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IN EXTRAPOLATION METHODS

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ABSTRACT

Standard trigonometric interpolation formulae are modified for use in extrapolation methods. Some applications are indicated.

Keywords: Trigonometric, interpolation, extrapolation, method.

Table 1
Basic trigonometric interpolation formulae for even functions

End-point approximation	Mid-point approximation
$g(x) = t_n(x) + \varepsilon_n(x), \quad (1)$	$g(x) = u_n(x) + \eta_n(x), \quad (7)$
$t_n(x) = \sum_{k=0}^n a_k^{(n)} \cos kx, \quad (2)$	$u_n(x) = \sum_{k=0}^{n-1} b_k^{(n)} \cos kx, \quad (8)$
$\varepsilon_n(x) = -2 \sum_{q=1}^{\infty} \left[\sum_{k=1}^{2n} a_{(2q-1)n+k} \sin \{(q-1)n+k\}x \right] \sin qnx, \quad (3)$	$\eta_n(x) = 2 \sum_{q=1}^{\infty} \left[\sum_{k=0}^{2n} a_{(4q-3)n+k} \cos \{(2q-2)n+k\}x \right] \cos (2q-1)nx - \quad (9)$
$a_k^{(n)} = \frac{2}{n} \sum_{j=0}^n g(j\pi/n) \cos (jk\pi/n), \quad (4)$	$-2 \sum_{q=1}^{\infty} \left[\sum_{k=0}^{2n} a_{(4q-1)n+k} \sin \{(2q-1)n+k\}x \right] \sin 2qnx, \quad (9)$
$a_k^{(n)} = a_k + \sum_{q=1}^{\infty} \left[a_{2qn-k} + a_{2qn+k} \right], \quad (5)$	$b_k^{(n)} = \frac{2}{n} \sum_{j=1}^n g\{(j-\frac{1}{2})\pi/n\} \cos \{(j-\frac{1}{2})k\pi/n\}, \quad (10)$
$a_k^{(2n)} = a_k^{(n)} - a_{2n-k}^{(2n)}, \quad k = 0 \quad (1) \quad (n-1), \quad (6)$	$b_k^{(n)} = a_k + \sum_{q=1}^{\infty} (-1)^q \left[a_{2qn-k} + a_{2qn+k} \right] \quad (11)$
$a_n^{(2n)} = \frac{1}{2} a_n^{(n)},$	
$a_{2n-k}^{(2n)} = \left[a_k^{(n)} - b_k^{(n)} \right] / 2, \quad k = 0 \quad (1) \quad (n-1).$	

Table 2

Basic trigonometric interpolation formulae for odd functions

End-point approximation	Mid-point approximation
$h(x) = v_n(x) + \xi_n(x), \quad (12)$	$h(x) = w_n(x) + \delta_n(x), \quad (18)$
$v_n(x) = \sum_{k=1}^{n-1} c_k^{(n)} \sin kx, \quad (13)$	$w_n(x) = \sum_{k=1}^{n-1} d_k^{(n)} \sin kx + \frac{1}{2} d_n^{(n)} \sin nx, \quad (19)$
$\xi_n(x) = \sum_{q=1}^{\infty} d_{(2q-1)n} \sin nx +$ $+ 2 \sum_{q=1}^{\infty} \left[\sum_{k=1}^{2n} d_{(2q-1)n+k} \cos \{(q-1)n+k\}x \right] \sin qnx \quad (14)$	$\delta_n(x) = 2 \sum_{q=1}^{\infty} \left[\sum_{k=1}^{2n} d_{(4q-3)n+k} \sin \{(2q-2)n+k\}x \right] \cos (2q-1)nx +$ $+ 2 \sum_{q=1}^{\infty} \left[\sum_{k=1}^{2n} d_{(4q-1)n+k} \cos \{(2q-1)n+k\}x \right] \sin 2qnx \quad (20)$
$c_k^{(n)} = \frac{2}{n} \sum_{j=1}^{n-1} h(j\pi/n) \sin (jk\pi/n), \quad (15)$	$d_k^{(n)} = \frac{2}{n} \sum_{j=1}^n h\{(j - \frac{1}{2})\pi/n\} \sin \{(j - \frac{1}{2})k\pi/n\}, \quad (21)$
$c_k^{(n)} = d_k^{(n)} - \sum_{q=1}^{\infty} \left[d_{2qn-k} - d_{2qn+k} \right], \quad (16)$	$d_k^{(n)} = d_k^{(n)} - \sum_{q=1}^{\infty} (-1)^q \left[d_{2qn-k} - d_{2qn+k} \right], \quad (22)$
$c_k^{(2n)} = c_k^{(n)} + c_{2n-k}^{(2n)}, \quad k = 1(1)(n-1),$ $c_n^{(2n)} = \frac{1}{2} d_n^{(n)},$ $c_{2n-k}^{(2n)} = \left[d_k^{(n)} - c_k^{(n)} \right] / 2, \quad k = 1(1)(n-1). \quad (17)$	

1. INTRODUCTION

Extrapolation methods based on polynomial and rational approximations have been discussed by several authors. For an introduction to the basic theory and applications see papers by Burlisch and Stoer¹⁻³⁾ and Smidt^{4,5)},

In this paper we will discuss the possibility of constructing extrapolation methods by minor modifications of some standard trigonometric interpolation formulae. In a previous paper⁶⁾ the author has given one application using trigonometric approximations of even functions.

Let $f(x)$, $x \in [-\pi, \pi]$ be a function for which a convergent Fourier expansion exists. We write $f(x) = g(x) + h(x)$ where $g(x) = g(-x)$ is an even function and $h(x) = -h(-x)$ is an odd function. In Tables 1 and 2 we have collected the basic formulae related to trigonometric interpolation of even and odd functions, respectively, obtained by dividing the range $[0, \pi]$ into n equal sub-intervals and using the function values at the interval end-points (end-point formulae) and at the interval mid-points (mid-point formulae). In the tables we use the index n to indicate the number of sub-intervals. The coefficients a_k , d_k are the exact Fourier cosine and sine coefficients. A single prime in the sum indicates that the first term has weight one half, a double prime indicates that the first and last term have weight one half.

It follows from Eqs. (6) and (17) that the end-point coefficients related to $2n$ sub-intervals are expressed in a simple way in terms of the end- and mid-point coefficients related to n sub-intervals. We also have asymptotically, for fixed values of $k < n$ and for increasing values of n ,

$$\left| a_k - a_k^{(2n)} \right| \leq \left| a_{2n-k}^{(2n)} \right|, \quad (23)$$

with a similar relation for the odd coefficients.

Equations (6) and (17) have some advantages in numerical calculations. In a scheme based on repeated halving of the sub-intervals we use at each step only the function values needed for the calculation of the mid-point coefficients, and these function values do not have to be stored from one step to the next. It is also rather simple to control when to terminate the calculation. The formulae described above have been used by the author⁷⁾ in a program for Chebyshev approximations of functions and by Lyness⁸⁾ in a scheme for calculating derivatives using contour integration.

2. EXTRAPOLATION PROPERTIES OF TRIGONOMETRIC APPROXIMATIONS

Looking at Tables 1 and 2 we observe that the end- and mid-point approximations do not have error terms of the same order. To get approximations with error terms of the same order we define two new approximations.

Even functions, modified mid-point approximation:

$$\tilde{u}_n(x) = u_n(x) + \frac{1}{2}a_n^{(n)} \cos nx, \quad (24)$$

with error term

$$\begin{aligned} \tilde{\eta}_n(x) = & 2 \sum_{q=1}^{\infty} \left[\sum_{k=1}^{2n-1} a_{(4q-3)n+k} \cos \{(2q-2)n+k\}x \right] \cos (2q-1)nx - \\ & - 2 \sum_{q=1}^{\infty} \left[\sum_{k=0}^{2n} a_{(4q-1)n+k} \sin \{(2q-1)n+k\}x \right] \sin 2qnx. \end{aligned} \quad (25)$$

Odd functions, modified end-point approximation:

$$\tilde{v}_n(x) = v_n(x) + \frac{1}{2}d_n^{(n)} \sin nx, \quad (26)$$

with error term

$$\begin{aligned} \tilde{\xi}_n(x) = & 2 \sum_{q=1}^{\infty} d_{(4q-1)n} \sin nx + \\ & + 2 \sum_{q=1}^{\infty} \left[\sum_{k=1}^{2n} d_{(2q-1)n+k} \cos \{(q-1)n+k\}x \right] \sin qnx. \end{aligned} \quad (27)$$

So far we have obtained the following approximations having error terms of the same order

$$t_n(x), \tilde{u}_n(x) \quad \text{for even functions}$$

and

$$\tilde{v}_n(x), w_n(x) \quad \text{for odd functions.}$$

These approximations do not, however, have the bounding properties with respect to the given function, $f(x)$, which we are looking for. As an example, if we consider an even function, then the approximations $t_n(x)$, $\tilde{u}_n(x)$ are exact at the points $x_j = j\pi/n$, $j = 0(1)n$ and $x_j = (j - \frac{1}{2})\pi/n$, $j = 1(1)n$, respectively. Thus the

error terms $\varepsilon_n(x)$, $\tilde{\eta}_n(x)$ get equal sign in about half of the interval $[0, \pi]$, since the approximations $t_n(x)$, $\tilde{u}_n(x)$ are oscillating around the exact function.

Nevertheless, in some applications bounds may be constructed using these approximations, if the exact Fourier expansion of the given function is sufficiently convergent.

In the following we will derive two other sets of approximations having the required properties.

2.1 The case of an even function

Let us define the approximations

$$g(x) = t_n^*(x) + \varepsilon_n^*(x) = u_n^*(x) + \eta_n^*(x) , \quad (28)$$

where

$$t_n^*(x) = t_n(x) + \Delta_n(x) , \quad u_n^*(x) = \tilde{u}_n(x) + \Delta_n(x) , \quad (29)$$

and

$$\Delta_n(x) = \sum_{k=0}^{n-1} ' a_{2n-k}^{(2n)} \cos (2n - k)x . \quad (30)$$

Then we may easily prove

$$t_{2n}(x) = \left[t_n^*(x) + u_n^*(x) \right] / 2 , \quad (31)$$

and

$$\varepsilon_n^*(x) = -\gamma_n(x) + \varepsilon_{2n}(x) , \quad \eta_n^*(x) = \gamma_n(x) + \varepsilon_{2n}(x) , \quad (32)$$

where $\varepsilon_{2n}(x)$ is the error term related to $t_{2n}(x)$, given by Eq. (3), when n is replaced by $2n$. For $\gamma_n(x)$ we get

$$\gamma_n(x) = \left[t_n(x) - \tilde{u}_n(x) \right] / 2 = \left[\tilde{\eta}_n(x) - \varepsilon_n(x) \right] / 2 , \quad (33)$$

and, after a small calculation using Eqs. (3) and (25),

$$\gamma_n(x) = \sum_{q=1}^{\infty} \sum_{k=1}^{2n-1} a_{(4q-3)n+k} \cos (n - k)x \quad (34)$$

From Eqs. (28) to (32) it follows that the validity of the inequality

$$\left| \varepsilon_{2n}(x) \right| \leq \left| \gamma_n(x) \right| , \quad (35)$$

implies the validity of the inequality

$$u_n^*(x) \leq g(x) \leq t_n^*(x) , \quad (36)$$

(or both inequalities reversed). Equation (36) may also be written

$$\left| g(x) - t_{2n}(x) \right| \leq \left| \left[t_n(x) - \tilde{u}_n(x) \right] / 2 \right| \quad (37)$$

Since both $\varepsilon_{2n}(x)$ and $\gamma_n(x)$ are oscillating functions, the leading terms in their expansions being $-2a_{2n+1} \sin x \sin 2nx$ and $a_{n+1} \cos (n-1)x$, respectively, we may expect that the inequality (35) will be valid except in small sub-intervals around the zeros of $\gamma_n(x) = 0$. Further, the length of these sub-intervals will be decreasing with increasing convergence of the exact Fourier expansion.

2.2 The case of an odd function

Let us define the approximations

$$h(x) = v_n^*(x) + \xi_n^*(x) = w_n^*(x) + \delta_n^*(x) , \quad (38)$$

where

$$v_n^*(x) = \tilde{v}_n(x) + \omega_n(x) , \quad w_n^*(x) = w_n(x) + \omega_n(x) , \quad (39)$$

and

$$\omega_n(x) = \sum_{k=1}^{n-1} c_{2n-k}^{(2n)} \sin (2n-k)x . \quad (40)$$

Then we may easily prove

$$v_{2n}(x) = \left[v_n^*(x) + w_n^*(x) \right] / 2 , \quad (41)$$

and

$$\xi_n^*(x) = -\theta_n(x) + \xi_{2n}(x) , \quad \delta_n^*(x) = \theta_n(x) + \xi_{2n}(x) , \quad (42)$$

where $\xi_{2n}(x)$ is the error term related to $v_{2n}(x)$, given by Eq. (14) when n is replaced by $2n$. For $\theta_n(x)$ we get

$$\theta_n(x) = \left[\tilde{v}_n(x) - w_n(x) \right] / 2 = \left[\delta_n(x) - \tilde{\xi}_n(x) \right] / 2 , \quad (43)$$

and, after a small calculation using Eqs. (20) and (27),

$$\theta_n(x) = - \sum_{q=1}^{\infty} \sum_{k=1}^{2n-1} d_{(4q-3)n+k} \sin(n-k)x \quad (44)$$

From Eqs. (38) and (42) it follows that the validity of the inequality

$$\left| \xi_{2n}(x) \right| \leq \left| \theta_n(x) \right| \quad (45)$$

implies the validity of the inequality

$$w_n^*(x) \leq h(x) \leq v_n^*(x) \quad (46)$$

(or both inequalities reversed). Equation (46) may also be written

$$\left| h(x) - v_{2n}(x) \right| \leq \left| \left[\tilde{v}_n(x) - w_n(x) \right] / 2 \right| . \quad (47)$$

The odd case is thus quite similar to the even case.

3. NUMERICAL EXAMPLES, APPLICATIONS

We first give two simple examples illustrating the enclosing properties of the modified trigonometric approximations.

Example 1:

$$g(x) = \left| \sin(x/2) \right| , \quad x \in [-\pi, \pi] , \quad (48)$$

i.e. an even function with the exact Fourier coefficients

$$a_k = - \frac{1}{\pi} \frac{4}{4k^2 - 1} \quad (49)$$

Due to discontinuity in the first order derivative, we have rather slow convergence. In Figs. 1 and 2 we have plotted the errors $\varepsilon_n^*(x)$, $\eta_n^*(x)$ and $\varepsilon_{2n}(x)$ for $n = 4$ and $n = 8$. Even in this case of slow convergence the inequality (36) is valid for the major part of the x -range.

Example 2:

$$f(x) = (1 + \sin x) / \{1 + \cos^2 (x/2)\} , \quad x \in [-\pi, \pi] , \quad (50)$$

with the exact Fourier coefficients

$$a_k = (-1)^k \sqrt{2} \gamma^k , \quad d_k = (a_{k-1} - a_{k+1})/2 , \quad (51)$$

with

$$\gamma = 3 - 2\sqrt{2} .$$

Figure 3 gives the errors $\varepsilon_n^*(x) + \xi_n^*(x)$, $\eta_n^*(x) + \delta_n^*(x)$ and $\varepsilon_{2n}(x) + \xi_{2n}(x)$ for $n = 4$ (i.e. the range $[-\pi, \pi]$ is divided into eight equal sub-intervals). The fast convergence is also expressed by the strong symmetry of the functions $\varepsilon_n^*(x) + \xi_n^*(x)$ and $\eta_n^*(x) + \delta_n^*(x)$. Within the accuracy of the plot we are not able to observe those regions where these functions have the same sign.

For both examples the bounds for the Fourier coefficients as expressed by Eq. (23) (and the similar relation for the odd coefficients) are valid. However, in both examples the absolute values of the coefficients form monotonically decreasing sequences for increasing order of the coefficients. If the absolute values of the coefficients are increasing up to a certain order, then using the computational scheme indicated by Eqs. (6) and (17), we would expect the end- and mid-point approximations to have partly opposite sign -- at least until a first approximation is obtained for the coefficient having the largest absolute value.

In the field of numerical integration one application has already been given by the author⁶⁾. Using this method we compute an integral over the range $[0, \pi]$ with the integrand $f(\cos \phi) \sin \phi$, approximating $f(\cos \phi)$ with cosine expansions of the type discussed in this paper.

We will also indicate some other possibilities. However, more research has to be done in order to determine the most useful methods.

If $f(x)$ is given in the interval $[0, \pi]$ and the integral over this range is wanted, then we may continue $f(x)$ as an odd function in the interval $[-\pi, 0]$ and

FIG. 2

FIG. 3