Archives

CONSTRAINTS ON THE PHASE OF SCATTERING AMPLITUDES
DUE TO POSITIVITY

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A B S T R A C T

From the positivity of Legendre expansion coefficients of the absorptive part of an elastic scattering amplitude, we get bounds on the phase of the scattering amplitude such as

$$\left| \text{Arg} F(s, \cos \theta_M) - \frac{\pi}{2} \right| < \pi + \frac{(\Gamma(\frac{1}{2}))^4}{\pi^3} \max_{0 \leq \theta \leq \theta_M} \left| \frac{\text{Im} F(s, \cos \theta = 1)}{F(s, \cos \theta)} \right|^2.$$

We also get bounds on the local variation of the phase; in the best case, i.e., if the modulus of the amplitude is Hölder continuous of order $\geq \frac{1}{2}$ we find a Hölder continuity condition of order $\frac{1}{4}$ for the phase, in which the coefficient is explicitly known from the modulus of the amplitude.

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I. INTRODUCTION

Information on the phase of scattering amplitudes is useful in several problems. One of them is the problem of reconstructing the scattering amplitude from the knowledge of the differential cross-section at a given energy ¹⁾. One also needs limitations on the phases to prove the asymptotic equality of differential cross-sections for line reversed processes ²⁾.

In principle the phase of a scattering amplitude is an observable because one can look at coherent scattering of a particle from an assembly of identical scattering centres (we are grateful to L. Van Hove for this remark). However, this cannot be done in practice. Actually, the best one can do in this direction, in the case of high energy hadron scattering, is to study scattering on deuteron targets ³⁾; but a theoretical model is needed to evaluate multiple scattering corrections ! Therefore, even if in principle the phase of a hadron-hadron scattering amplitude is observable, in practice it is not, except in the forward direction when one can use Coulomb interference.

Here we want to stress that fortunately, at least in the case of an elastic amplitude describing $A+B \rightarrow A+B$, the phase is not arbitrary once the modulus of the scattering amplitude and the total cross-section are known at a given energy with a sufficiently good but finite accuracy. This is a consequence of the positivity of the Legendre expansion coefficients of the absorptive part of the amplitude. In other words, our results are linked in some mysterious way to invariance with respect to the three-dimensional rotation group. Invariance with respect to the two-dimensional rotations inside the scattering plane is not sufficient. The proofs we shall give are based on rather fine properties of Legendre polynomials (some of which do not seem to be well known). We apologize to our friends who prefer general methods of proof.

In this paper we get two sorts of bounds :

- i) global bounds on the phase of the scattering amplitude in an angular interval $\cos\theta_0 \leq \cos\theta \leq 1$. These depend on the minimum of the quantity

$$\frac{|F(s, \cos \theta)|}{\text{Im } F(s, \cos \theta = 1)}$$

in the relevant interval;

- ii) local bounds on the variation of the phase around a given angle. It can be shown that if the modulus of the scattering amplitude satisfies a Hölder condition of order $\gg \frac{1}{2}$ and is strictly larger than a positive number, the phase of the scattering amplitude satisfies a Hölder condition of order $1/4$ in which the coefficient depends only on the forward absorptive part, on the local value of the modulus, and the coefficient in the Hölder condition for the modulus. Weaker conditions on the modulus lead to weaker conditions on the phase.

In all these considerations we need only the uniform convergence of the Legendre polynomial expansion of the absorptive part for $-1 < \cos\theta < +1$. The existence of an ellipse of convergence for the absorptive part and the full amplitude are not needed and, unfortunately not useful, because they give only qualitative results on the phase variation, while we need quantitative, numerical results.

II. GLOBAL BOUNDS ON THE PHASE OF THE SCATTERING AMPLITUDE

In this section we shall try to generalize a remark which has been used ⁴⁾ in order to prove the asymptotic equality of the widths of the diffraction peaks for reactions $A+B \rightarrow A+B$ and $\bar{A}+B \rightarrow \bar{A}+B$. We noticed that if in an interval $0 < \cos\theta_0 < \cos\theta < 1$ we have

$$\left| \frac{F(s, \cos\theta)}{\text{Im } F(s, \cos\theta=1)} \right| > \frac{1}{2}$$

the phase of the scattering amplitude cannot take the value $-\pi/2$. This was due to the inequality

$$\text{Im } F(s, \cos\theta) \geq -\frac{1}{2} \text{Im } F(s, \cos\theta=1) \quad (1)$$

for $0 \leq \cos\theta \leq 1$, which follows from the property

$$P_\ell(\cos\theta) \gg -\frac{1}{2} \quad (2)$$

for $0 \leq \cos\theta \leq 1$, where P_ℓ is the ℓ^{th} Legendre polynomial. Since F is continuous, since $0 \leq \text{Arg } F \leq \pi$ for $\cos\theta=1$, this implies that in the interval $\cos\theta_0 < \cos\theta < 1$ when

$$\frac{|F|}{\text{Im } F(s, 1)} > \frac{1}{2}$$

we have

$$-\frac{\pi}{2} < \text{Arg. } F < \frac{3\pi}{2} \quad (3)$$

The problem is to generalize (3) to the case where $(|F(s, \cos \theta)|)/(\text{Im } F(s, 1))$ is larger than some constant at a given energy, this constant being possibly less than $\frac{1}{2}$. First one might hope to improve the number in the right-hand side of inequality (2). Noticing that for large ℓ we have $P_\ell(\cos \theta) \simeq \sqrt{\theta/\sin \theta} J_0(\ell \theta)^{5)}$, uniformly in θ , we see that

$$P_\ell(\cos \theta) > -0.4027 \dots - \epsilon \quad (4)$$

for any arbitrarily small ϵ provided $1 - \gamma < \cos \theta \leq 1$, γ sufficiently small. $-0.4027 \dots$ represents the first minimum of the Bessel function J_0 . Unfortunately one cannot improve (4) any further and a more sophisticated trick must be found.

We shall try to replace (1) by an inequality of the following type :

$$\text{Im } F(x_1) - \text{Im } F(x_2) \dots - (-1)^n \text{Im } F(x_n) > -\text{const. Im } F(1) \quad (5)$$

with $1 \gg x_1 \gg x_2 \gg \dots \gg x_n \gg -1$

To establish an inequality of the type (5), it is sufficient to establish

$$P_\ell(x_1) - P_\ell(x_2) + P_\ell(x_3) \dots - (-1)^n P_\ell(x_n) > -\text{const} \quad (6)$$

with $1 \gg x_1 \gg x_2 \dots \gg x_n \gg -1$

where the constant is independent of ℓ . If we succeed in doing that, we can get inequality (5) by using the Legendre polynomial expansion of the absorptive part :

$$\text{Im } F(s, x = \cos \theta) = \sum (2\ell+1) \text{Im } f_\ell(s) P_\ell(x) \quad (7)$$

with $\text{Im} f_\ell(s) \geq 0$.

What we have to do now is a study of the sums of type (6). Two rather important auxiliary results are needed for this :

i) if $x_{i,\ell}$ is the i^{th} extremum of $P_\ell(x)$ with

$$1 > x_{1,\ell} > x_{2,\ell} \dots \quad (8)$$

and

$$m_{i,\ell} = |P_\ell(x_{i,\ell})| \quad (9)$$

then, for $x_{i,\ell} > 0$, i.e., for $i \leq \ell/2$, we have

$$m_{1,\ell} > m_{2,\ell} > \dots > m_{i-1,\ell} > m_{i,\ell} \quad (10)$$

this property is well known ⁵⁾;

ii) the value of $|P_\ell|$ at the i^{th} maximum is a decreasing function of ℓ , for $i < \ell/2$:

$$\mu_i < m_{i,\ell+1} < m_{i,\ell} < m_{i,2i} = |P_{2i}(0)| \quad (11)$$

where $\mu_i = m_{i,\infty}$ is the value of $|J_0|$ at its i^{th} extremum. This property is established in Appendix A.

With these two results, we establish the following results in Appendix B.

A) If like in (2) we add the restriction

$$1 \gg x_1 \gg x_2 \dots \gg x_n \gg 0 \quad (12)$$

we call

$$A_{n,\ell} = \sup_{x_1, x_2, \dots, x_n} [-P_\ell(x_1) + P_\ell(x_2) \dots + (-1)^n P_\ell(x_n)] \quad (13)$$

Then one finds, using (11) :

$$A_{n,e} \geq A_{n,2n} \quad \text{for } e \geq 2n \quad (14)$$

with

$$A_{n,2n} = m_{1,2n} + m_{2,2n} + \dots + m_{n,2n} \quad (14')$$

i.e., the sum of the extremal values of P_{2n} in modulus, for $x > 0$.

For $e \leq 2n$ we find

$$A_{n,e} = A_{\frac{e}{2}, e} \quad (15)$$

if e is even, and

$$A_{n,e} < A_{\frac{e-1}{2}, e-1} \quad (16)$$

if e is odd.

If we use (13), (14), (14'), (15) and (16), we get

$$\sup_{x_1, x_2, \dots, x_n, e} [-P_e(x_1) + P_e(x_2) - \dots + (-1)^n P_e(x_n)] \leq \quad (17)$$

$$\leq \sup_{p=1}^{p=n} A_{p, 2p}$$

where $A_{p, 2p}$ is given by (14'). If $A_{p, 2p}$ is an increasing function of p the right-hand side of (17) is just $A_{n, 2n}$. We suspect that this is the case, but we could not prove it. Instead, we could find two upper bounds of $A_{p, 2p}$ which possess precisely this monotony property, and this is sufficient for our purpose.

— From the monotony property (11), one gets

$$m_{k,e} < |P_{2k}(0)| < \sqrt{\frac{2}{\pi(2k + \frac{1}{2})}}$$

then

$$A_{p, 2p} \leq \sqrt{\frac{2}{\pi}} \left[\sqrt{2p + \frac{1}{2}} - \sqrt{\frac{5}{2}} \right] + \frac{1}{2} = C_p^{(0)}$$

and hence

$$P_e(x_1) - P_e(x_2) \dots - (-1)^n P_e(x_n) > -C_n^{(0)} \quad (18)$$

$$1 \geq x_1 \geq x_2 \dots \geq x_n \geq 0$$

with

$$C_n^{(0)} = \sqrt{\frac{2}{\pi}} \left[\sqrt{2n + \frac{1}{2}} - \sqrt{\frac{5}{2}} \right] + \frac{1}{2}$$

— A more careful estimate of (14') can be made, using the upper bound

$$|P_e(\cos \theta)| < \sqrt{\frac{2}{\pi(l + \frac{1}{2})}} \frac{1}{\sqrt{\sin \theta}} \quad (19)$$

in combination with information of the location of the maxima :

$$x_{k,l} = \cos \vartheta_{k,l} \quad \vartheta_{k,l} > \frac{k\pi}{l} \quad k \leq \frac{l}{2} \quad (20)$$

This gives again a monotonous increasing bound of $A_{p,2p}$ and one gets

$$P_e(x_1) - P_e(x_2) \dots - (-1)^n P_e(x_n) > -C_n^{(1)}$$

with

$$C_n^{(1)} = \frac{n}{\sqrt{2n + \frac{1}{2}}} \frac{\left(T\left(\frac{1}{4}\right)\right)^2}{\pi^2} - \frac{2}{\pi} \sqrt{\frac{2n}{2n + \frac{1}{2}}} + \frac{1}{\sqrt{2\pi(2n + \frac{1}{2})}} \quad (21)$$

the detailed derivation of these results is given in Appendix C.

The case of n small can be treated exactly :

$n=1$: $P_l(x) > -\frac{1}{2}$, while (21) gives -0.527 ;

$n=2$: we have $A_{2,4} = \frac{3}{8} + \frac{3}{7} > A_{1,2} = \frac{1}{2}$ hence

$$P_e(x_1) - P_e(x_2) > -\frac{45}{56} \simeq -0.803$$

while (21) gives -0.848 .

What really matters, in fact, as we shall soon find out is that $c_n^{(0)}$ as well as $c_n^{(1)}$ behave like $\sqrt{n}$ for large n .

B) We can also obtain an inequality in which $x_1, x_2, \dots, x_n$ are not any more constrained to lie in the forward hemisphere. So we have only

$$1 \geq x_1 \geq x_2 \dots \geq x_n \geq -1$$

Here we define

$$B_{n,l} = \sup_{1 \geq x_1 \geq x_2 \dots \geq x_n \geq -1} [-P_e(x_1) + P_e(x_2) + \dots + (-1)^n P_e(x_n)] \quad (22)$$

and again we find that we must distinguish $n \leq l$ and $n > l$. However, here the possibility that x_n takes the value -1 complicates things a little more (see Appendix B) :

a) $n \leq l$, n and l have the same parity, then

$$B_{n,l} \leq B_{n,n}$$

$$B_{n,n} = m_{1,n} + m_{2,n} + \dots + m_{n-1,n} + 1$$

(notice $m_{k,n} = m_{n-k,n}$).

b) $n \leq l$, n and l have opposite parity, then

$$B_{n,l} \leq B_{n,n+1}$$

$$B_{n,n+1} = m_{1,n+1} + m_{2,n+1} + \dots + m_{n,n+1}$$

c) $n > l$, n and l have the same parity, then

$$B_{n,l} = B_{l,l}$$

d) $n > l$, n and l have opposite parity, then

$$B_{n,l} = B_{l-1,l}$$

However, in fact, one can also prove by using (11) that

$$B_{l-1,l} < B_{l-1,l-1}$$

Therefore we get

$$\sup_{\substack{e, 1 \gg x_1 \gg x_2 \dots \gg x_n \gg -1}} [-P_e(x_1) + P_e(x_2) + \dots + (-1)^n P_e(x_n)] < \sup_{p=1}^n B_{p,p}$$

We derive in Appendix C bounds of $B_{p,p}$ similar to those obtained for the case $0 < x < 1$. Here, for $-1 < x < +1$ we find $B_{p,p} < D_p^{(0)}$, where $D_p^{(0)}$ is the analogue of $C_p^{(0)}$, and we get

$$D_p^{(0)} = 1 + \sqrt{\frac{2}{\pi}} \left[\sqrt{p - \frac{1}{2}} - \sqrt{\frac{3}{2}} + \sqrt{\frac{2}{5}} \right] \quad (23)$$

We find also

$$B_{p,p} < D_p^{(1)}$$

$$D_p^{(1)} = \frac{p}{\sqrt{p + \frac{1}{2}}} \frac{\left(\Gamma\left(\frac{1}{4}\right)\right)^2}{\pi^2} + 1 - \frac{4}{\pi} \sqrt{\frac{p}{p + \frac{1}{2}}} \quad (24)$$

Both $D_p^{(0)}$ and $D_p^{(1)}$ are increasing functions of p . Therefore we conclude

$$\left\{ \begin{aligned} & \sup_{\substack{e, 1 \gg x_1 \gg x_2 \dots \gg x_n \gg -1}} [-P_e(x_1) + P_e(x_2) \dots + (-1)^n P_e(x_n)] \\ & \leq 1 + \sqrt{\frac{2}{\pi}} \left[\sqrt{n - \frac{1}{2}} - \sqrt{\frac{3}{2}} + \sqrt{\frac{2}{5}} \right] \quad (25) \\ & \leq \frac{n}{\sqrt{n + \frac{1}{2}}} \frac{\left(\Gamma\left(\frac{1}{4}\right)\right)^2}{\pi^2} + 1 - \frac{4}{\pi} \sqrt{\frac{n}{n + \frac{1}{2}}} \quad (25') \end{aligned} \right.$$

Both estimates are very good in the limit of small n . For instance :

- for $n=1$ (25') gives 1.06, while the exact answer is of course 1;
- for $n=2$ the exact answer is $P_2(0) - P_2(-1) = \frac{3}{2}$, while (25') gives 1.55.

C) Naturally, if one likes, one can try to solve problems with more complicated constraints, for instance try to find

$$\sup [-P_e(x_1) + P_e(x_2) + \dots + (-1)^n P_e(x_n)]$$

with $1 \geq x_{\max} \geq x_1 \geq x_2 \dots \geq x_n \geq x_{\min} \geq -1$

However, we shall not do this here. We notice that in the limit of large n the case $x_{\max}=1$, $x_{\min}=-1$, differs only by a factor $\sqrt{2}$ from the case $x_{\max}=1$, $x_{\min}=0$. Similarly, one can anticipate from Hilb's formula

$$P_e(\cos \theta) \approx \sqrt{\frac{\theta}{\sin \theta}} J_0(\ell \theta)$$

that in the case $x_{\max}=1$, $x_{\min}=1-\epsilon$, ϵ very small, the improvement with respect to $x_{\max}=1$, $x_{\min}=0$ cannot be more than dividing the bound by a factor $\sqrt{\pi/2} \approx 1.25$.

Now we can proceed and get a bound on the phase of the scattering amplitude for $\cos \theta_0 \leq \cos \theta \leq 1$, if we know the minimum of the modulus of the scattering amplitude in this interval. Naturally we have to distinguish $\cos \theta_0 > 0$ and $\cos \theta_0 < 0$.

From the optical theorem, we have $\text{Im} F(s, \cos \theta = 1) > 0$ and we can make the convention

$$0 < \text{Arg } F(s, \cos \theta = 1) < \pi \quad (26)$$

Suppose now that the phase at $\cos \theta = \cos \theta_M$ exceeds $(\pi/2) + n\pi$. This means that when θ varies from 0 to θ_M , $\text{Arg } F$ takes at least once the values $3\pi/2, 5\pi/2, \dots, (\pi/2) + n\pi$, at angles $0 < \theta_1 < \theta_2, \dots, < \theta_n < \theta_M$. At these points we have

$$\begin{aligned} |F(s, \cos \theta_1)| &= -\text{Im } F(s, \cos \theta_1) \\ |F(s, \cos \theta_2)| &= +\text{Im } F(s, \cos \theta_2) \\ |F(s, \cos \theta_p)| &= (-1)^p \text{Im } F(s, \cos \theta_p) \\ &\dots \end{aligned}$$

Hence we get

$$\sum_{p=1}^n (-1)^p \operatorname{Im} F(s, \cos \theta_p) = |F(s, \cos \theta_1)| + |F(s, \cos \theta_2)| + \dots + |F(s, \cos \theta_n)|$$

$$\geq n \cdot \text{minimum of } |F| \quad (27)$$

$$0 < \theta < \theta_M$$

On the other hand, the left-hand side of (27) is less or equal to C_n or D_n times $\operatorname{Im} F(s, \cos \theta = 1)$ if, respectively $\cos \theta_0 \geq 0$, $\cos \theta_0 \leq 0$, according to inequalities (18), (21), (25) and (25'). So we get :

$$i) \quad 0 \leq \theta_M \leq \theta_0 \leq \frac{\pi}{2}$$

$$\frac{\operatorname{Im} F(s, \cos \theta = 1)}{\min_{0 < \theta \leq \theta_M} |F|} > \frac{n}{C_n^{(10,1)}} \quad (28)$$

if $\operatorname{Arg} F > n\pi + (\pi/2)$ at θ_M . Equation (28) gives a bound on n . However, an expression like (21) is too complicated to allow a simple solution. Therefore we shall just use

$$C_n^{(11)} < \frac{(T(\frac{1}{4}))^2}{\pi^2} \sqrt{\frac{n}{2}} \quad (29)$$

In this way one gets

$$n < \frac{(T(\frac{1}{4}))^4}{2\pi^4} \max_{0 < \theta < \theta_M} \left| \frac{\operatorname{Im} F(s, \cos \theta = 1)}{F(s, \cos \theta)} \right|^2 \quad (30)$$

if n is the largest integer such that inequality (29) is satisfied, it is clear that we cannot substitute $n+1$ for n in (29). Therefore we get

$$\left| \text{Arg } F - \frac{\pi}{2} \right| < \pi + \frac{(\tau(\frac{1}{4}))^4}{2\pi^3} \text{Max} \left| \frac{\text{Im } F(s, \cos \theta = 1)}{F(s, \cos \theta)} \right|^2 \quad (31)$$

$$0 < \theta < \theta_M < \frac{\pi}{2}$$

This expression has all desirable qualitative features (we leave as an exercise to check that it applies also to $\text{Arg } F < 0$). However, when $\max |\text{Im } F/F|$ is small it is a strong overestimate because replacing (21) by (29) is very damaging. For small n one can find the "exact solution of the problem" : if

$$\text{Max} \left| \frac{\text{Im } F(s, 1)}{F(s, \cos \theta)} \right| < 2$$

we already know that the phase cannot take the value $-\pi/2$. If

$$\text{Max} \left| \frac{\text{Im } F(s, 1)}{F(s, \cos \theta)} \right| < \frac{2}{\frac{3}{8} + \frac{3}{7}}$$

since we have

$$\frac{\text{Im } F(s, \cos \theta_1) - \text{Im } F(s, \cos \theta_2)}{\text{Im } F(s, 1)} > P_4(x_{1,4}) - P_4(0) = -\left(\frac{3}{8} + \frac{3}{7}\right)$$

the phase has to stay between $-3(\pi/2)$ and $+5(\pi/2)$. In intermediate situations one can slightly improve this. All this is described in Fig.1 where the allowed region for $\text{Arg } F$ is given as a function of $R = \min(|F|/|\text{Im } F|)$. Notice that for $R > 1$ we have two disconnected allowed regions for the phase depending on whether $0 < \text{Arg } F(\cos \theta = 1) < \frac{\pi}{2}$ or $\frac{\pi}{2} < \text{Arg } F(\cos \theta = 1) < \pi$.

- ii) $0 \leq \theta_M \leq \theta_0 < \pi$, the reasoning is the same, except for the fact that we replace C_n by D_n . In this way if we overestimate D_n by

$$D_n^{(1)} < \sqrt{n} \frac{(\tau(\frac{1}{4}))^2}{\pi^2} \quad (32)$$

we get

$$\left| \text{Arg } F - \frac{\pi}{2} \right| < \pi + \frac{(\tau(\frac{1}{4}))^4}{\pi^3} \max_{0 < \theta < \theta_M} \left| \frac{\text{Im } F(s, 1)}{F(s, \cos \theta)} \right|^2 \quad (33)$$

Again, this formula is very poor when $\text{Im } F/|F|$ is small. A direct calculation can be done. For instance

$$\left| \text{Arg } F - \frac{\pi}{2} \right| < 2\pi$$

(34)

$$\text{for } \sup \left| \frac{\text{Im } F(s, 1)}{F(s, \cos \theta)} \right| < 4/3$$

We do not know how much these bounds can be quantitatively improved. However, we cannot expect any qualitative improvement as shown by the following example : consider

$$F(s, \cos \theta) = i P_L(\cos \theta) + \frac{\sqrt{(\sin \theta)^2 + \lambda^2}}{\sqrt{L(L+1)}} P_L'(\cos \theta) \quad (35)$$

we have

$$\text{Im } F(s, \cos \theta = 1) = 1 \quad (36)$$

and

$$|F(s, \cos \theta)|^2 \gg |P_L(\cos \theta)|^2 + \frac{\sin^2 \theta |P_L'(\cos \theta)|^2}{L(L+1)} \quad (37)$$

where the equality sign holds at zeros of $P_L^!$. According to Szegö⁵⁾, the right-hand side of (37) is a monotonous function of $\cos\theta$ for $0 < \cos\theta < 1$. We have

$$\min_{\cos\theta_M < \cos\theta < 1} |F(s, \cos\theta)| \simeq \sqrt{\frac{2}{\pi L \sin\theta_M}}$$

for $L\theta$ large.

On the other hand, the phase variation of (35) is of the order of $L\theta$ since the zeros of P_L and $P_L^!$ are intertwined, and since we know their location from inequalities (20). Therefore, for large L we have

$$|\text{Arg } F(\cos\theta_M)| \sim \frac{2\theta_M}{\pi \sin\theta_M} \quad \text{Max.} \quad \left| \frac{\text{Im } F(1)}{F(\cos\theta)} \right|^2 \quad (38)$$

which agrees with the qualitative behaviour of (31) or (33).

III. LOCAL BOUNDS ON THE PHASE VARIATION

Positivity also allows to get bounds on the phase variation of the scattering amplitude from one angle to the other, in such a way that when the variation of angle goes to zero, the phase variation goes to zero; this is true in the neighbourhood of any angle different from 0 or π .

This is possible because the absorptive part satisfies a Hölder condition of order $\frac{1}{2}$ in the physical region if the forward absorptive part and hence the total cross-sections are finite. The reader may think that we have a much stronger information coming from the convergence of the Legendre polynomial expansion of the absorptive part inside an ellipse in the $\cos\theta$ plane; it must be realized that this convergence property only supplied qualitative results unless one knows strict bounds of the absorptive part outside the physical region. On the contrary in our Hölder condition, the only thing which enters is the forward absorptive part which is a physical quantity. We still assume the convergence of the Legendre polynomial

expansion of the absorptive part, but only in the real interval $-1 \leq \cos\theta \leq +1$. This, anyway, seems to be unavoidably correct if the total cross-section is finite.

We start from the following Hölder inequality independent of l for Legendre polynomials :

$$|P_l(\cos\theta_1) - P_l(\cos\theta_2)| < \frac{2}{\sqrt{\pi}} \frac{\sqrt{|\theta_1 - \theta_2|}}{(\sin\theta_1)^{1/4} (\sin\theta_2)^{1/4}} \quad (39)$$

This inequality has already been derived in the Appendix of Ref. 1), however without the value of the numerical coefficient. In (39), the numerical coefficient is almost the best possible one, because one sees, on the asymptotic form of the P_l 's for large l that the only possible improvement is a factor

$$\sup \left| \frac{\sin x}{\sqrt{x}} \right| \simeq 0.852$$

We now insert (39) into the expansion (7) and we get

$$\frac{|Im F(s, \cos\theta_1) - Im F(s, \cos\theta_2)|}{Im F(s, 1)} < \frac{2}{\sqrt{\pi}} \frac{\sqrt{|\theta_1 - \theta_2|}}{(\sin\theta_1)^{1/4} (\sin\theta_2)^{1/4}} \quad (40)$$

Consider now an angular interval $\theta_m < \theta < \theta_M$ in which

$$\frac{|F(s, \cos\theta)|}{Im F(s, 1)} \gg R(s)$$

Consider a smaller interval

$$\theta_m < \theta_1 \leq \theta \leq \theta_2 \leq \theta_M$$

which is chosen in such a way that

$$\frac{2}{\sqrt{\pi}} \frac{\sqrt{|\theta_1 - \theta_2|}}{(\sin\theta_1)^{1/4} (\sin\theta_2)^{1/4}} < 2 R \quad (41)$$

For any pair θ_1', θ_2' such that $\theta_1 < \theta_1' < \theta_2' < \theta_2$, we have

$$|\operatorname{Im} F(s, \cos \theta_1') - \operatorname{Im} F(s, \cos \theta_2')| < 2R \operatorname{Im} F(s, 1) < 2 \min |F| \quad (42)$$

$$\theta_1' < \theta < \theta_2'$$

This means that inside this interval the phase variation does not exceed 2π . Otherwise the phase has to go at least once through $(\pi/2) + 2k\pi$ and $-(\pi/2) + 2k'\pi$, which contradicts (42).

A finer result can be obtained if the modulus of the amplitude is itself uniformly continuous. Consider again the interval defined by (41). Inside this interval, the phase cannot take both values $(\pi/2) + 2k\pi$ and $-(\pi/2) + 2k'\pi$. We now consider a smaller interval

$$\theta_1 < \theta_1'' < \theta < \theta_2'' < \theta_2$$

such that inside this interval

$$|\Delta|F|| < \frac{R}{10} \operatorname{Im} F(s, 1) < \frac{1}{10} \min |F| \quad (43)$$

$$|\Delta \operatorname{Im} F| < \frac{R}{10} \operatorname{Im} F(s, 1) < \frac{1}{10} \min |F| \quad (43')$$

The latter condition is easily ensured by using (40). The former can be satisfied as a consequence of the uniform continuity of $|F|$.

Then two situations can occur :

- i) somewhere in the interval $\theta_1'' < \theta < \theta_2''$ the phase goes through $\pi/2$ or $-\pi/2$ (not both, this being impossible in $\theta_1\theta_2$). Let $\bar{\theta}$ be the angle where this takes place. Consider for instance the case where the phase goes through $\pi/2$. We have

$$||F(\cos \bar{\theta})| - |F(\cos \theta_1')|| \sin [\operatorname{Arg} F(\cos \theta_1')] < |\Delta \operatorname{Im} F|$$

where $|\Delta \operatorname{Im} F|$ is defined by (40) for the whole interval $\theta_1'' \theta_2''$, and hence

$$|1 - \sin(\operatorname{Arg} F(\cos \theta_1''))| < \frac{|\Delta \operatorname{Im} F| + |\Delta |F||}{\min |F|}$$

hence, using (43) and (43')

$$|\frac{\pi}{2} - \operatorname{Arg} F(\cos \theta_1'')| < 2 \sqrt{\frac{|\Delta \operatorname{Im} F| + |\Delta |F||}{\min |F|}}$$

Since the same inequality can be written for θ_2'' we get

$$|\operatorname{Arg} F(\cos \theta_2'') - \operatorname{Arg} F(\cos \theta_1'')| < 4 \sqrt{\frac{|\Delta |F|| + |\Delta \operatorname{Im} F|}{\min |F|}} \quad (44)$$

since $\Delta \operatorname{Im} F$ satisfies a Hölder condition of order $\frac{1}{2}$ we see that at best we can get a Hölder condition of order $1/4$ if $|F|$ satisfies at least a Hölder condition of order $\frac{1}{2}$.

ii) if the phase does not go through $\pi/2$ we can for instance assume that it stays between $-\pi/2$ and $+\pi/2$. Then we write

$$\begin{aligned} |F(\cos \theta_1'')| \left| \sin(\operatorname{Arg} F(\cos \theta_1'')) - \sin(\operatorname{Arg} F(\cos \theta_2'')) \right| < \\ < |\Delta \operatorname{Im} F| + |\Delta |F|| \end{aligned}$$

We use then the inequality

$$\begin{aligned} \sin \beta - \sin \alpha &> \sin \beta (\sin \beta - \sin \alpha) + \cos \beta (\cos \beta - \cos \alpha) = \\ &= 1 - \cos(\alpha - \beta) > \frac{(\alpha - \beta)^2}{4} \end{aligned}$$

for $-|\beta| \leq \alpha < \beta < \pi/2$, and hence

$$|\operatorname{Arg} F(\cos \theta_1) - \operatorname{Arg} F(\cos \theta_2)| < 2 \sqrt{\frac{|\Delta \operatorname{Im} F| + |\Delta |F||}{\min |F|}} \quad (45)$$

The result is qualitatively the same as in (44).

To summarize both cases we get, for instance if $|F|$ satisfies a Hölder condition of order $\frac{1}{2}$

$$||F(\cos \theta_1) - F(\cos \theta_2)|| < K \sqrt{|\theta_1 - \theta_2|} \quad (46)$$

$$|\operatorname{Arg} F(\cos \theta_1) - \operatorname{Arg} F(\cos \theta_2)| < 4 |\theta_1 - \theta_2|^{\frac{1}{4}} \sqrt{\frac{K + \frac{\operatorname{Im} F(\cos \theta=1)}{\sqrt{\pi}} \frac{2}{(\sin \theta_1)^{\frac{1}{4}} (\sin \theta_2)^{\frac{1}{4}}}}{\min |F|}} \quad (47)$$

as long as $|\theta_1 - \theta_2|$ is sufficiently small to make the right-hand side less than $\pi/2$.

IV. CONCLUDING REMARKS

The results we have obtained cannot be considered as final because they have not been obtained in a systematic way. However, as they are they exhibit the power of positivity properties of the absorptive part since they lead to an effective control of the phase of the scattering amplitude from the knowledge of the modulus. It must be noticed that contrary to what often happens when one uses analyticity, it is not necessary to know the modulus with infinite accuracy to get bounds on the phase in the physical region. It must also be noticed that even though our results are unsystematic, some qualitative aspects of these cannot be improved in particular the fact that the bound on the phase behaves like $(\operatorname{Im} F(\cos \theta=1)/(\min |F|))^2$ in the limit where this quantity is large.

We have considered the spinless case only. A superficial look seems to indicate that a generalization to higher spins is possible. In particular it seems that in the spin 0 - spin $\frac{1}{2}$ case one does not need a complete set of measurements to get bounds on the phases of the no-spin flip and spin flip amplitude. The no-spin flip amplitude has the same positivity properties at fixed energy as a spin 0 amplitude. A lower bound on its modulus can be obtained from lower bounds on the differential cross-section and the transverse polarization. In this way a bound on the phase of the no-spin flip amplitude is obtained. The phase of the spin flip amplitude, on the other hand, must follow (within an interval of π) the phase of the no-spin flip amplitude as long as the polarization does not vanish.

A P P E N D I X A

MONOTONY PROPERTIES OF EXTREMA OF LEGENDRE POLYNOMIALS

Let $x_{n,\ell}$ be the extrema of Legendre polynomials, given by

$$P'_\ell(x_{n,\ell}) = 0 \quad (\text{A.1})$$

with

$$1 > x_{1,\ell} > x_{2,\ell} > \dots > x_{n-1,\ell} > x_{n,\ell} \dots > -1 \quad (\text{A.2})$$

We define

$$m_{n,\ell} = m_{\ell-n,\ell} = |P_\ell(x_{n,\ell})|$$

A classical property to be found in Szegő⁵⁾ is

$$m_{n,\ell} > m_{n+1,\ell} \quad n < \ell/2 \quad (\text{A.3})$$

$$\text{for } 1 > x_{n,\ell} > x_{n+1,\ell} \geq 0$$

so that when one goes from $x=0$ to $x=1$, the moduli of the values at the extrema form an increasing sequence.

Here we want to prove a more subtle property linking different ℓ 's :

$$m_{n,\ell} > m_{n,\ell+1} \quad (\text{A.4})$$

for $n \leq \ell/2$.

If we succeed in proving (A.4), we prove for instance

$$|P_{2n}(0)| > m_{n,\ell} > \mu_n \quad (\text{A.5})$$

for $\ell > 2n$, where μ_n is the value taken at its n^{th} extremum by the Bessel function $|J_0|$.

By using (A.3) and (A.5) for $n=1$ one gets

$$P_e(x) > -\frac{1}{2} \quad 0 < x < 1 \quad (\text{A.6})$$

which was already known ⁶⁾.

To prove (A.4) we compare the values of $P_\ell(x)$ at $x=x_\ell$, where $0 \leq x_\ell$ is an extremum of P_ℓ , and of $P_{\ell+1}(x)$ at $x=x_{\ell+1}$, where $x_{\ell+1}$ is the smallest value $x_{\ell+1} > x_\ell$ such that $P_{\ell+1}$ has an extremum.

1) Basic equations :

Legendre differential equation :

$$\begin{cases} (1-x^2) P_e'' - 2x P_e' + \ell(\ell+1) P_e = 0 \end{cases} \quad (\text{A.7})$$

$$\begin{cases} (1-x^2) P_{\ell+1}'' - 2x P_{\ell+1}' + (\ell+1)(\ell+2) P_{\ell+1} = 0 \end{cases} \quad (\text{A.8})$$

Recursion relations :

$$(1-x^2) P_e' = (\ell+1) [x P_e - P_{\ell+1}] \quad (\text{A.9})$$

$$(1-x^2) P_{\ell+1}' = (\ell+1) [P_e - x P_{\ell+1}] \quad (\text{A.10})$$

2) Values taken at $x=x_\ell$ and $x_{\ell+1}$

We can assume, without loss of generality $P_\ell(x_\ell) > 0$. Then we have, from (A.9), (A.10) and (A.7) and $P_\ell'(x_\ell) = 0$

$$\left. \begin{aligned} P_{\ell+1}(x_\ell) &= x_\ell P_e(x_\ell) > 0 \\ P_{\ell+1}'(x_\ell) &= (\ell+1) P_e(x_\ell) > 0 \\ (1-x_\ell^2) P_e''(x_\ell) &= -\ell(\ell+1) P_e(x_\ell) < 0 \\ (1-x_\ell^2) P_{\ell+1}''(x_\ell) &= -\ell(\ell+1) P_{\ell+1}(x_\ell) < 0 \end{aligned} \right\} \quad (\text{A.11})$$

Similarly

$$\left. \begin{aligned} P_e(x_{e+1}) &= x_{e+1} P_{e+1}(x_{e+1}) \\ P'_e(x_{e+1}) &= -(e+1) P_{e+1}(x_{e+1}) \\ (1-x_{e+1}^2) P''_{e+1}(x_{e+1}) &= -(e+1)(e+2) P_{e+1}(x_{e+1}) \\ (1-x_{e+1}^2) P''_e(x_{e+1}) &= -(e+1)(e+2) P_e(x_{e+1}) \end{aligned} \right\} \quad (A.12)$$

3) Sign of the various quantities for $x_e \leq x \leq x_{e+1}$

From (A.11) we see that since x_{e+1} is the first extremum of P_{e+1} to the right of x_e , this extremum is necessarily a maximum and

$$\left. \begin{aligned} P_{e+1}(x) &> 0 \\ P'_{e+1}(x) &\geq 0 \end{aligned} \right\} \text{ for } x_e \leq x \leq x_{e+1} \quad (A.13)$$

then from (A.10) we get

$$P_e(x) \gg x P_{e+1}(x) \gg 0 \quad (A.14)$$

Now since P'_e can only vanish once in an interval where P_e has a constant sign, because it is a polynomial with real zeros, and since P'_e is negative at $x = x_{e+1}$ we have

$$P'_e(x) < 0 \quad (A.15)$$

Similarly P''_e can only vanish once in an interval where P'_e has a constant sign. Therefore since P''_e is negative as a consequence of (A.11) and (A.12) at the extremities of the interval, we get $P''_e(x) < 0$ and, similarly, $P''_{e+1}(x) < 0$. To summarize, we have

$$\begin{aligned} P_e > 0, P_{e+1} > 0, P'_e \geq 0, P'_{e+1} \leq 0, P''_e < 0, P''_{e+1} < 0 \\ x_e \leq x \leq x_{e+1} \end{aligned} \quad (A.16)$$

4) Comparison of $P_\ell(x_\ell)$ and $P_{\ell+1}(x_{\ell+1})$ by integration of the differential equation

If we multiply (A.7) by $P_{\ell+1}$ and (A.8) by P_ℓ , subtract and integrate, we get

$$\frac{(1-x^2)}{\ell+1} (P_{\ell+1}' P_\ell' - P_\ell P_{\ell+1}') - 2 \int_{x_\ell}^x P_\ell(x') P_{\ell+1}(x') dx = \text{const.} \quad (\text{A.17})$$

Applying this equation to the interval $x_\ell, x_{\ell+1}$ and substituting the values of P_ℓ' and $P_{\ell+1}'$ given by (A.11) and (A.12), we get

$$(1-x_{\ell+1}^2) (P_{\ell+1}(x_{\ell+1}))^2 - (1-x_\ell^2) (P_\ell(x_\ell))^2 + 2 \int_{x_\ell}^{x_{\ell+1}} P_\ell(x) P_{\ell+1}(x) dx = 0 \quad (\text{A.18})$$

which already proves

$$(1-x_{\ell+1}^2) (P_{\ell+1}(x_{\ell+1}))^2 < (1-x_\ell^2) (P_\ell(x_\ell))^2$$

This, however, is not enough, and we have to take into account the integral in (A.18). We notice that for $x_\ell < x < x_{\ell+1}$

$$\frac{d^2}{dx^2} [P_\ell(x) P_{\ell+1}(x)] = P_\ell'' P_{\ell+1} + P_\ell P_{\ell+1}'' + 2 P_\ell' P_{\ell+1}' < 0 \quad (\text{A.19})$$

as a consequence of inequalities (A.16), so that the function $P_\ell(x) P_{\ell+1}(x)$ is concave. Hence we have

$$\begin{aligned} \int_{x_\ell}^{x_{\ell+1}} P_\ell(x) P_{\ell+1}(x) dx &> \left(\frac{x_{\ell+1} - x_\ell}{2} \right) \left[P_\ell(x_\ell) P_{\ell+1}(x_\ell) + P_\ell(x_{\ell+1}) P_{\ell+1}(x_{\ell+1}) \right] = \\ &= \left(\frac{x_{\ell+1} - x_\ell}{2} \right) \left[x_\ell (P_\ell(x_\ell))^2 + x_{\ell+1} (P_{\ell+1}(x_{\ell+1}))^2 \right] \quad (\text{A.20}) \end{aligned}$$

If we substitute this into (A.18), we get

$$(1-x_e x_{e+1}) P_{e+1}^e(x_{e+1}) - (1-x_e x_{e+1}) P_e^2(x_e) < 0 \quad (\text{A.21})$$

which proves

$$|P_{e+1}(x_{e+1})| < |P_e(x_e)| \quad (\text{A.22})$$

Obviously, the proof works as well for the case $P_l(x_l) < 0$. We shall now show that if $x_l = x_{n,l}$, i.e., if x_l is the n th extremum (in decreasing order), then $x_{l+1} = x_{n,l+1}$.

We notice that if we consider the sequence

$$1 > x_{1,e} > x_{2,e} > \dots > x_{[\frac{e}{2}],e}$$

the values of P_l take alternate signs at these points, $P_l(x_{1,l}) < 0$, $P_l(x_{2,l}) > 0$, etc.

Similarly, by associating to each x_l the first $x_{l+1} > x_l$ we form a sequence of values where $P_{l+1}(x)$ takes alternate signs. Therefore, we exhaust completely the minima and maxima of P_{l+1} for $0 < x < 1$, and we prove two things

$$1 > x_{1,l+1} > x_{1,e} > x_{2,l+1} > x_{2,e} \dots > x_{n,e} > x_{n,l+1} \quad (\text{A.23})$$

and, from (A.22)

$$|P_{l+1}(x_{n,l+1})| < |P_e(x_{n,e})| \quad (\text{A.24})$$

for $l \geq 2n$.

A P P E N D I X B

UPPER BOUNDS FOR ALTERNATE SUMS OF LEGENDRE POLYNOMIALS WITH THE SAME INDEX AND ORDERED VARIABLES

Let us define for ℓ fixed :

$$A_{n,\ell}(x_1, \dots, x_n) = -P_\ell(x_1) + P_\ell(x_2) + \dots + (-1)^n P_\ell(x_n) \quad (B.1)$$

$$1 \geq x_1 \geq x_2 \geq \dots \geq x_n \geq 0$$

$$A_{n,\ell} = \sup_{\forall x_i} A_{n,\ell}(x_1, \dots, x_i, \dots, x_n) \quad (B.2)$$

We want to determine the configuration of the x_i 's leading to this $\sup A_{n,\ell}$. For this we take into account the property $m_{i,\ell} > m_{j,\ell}$ for $i < j$ and $m_{i,\ell} = |P_\ell(x_{i,\ell})|$ are the extrema of P_ℓ corresponding to $x_{i,\ell} > 0$. First we note that an even number of equal x_i leads to a contribution zero in (B.1) whereas an odd number of them contribute to one term. This means that if n is the number of x_i , only ν of them (corresponding to different x_i values) contribute effectively to the sum (A.1), ($\nu \leq n$, $n - \nu$ being even). Secondly, we remark that the ν different x_i values correspond to the extrema of P_ℓ because from $(\partial/\partial x_i)A_{n,\ell}(x_1, \dots, x_i, \dots) = 0$ we get $(\partial/\partial x_i)P_\ell(x_i) = 0$.

Let us therefore consider these ν effective x_i values ($i = 1, \dots, \nu$) located at $x_{j,\ell}$. From the fact that the $m_{i,\ell}$ are decreasing when i increases, we see that a configuration corresponding to the maximum of (B.1) is $x_1 = x_{1,\ell}, x_2 = x_{2,\ell}, \dots, x_\nu = x_{\nu,\ell}$ leading to a contribution $\sum_1^\nu m_{i,\ell}$ in (B.1). Then two cases occur :

i) $\ell \geq 2n$.

In order to obtain the $\sup$ in (B.2), we must use all the x_i 's at our disposal and $\nu = n$. We get

$$A_{n,\ell} = \sum_1^n m_{i,\ell} \leq \sum_1^n m_{i,2n} = A_{n,2n} \quad (B.3)$$

where we have taken into account the property $m_{i,\ell'} < m_{i,\ell}$, $\ell' > \ell$ shown in Appendix A.

ii) $\ell < 2n$.

Here n exceeds necessarily ν and the sup in (B.2) is obtained by locating x_i ($i \leq \nu$) at $x_{i,\ell} \geq 0$ and the $(n-\nu)$ remaining x_i give a contribution zero. If ℓ is even, we find $\nu = \ell/2$ and obtain

$$A_{n,\ell} = \sum_i^{\ell/2} m_{i,\ell} = A_{\frac{\ell}{2},\ell} \quad (\text{B.4})$$

whereas if ℓ is odd, $\nu = (\ell-1)/2$ and

$$A_{n,\ell} = \sum_i^{\frac{\ell-1}{2}} m_{i,\ell} < \sum_i^{\frac{\ell-1}{2}} m_{i,\ell-1} = A_{\frac{\ell-1}{2},\ell-1} \quad (\text{B.5})$$

where we still used the result of Appendix A.

We now call $B_{n,\ell}(x_1, \dots, x_n)$ the same sum as in (B.1), but with the restriction

$$1 \gg x_1 \gg x_2 \dots \gg x_i \dots \gg x_n \gg -1$$

and define

$$B_{n,\ell} = \sup_{\forall x_i} B_{n,\ell}(x_1, \dots, x_i, \dots, x_n)$$

We also have two cases :

i) $n \leq \ell$.

Let us still call ν the number of different x_i values. It is clear of course that in order to get the sup, then each x_i must be such that the corresponding term $(-1)^i P_\ell(x_i)$ gives a non-negative contribution in (B.1). Due to this remark, we see that if n and ℓ have the same parity, the last x_i , x_ν is located at -1 whereas if n and ℓ have opposite parity, then -1 is not occupied by an x_i . (Otherwise it would give a negative term in the sum.) Of course in the last case an even number of x_i terms can be located at -1 . For the $\nu - 1$ (or ν) effective x_i values remaining at our disposal, we take into account the fact that the moduli of the extrema are decreasing when x goes from 1 to 0 or -1 to 0 and equal for opposite x_i values. So we put the x_i symmetrically with respect to the origin : $x_{1,\ell}$ and $-x_{1,\ell}$, $x_{2,\ell}$ and $-x_{2,\ell}$, It can happen that at the end we still have one x_i value. In this case we place it at one of the highest modulus extrema unoccupied. Finally we obtain :

$$\begin{aligned}
 & \left. \begin{aligned}
 & n \text{ odd, } l \text{ odd} \\
 & B_{n,l} = 1 + 2 \sum_{i=1}^{\frac{n-1}{2}} m_{i,l} \leq B_{n,n} \\
 & n \text{ even, } l \text{ even} \\
 & B_{n,l} = 1 + 2 \sum_{i=1}^{\frac{n}{2}-1} m_{i,l} + m_{\frac{n}{2},l} \leq B_{n,n} \\
 & n \text{ even, } l \text{ odd} \\
 & B_{n,l} = 2 \sum_{i=1}^{\frac{n}{2}} m_{i,l} \leq B_{n,n+1} < B_{n,n} \\
 & n \text{ odd, } l \text{ even} \\
 & B_{n,l} = 2 \sum_{i=1}^{\frac{n-1}{2}} m_{i,l} + m_{\frac{n+1}{2},l} \leq B_{n,n+1} < B_{n,n} \\
 & B_{n,n} = 1 + \sum_{i=1}^{n-1} m_{i,n}
 \end{aligned} \right\} \quad (B.6)
 \end{aligned}$$

ii) $l < n$.

Here also n exceeds necessarily γ . The sup in (B.2) is obtained by locating each effective x_i ($i \leq \gamma$) at a corresponding $x_{i,l}$. All $x_{i,l}$ are occupied, but not -1 if n and l have opposite parity (otherwise it gives a negative contribution to the sum). We obtain, if n and l have the same parity :

$$B_{n,l} = B_{l,l}$$

and $B_{l,l}$ is given in (B.6). On the contrary, if n and l have opposite parity, we get

$$B_{n,l} = B_{l-1,l} < B_{l-1,l-1}$$

with $B_{l-1,l}$ given by (B.6) and where we use again the result of Appendix A for the inequality.

Obviously what has been done above is the generalization of the inequality $-P_l < \frac{1}{2}$. However, there exists also an upper bound $P_l < 1$ and the question arises : does a similar generalized upper bound exist for such alternate sums of Legendre polynomials ? The answer is yes and we shall give only the results because the method is similar to the previous one.

Let us define

$$E_{n,l} = \sup_{\forall x_i} [P_e(x_1) - P_e(x_2) \dots - (-1)^n P_e(x_n)] \quad (B.7)$$

$$1 \gg x_1 \gg x_2 \dots \gg x_n \gg 0$$

One finds easily :

i) $l \geq 2n-2$

$$E_{n,l} = 1 + \sum_{i=1}^{n-1} m_{i,l} \leq 1 + \sum_{i=1}^{n-1} m_{i,2n-2} = E_{n,2n-2}$$

ii) $l < 2n-2$. If l is even, one has $E_{n,l} = E_{(l/2)+1,l}$, whereas if l is odd,

$$E_{n,l} = E_{\frac{l+1}{2},l} < E_{\frac{l+1}{2},2(\frac{l+1}{2})-2}$$

Finally

$$\sup_{\forall l} (E_{n,l} - 1) = \sup_{p=2}^{p=n} \sum_{i=1}^{i=p-1} m_{i,2(p-1)} = \sup_{q=1}^{q=n-1} A_{q,2q} \quad (B.8)$$

$$< \sum_{i=1}^{n-1} m_{i,2i}$$

Of course, we can also consider in the alternate sum (B.7) that the x_i satisfy other conditions; for instance $1 \geq x_1 \geq x_2 \dots \geq x_n \geq -1$. With the method developed in this Appendix, it is an easy exercise to determine the corresponding upper bounds. In (B.7) let us call

$$F_{n,l} = \sup_{\forall x_i} [P_e(x_1) - P_e(x_2) + \dots - (-1)^n P_e(x_n)]$$

when the x_i satisfy $1 \geq x_1 \geq x_2, \dots, \geq x_n \geq -1$. The corresponding result is

$$\sup_{\forall l} (F_{n,l} - 1) \leq 1 + \sup_{q=1}^{q=n} \sum_{i=1}^{q-1} m_{i,q} = \sup_{q=1}^{q=n} B_{q,q} \quad (B.9)$$

where $B_{q,q}$ is given in (B.6).

A P P E N D I X C

UPPER BOUNDS FOR SUMS OF MODULI OF EXTREMA CORRESPONDING TO LEGENDRE POLYNOMIALS

We want to prove

$$A_{n,2n} \leq C_n^{(0)} = \sqrt{\frac{2}{\pi}} \left[\sqrt{2n+\frac{1}{2}} - \sqrt{\frac{5}{2}} \right] + \frac{1}{2} \quad (C.1)$$

$$A_{n,2n} = \sum_{i=1}^{i=n} m_{i,2n}$$

and from $m_{k,2n} \leq |P_{2k}^{(0)}|$

$$A_{n,2n} \leq \sum_{k=1}^n |P_{2k}^{(0)}|$$

Now one has

$$|P_{2k}^{(0)}| \leq \sqrt{\frac{2}{\pi(2k+\frac{1}{2})}} \quad (C.2)$$

This can be established by showing that $P_{2k}^{(0)} \sqrt{2k+\frac{1}{2}}$ is an increasing function of k .

More generally

$$|P_\ell(\cos \theta)| \leq \sqrt{\frac{2}{\pi(\ell+\frac{1}{2}) \sin \theta}} \quad (C.3)$$

This can be established for even ℓ by proving that

$$|P_\ell(\cos \theta)| < \frac{1}{\sqrt{\sin \theta}} |P_\ell(0)| \quad (C.4)$$

while for odd ℓ one uses

$$|P_\ell(\cos \theta)| < |P'_\ell(0)| \left(\frac{1}{4} + (\ell+\frac{1}{2})^2 \right)^{-\frac{1}{2}} \quad (C.5)$$

together with a bound on $|P'_\ell(0)|$ analogous to (C.3). Then for $n \geq 1$

$$\begin{aligned} A_{n,2n} &\leq \frac{1}{2} + \sqrt{\frac{2}{\pi}} \sum_{k=2}^{k=n} \frac{1}{\sqrt{2k+\frac{1}{2}}} \\ &\leq \frac{1}{2} + \sqrt{\frac{2}{\pi}} \int_{k=1}^n \frac{dk}{\sqrt{2k+\frac{1}{2}}} = C_n^{(1)} \end{aligned} \quad (C.6)$$

which gives readily (C.1).

We also prove

$$A_{n,2n} \leq C_n^{(1)}$$

Here $C_n^{(1)}$ comes from an asymptotic estimate. Let $\cos \theta_{i,2n} = x_{i,2n}$ be the extrema of P_{2n} . We have

$$A_{n,2n} = \sum_{i=1}^n |P_{2n}(\cos \theta_{i,2n})| = \sum_{i=1}^n m_{i,2n} \quad (C.7)$$

and from (C.3)

$$A_{n,2n} \leq \sqrt{\frac{2}{\pi(2n+\frac{1}{2})}} \sum_{i=1}^n \frac{1}{\sqrt{\sin \theta_{i,2n}}} \quad (C.8)$$

Here we insert some information about the location of the extrema.

The extrema $\cos \theta_{i,l}$ of $P_l(\cos \theta)$ are the zeros of $P'_l(\cos \theta)$ or, equivalently, the zeros of the Jacobi polynomial $P_{l-1}^{1,1}(\cos \theta)$, which is of ultraspherical type. The k th zero of $P_{l-1}^{\alpha,\alpha}(\cos \theta)$, $\theta_{k,l-1}^\alpha$ possesses the property ⁵⁾

$$\frac{\partial}{\partial \alpha} \theta_{k,l-1}^\alpha > 0$$

for $k < (l-1)/2$. However, we know the location of the zeros of

$$P_{l-1}^{\frac{1}{2},\frac{1}{2}}(\cos \theta) = \frac{\sin l\theta}{\sin \theta}$$

it is $\theta_{k,l-1}^{\frac{1}{2}} = \frac{k\pi}{l}$

Hence

$$\theta_{k,l} \equiv \theta'_{k,l-1} > \frac{k\pi}{l} \quad (C.9)$$

Therefore, since $1/\sqrt{\sin\theta}$ is a decreasing function of θ , we get

$$A_{n,2n} \leq \sqrt{\frac{2}{\pi(2n+\frac{1}{2})}} \sum_{k=1}^n \frac{1}{\sqrt{\sin \frac{k\pi}{2n}}} \quad (C.10)$$

Now, from the convexity of $1/\sqrt{\sin\theta}$:

$$\frac{d^2}{d\theta^2} \left[\frac{1}{\sqrt{\sin\theta}} \right] > 0$$

we get, for $\theta_a < \theta_b$

$$\frac{\theta_b - \theta_a}{\sqrt{\sin(\frac{\theta_a + \theta_b}{2})}} < \int_{\theta_a}^{\theta_b} \frac{d\theta}{\sqrt{\sin\theta}} \quad (C.11)$$

This allows us to replace (C.10) by an integral :

$$\begin{aligned} A_{n,2n} &\leq \sqrt{\frac{2}{\pi(2n+\frac{1}{2})}} \left[\frac{1}{2} + \frac{2n}{\pi} \int_{\frac{\pi}{4n}}^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{\sin\theta}} \right] \\ &\leq \sqrt{\frac{2}{\pi(2n+\frac{1}{2})}} \left[\frac{1}{2} + \frac{2n}{\pi} \int_0^{\frac{\pi}{2}} \frac{\frac{n}{4n} d\theta}{\sqrt{\sin\theta}} - 2 \sqrt{\frac{n}{\pi}} \right] \end{aligned}$$

i.e.,

$$A_{n,2n} < C_n^{(1)} = \frac{n}{\sqrt{2n+\frac{1}{2}}} \left(\frac{\Gamma(\frac{1}{4})}{\pi^2} \right)^2 - \frac{2}{\pi} \sqrt{\frac{2n}{2n+\frac{1}{2}}} + \frac{1}{\sqrt{2n(2n+\frac{1}{2})}} \quad (C.12)$$

Now we derive the analogous bounds for

$$B_{n,n} = m_{1,n} + m_{2,n} + \dots + m_{n-1,n} + 1 \quad (C.13)$$

Here we must distinguish n even and n odd to compute $D_n^{(0)}$:

$$\begin{aligned}
 n \text{ even :} \\
 B_{n,n} &< 1 + 2 \sum_1^{\frac{n-2}{2}} \sqrt{\frac{2}{\pi(2k+\frac{1}{2})}} + \sqrt{\frac{2}{\pi(n+\frac{1}{2})}} \\
 &\leq 1 + \sqrt{\frac{2}{\pi}} \left[\sqrt{n-\frac{3}{2}} - \sqrt{\frac{1}{2}} + \frac{1}{\sqrt{n+\frac{1}{2}}} \right] \\
 &\quad \text{by integration}
 \end{aligned}$$

n odd :

$$B_{n,n} < 1 + 2 \sum_1^{\frac{n-1}{2}} \sqrt{\frac{2}{\pi(2k+\frac{1}{2})}}$$

which gives by integration

$$B_{n,n} < 1 + \sqrt{\frac{2}{\pi}} \left[\sqrt{n-\frac{1}{2}} - \sqrt{\frac{1}{2}} \right]$$

A universal formula fitting both n even and n odd is :

$$B_{n,n} < D_n^{(0)} = 1 + \sqrt{\frac{2}{\pi}} \left[\sqrt{n-\frac{1}{2}} - \sqrt{\frac{3}{2}} + \sqrt{\frac{2}{5}} \right] \quad (C.14)$$

The analogue of (C.12) is now obtained by noticing that

$$\begin{aligned}
 B_{n,n} &< \sum_{k=1}^n \sqrt{\frac{2}{\pi(n+\frac{1}{2})}} \frac{1}{\sqrt{\sin \frac{k\pi}{n}}} + 1 \\
 &\leq \sqrt{\frac{2}{\pi(n+\frac{1}{2})}} \frac{n}{\pi} \int_{\frac{\pi}{2n}}^{\pi-\frac{n}{2n}} \frac{d\theta}{\sqrt{\sin \theta}} + 1 \\
 &\leq \sqrt{\frac{2}{\pi(n+\frac{1}{2})}} \left[\frac{n}{\pi} \int_0^{\pi} \frac{d\theta}{\sqrt{\sin \theta}} - 2\sqrt{\frac{2n}{\pi}} \right] + 1
 \end{aligned}$$

and hence

$$B_{n,n} < D_n^{(1)}$$

$$D_n^{(1)} = \frac{n}{\sqrt{n+\frac{1}{2}}} \left(\frac{T(\frac{1}{4})}{\pi^2} - \frac{4}{\pi} \sqrt{\frac{n}{n+\frac{1}{2}}} + 1 \right) \quad (C.15)$$

notice that

$$\frac{4}{\pi} \sqrt{\frac{n}{n+\frac{1}{2}}} > 1$$

for all $n \geq 1$.

We can also determine l independent upper bounds for the quantities $E_{n,l}$, $F_{n,l}$ defined in (B.7) at the end of Appendix B. From (B.8)-(B.9) we get

$$\sup_{\forall e} (E_{n,e} - 1) < C_{n-1}^{(i)}, \sup_{\forall e} (F_{n,e} - 1) < D_n^{(i)}$$

where the expressions corresponding to $i=0$ and $i=1$ are given in (C.1)-(C.12), (C.14) and (C.15). Finally we get

$$-C_n^{(i)} < P_e(x_1) - P_e(x_2) \dots - (-1)^n P_e(x_n) < 1 + C_{n-1}^{(i)}; 1 \gg x_1 \gg x_2 \dots \gg x_n \gg 0 \quad (C.16)$$

$$-D_n^{(i)} < P_e(x_1) - P_e(x_2) \dots - (-1)^n P_e(x_n) < 1 + D_n^{(i)}; 1 \gg x_1 \gg \dots \gg x_n \gg -1 \quad (C.17)$$

and the corresponding formula for $\text{Im } F(s, x = \cos \theta)$:

$$\begin{cases} -C_n^{(i)} < \frac{\text{Im } F(s, x_1) - \text{Im } F(s, x_2) + \dots - (-1)^n \text{Im } F(s, x_n)}{\text{Im } F(s, 1)} < 1 + C_{n-1}^{(i)} \\ 1 \gg x_1 \gg x_2 \dots \gg x_n \gg 0 \quad i = 0, 1 \end{cases} \quad (C.18)$$

$$\begin{cases} -D_n^{(i)} < \frac{\text{Im } F(s, x_1) - \text{Im } F(s, x_2) \dots - (-1)^n \text{Im } F(s, x_n)}{\text{Im } F(s, 1)} < 1 + D_n^{(i)} \\ 1 \gg x_1 \gg x_2 \dots \gg x_n \gg -1 \quad i = 0, 1 \end{cases} \quad (C.19)$$

From (C.17) we see that the sum of alternate terms with ordered variables x_i has a modulus bounded by $\text{const.} \sqrt{n}$ whereas if the x_i were not constrained the bound would be n . Thus with ordered variables we have gained a factor $\sqrt{n}$ in the bound.

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FIGURE CAPTION

The allowed region for the phase of the amplitude in interval $0 < \cos \theta_M < \cos \theta < 1$ as a function of

$$R = \min \left| \frac{F(s, \cos \theta)}{\operatorname{Im} F(s, 1)} \right|$$