding and neglecting eddy current effects, the excitation current would amount to:

\[
I_{\text{lin}}(t) = \frac{E_0}{L_{\text{air}} + L_i} \cdot t. \tag{31}
\]

Due to eddy current effects the end plate model will draw more current, the total excitation current becoming:

\[
I(t) = I_{\text{lin}}(t) + \Delta I(t). \tag{32}
\]

It is interesting to note that the magnetic length or flux of our model is not influenced by the eddy current phenomenon, since

\[
\Phi(t) = \int_0^t E_0 dt = E_0 t \tag{33}
\]

at an applied step voltage (or any other voltage form) our end plate model will have the same magnetic flux or magnetic length. Due to eddy currents the model will draw more currents than \( I_{\text{lin}}(t) \) as shown in Fig. 2.
with the solution:

$$B(x, p) = \frac{\text{ch}[\gamma \sqrt{p} (x-a)]}{\text{ch}(\gamma \sqrt{p} a)} B(p) .$$  \hspace{1cm} (38)$$

Introducing (37) into (34):

$$E_0 \cdot \frac{1}{p} = W \int_0^2 \frac{2a \text{ch}[\gamma \sqrt{p} (x-a)]}{\text{ch}(\gamma \sqrt{p} a)} \, dx \cdot B(p) =$$

$$= 2aWB(p) \frac{\text{th}[\gamma \sqrt{p} a]}{\gamma \sqrt{p} a} .$$  \hspace{1cm} (39)$$

Deriving $B(p)$ from this equation and considering Eq. (34) one can write Eq. (35) as:

$$I(p) = \frac{E_0}{p} \left[ \frac{1}{\mu_0} \frac{1}{2a \cdot W} \frac{\gamma \sqrt{p} \cdot a}{\text{th}(\gamma \sqrt{p} a)} + \frac{h}{\mu_0} \frac{1}{2a \cdot W} \right] .$$ \hspace{1cm} (40)$$

The second part of Eq. (40) is easily recognized as the excitation current required to establish the flux $\Phi(t)$ in the air gap only: (at $N = 1)$

$$I_{II}(t) = \frac{E_0}{L_{\text{air}}} \cdot t .$$ \hspace{1cm} (41)$$

The first part is the excitation current required for driving the same flux in the iron part:

$$I_I(p) = \frac{E_0}{p} \cdot \frac{1}{L_1} \cdot \frac{\gamma \sqrt{p} a}{\text{th}(\gamma \sqrt{p} a)} = \frac{E_0}{p} \left[ \frac{1}{L_1} + \frac{1}{L_1} \frac{1}{\text{th}(\gamma \sqrt{p} a)} \frac{\text{th}(\gamma \sqrt{p} a)}{\gamma \sqrt{p} a} \right] .$$ \hspace{1cm} (42)$$
Introducing the time constant

\[ T = \left( \frac{\alpha \nu}{\pi} \right)^2 \text{[s]} \]  

(50)

and taking into account Eq. (41), (43) and (44), one finally obtains for the total excitation current for the end plate model:

\[ I(t) = E_0 \cdot t \left\{ \frac{1}{L_{\text{air}}} + \frac{1}{L_{\text{I}}} \left[ 1 + \frac{T_n^2}{t} \left( \frac{1}{3} - \sum_{n=1}^{\infty} \frac{2}{n^2 \pi^2} e^{-\frac{tn^2}{T}} \right) \right] \right\} \]  

(51)

so

\[ \Delta I(t) = E_0 \cdot \frac{1}{L_{\text{I}}} T_n^2 \left( \frac{1}{3} - \sum_{n=1}^{\infty} \frac{2}{n^2 \pi^2} e^{-\frac{tn^2}{T}} \right). \]  

(52)

The relative error in the equivalent magnetic length will be equal to the ratio \( \frac{\Delta I(t)}{I_{\text{II}}(t)} \) taking, of course, the end plate and laminated part length into account:

\[ \frac{\Delta L_{\text{eq}}}{L_{\text{eq}}} = \frac{4a}{L} \cdot \frac{\Delta I(t)}{I_{\text{II}}(t)} = \frac{4a}{L} \cdot \frac{L_{\text{air}} T_n^2}{L_{\text{I}}} \frac{t}{T} \left( \frac{1}{3} - \sum_{n=1}^{\infty} \frac{2}{n^2 \pi^2} e^{-\frac{tn^2}{T}} \right) \]

\[ = \frac{4a}{L} \cdot \frac{\rho_1}{h \mu_r} \frac{T_n^2}{T} \left( \frac{1}{3} - \sum_{n=1}^{\infty} \frac{2}{n^2 \pi^2} e^{-\frac{tn^2}{T}} \right). \]  

(53)

Assuming as in part I of this report \( \mu_r = 1000, \rho = 10^{-7}[\Omega \text{m}], \) \( L = 1.66 \text{ m}, 2a = 0.025 \text{ m} \) and \( \rho_1 = 10 \cdot h = 0.7 \text{ m}, \) one obtains the following numerical figures for \( \frac{\Delta L_{\text{eq}}}{L_{\text{eq}}}(t) : \)
(z_0 = \frac{b}{2}).

Equation (58) satisfies Maxwell's relation (36) if one writes:

$$B_x(z, p) = B_{is}^x(p) e^{-\sqrt{\mu}(z - z_0)}.$$  (59)

The magnetic field $B_x(z, p)$ has its maximum value $B_{is}^x(p)$ at the edge of the magnet for $z = z_0$. It also takes care of eddy current effects, satisfies Eq. (37) and provides a reasonable approximation for the spatial distribution of $B_x$ along the z-axis.

In order to find the equivalent magnetic field length error we proceed in a similar way as in part 1 and assume a fictitious resistance free winding exciting the horizontal end field at an applied constant voltage $E_0 \cdot e(t)$.

Introducing Eq. (59) into (39) one obtains:

$$I(p) = \frac{E_0}{p} \left\{ \frac{\sqrt{\mu}(z_n - z_0)}{W(z_n - z_0)} \cdot \frac{l_{ix}}{\mu_0} \cdot \frac{1}{1 - e^{\sqrt{\mu}(z_n - z_0)}} + \frac{l_{air}}{F \mu_0} \right\}.$$  (61)

$F[m^2]$ being the end flux effective surface.

The second part of Eq. (61) yields the excitation current for the end flux in the air:

$$I(2)(t) = \frac{E_0}{L_{air end}} \cdot t.$$  (62)

For the first part one can write:
where $z_n [m]$ is the distance of the coil end part symmetry line from the gap center and $z_0 = \frac{h}{2} [m]$ half the gap height (Fig. 4).

iii) The horizontal field component distribution along the vertical axis at the magnet end $B_x = f(y)$ is estimated as follows:

At a D.C. field with no eddy currents and neglecting for the moment the coil end effect, the magnetic field distribution at the corner of a bending magnet (see Fig. 5) is given by the conformal transformation:

$$ q = x + iz = \xi^2 $$

the magnetic field being proportional to:

$$ B \equiv \mu_0 \cdot \frac{2}{h} \cdot \frac{1}{\sqrt{\xi^2 + 1}} $$. (56)

The relation between the original and transformed region is given by:

$$ q = \frac{h}{2\pi} \left[ 2\sqrt{\xi^2 + 1} + \ln \frac{\sqrt{\xi^2 + 1} - 1}{\sqrt{\xi^2 + 1} + 1} \right]. $$ (57)

For $\xi = -1$, i.e. for the magnet edge $B = \infty$; for $-\infty < \xi < -1$ one obtains the values for $B_x$ along the y-axis; the values are imaginary which only means that $B_x \equiv j B_z$ and that the horizontal field component $B_x$ is perpendicular to the normal gap field $B_z$.

It is interesting to mention that the horizontal field component $B_x$ has in the immediate neighbourhood of the edge for $\xi = -2$ or $z = 0.57 \cdot h$ the absolute value of $B_z$ in the gap $|B_x| = |B_z|$. As to the influence of the coil ends, we shall assume that it will result in making $B_x \approx 0$ at the distance $z_n$ corresponding to the coil symmetry line, as shown in Fig. 6. The corresponding field distribution $B_x = f(z)$ is drawn in Fig. 6, curve 1. It is fairly close approximated by an exponential curve 2.

$$ B_x = B_z \cdot e^{-\frac{1}{z_0} \cdot (z - z_0)} $$ (58)
\[ I(1)(p) = \frac{E'_0}{L_1 \text{end}} \cdot \frac{1}{p} \cdot \frac{\sqrt{p}(z_n - z_0)}{1 - e^{-\sqrt{p}(z_n - z_0)}} \cdot (63) \]

Introducing new variables:

\[ s \rightarrow [\gamma(z_n - z_0)]^2 \cdot p \quad (64) \]

\[ t \rightarrow [\gamma(z_n - z_0)]^2 \cdot \tau \quad (65) \]

\[ I(1)(s) = \frac{E'_0}{L_1 \text{end}} \cdot \frac{1}{s} [\gamma(z_n - z_0)]^2 \cdot \frac{\sqrt{s}(1 + e^\sqrt{s})}{(1 - e^{-\sqrt{s}})(1 + e^\sqrt{s})} \]

\[ = \frac{E'_0}{L_1 \text{end}} [\gamma(z_n - z_0)]^2 \left\{ \frac{1}{2\sqrt{s}} + \frac{1}{s} \frac{\sqrt{s} \text{ch} \frac{\sqrt{s}}{2}}{2 \text{sh} \frac{\sqrt{s}}{2}} \right\}. (66) \]

The first part of (66) can be transformed into the original t-region:

\[ I(1)(t) = \frac{E'_0}{L_1 \text{end}} [\gamma(z_n - z_0)] \sqrt{\frac{t}{\pi}}. (67) \]

If a second transformed variable \( \frac{\sqrt{s}}{2} = \sqrt{\mu} \) is introduced, the second part of Eq. (66) corresponds to Eq. (42) and can be solved accordingly. The solution is:

\[ I(t) = E_o \left[ \frac{-t}{L_{1 \text{end}}} + \frac{1}{L_1 \text{end}} \left[ \frac{2}{\pi} \sqrt{t}T' + t[1 + \frac{T'_1}{t} (\frac{1}{3} - \sum_{n=1}^{\infty} \frac{2}{n^2 \pi^2 e^{\frac{T'_1}{n}}})] \right] \right] \} (68) \]
4. **Power dissipation**

For a number of reasons, it is often required to know the power dissipated in the iron because of eddy currents.

As it has been done in what proceeds, we shall consider separately the case of the vertical field component in the thick end plates and the one of the horizontal field component in the end regions.

4.1 Eddy current power due to the vertical field component in the end plates

The field $H(x,t)$ in the end plates is given by (27) and is repeated here for convenience:

$$H(x,t) = \frac{E_0}{N(k_3 + k_4)} - \frac{2E_0}{N} \sum_{m=1}^{\infty} \frac{-y_m^2}{y_m^2 D_m} \frac{e}{\cos \gamma_y y_m x} \cos \gamma_y y_m a$$

$$+ \frac{E_1}{N(k_3 + k_4)} \left( t + \frac{1}{2} \gamma_y 2x^2 - \frac{k_1 + k_2 + \left( \frac{k_3}{6} + \frac{k_4}{2} \right) \gamma_y 2a^2}{k_3 + k_4} \right)$$

$$+ \frac{2E_1}{N} \sum_{m=1}^{\infty} \frac{-y_m^2}{y_m^4 D_m} \frac{e}{\cos \gamma_y y_m x} \cos \gamma_y y_m a$$
By inserting the values of \( \gamma_0 \), \( k_3 \) and \( k_4 \) and by remembering that in our circuit \( i = \frac{E_1}{R} \) in the absence of eddy currents, one has:

\[
\bar{F}_0 = \frac{2}{3} \frac{w L}{\rho} B^2 a^3 ,
\]

where in \( B \) the variation of the field due to eddy currents is neglected.

An evaluation of the error made by such a procedure is readily obtained by considering the first term of the series in (73). This alone gives rise to a power dissipation:

\[
\Delta P = 2 \rho \, \omega \, \frac{1 - e^{-y_1^2 T}}{T} \left( \frac{2 \alpha \beta_1}{\gamma_0^2} \sin \gamma_0 y_1 a - \frac{2 \alpha \beta_1 a}{\gamma_0} \cos \gamma_0 y_1 a \right)
\]

\[
+ \frac{1}{2} \beta_1^2 a - \frac{\beta_1^2}{4} \frac{\gamma_0^2 y_1}{y_1^2 D_1} \sin 2 \gamma_0 y_1 a
\]

(75)

where

\[
\alpha = \frac{E_1}{N(k_3 + k_4)} \quad \beta_1 = \frac{2 \gamma_0 (y_1^2 E_0 - E_1)}{N y_1^2 D_1 y_1 \cos \gamma_0 y_1 a} .
\]

As an example, we treat the case of the Booster Bending Magnet (see also page 10). The numerical values are as follows:

\( \rho = 10^{-7} \) \( \Omega \)m, \( \omega = 0.2 \) m, \( L = 0.7 \) m, \( a = 1.25 \times 10^{-2} \) m, \( \gamma_0 = 112 \) sec\(^{-1}\)m\(^{-2}\), \( k_3 = 1.152 \times 10^{-3} \) \( \Omega \)m, \( k_4 = 1.152 \times 10^{-5} \) \( \Omega \)m, \( T = 0.6 \) sec, \( y_1 = 1.434 \) sec\(^{-1}\).

One obtains

\[
\bar{F}_0 = 1.42 \text{ watt} \quad \Delta P = 4 \times 10^{-4} \text{ watt} .
\]

One sees that \( \Delta P \) is completely negligible with respect to \( \bar{F}_0 \), as it could be expected since the distortion of the field due to eddy currents is in average only of a few per cent and it enters into the power dissipation formula squared.
5. Overall conclusions

The eddy current associated with both the vertical and the horizontal field components at the ends of the Booster Bending Magnet makes the equivalent length $L_{eq}$ slightly smaller than in static conditions and increasing from injection to top energy. The variation between these two extreme energies is of the order of $\sim 5\%$, obtained by combining the results of chapter 1 (or 2) and of chapter 3. This is only approximately correct since the vertical field component has been assumed equal to the one in the centre of the magnet and independent of the horizontal field component. The $L_{eq}$ at top energy differs only by $\sim 0.4\%$ from the one in static conditions.

The power dissipated is in total less than 1 kW.
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Distribution:
List MPS-SI/1
\[ \frac{B_x(z)}{B_{z\text{gap}}} \]

**Conformal mapping** ①

**Approximation** ②

\[ B_x(z) = B_{z\text{gap}} e^{\frac{z-z_e}{Z_e}} \]