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INCLUSIVE CURRENT INDUCED REACTIONS AND LIGHT
CONE SINGULARITIES ^{*)}

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A B S T R A C T

We analyze inclusive reactions of the type

$$\text{current} + t \rightarrow p + \text{anything}$$

where t denotes either the vacuum or a hadronic target and p a produced hadron state. Conditions in momentum space are derived which enforce the scaling behaviour due to a canonical leading light cone singularity structure.

The multiplicity for the production of p off a hadron target grow less fast than the centre-of-mass energy. The problem of continuation of deep inelastic scattering amplitudes to the annihilation channel is investigated. Finally we outline how hadronic scaling is obtained in the limit of finite current mass.

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We would like to dedicate this work to
the memory of

Bruno Renner and Ciaran Ryan

who tragically lost their lives in the
mountains of the Haute Savoie.

1. INTRODUCTION

In this paper we consider inclusive reactions of the type :

$$\begin{aligned} \text{current} &\rightarrow \text{hadron} + \text{anything, and} \\ \text{current} + \text{hadron} &\rightarrow \text{hadron} + \text{anything} \end{aligned} \quad 1)$$

The simplest process is e^+e^- annihilation, which is considered in more detail in Section 2. The scaling behaviour as conjectured in Ref. 2) is found to be the consequence of a canonical singularity structure on the light cone. The matrix element for the inclusive process is not local in co-ordinate space. The spectrum condition, on the other hand, is more stringent in the annihilation process than in deep inelastic scattering. This implies a light cone expansion in terms of the basic singularity $1/(x^2 - i\epsilon x_0)$ only. At the end of Section 2 we comment on the problem of continuing the electro-production scaling functions to the annihilation region 2). Even if the light cone dominates in the annihilation region, we find no evidence for any relation between the two sets of scaling functions.

In Section 3 we generalize the light cone algebra of current commutators 3). The inclusive reactions are determined by matrix elements like

$$\sum_n \left\{ \langle t | \widehat{F}_{i\mu}(x) | p, n \rangle \langle n, p | \widehat{F}_{j\nu}(y) | t \rangle \right\} \quad (1.1)$$

where t and p are arbitrary hadron states. The algebra of current commutators gives no direct information about matrix elements like (1.1). Nevertheless, one may abstract from field theoretic models like the free quark model or the formal quark gluon interaction model simple algebraic properties of matrix elements like (1.1). These abstractions are described in Section 3. We assume that our relations are right for the real world of hadrons, even if there are no quarks in the physical spectrum. It has to be noted that these abstractions are much more restrictive than the light cone algebra of current commutators. They might be wrong even if the latter are right. To prove or disprove the compatibility of the above abstraction with the absence of physical quarks amounts to the formidable task of constructing a non-trivial field theory model.

Our assumptions lead to the existence of operators $\phi_{k\sigma}(x, y, p)$ depending on two space-time co-ordinates x and y with $(x-y)^2 = 0$ and on the hadron state $|p\rangle$ produced in the inclusive reaction. The operators $\phi_{k\sigma}$ are generalizations of bilocal current operators

$$\phi_{k\sigma}(x, y, p) \leftrightarrow \tilde{F}_{k\sigma}(x, y) \quad (1.2)$$

The presence of the state $|p\rangle$ greatly influences the properties of these operators. In particular, they might be singular in the limit $x \rightarrow y$, where the bilocal currents by definition tend to local current operators.

In Section 4 we consider current annihilation processes. They are described by vacuum expectation values of the operators $\phi_{k\sigma}$. We obtain symmetry relations and inequalities for cross-sections similar to corresponding relations derived for deep inelastic scattering. The spectral conditions enforce a non-local leading light cone singularity of the type $(1/(x^2 - i\epsilon x_0))\phi_{k\sigma}(x, y, p)$. This non-local singularity allows for an arbitrary growth of the corresponding scaling functions with $\xi = (q^2/2\nu)$ for $\xi \rightarrow +\infty$ *). The spectrum condition yields the following restrictions on

$$\begin{aligned} \phi_{k\sigma}(x, 0, p) &= \tilde{F}_k(xp) p_\sigma + \dots \\ \tilde{F}_k(xp) &= \int d\alpha e^{-i\alpha(p \cdot x)} F(\alpha) \\ F(\alpha) &= 0 \quad \text{for } \alpha < 1 \end{aligned} \quad (1.3)$$

In Section 5 we apply the same technique to inclusive current induced reactions, which are described by one particle matrix elements of the operators $\phi_{k\sigma}$. We discuss a characteristic set of symmetry relations and inequalities for the structure functions following from our algebraic abstractions. We further outline a natural connection with hadronic scaling generalizing the exchange of scaling and Regge limits in deep inelastic scattering. The multiplicities increase logarithmically, if the scaling functions go to a finite limit at $x \rightarrow 0$. There is no indication that one should obtain $SU(3)$ symmetric scaling functions and scaling multiplicities. The symmetry of the currents is not the symmetry of the states.

*) This controversial point was cleared up after a prolonged discussion with Y. Frishman and J. Ellis [Caltech preprint, CALT 68-377 (1973)]; see also the corresponding conclusion drawn by Callan and Gross ¹⁾.

2. KINEMATICS, CONNECTION TO DEEP INELASTIC SCATTERING AND LIGHT-CONE ANALYSIS

First we consider the kinematics of the process

$$\text{current} \rightarrow \text{hadron} + \text{anything} \quad . \quad (2.1)$$

In particular, we concentrate on the reaction

$$e^+ + e^- \rightarrow \text{hadron} + \text{anything} \quad (2.2)$$

in the one-photon approximation. This process is related via crossing to virtual photon-hadron scattering

$$e^- + \text{hadron} \rightarrow \text{anything} \quad . \quad (2.3)$$

The latter is usually described by the structure functions W_1, W_2 or V_1, V_2 defined as follows ^{4),5)}.

The current correlation function is given by

$$W_{\mu\nu}(P, q) = \frac{1}{4\pi} \int dx e^{iqx} \langle P | [\bar{j}_\mu(x), j_\nu(0)] | P \rangle \quad (2.4)$$

where the connected matrix element of the current commutator is taken. We define the structure functions by the following decomposition

$$W_{\mu\nu} = \left(g_{\mu\nu} q^2 - g_{\mu\nu} q^2 \right) V_1(q^2, \nu) + \left(P_\mu P_\nu q^2 - \nu (P_\mu q_\nu + P_\nu q_\mu) + g_{\mu\nu} \nu^2 \right) \cdot V_2(q^2, \nu) \quad (2.5)$$

where $\nu = p \cdot q$. The functions V_1 and V_2 are related to the structure functions W_1 and W_2 by

$$\begin{aligned} V_1(q^2, \nu) &= \frac{1}{q^2} \left\{ W_1(q^2, \nu) + \frac{\nu^2}{q^2} W_2(q^2, \nu) \right\} \\ V_2(q^2, \nu) &= \frac{1}{q^2} W_2(q^2, \nu) \end{aligned} \quad (2.6)$$

The usual scaling assumptions in the deep inelastic region $q^2 \rightarrow -\infty, \nu \rightarrow \infty, -q^2/2\nu = \xi$ fixed, are

$$V_1 \xrightarrow[\xi]{\frac{1}{2\nu}} \frac{F_2(\xi)}{\xi} \quad V_2 \rightarrow - \frac{1}{2\nu^2} \frac{F_2(\xi)}{\xi} \quad (2.7)$$

or

$$W_1 \rightarrow F_1(\xi) \quad W_2 \rightarrow \frac{1}{\nu} F_2(\xi), \quad F_e(\xi) = -F_1(\xi) + \frac{F_2(\xi)}{2\xi}$$

Note also the following crossing and support properties of the scaling functions :

$$F_e(\xi) = -F_e(-\xi) \quad F_2(\xi) = F_2(-\xi)$$

$$F_e(\xi) = F_2(\xi) = 0 \quad \text{for } |\xi| > 1$$

The annihilation process (2.2) is described by the matrix element ²⁾

$$\begin{aligned} \bar{W}_{\mu\nu}(p, q) &= \frac{1}{4\pi} (2\pi)^4 \sum_n \delta^4(p+n-q) \langle 0 | j_\mu(0) | p, n \rangle \times \\ &\quad \times \langle n, p | j_\nu(0) | 0 \rangle = \\ &= \frac{1}{4\pi} \int dx e^{i \cdot q \cdot x} \sum_n \langle 0 | j_\mu(x) | p, n \rangle \langle n, p | j_\nu(0) | 0 \rangle = (2.8) \\ &= (q_\mu q_\nu - g_{\mu\nu} q^2) \bar{V}_1(q^2, \nu) + \left(p_\mu p_\nu q^2 - \nu (p_\mu q_\nu + p_\nu q_\mu) + g_{\mu\nu} \nu^2 \right) \times \\ &\quad \times \bar{V}_2(q^2, \nu) \end{aligned}$$

where $|p\rangle$ is the one-particle state selected with four-momentum p_μ . The process is described by two structure functions $\bar{V}_1$ and $\bar{V}_2$ related to the previously defined functions V_1 and V_2 by crossing. It was first conjectured in Ref. 2) that the crossed functions also behave scale invariantly in the limit $q^2 \rightarrow \infty, \nu \rightarrow \infty, q^2/2\nu = \xi$ fixed

$$\bar{V}_1 \xrightarrow[\text{by}]{} \frac{1}{2\nu} \frac{\bar{F}_e(\xi)}{\xi} \quad \bar{V}_2 \xrightarrow[\text{by}]{} \frac{1}{2\nu^2} \frac{\bar{F}_2(\xi)}{\xi}$$

(2.9)

$$\bar{F}_1, -\bar{F}_2 > 0 \quad \text{for } \xi > 1$$

$$\bar{F}_e(\xi) = \bar{F}_2(\xi) = 0 \quad \text{for } \xi < 1 \quad \text{support property}$$

The support property of the scaling functions is a consequence of the spectrum condition. It was also conjectured by the same authors ²⁾ that the scaling functions $\bar{F}_1$ (or $\bar{F}_L$) and $\bar{F}_2$ are simply related to the corresponding unbarred functions by analytic continuation. This is essentially a consequence of crossing symmetry in the particular parton model of Ref. 2). We shall indicate later in this section that this is not just a consequence of crossing symmetry alone, as one might suppose, but a very special assumption. In particular, the scaling itself as described by Eq. (2.9) might break down in the annihilation case but still work in the case of deep inelastic scattering without violating any principles of relativistic field theory as crossing symmetry. Before going into these details, we should like to demonstrate the connection between the scaling hypothesis (2.9) and the light-cone behaviour of the matrix element

$$\begin{aligned} \bar{W}_{\mu\nu}(x, P) &= \frac{2}{(2\pi)^3} \int d^4q e^{-iqx} \bar{W}_{\mu\nu}(q, P) \\ &= \sum_n \left\{ \left(\langle 0 | J_\mu(x) | P, n \rangle \langle n, P | J_\nu(0) | 0 \rangle \right) \right\} \end{aligned} \quad (2.10)$$

In electroproduction, the scaling in the deep inelastic region may be described by the following behaviour of the functions $V_1(x^2, px)$ and $V_2(x^2, px)$ [which are just the Fourier transforms of the functions V_1 and V_2 as defined in (2.5)] near the light-cone ^{5), 6)}:

$$\begin{aligned} V_1(x^2, px) &\xrightarrow{x^2 \rightarrow 0} \frac{1}{2\pi i} \varepsilon(x^0) \delta(x^2) F_1(px) \\ V_2(x^2, px) &\xrightarrow{x^2 \rightarrow 0} \frac{1}{2\pi i} \varepsilon(x^0) \theta(x^2) F_2(px) \end{aligned} \quad (2.11)$$

where the functions $F(px)$ and $F(\frac{x}{2})$ are related by a simple Fourier transformation. One may suppose that a similar behaviour near the light cone also holds in the annihilation case for the corresponding functions $\bar{V}_1(x^2, px)$ and $\bar{V}_2(x^2, px)$, giving the scaling behaviour as in (2.9) ⁷⁾. Such a simple light cone analysis is not without troubles. One problem one is facing here is that the matrix element (2.10) is not local and therefore does not have to vanish at space-like distances, in contrast to the case of electroproduction. As a consequence, the structure functions in co-ordinate space $\bar{V}_1(x^2, px)$ and $\bar{V}_2(x^2, px)$ do not have to vanish at space-like x . Following the light cone assumption of (2.11), we may suppose the following behaviour of the structure functions near the light cone:

$$\begin{aligned} \bar{V}_1(x^2, p_x, \varepsilon(x)) &\xrightarrow{x^2 \rightarrow 0} \frac{1}{2\pi i} \left\{ \varepsilon(x^0) \delta(x^2) \bar{F}_1'(p_x) + \frac{1}{x^2} \bar{F}_1''(p_x) \right\} \\ \bar{V}_2(x^2, p_x, \dots) &\xrightarrow{x^2 \rightarrow 0} \frac{1}{2\pi i} \left\{ \varepsilon(x^0) \delta(x^2) \bar{F}_2'(p_x) + \log|x^2| \bar{F}_2''(p_x) \right\} \end{aligned} \quad (2.12)$$

Instead of the simple singularities of the types $\varepsilon(x) \delta(x^2)$ or $\varepsilon(x) \theta(x^2)$ as in (2.11) dictated by scaling and locality all singular functions of the same dimension are allowed in (2.12). Due to this more complicated structure, the light cone analysis of the annihilation process becomes more involved. Another complication also connected with the lack of locality in the annihilation process is the following :

$$\bar{V}_1(x^2, p_x) \rightarrow - \frac{1}{4\pi^2} \frac{1}{x^2 - i\varepsilon x^0} \bar{F}_e(p_x) + \frac{1}{4\pi^2} \frac{1}{x^2 + i\varepsilon x^0} \bar{G}_e(p_x) \quad (2.13)$$

In momentum space we have

$$\begin{aligned} \bar{V}_1(q^2, \nu) \rightarrow \frac{1}{16\pi^3} \int dx e^{i(q-\alpha p)x} \left\{ - \frac{1}{x^2 - i\varepsilon x^0} \tilde{F}_e(\alpha) + \right. \\ \left. + \frac{1}{x^2 + i\varepsilon x^0} \tilde{G}_e(\alpha) \right\} \end{aligned} \quad (2.14)$$

where the functions $\tilde{F}(\alpha)$, $\tilde{G}(\alpha)$ are defined by

$$\begin{pmatrix} \bar{F}_e(p_x) \\ \bar{G}_e(p_x) \end{pmatrix} = \int d\alpha e^{-i\alpha/p_x} \begin{pmatrix} \tilde{F}_e(\alpha) \\ \tilde{G}_e(\alpha) \end{pmatrix} \quad (2.15)$$

Noting the results of the Fourier transformation

$$- \frac{1}{(2\pi)^3} \frac{1}{x^2 \mp i\varepsilon x^0} \xrightarrow{\mathcal{F}} \nu(\pm q^0) \delta(q^2)$$

we obtain

$$\bar{V}_1(q^2, \nu) \rightarrow \frac{1}{2} \int d\alpha \left\{ \begin{aligned} & \mathcal{V}(\nu - \alpha p^2) \delta(q^2 - 2\alpha\nu + \alpha^2 p^2) \tilde{F}_e(\alpha) \\ & - \mathcal{V}(-\nu + \alpha p^2) \delta(q^2 - 2\alpha\nu + \alpha^2 p^2) \tilde{G}_e(\alpha) \end{aligned} \right\} \quad (2.16)$$

It is instructive to compare the above asymptotic expression with the analogous one for deep inelastic scattering

$$V_1(q^2, \nu) \rightarrow \frac{1}{2} \int d\alpha \, \varepsilon(\nu - \alpha p^2) \delta(q^2 - 2\alpha\nu + \alpha^2 p^2) \times \hat{F}_e(\alpha) \quad (2.17)$$

with

$$\begin{aligned} F_e(p \cdot x) &= \int d\alpha \, e^{-i\alpha(p \cdot x)} \hat{F}_e(\alpha) \\ \hat{F}_e(\alpha) &= \hat{F}_e(-\alpha) \end{aligned} \quad (2.18)$$

The argument of the δ function in (2.16) and (2.17) has two roots which, in the Bjorken limit, are given by

$$\alpha_1 = \frac{q^2}{2\nu}, \quad \alpha_2 = \frac{2\nu}{p^2} \quad (2.19)$$

The step functions in (2.16) and (2.17) reach the limits

$$\begin{aligned} \mathcal{V}(\pm \nu \mp \alpha_1 p^2) &\rightarrow \mathcal{V}(\pm \nu), \quad \varepsilon(\nu - \alpha_1 p^2) \rightarrow \varepsilon(\nu) \\ \mathcal{V}(\pm \nu \mp \alpha_2 p^2) &\rightarrow \mathcal{V}(\mp \nu), \quad \varepsilon(\nu - \alpha_2 p^2) \rightarrow -\varepsilon(\nu) \end{aligned}$$

Then one obtains

$$4\nu/V_1(q^2, \nu) \rightarrow \mathcal{F}(\nu) \left\{ \hat{F}_e\left(\frac{q^2}{2\nu}\right) - \hat{G}_e\left(\frac{2\nu}{p^2}\right) \right\} - \mathcal{F}(-\nu) \left\{ \hat{G}_e\left(\frac{q^2}{2\nu}\right) - \hat{F}_e\left(\frac{2\nu}{p^2}\right) \right\} \quad (2.20)$$

$$4\nu/V_1(q^2, \nu) \rightarrow \varepsilon(\nu) \left\{ \hat{F}_e\left(\frac{q^2}{2\nu}\right) - \hat{F}_e\left(\frac{2\nu}{p^2}\right) \right\}$$

Light cone dominance demands that in the relevant limits ($\nu \rightarrow \pm\infty$) the functions depending on α_2 vanish ^{*})

$$\lim_{\nu \rightarrow +\infty} \hat{G}_e\left(\frac{2\nu}{p^2}\right) = \lim_{\nu \rightarrow -\infty} \tilde{F}_e\left(\frac{2\nu}{p^2}\right) = \lim_{\nu \rightarrow \pm\infty} \hat{F}_e\left(\frac{2\nu}{p^2}\right) = 0 \quad (2.21)$$

We note that no asymptotic condition for $\tilde{F}(\xi)$ for $\xi \rightarrow +\infty$ results. The spectral conditions are :

$$\bar{V}_1 = 0 \quad \text{for } q \notin V_+ = \{q / q_0 \geq 0, q^2 \geq 0\} \quad \text{and } q-p \notin V_+ \quad (2.22)$$

$$V_1 = 0 \quad \text{for } q+p \notin V_+ \text{ or } -q+p \notin V_+$$

^{*}) A non-leading singularity can contribute competing constants if the limits in (2.21) are finite but different from zero. See also the results obtained by Y. Frishman and J. Ellis [Caltech preprint, CALT 68-377 (1973)].

Therefore we have in the scaling limit

$$\begin{aligned} \tilde{G}_e(\xi) \equiv 0, \quad \tilde{F}_e(\xi) = 0 \quad \xi < 1 \\ \hat{F}_e(\xi) = 0 \quad |\xi| > 1 \end{aligned} \quad (2.23)$$

$$\xi = q^2/2\nu$$

Equation (2.23) implies that $\tilde{F}_e(px)$ is the boundary value of an analytic function in the half-plane

$$\text{Im}(px) < 0$$

whereas $F_e(px)$ is an entire function (of exponential type 1).

An analogous discussion can be carried out for $\bar{V}_2(x^2, px)$ in terms of the singular functions $\log(x^2 - i\varepsilon x^0)$, $\log(x^2 + i\varepsilon x^0)$:

$$\begin{aligned} \bar{V}_2(x^2, px) \rightarrow -\frac{1}{4\pi^2} \log(x^2 - i\varepsilon x^0) \bar{F}_4(px) \\ + \frac{1}{4\pi^2} \log(x^2 + i\varepsilon x^0) \bar{G}_4(px) \end{aligned} \quad (2.24)$$

$$\begin{aligned} V_2(x^2, px) \rightarrow -\frac{1}{4\pi^2} [\log(x^2 - i\varepsilon x^0) - \log(x^2 + i\varepsilon x^0)] F_4(px) \\ F_4(px) = F_4(-px) \end{aligned}$$

In Eq. (2.24) the arguments of $x^2 \mp i\varepsilon x^0$ have to be chosen in the following way:

$$\arg(x^2 \mp i\varepsilon x^0) = \begin{cases} \pm \pi & x^0 > 0, x^2 > 0 \\ 0 & x^2 < 0 \\ \mp \pi & x^0 < 0, x^2 > 0 \end{cases} \quad (2.25)$$

The spectrum conditions again imply

$$\begin{aligned} \bar{G}_t(p, x) &\equiv 0, & \hat{F}_t(\xi) &= 0 & \xi < 1 \\ \hat{F}_t(\xi) &= 0 & |\xi| &> 1 \end{aligned} \quad (2.26)$$

It is useful to compare the light cone singularities for the annihilation process with the corresponding ones for electroproduction, replacing the current commutator by the current product in describing the latter :

$$\begin{aligned} W_{\mu\nu}^{(P)} &= \frac{1}{4\pi} \int d^4x e^{+i q \cdot x} \langle P | J_\mu(x) J_\nu(0) | P \rangle \\ &= (g_\mu g_\nu - g_{\mu\nu} q^2) V_1^{(P)}(q^2, \nu) \\ &\quad + (\bar{P}_\mu \bar{P}_\nu q^2 - \nu (\bar{P}_\mu q_\nu + \bar{P}_\nu q_\mu) + g_{\mu\nu} \nu^2) V_2^{(P)}(q^2, \nu) \end{aligned} \quad (2.27)$$

gathering the results of Eqs. (2.13), (2.23), (2.24) and (2.26), one obtains :

$$\begin{aligned} V_1^{(P)}(x^2, p, x) &\longrightarrow - \frac{1}{4\pi^2} \frac{1}{x^2 - i\varepsilon x^0} \bar{F}_e(p, x) \\ \bar{V}_1(x^2, p, x) &\longrightarrow - \frac{1}{4\pi^2} \frac{1}{x^2 - i\varepsilon x^0} \bar{\bar{F}}_e(p, x) \\ V_2^{(P)}(x^2, p, x) &\longrightarrow - \frac{1}{4\pi^2} \log(x^2 - i\varepsilon x^0) \bar{F}_t(p, x) \\ \bar{V}_2(x^2, p, x) &\longrightarrow - \frac{1}{4\pi^2} \log(x^2 - i\varepsilon x^0) \bar{\bar{F}}_t(p, x) \end{aligned} \quad (2.28)$$

Locality implies

$$\bar{F}_{e,t}(p_x) = \bar{F}_{e,t}(-p_x) \quad (2.29)$$

but no symmetry relations for $\bar{F}_{l,t}(p_x)$ hold.

The crossing operation which connects

$$V_i(q^2, \nu) \text{ to } \bar{V}_i(q^2, \nu)$$

does not affect the form of the leading light cone singularity, but just transforms the corresponding matrix elements of the bilocal quantity for electroproduction into an associated bilinear (not bilocal) quantity for the annihilation process. In the annihilation case the function $\bar{F}(p_x)$ is not an expectation value of a bilocal operator, it might therefore be singular at $p_x \rightarrow 0$. In particular, the behaviour

$$\bar{F}(\xi) \xrightarrow{\xi \rightarrow \infty} \text{const. } \xi^{-\delta} \quad \delta > 0$$

would give

$$\bar{F}_i(p_x) \xrightarrow{p_x \rightarrow 0} \text{const. } (p_x - i\varepsilon)^{-\delta-1} \quad (2.30)$$

Finally we should like to comment on the conjecture that the scaling functions $F(\xi)$ and $\bar{F}(\xi)$ are pieces of one analytic function²⁾. We first consider the current correlation function (2.4) at time-like q . Here one has to take into account besides the ordinary diagram, Fig. 1^{*)}

*) We should like to mention that the support properties of the structure functions $\bar{V}(q^2, \nu)$ at $q^2 > 0$ are quite different from the ones at space-like q^2 . The physical region is delimited by : $q^2, \nu > 0$

$$q^2 - 2\nu \geq (m^*)^2 - p^2$$

m^* : smallest mass of a state $|X\rangle$ produced together with $|p\rangle$ in the reaction $e^+e^- \rightarrow p+X$

$$\nu^2 \geq p^2 \cdot q^2$$

Fig.1

the additional diagrams, Figs. 2, 3 and 4

Fig.2

Fig.3

Fig.4

The matrix element (2.4) is a sum of terms described by the various diagrams, Figs. 1-4. The Z diagram of Fig. 2 in particular describes the annihilation process, but the information one has from the matrix element (2.4) is not sufficient to determine the annihilation structure functions $\bar{V}(q^2, \nu)$, since one cannot separate the term described by Fig. 2 from the sum. But the light cone structure, as assumed in (2.11), and the fact that the scaling functions $F(\xi)$ vanish for $|\xi| > 1$ give information about the behaviour of the structure functions $V(q^2, \nu)$ in the deep inelastic limit at positive q^2 . In particular these functions have to scale to 0 at $|\xi| > 1$:

$$\left. \begin{array}{l} \nu q^2 V_2(q^2, \nu) \\ q^2 V_1(q^2, \nu) \end{array} \right\} \xrightarrow{b_j} 0 \quad \left| \frac{q^2}{2\nu} \right| = |\xi| > 1 \quad (2.31)$$

Since the annihilation process is partly determined by one of the diagrams describing the current correlation function (2.4) at positive q^2 (Fig.2), e.g.,

$$V(q^2, \nu) = \bar{V}(q^2, \nu) + \text{other terms}$$

one has to assume that in the remaining terms described by the diagrams, Figs. 1, 3 and 4, there are pieces which in the deep inelastic limit also give contributions at $|\xi| > 1$ and cancel the corresponding ones of $\bar{V}(q^2, \nu)$ in order to ensure (2.31). This non-trivial information about the behaviour of the structure functions $V(q^2, \nu)$ at time-like q^2 may be used together with dispersion relations in order to obtain information about the annihilation scaling functions $\bar{F}(\xi)$. In the Appendix we demonstrate how this could be done. The main conclusions are as follows.

If we assume the existence of dispersion relations in q^2 and ν for the virtual Compton amplitude and also that we are allowed to interchange scaling limits and dispersion integrations, we arrive at the following integral representation for $\bar{F}_2(\xi)$:

$$\bar{F}_2(\xi) = - \frac{1}{4\pi} \int_0^\infty dz \frac{\tau(z) \cdot z}{(1-\xi)z + \xi} \quad (2.32)$$

$$0 \leq \xi \leq 1$$

where the function $\tau(z)$ is defined in the Appendix, (A.20). The annihilation scaling function $\bar{F}_2(\xi)$ is given by the following representation:

$$\bar{F}_2(\xi) = \frac{1}{2} (\phi(\xi) - \phi(-\xi)) \quad (2.33)$$

$$\phi(\xi) = \frac{1}{4\pi} \int_0^{1/2} dz \frac{\tau(z) \cdot z}{(1-\xi)z + \xi} \quad \xi \geq 1$$

Corresponding representations exist for F_1 and $\bar{F}_1$. Since the integration range in (2.32) is different from the one in (2.33) one does not find any trace of the conjecture in Ref. 2) that $F(\xi)$ and $\bar{F}(\xi)$ are different pieces of one analytic function ^{*}). In particular it might be that the function $\mathcal{T}(z)$ vanishes in the interval $0 \leq z \leq \frac{1}{2}$, which would mean that the annihilation structure function $\bar{V}_2(q^2, \nu)$ does not scale according to (2.9). We cannot conclude from a general light cone and dispersion analysis any connection between scaling and scaling functions in electroproduction and e^+e^- annihilation. We note that a characteristic term like

$$F_2(\xi) \equiv \bar{F}_2(\xi) = \text{const.} (1-\xi)^3$$

which would yield a logarithmic increase of multiplicities for particles produced in the annihilation process and does describe the behaviour of W_2 in deep inelastic scattering fairly well, constitutes a subtraction polynomial to the representations in Eqs. (2.32) and (2.33).

3. LIGHT CONE RELATIONS ABSTRACTED FROM A QUARK PICTURE

The scaling behaviour observed for the electroproduction structure functions

$$\begin{aligned} V_1(q^2, \nu) &\xrightarrow{b_j} 0 \quad (\text{large errors}) \\ V_2(q^2, \nu) &\xrightarrow{b_j} -\frac{1}{2\nu^2} \frac{F_2(\xi)}{\xi} \end{aligned} \quad (3.1)$$

has given support for a simple light cone algebra of the observed electromagnetic and weak currents ³⁾. The idea is to abstract from a field theoretic quark model algebraic relations for the leading light cone singularity of a current commutator. Such relations are supposed to be true in the real world of hadrons, even if quarks do not exist as real particles. The light cone relations, which one might abstract from a renormalizable quark-gluon field theory model by considering the perturbation expansion term by term, contain powers of unwanted logarithmic factors which destroy the scaling behaviour in the deep inelastic region. Therefore one abstracts the light cone algebra

^{*}) Such a connection exists in specific models [see, Ref. 2), and P.V. Landshoff, J.C. Polkinghorne, DAMPT preprint 72/16 (1972); P. Menotti, Talk presented at the Informal Meeting on Electromagnetic Interactions, Frascati, May (1972)]. There one assumes that only certain diagrams survive in the scaling limit. Such models violate our assumptions about dispersion relations, made in the Appendix.

of currents either from a free quark model or from a formal interacting quark-gluon theory neglecting the logarithmic terms and also unwanted additional non-canonical singularities ("anomalies") coming from renormalization ⁸⁾.

The simplest abstraction one can make is the algebraic structure of the leading light cone singularity of a current commutator, for example

$$\begin{aligned} [\widehat{F}_{i\mu}(x), \widehat{F}_{j\nu}(y)] &\cong [\widehat{F}_{i\mu}^{(5)}(x), \widehat{F}_{j\nu}^{(5)}(y)] \cong \\ &\frac{1}{4\pi} \partial^\rho (\varepsilon_{\rho\sigma} \delta(z^2)) \left\{ i f_{ijk} \left[(\widehat{F}_{k\sigma}(x,y) + \widehat{F}_{k\sigma}(y,x)) S_{\mu\nu\rho\sigma} \right. \right. \\ &\quad \left. \left. - i \varepsilon_{\mu\nu\rho\sigma} (\widehat{F}_{k\sigma}^{(5)}(x,y) - \widehat{F}_{k\sigma}^{(5)}(y,x)) \right] + \right. \\ &\quad \left. + d f_{ijk} \left[S_{\mu\nu\rho\sigma} (\widehat{F}_{k\sigma}(x,y) - \widehat{F}_{k\sigma}(y,x)) - \right. \right. \\ &\quad \left. \left. - i \varepsilon_{\mu\nu\rho\sigma} (\widehat{F}_{k\sigma}^{(5)}(x,y) + \widehat{F}_{k\sigma}^{(5)}(y,x)) \right] \right\} \end{aligned} \quad (3.2)$$

$$S_{\mu\nu\rho\sigma} = g_{\mu\rho} g_{\nu\sigma} + g_{\mu\sigma} g_{\nu\rho} - g_{\mu\nu} g_{\rho\sigma}, \quad z = x - y$$

The bilocal current operators $\widehat{F}_{k\sigma}^{(5)}(x,y)$ as defined by (3.2) reduce to local currents in the limit $x \rightarrow y$. Their expectation values determine the corresponding scaling functions in the deep inelastic region. Because of the spectrum condition and locality, these expectation values are entire analytic functions in pz (of exponential type 1).

Let us now consider matrix elements of the following kind :

$$\begin{aligned} \overline{W}_{\mu\nu}^{ij}(p, x) &= \\ \sum_n \left(\langle 0 | \widehat{F}_{i\mu}(x) | p, n \rangle \langle n, p | \widehat{F}_{j\nu}(y) | 0 \rangle \right) \end{aligned} \quad (3.3)$$

or, more generally,

$$\begin{aligned} \overline{W}_{\mu\nu}^{ij}(p, x, t) &= \\ \sum_n \left(\langle t | \widehat{F}_{i\mu}(x) | p, n \rangle \langle n, p | \widehat{F}_{j\nu}(y) | t \rangle \right) \end{aligned} \quad (3.4)$$

where $|p\rangle$ and $|t\rangle$ are definite hadron states (for example, one-particle states) with momenta p_μ and t_μ . In the special case of the electromagnetic current, the matrix element (3.3) is related to the inclusive process

$$e^+ + e^- \rightarrow p + \text{anything}$$

In general, one might consider the operator

$$O_{\mu\nu}^{ij}(x, p) = \sum_n \left\{ \begin{aligned} & \widehat{f}_{i\mu}(x) / p, n \rangle \langle n, p / \widehat{f}_{j\nu}(y) - \\ & - \widehat{f}_{j\nu}(y) / p, n \rangle \langle n, p / \widehat{f}_{i\mu}(x) \end{aligned} \right\} \quad (3.5)$$

The question now arises if we can develop a similar abstraction framework for operators like (3.5) as the light cone algebra for ordinary current commutators. Before we consider operators like (3.5) in general, we return to the vacuum matrix element (3.3) and treat the special case when the particle detected is a vector meson or a pseudoscalar meson. Then the matrix element (3.3) is determined by a four-current function as

$$\langle 0 / T(\widehat{f}_{i\mu}(x) j(z)) \cdot T^*(\widehat{f}_{j\nu}(y) j(t)) / 0 \rangle \quad (3.6)$$

where $j(z)$ denotes a current or the divergence of an axial current, respectively ^{*}. In particular, one has :

$$W_{\mu\nu}^{ij}(x, p) = \int d^4z d^4t e^{ip(t-z)} \cdot \left\{ \begin{aligned} & (\square_z + p^2)(\square_t + p^2) \langle 0 / T(\widehat{f}_{i\mu}(x) j(z)) T^*(\widehat{f}_{j\nu}(y) j(t)) / 0 \rangle \\ & - \{ \mu \leftrightarrow \nu, i \leftrightarrow j, x \leftrightarrow y \} \end{aligned} \right\} \quad (3.7)$$

Note that we can, by using a field theoretic model, also treat baryons the same way, if we replace the current $j(z)$ by an interpolating field with the quantum number of the corresponding baryon, e.g., $q(z)q(z)q(z)$. The choice of this interpolating field is somehow ambiguous, but the matrix element (3.7) is independent of any ambiguities since the baryon is on its mass shell.

We are interested in the behaviour of the function (3.6) in the vicinity of the light cone $(x-y)^2 \rightarrow 0$. All other distances involved : $(x-z)$, $(x-t)$, $(y-z)$, $(y-t)$, $(z-t)$ are not constrained since we later integrate over the mass-shell as denoted by (3.7). Note that in (3.6), one is not testing a current product near the light cone since the T product in (3.6) contains terms in which the two currents, whose separation becomes light-like, are not next to each other, e.g., there are pieces like

$$\langle 0 / \widehat{f}_{i\mu}(x) j(z) j(t) \widehat{f}_{j\nu}(y) / 0 \rangle \quad (3.8)$$

*)

One may also consider mesons of higher spin. In this case one uses instead of $j(z)$ one of the local "generalized currents" appearing in the Taylor expansion of a bilocal current.

Such a function is supposed to be known and identical to the one obtained from a free quark model with quark statistics if all separations are light-like (tend to a light-like line)⁹⁾. This is not the case here, since the distances involving z and t are still arbitrary. The question now arises: is it possible to obtain information about vacuum expectation values like (3.7) or, more generally, about current products like

$$\hat{T}_{\mu}(x) j(z) j(t) \hat{T}_{\nu}(y) \quad (3.9)$$

where $(x-y)^2 \rightarrow 0$, but all other distances are arbitrary? We note that a Wightman function of currents like (3.7) is a boundary value of an analytic function in co-ordinate space. In particular the function (3.8) and the permuted function

$$\langle 0 | j(z) \hat{T}_{\mu}(x) \hat{T}_{\nu}(y) j(t) | 0 \rangle \quad (3.10)$$

are connected by analytic continuation, according to the theorem of Bargmann, Hall and Wightman¹⁰⁾. The permuted function (3.10) is given in the limit $(x-y)^2 \rightarrow 0$ as a singular function times an expectation value of a bilocal current:

$$S^{\rho}(x-y) \langle 0 | j(z) \hat{T}_{\mu}(x,y) j(t) | 0 \rangle \quad (3.11)$$

The question now arises, what happens if we carry out the analytic continuation from the function (3.10) to the permuted one (3.8)? Do we still get the same type of singularity in $(x-y)^2$ for (3.8) as in (3.10)? In other words, is it allowed just to take the asymptotic expansion (3.11) and to carry out the analytic continuation in the matrix element of the bilocal operator? From general arguments, we cannot give an answer to these questions. Instead, we proceed similarly as in current algebra and consider first a relativistic quark field theory, abstract simple algebraic relations and then postulate these relations as being true for the real world of hadrons, even if there are no quarks or particles with triality non-zero in the physical spectrum. For simplicity, we use the free quark model for our abstraction, but we could also consider formal interacting gluon models, which lead to the same algebraic relations. In doing so, we again are faced with the question if such abstract algebraic relations already imply the existence of triality non-zero states in the particle spectrum. We assume that this is not the case. Since we always work with bilinear densities in the formal quark fields, there is no obvious contradiction to this assumption. A possible candidate for a quark field theory without quark particles would be a field theory, where any field with triality non-zero quantum number does not produce asymptotic states.

We can write formally :

$$T(\tilde{f}_{i\mu}(x) j(z)) T^*(\tilde{f}_{j\nu}(y) j(t)) \sim \\ T(\bar{q}(x) \frac{1}{2} \lambda_i \delta_\mu q(x) j(z)) T^*(\bar{q}(y) \frac{1}{2} \lambda_j \delta_\nu q(y) j(t)) \quad (3.12)$$

In the free theory and also in formal interacting gluon models, the leading singularity in $(x-y)^2$ is given by a single quark contraction (connected part only). We assume that we can abstract this behaviour. Our assumption makes sure that one is allowed just to take the most singular term in (3.9) as given by (3.10) and carry out the analytic continuation in co-ordinate space to arrive at the strongest light cone singularity for the function (3.8). After integrating over z and t we obtain the following algebraic relations :

$$\int dz dt e^{i p(t-z)} \left\{ (\square_z + p^2)(\square_t + p^2) T(\tilde{f}_{i\mu}(x) j(z)) T^*(\tilde{f}_{j\nu}(y) j(t)) - \right. \\ = \sum_n \left(\tilde{f}_{i\mu}(x) |p, n\rangle \langle n, p| \tilde{f}_{j\nu}(y) \right) \\ \left. \frac{-i}{8\pi^2} \int_{z'}^0 \frac{1}{z'^2 - i\epsilon z'^0} \left[i f_{ijk} \left\{ \int_{\mu\nu\rho\sigma} (\phi_{k\sigma}(x, y, p) + \tilde{\phi}_{k\sigma}(y, x, p)) \right. \right. \right. \\ \left. \left. \left. - i \varepsilon_{\mu\nu\rho\sigma} (\phi_{k\sigma}^{(5)}(x, y, p) - \tilde{\phi}_{k\sigma}^{(5)}(y, x, p)) \right\} \right. \right. \\ \left. \left. + d_{ijk} \left\{ \int_{\mu\nu\rho\sigma} (\phi_{k\sigma}(x, y, p) - \tilde{\phi}_{k\sigma}(y, x, p)) \right. \right. \right. \\ \left. \left. \left. - i \varepsilon_{\mu\nu\rho\sigma} (\phi_{k\sigma}^{(5)}(x, y, p) + \tilde{\phi}_{k\sigma}^{(5)}(y, x, p)) \right\} \right] \right] \\ z' = x - y$$

The quantities $\phi^{(5)}(x, y, p)$ defined by relation (3.8) can be written in the model as :

$$\tilde{\phi}_{k\sigma}(y, x, p) = - \int dz dt (\square_z + p^2)(\square_t + p^2) T(q_\beta(x) j(z)) T^*(\bar{q}(y) j(t)) \frac{\lambda_k}{2} \delta_\sigma^\beta \\ \phi_{k\sigma}(x, y, p) = \int dz dt (\square_z + p^2)(\square_t + p^2) \times \\ \times (T(\bar{q}(x) j(z)) \frac{1}{2} \lambda_k \delta_\sigma (T^* q(y) j(t))) \quad (3.14)$$

We abstract just the Lorentz - and $SU(3)$ - properties of these operators, but reject their specific structure in terms of the underlying quark fields. The algebraic relations (3.8) are very similar to the light cone relations for ordinary current commutators ³⁾, where instead of the quantities ϕ the bilocal current operators appear. But the operators ϕ have different properties than the bilocal currents :

- 1) they do not fulfil a closed algebra;
- 2) they contain information about the state p ;
- 3) especially they have additional Lorentz - $SU(3)$ - indices characterizing the transformation properties of the hadron state $|p\rangle$;
- 4) they may be singular in the limit $x \rightarrow y$.

In Section 2 we have emphasized that matrix elements like (3.3) or in general operators like (3.5) or (3.13) need not vanish at space-like distances $((x-y)^2 < 0)$, since locality does not apply to this situation.

The algebraic relations (3.13) are assumed to be true for all hadron states $|p\rangle$. If $|p\rangle$ is a baryon state, the operators ϕ have now two additional Dirac indices. One also can see easily that it is not essential that the state p detected in the final state is a one-particle state. It may also be a multiparticle state.

Note that it is essential to integrate in (3.8) over z and t . In such a way we take into account that the hadron state p is a smeared out object. If z or t happen to be identical with x or y , one gets a stronger singularity as denoted in (3.13) for the four current product (3.9). Such stronger singularities are suppressed if one integrates over z and t with the phase $e^{-ip(z-t)}$ where $|p\rangle$ is constrained on the mass shell of the corresponding particle.

We should like to note that the algebraic relations (3.13) cannot be directly related to the light cone algebra of current commutators, although the analogy is obvious. But nevertheless both abstractions are different, and the ansatz expressed in (3.13) might be wrong, even if the light cone relations for ordinary current commutators are right. Let us finally study the phenomenological structure of the relations (3.13). The tensor- and singularity-structure of these relations leads, for example, to the conclusion that the scaling function $\bar{F}_\ell(\xi)$ vanishes. This, of course, is analogous to the fact that the tensor- and singularity-structure of the quark light cone algebra of currents leads to the vanishing of the cross-section for longitudinal photons in the scaling limit.

The algebraic relations (3.13) are completely $SU(3) \times SU(3)$ symmetric with respect to the current indices, but one should be beware of transforming the hadron state $|p\rangle$. $SU(3) \times SU(3)$ breaking will just affect the lower lying singularities on the light cone. We emphasize again that the operators ϕ cannot be regarded as bilocal operators in the usual sense. In particular their Taylor expansion in $(x-y)$ need not exist and certainly does not start with a local current operator. Therefore there is no possibility of obtaining sum rules as in deep inelastic scattering. However, one obtains symmetry relations and inequalities for the structure functions, which arise from the $SU(3) \times SU(3)$ structure of the relations (3.13). In the following two sections we consider several immediate consequences of our assumptions.

4. CURRENT ANNIHILATION PROCESSES

Current annihilation processes are described by vacuum expectation values of the relations (3.13). They correspond to reactions of the type described in Fig. 5 :

Fig.5

In particular we concentrate on the special case where the state $|p\rangle$ is just a one-particle state. Since the formalism is analogous to the case of deep inelastic scattering, we can use the same technique as in Ref. 3) to obtain information about the structure functions. We take the usual expressions for the weak and electromagnetic currents in terms of the current-octet (in particular, we assume that the electromagnetic current transforms as an octet operator, without any charmed component). The scaling functions are given as Fourier transforms of the vacuum expectation values of the quantities ϕ_{kp} . We define :

$$\langle 0 | \phi_{kp}(x, y, p) \pm \tilde{\phi}_{kp}(y, x, p) / 0 \rangle = \frac{2}{2} \frac{\bar{S}_k(p^2) \cdot P_p}{\bar{A}_k(p^2) \cdot P_p} + \text{terms proportional to } p$$

$$\bar{S}_k(p^2) = \int d\alpha e^{-i\alpha(p^2)} \tilde{S}_k(\alpha), \quad \bar{A}_k(p^2) = \int d\alpha e^{-i\alpha(p^2)} \tilde{A}_k(\alpha) \quad (4.1)$$

Using these definitions, we obtain for the electromagnetic case :

$$\begin{aligned}\bar{F}_1^e(\xi) &= \frac{1}{2} \left(\frac{2}{3} \sqrt{\frac{2}{3}} \hat{A}_0(\xi) + \frac{1}{3\sqrt{3}} \hat{A}_8(\xi) + \frac{1}{3} \hat{A}_3(\xi) \right) \\ \bar{F}_0^e(\xi) &= 0\end{aligned}\quad (4.2)$$

Similarly, one gets relations for the weak processes which are formally identical to the corresponding unbarred relations in the case of deep inelastic scattering [see Ref. 3]. In particular, we also obtain symmetry relations, for example

$$12 \left(\bar{F}_1^{eP}(\xi) - \bar{F}_1^{eN}(\xi) \right) = \left(\bar{F}_3^{\nu P}(\xi) - \bar{F}_3^{\nu N}(\xi) \right) \quad (4.3)$$

$|p\rangle$ and $|n\rangle$ denote proton and neutron states, respectively. Unfortunately, one cannot easily produce weak currents with time-like four-momenta, so these relations have no immediate practical application. We note again that it is impossible to obtain sum rules for the scaling functions. From (4.2) we also obtain relations for arbitrary isospin multiplets with total isospin $> \frac{1}{2}$:

$$\frac{\bar{F}_i^{(e) P_i^I} - \bar{F}_i^{(e) P_j^I}}{\bar{F}_i^{(e) P_i^I} - \bar{F}_i^{(e) P_j^I}} = \frac{i-j}{i'-j'}$$

Here the indices I and $i, j, \dots$ denote the total isospin of the multiplet and the third components, respectively. In particular, we have :

$$\begin{aligned}\bar{F}_1^{e\pi^0} &= \bar{F}_1^{e\pi^+} = \bar{F}_1^{e\pi^-} \\ 2 \bar{F}_1^{e\Sigma^0} &= \bar{F}_1^{e\Sigma^+} + \bar{F}_1^{e\Sigma^-}\end{aligned}$$

The relations (3.13) also imply, together with positivity, inequalities like the following one :

$$\frac{1}{4} \leq \frac{\bar{F}_1^{eN}(\xi)}{\bar{F}_1^{eP}(\xi)} \leq 4 \quad (4.4)$$

Note that the same inequality holds also for the (unbarred) electro-production functions. The simplest proof is the following ^{11),12)}. The electromagnetic current is written as

$$j_\mu = \widehat{f}_{3\mu} + \frac{1}{\sqrt{3}} \widehat{f}_{8\mu} = \frac{2}{3} \widehat{f}_{u\mu} - \frac{1}{3} \widehat{f}_{d\mu} - \frac{1}{3} \widehat{f}_{s\mu} \quad (4.5)$$

where the currents $\widehat{f}_{u\mu} \dots$ are certain combinations of the octet currents $\widehat{f}_{i\mu}$, with transformation properties as given by the following formal quark representation :

$$\begin{aligned} \widehat{f}_{u\mu} &\sim \bar{q} \gamma_\mu \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} q, \quad \widehat{f}_{d\mu} \sim \bar{q} \gamma_\mu \begin{pmatrix} 0 & & \\ & 1 & \\ & & 0 \end{pmatrix} q, \\ \widehat{f}_{s\mu} &\sim \bar{q} \gamma_\mu \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix} q \end{aligned} \quad (4.6)$$

The currents $\widehat{f}_{u\mu} \dots$ are Hermitean. From the current algebra at light-like distances, one finds :

$$[j_\mu, j_\nu] \cong \frac{4}{9} (\widehat{f}_{u\mu} \widehat{f}_{u\nu}) + \frac{1}{9} (\widehat{f}_{d\mu} \widehat{f}_{d\nu}) + \frac{1}{9} (\widehat{f}_{s\mu} \widehat{f}_{s\nu}) \quad (4.7)$$

Note that these relations are right only with respect to the leading singularity on the light cone. A similar decomposition as in (4.7) also holds for algebraic relations of the type (3.13). We therefore obtain :

$$\bar{F}_1^{0P} = \frac{4}{9} \bar{F}_1^{uP} + \frac{1}{9} \bar{F}_1^{dP} + \frac{1}{9} \bar{F}_1^{sP} \quad (4.8)$$

and, using SU(2) symmetry :

$$\bar{F}_1^{en} = \frac{1}{9} \bar{F}_1^{uP} + \frac{4}{9} \bar{F}_1^{dP} + \frac{1}{9} \bar{F}_1^{sP}$$

where the functions $\bar{F}_1^{\text{up}}(\xi) \dots$ describe the correlation functions $\langle 0 | \widehat{f}_{u\mu}(x) | p, n \rangle \langle n, p | \widehat{f}_{u\nu}(y) | 0 \rangle \dots$ in the light cone limit. Since these functions are positive definite because of the Hermiticity properties of the currents (4.6), one immediately arrives at the inequality (4.4). Similar relations one may obtain for any isospin multiplet. We denote the detected particle by $\begin{smallmatrix} I \\ i \end{smallmatrix}$ where I is the total isospin of the multiplet and i, j stand for its third component. Then we find :

$$\frac{1}{4} \leq \frac{1 - \frac{3}{5} \left| \frac{i}{I} \right|}{1 + \frac{3}{5} \left| \frac{j}{I} \right|} \leq \frac{\bar{F}_1 e^{P_1^I}}{\bar{F}_1 e^{P_2^I}} \leq \frac{1 + \frac{3}{5} \left| \frac{i}{I} \right|}{1 - \frac{3}{5} \left| \frac{j}{I} \right|} \leq 4$$

In particular, we obtain :

$$\frac{1}{4} \leq \frac{\bar{F}_1 e^{\Sigma^-}}{\bar{F}_1 e^{\Sigma^+}} \leq 4, \quad \frac{2}{5} \leq \frac{\bar{F}_1 e^{\Sigma^+}}{\bar{F}_1 e^{\Sigma^0}} \leq \frac{8}{5}$$

$$\frac{2}{3} \leq \frac{\bar{F}_1 e^{\Delta^+}}{\bar{F}_1 e^{\Delta^0}} \leq \frac{3}{2} \quad \text{etc.} \quad (4.9)$$

Let us finally consider the consequences of our assumptions for the multiplicity of the outgoing hadrons. Using the variable $\omega = 1/\xi$ we find for the scaling contribution to the multiplicity, provided the total annihilation cross-section behaves like const/q^2 ¹³⁾ :

$$n^{\text{scaling}}(q^2) \sim \text{const.} \cdot \int_{\sqrt{\frac{4p^2}{q^2}}}^1 d\omega \cdot \omega \bar{F}_1(\omega) \quad (4.10)$$

A logarithmic increase of multiplicity with q^2 would imply ^{*)}

$$\bar{F}_1(\omega) \xrightarrow{\omega \rightarrow 0} \frac{\text{const.}}{\omega^2} \quad (4.11)$$

*) $n^{\text{scaling}}(q^2) \rightarrow \text{const.}$ was obtained by Callan and Gross, Ref. 1) from a different and probably wrong assumption.

Of course, the total multiplicity is not at all determined by the scaling contribution $n^{\text{scaling}}(q^2)$, and one can easily construct models in which the main part of the total multiplicity comes from terms which vanish in the scaling limit.

5. INCLUSIVE ELECTRO- AND NEUTRINO-PRODUCTION

In this section we apply the algebraic relations (3.13) to inclusive electro- and neutrino-production experiments off a hadronic target $|t\rangle$ where one detects a final particle $|p\rangle$ ^{*)} (Fig.6a)

Fig.6a

Fig.6b

These processes can be viewed as being the crossed ones of processes like

current + two-particle state $\rightarrow$ anything

(see Fig.6b).

The latter is described in the scaling limit by a two-particle expectation value of a bilocal current operator while the first one is given by a one-particle expectation value of the operators ϕ defined in (3.13). We describe inclusive reactions of the type (Fig.6a) by the following matrix elements :

*) The fragmentation region of the projectile cannot be studied with our methods. There one can easily check that the light cone does not dominate in the corresponding scaling limit, similarly as in the crossed situation, where one observes the production of heavy lepton pairs in hadron-hadron collisions. In a parton model, one still may obtain information about this region [see, M. Gronau, F. Ravndal and Y. Zarmi, Caltech preprint CALT-68-367 (1972)].

$$W_{\mu\nu}^{ij}(q, p, t) = \frac{1}{4\pi} \int dx e^{i q \cdot x} \left(\langle \psi | \hat{F}_{ij}(x) | p, n \rangle \langle n, p | \hat{F}_{ij}(0) | \psi \rangle \right) \quad (5.1)$$

This tensor is described in general by 16 independent structure functions, depending on the following variables :

$$\nu = t \cdot q \quad \nu' = p \cdot q \quad , \quad q^2 \quad , \quad u = (t - p)^2 \quad (5.2)$$

We also define four vectors R and Δ by : $R_\mu = t_\mu + p_\mu$, $\Delta_\mu = t_\mu - p_\mu$. Current conservation reduces the number of structure functions to nine. The deep inelastic limit is defined as :

$$\left. \begin{aligned} \nu, \nu', -q^2 &\rightarrow +\infty \\ \xi &= \frac{-q^2}{2\nu} \\ x &= \frac{\nu'}{\nu} \\ u &= \Delta^2 \end{aligned} \right\} \text{fixed} \quad (5.3)$$

The physical meaning of these variables can easily be seen in the centre-of-mass frame (see Fig.7), where one has :

Fig.7

$$\begin{aligned} t_\mu &= (\sqrt{m_t^2 + t_3^2}, t_3, 0, 0) & q_\mu &= (q_0, -t_3, 0, 0) \\ p_\mu &= (\sqrt{m_p^2 + p_3^2 + p_\perp^2}, p_3, p_\perp) \end{aligned} \quad (5.4)$$

We choose $t_3 > 0$. Note :

$$\begin{aligned}
 v = t q &= q_0 \sqrt{m_t^2 + t_3^2} + t_3^2 \\
 v' = p q &= q_0 \sqrt{m_p^2 + p_3^2 + p_1^2} + t_3 p_3 \\
 u = (t-p)^2 &= m_t^2 + m_p^2 + 2 t_3 p_3 - 2 t_3 p_1 \sqrt{\left(1 + \frac{m_t^2}{t_3^2}\right) \left(1 + \frac{m_p^2 + p_1^2}{p_3^2}\right)} \\
 \xi = \frac{-q^2}{2v} &\sim \frac{t_3 - q_0}{2 t_3} \\
 x = \frac{v'}{v} &\sim \frac{p_3}{t_3}
 \end{aligned}
 \tag{5.5}$$

$$q_0 \geq 0 \rightarrow 0 \leq \xi \leq \frac{1}{2}$$

$$q_0 < 0 \rightarrow \frac{1}{2} \leq \xi \leq 1$$

The scaling limit (5.3) corresponds to :

$$\begin{aligned}
 t_3 &\rightarrow +\infty & x &\rightarrow \frac{p_3}{t_3} \text{ fixed} \\
 u &\rightarrow (1-x) m_p^2 + (1-\frac{1}{x}) m_t^2 - \frac{1}{x} p_1^2 \text{ fixed}
 \end{aligned}$$

We emphasize that this scaling limit degenerates in the case of $-q^2$ fixed ($\xi \rightarrow 0$) into the hadronic scaling limit, where one considers the fragmentation of the target only. The range of our scaling variables is the following :

$$0 \leq \xi \leq 1, \quad 0 \leq x \leq 1 - \xi \tag{5.6}$$

In the scaling limit, one tests the light cone region in co-ordinate space and we may apply the relations (3.13). Because of their singularity structure which is abstracted from a quark model, we have in the scaling limit :

$$\varepsilon_l^\mu \mathcal{W}_{\mu\nu}^+ \rightarrow 0 \quad \varepsilon_l^\nu \mathcal{W}_{\mu\nu}^+ \rightarrow 0 \tag{5.7}$$

where ε_l^μ is the longitudinal polarization vector, normalized to one. Thus the hadronic medium is asymptotically transparent for the longitudinal current mode.

Due to the transversality conditions (5.7), the leading contributions to the tensor (5.1) can be expressed by two structure functions. In the Breit frame with

$$R_\mu = (R_0, 0, 0, 0) \quad \Delta_\mu = (\Delta_0, \Delta_1, 0, \Delta_3) \quad \Delta_1 > 0$$

$$g_\mu = (g_0, 0, 0, g_3) \quad \varepsilon_{\mu 1}^{tr} = (0, 1, 0, 0) \quad \varepsilon_{\mu 2}^{tr} = (0, 0, 1, 0)$$

we have :

$$W_{\mu\nu}^{ij} \rightarrow \varepsilon_{\mu m}^{tr} \varepsilon_{\nu n}^{tr} \bar{F}_{mn}^{ij} \quad m, n = 1, 2$$

$$\bar{F}_{mn}^{ij} = \begin{pmatrix} F_1^{ij} & i F_3^{ij} \\ -i F_3^{ij} & F_1^{ij} \end{pmatrix} \quad (5.8)$$

F_1 and F_3 are the scaling limits of structure functions W_1 and $-\frac{1}{2}W_3 \cdot \nu$ which are kinematically identical to the conventional structure functions W_1 and $-\frac{1}{2}W_3 \cdot \nu$. How are the scaling functions related to the expectation values of the operators ϕ ? We write :

$$\frac{1}{2} \langle t / \phi_{i\sigma}(\frac{\tau}{2}, -\frac{\tau}{2}, p) \pm \tilde{\phi}_{i\sigma}(-\frac{\tau}{2}, \frac{\tau}{2}, p) / t \rangle$$

$$= t_\sigma \left\{ \begin{matrix} \bar{S}_i^+ (t\tau, \frac{p^2}{t^2}, u) \\ \bar{A}_i^+ (t\tau, \frac{p^2}{t^2}, u) \end{matrix} \right\} + p_\sigma \left\{ \begin{matrix} \bar{S}_i^- (t\tau, \frac{p^2}{t^2}, u) \\ \bar{A}_i^- (t\tau, \frac{p^2}{t^2}, u) \end{matrix} \right\} \quad (5.9)$$

The four-dimensional Fourier transforms of the matrix elements (5.9), multiplied with the light cone singularity, are described by the following formula :

$$\frac{\nu}{4\pi^3 i} \int d^4 z e^{+i q \cdot z} f_2^{\rho} \left(\frac{1}{z^2 - i \varepsilon z^0} \right) \bar{S}_i^+ (t\tau, \frac{p^2}{t^2}, u)$$

$$\rightarrow q^\rho \tilde{S}_i^+ (\xi, x, u), \quad \dots \quad x = \frac{p^2}{t^2} \quad (5.10)$$

The asymptotic expression (5.10) holds only if the light cone dominates in the deep inelastic limit. The correctness of (5.10) combined with the spectrum condition yields the following information on the matrix elements (5.9) in analogy to (2.17) (with $\tau = t\tau$, $x = \frac{p^2}{t^2}$).

$\bar{S}_i^t(\tau, x, u)$ is the boundary value of an analytic function in the half-plane $\text{Im } \tau > 0$. The singularity at $x=0$ is restricted by :

$$x \cdot \bar{S}_i^t(\tau, x, u) \xrightarrow{x \rightarrow 0} 0, \dots \quad (5.11)$$

If (5.11) does not hold the leading light cone singularity will not dominate in the scaling region. The one-dimensional Fourier transforms of the matrix elements (5.9) with respect to the first variable $(tz) = \tau$ and fixed $x = \frac{pz}{tz}$ are defined by :

$$\tilde{S}_i^t(\xi, x, u) = \frac{1}{2\pi} \int d\tau e^{-i\tau \xi} \bar{S}_i^t(\tau, x, u), \dots \quad (5.12)$$

The scaling functions, as given in (5.8), depend only on the combinations :

$$\begin{aligned} \bar{F}_{S_i}(\xi, x, u) &= \tilde{S}_i^t(\xi, x, u) + x \tilde{S}_i^P(\xi, x, u) \\ \bar{F}_{A_i}(\xi, x, u) &= \tilde{A}_i^t(\xi, x, u) + x \tilde{A}_i^P(\xi, x, u) \end{aligned} \quad (5.13)$$

in the following way :

$$\begin{aligned} 2\bar{F}_1^{ij} &= i f_{ijk} \bar{F}_{Sk} + d_{ijk} \bar{F}_{ak} \\ 2\bar{F}_3^{ij} &= i f_{ijk} \bar{F}_{ak} + d_{ijk} \bar{F}_{Sk} \end{aligned} \quad (5.14)$$

We now define the scaling functions, which describe the scattering of a current $\tilde{T}_\mu^{(P)}(x)$ with quantum numbers $\lambda^{1/2} \cdot \rho_i$

$$\tilde{T}_{1,3}^{(P)} = \rho^* i \tilde{T}_{1,3}^{ij} \rho^j \quad (5.15)$$

Later we also use the symbols ρ^+ and ρ^- referring to the chiral currents $\tilde{T}_\mu^{\rho^+} = \tilde{T}_\mu^\rho + \tilde{T}_\mu^{5\rho}$, $\tilde{T}_\mu^{\rho^-} = \tilde{T}_\mu^\rho - \tilde{T}_\mu^{5\rho}$. We denote them in the cases of electro- and neutrino-production :

$$\begin{aligned}
 \bar{F}_1^e &= \frac{2}{3} \sqrt{\frac{2}{3}} \bar{F}_{a_0} + \frac{2}{3\sqrt{3}} \bar{F}_{a_8} + \frac{2}{3} \bar{F}_{a_3} \\
 \bar{F}_1^\nu &= 2 \sqrt{\frac{2}{3}} \bar{F}_{a_0} + \frac{4}{\sqrt{3}} \bar{F}_{a_8} + 4 \bar{F}_{s_3} \\
 \bar{F}_3^\nu &= 2 \sqrt{\frac{2}{3}} \bar{F}_{s_0} + \frac{4}{\sqrt{3}} \bar{F}_{s_8} + 4 \bar{F}_{a_3}
 \end{aligned} \tag{5.16}$$

The corresponding cross-sections can be obtained in the following way. Let the transverse polarization vector of the current $\hat{F}_\mu^{(p)}$ be given by :

$$\hat{F}_\mu^{(p)} = \alpha^{(p)} \hat{\Sigma}_{\mu 1}^{tr} + \beta^{(p)} \hat{\Sigma}_{\mu 2}^{tr} \tag{5.17}$$

$|\alpha|^2 + |\beta|^2 = 1$, α, β complex. The Stokes matrix can then be written as :

$$\begin{pmatrix} |\alpha|^2 & \alpha \beta^* \\ \alpha^* \beta & |\beta|^2 \end{pmatrix}^{(p)} \tag{5.18}$$

and the degree of transverse polarization is :

$$-1 \leq P_{tr}^{(p)} = \left(\frac{2 \operatorname{Im} \alpha \beta^*}{|\alpha|^2 + |\beta|^2} \right)^{(p)} \leq 1 \tag{5.19}$$

Then the inclusive cross-section of a polarized chiral current $\hat{F}_\mu^{(p) \pm}$ on a target t turns out to be :

$$E_p \frac{d\sigma^{(p \pm, P^p)}}{d^3p} \rightarrow \frac{1}{16\pi^2\nu} \left(\bar{F}_1^{PP^\pm} \pm \bar{F}_3^{PP^\pm} P_{tr}^{P^\pm} \right) / (x, z) \tag{5.20}$$

In the following we use for the expression in the brackets the short-hand notation $\bar{F}_1 \pm \bar{F}_3 P_{tr} = F^{PP^\pm}$.

From the algebraic relations (5.16), we arrive at the following relations between structure functions :

$$6 \left(\bar{F}_1 e^{p\pi^+}(\xi, x, u) - \bar{F}_1 e^{n\pi^-}(\xi, x, u) \right) = \left(\bar{F}_3^{\nu n\pi^-} - \bar{F}_3^{\nu p\pi^+} \right) / (\xi, x, u)$$

$$6 \left(\bar{F}_1 e^{p\pi^0} - \bar{F}_1 e^{n\pi^0} \right) = \left(\bar{F}_3^{\nu n\pi^0} - \bar{F}_3^{\nu p\pi^0} \right) \quad (5.21)$$

Similar relations are obtained for the production of other isospin multiplets like Σ , etc. The functions $\bar{F}_{is}^{(5)}(\xi, x, u)$ and $\bar{F}_{ia}^{(5)}(\xi, x, u)$ defined in (5.13) transform under SU(2) like

$$\langle t_p / Q_i / t_p \rangle$$

where Q_i denote the SU(3) charges. Therefore, we obtain also relations like the following ones :

$$\frac{\bar{F}_{1\langle t \rangle} e^{i^I} - \bar{F}_{1\langle t \rangle} e^{j^I}}{\bar{F}_{1\langle t \rangle} e^{i'^I} - \bar{F}_{1\langle t \rangle} e^{j'^I}} = \frac{i-j}{i'-j'} = \frac{\bar{F}_3^{\nu i^I} - \bar{F}_3^{\nu j^I}}{\bar{F}_3^{\nu i'^I} - \bar{F}_3^{\nu j'^I}} =$$

$$= \frac{\bar{F}_3^{\nu i^I} - \bar{F}_3^{\nu j^I}}{\bar{F}_3^{\nu i'^I} - \bar{F}_3^{\nu j'^I}} \quad (5.22)$$

where $\binom{I}{i}$ denote the total isospin and its third component of the particle produced. The symbol $\langle t \rangle$ means average over the target isospin. For example, we obtain ¹⁴⁾ :

$$\bar{F}_{1\langle t \rangle} e^{\pi^0} = \frac{1}{2} \left(\bar{F}_{1\langle t \rangle} e^{\pi^+} + \bar{F}_{1\langle t \rangle} e^{\pi^-} \right) \quad (5.23)$$

The same relation is right for neutrino-production. All these symmetry relations are analogous to the ones given in Section 2 for e^+e^- annihilation, since averaging over the targets yields the quantum numbers of the vacuum.

From positivity of the cross-sections in (5.19) and SU(2) symmetry, we also get inequalities. If we average over proton and neutron targets, we obtain analogous relations as in Section 2 :

$$\frac{1}{4} \leq \frac{1 - \frac{3}{5} \left| \frac{i}{I} \right|}{1 + \frac{3}{5} \left| \frac{i}{I} \right|} \leq \frac{\bar{F}_{1, \langle t \rangle}^{e p \bar{p}}(\xi, x, u)}{\bar{F}_{1, \langle t \rangle}^{e p \bar{p}}(\xi, x, u)} \leq \frac{1 + \frac{3}{5} \left| \frac{i}{I} \right|}{1 - \frac{3}{5} \left| \frac{i}{I} \right|} \leq 4 \quad (5.24)$$

Finally, we demonstrate the connection of our scaling results with the hadronic scaling suggested by Feynman. We have already noted that the Feynman scaling region is formally obtained in the limit $\xi \rightarrow 0$, where $-q^2$ is held fixed. Since $\xi = -q^2/2\nu$ coincides with the usual scaling variable for deep inelastic scattering (where $p_\mu \rightarrow 0$), the connection between the Feynman scaling region and the deep inelastic region is just the analogue of the connection between Regge region and deep inelastic region in electro- and neutrino-production. The variable x defined in (5.3) is just the Feynman scaling variable. We note that in the limit $p_\mu \rightarrow 0$ our functions $\bar{F}(\xi, x, u)$ degenerate into

$$\bar{F}(\xi, x, u) \xrightarrow{p \rightarrow 0} \bar{F}(\xi, 0, u = u_t^2) \quad (5.25)$$

In deep inelastic scattering, the interchange of Regge and scaling limits means that the scaling functions themselves are Regge behaved at small ξ . In particular, the Pomeranchuk trajectory gives :

$$\bar{F}_{a, 0}(\xi) \xrightarrow{\xi \rightarrow 0} \frac{\omega_{a, 1} \xi}{\xi} \quad (5.26)$$

where $\bar{F}_{a, 0}(\xi)$ is the Fourier transform of the expectation value of the anti-symmetrized bilocal baryon current $\hat{F}_{a, 0}(x, y)$ [we exclude, for simplicity, the possibility that the Pomeranchuk singularity transforms as a mixture of $\hat{F}_{a, 0}$ and $\hat{F}_{a, 8}$].

Let us now consider the function $\bar{F}_{a, 0}(\xi, x, u)$ at small values of ξ . According to (5.26), we assume the following behaviour

$$\bar{F}_{a, 0}(\xi, x, u) \xrightarrow{\xi \rightarrow 0} \frac{g(x, u)}{\xi} \quad (5.27)$$

Then the function $g(x,u) = \hat{g}(x, p_{\perp}^2)$ is our scaling function in the particular case $m^2 \ll -q^2 \ll \gamma$. Especially, we get for the inclusive electroproduction:

$$E_p \frac{d\sigma}{d^3p} \xrightarrow[\xi \rightarrow 0]{\gamma, -q^2 \rightarrow \infty} \frac{\alpha}{2\pi} \frac{g(x, u)}{(-q^2)} \quad (5.28)$$

We try to extrapolate to the case $-q^2$ fixed by the substitution $-q^2 \rightarrow \bar{\beta}(q^2)$ where $\bar{\beta}(q^2)$ describes the residue of the Pomanchuk singularity. This kind of extrapolation is suggested by the electroproduction experiments. We obtain just the usual hadronic scaling behaviour:

$$E_p \frac{d\sigma}{d^3p} \xrightarrow[\substack{\gamma \rightarrow \infty \\ -q^2 \text{ fixed}}]{\gamma, -q^2 \rightarrow \infty} \frac{\alpha}{2\pi} \bar{\beta}(q^2) \hat{g}(x, p_{\perp}^2) \quad (5.29)$$

Further we assume that the parametrization (5.29) works even at small q^2 , as suggested by the electroproduction experiments. Then we may also study purely hadronic reactions. The residue function $\bar{\beta}(q^2)$ has double poles at $q^2 > 0$, which describe the particles produced in the current channel. The corresponding residues of the amplitude describe purely hadronic reactions. In the case of the electromagnetic current, we have for example:

$$E_p \frac{d\sigma}{d^3p} \xrightarrow{pt \rightarrow p^4} \text{const.} \hat{g}(x, p_{\perp}^2) \quad (5.30)$$

Our analysis does not give an explanation why the functions $\bar{F}(\xi, x, p_{\perp}^2)$ and also the hadronic scaling functions $\hat{g}(x, p_{\perp}^2)$ fall off sharply at increasing transverse momenta. The transverse momentum cut-off comes from the specific structure of the matrix elements of the operators $\phi_{i\sigma}$.

Further, we discuss the multiplicity of the produced particles. When one scatters a current $J_{\mu}(s)$ off a hadronic target t , the differential multiplicity of producing a particle p is defined by

$$\frac{dn}{dx d^2p_{\perp}^2}(\xi, x, p_{\perp}^2) = \lim_{\substack{-q^2 \rightarrow \infty \\ \gamma \rightarrow \infty}} \frac{1}{\sigma^{tp}(\gamma^2, \nu)} \frac{d\sigma}{dx d^2p_{\perp}^2} \quad (5.31)$$

Note that the total cross-sections $\sigma^{tP}(q^2, \nu)$ in the deep inelastic limit are determined by the scaling functions $F^{tP\pm}$ which stand for [see (5.20)] :

$$F^{tP\pm} = F_1 \pm F_2 P^P$$

$$\sigma^{tP}(q^2, \nu) \rightarrow \frac{\pi}{\nu} F^{tP}(\xi) \quad (5.32)$$

Therefore we get, using (5.20) :

$$\frac{d h^{PP}(\xi, x, P_\perp^2)}{dx dP_\perp^2} = \frac{1}{x} D^{PP}(\xi, x, P_\perp^2) \quad (5.33)$$

where we have defined the function D by :

$$D^{PP}(\xi, x, P_\perp^2) = \frac{1}{16\pi^3} \frac{\overline{F}^{PP}(\xi, x, P_\perp^2)}{F^P(\xi)} \quad (5.34)$$

At finite energies, we have to use non-asymptotic kinematics to evaluate (5.31) :

$$\frac{d h^{PP}}{dx dP_\perp^2} = \frac{t_3}{E_P} D^{PP} = \frac{1}{\sqrt{\frac{m_P^2 + P_\perp^2}{t_3^2} + x^2}} D^{PP} \quad (5.35)$$

We integrate over the transverse momenta and use the mean value theorem to define a formal mass parameter $\mathcal{E}$ in the following way (this definition is only possible if the integral over the transverse momenta exists, which we assume) :

$$\begin{aligned} \frac{d h^{PP}}{dx} &= \frac{1}{\sqrt{\frac{\mathcal{E}^2}{t_3^2} + x^2}} \hat{D}^{PP}(\xi, x) \\ \hat{D}^{PP}(\xi, x) &= \int_0^\infty dP_\perp^2 D^{PP}(\xi, x, P_\perp^2) \end{aligned} \quad (5.36)$$

The formal mass parameter $\mathcal{E}$ depends still on the remaining variables ξ , x and t_3 . A cut-off in the transverse momenta amounts to

$$\lim_{t_3 \rightarrow \infty} \mathcal{E}(\xi, x, t_3) = \mathcal{E}(\xi, x) < \infty$$

In that case the main contribution to the multiplicity for $t_3 \rightarrow \infty$ comes from the central region ($x \approx 0$). We evaluate the multiplicity to :

$$h^{PP}(t_3, \xi) = \int_0^{1-\xi} \frac{dx}{\sqrt{\frac{\xi^2}{t_3^2} + x^2}} \hat{D}^{PP}(\xi, x) \quad (5.37)$$

The divergent part of this integral may be evaluated replacing $\mathcal{E}(\xi, x)$ by $\mathcal{E}(\xi, 0)$. It seems natural that the function $\hat{D}^{PP}(\xi, x)$ approaches a finite limit for $x \rightarrow 0$. In that case we find :

$$\begin{aligned} h^{PP}(t_3, \xi) &\rightarrow \log\left(\frac{t_3}{\xi}\right) \hat{D}^{PP}(\xi, 0) \\ &= \frac{1}{2} \log\left(\frac{\nu}{\xi^2}\right) \hat{D}^{PP}(\xi, 0) \end{aligned} \quad (5.38)$$

The light cone still tolerates a behaviour like $\hat{D}^{PP}(\xi, x) \rightarrow x^{\delta-1} d^{PP}(\xi)$ with $\delta > 0$ [see (5.11)]. This amounts to a multiplicity like

$$h(\nu, \xi) \rightarrow \text{const. } d^{PP}(\xi) (\sqrt{\nu})^{1-\delta} \quad (5.39)$$

Thus the upper limit just forbidden by light cone dominance corresponds to the equipartition case of a finite characteristic energy among the particles produced.

In the particular case $m_t^2 \ll -q^2 \ll \nu$ the multiplicity is given by the limit $\xi \rightarrow 0$ in (5.38) or (5.39)

$$\begin{aligned} h^{PP}(s, \xi=0) &\rightarrow \frac{1}{2} \log\left(\frac{s}{\xi^2}\right) \hat{D}^{PP}(0, 0) \\ s &= (q+t)^2 \rightarrow 4t_3^2 \end{aligned} \quad (5.40)$$

At finite q^2 we use

$$\sigma^{TP}(q^2, \nu) \xrightarrow{\nu \rightarrow \infty} 2\pi \beta(q^2) \left(\xi F^{TP}(\xi) \right)_{\xi=0}$$

and get

$$h^{PP}(q^2, s, \xi=0) \rightarrow \frac{\bar{\beta}(q^2)}{\beta(q^2)} \frac{1}{2} \log\left(\frac{s}{\xi^2}\right) \hat{D}^{PP}(0, 0) \quad (5.41)$$

Note that all multiplicities become target independent, since the function (5.34) does not depend on the target in the limit $\xi \rightarrow 0$. It is reasonable to assume that the functions $\bar{\beta}(q^2)$ and $\beta(q^2)$ both describing the off-shell dependence of the Pomanchuk residue are identical. Thus (5.41) strongly suggests that the multiplicity becomes independent of q^2 as the energy goes to infinity.

6. CONCLUDING REMARKS

We have considered a light cone approach to inclusive reactions. The abstraction framework used is a speculation about the simplicity of the strong interactions at light-like distances. It is essentially based on the assumption that the algebraic structure of Green's functions in the field theory model of strong interactions resembles the one in free quark theory as far as the leading light cone singularity is concerned. The problems we encounter in describing inclusive reactions is the lack of a causal representation in contrast to the case of deep inelastic scattering. In our approach the problem of multiplicity in current annihilation is quite different from the corresponding one in deep inelastic scattering. This is connected to the fact that the e^+e^- annihilation proceeds in the $L=0,1$ channels only, while in deep inelastic scattering the angular momentum distribution depends on the energy. Our algebraic abstractions are based on the quark model which has the property that leading light cone singularities are $SU(3) \times SU(3)$ invariant. This exact invariance leads to numerous symmetry relations and inequalities, as described in Sections 4 and 5. We should like to remark that these relations may also be obtained from the assumption that there are no exotic exchanges in the t channel, just as in ordinary deep inelastic scattering. In fact, it is easy to see that the algebraic relations discussed in Section 3 lead immediately to the absence of t channel exotics in the scaling limit. The advantage of our relations in coordinate space lies in the fact that they are supposed to be exact and therefore give us information under which conditions the symmetry relations become exact.

Finally, we should like to comment on the connection between hadronic (Feynman) scaling and deep inelastic (Bjorken) scaling. The scaling behaviour denoted in (5.8) may be considered as mixed scaling in the sense that both types of scaling are present. The scaling behaviour in the variable ξ is just the consequence of the simple light cone structure, like in deep inelastic scattering :

$$W_{\mu\nu}(t, p, z) \xrightarrow{z^2 \rightarrow 0} (\text{singular function of } z^2) \cdot F(tz, pz, u)$$

Once we have made this assumption, the scaling in the variable x arises naturally, since the residue of the leading light cone singularity $F(tz, pz, u)$ can also be written as $\hat{F}(tz, \frac{pz}{tz}, u) = \hat{F}(tz, x, u)$. The scaling parameter x just serves to describe the hadronic two-particle state consisting of the target $|t\rangle$ and the produced particle $|p\rangle$ (see Figs. 6a and 6b). Therefore the scaling in x has nothing to do with the Bjorken scaling in ξ , but only expresses the fact that one has to deal with fixed hadronic states $|t\rangle$ and $|p\rangle$ described by x and $p_\perp$ in going to the Bjorken limit. In Eqs. (5.28) - (5.30), we have indicated how to arrive at the ordinary hadronic scaling limit (q^2 fixed) by interchanging the generalized Bjorken and Regge limits. The hadronic scaling then arises as a consequence of Pomeron exchange between current and hadronic two-particle state $|tp\rangle$. This way we arrive at the description of hadronic scaling, given by Mueller¹⁵⁾.

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A P P E N D I X

We analyze the consequences of dispersive representations for the amplitudes

$$\begin{aligned} T_{\mu\nu} &= i \int d^4x e^{+iqx} \langle p | T(j_\mu(x) j_\nu(0)) | p \rangle \\ T_{\mu\nu}^{ret} &= i \int d^4x e^{+iqx} j(x) \langle p | [j_\mu(x), j_\nu(0)] | p \rangle \end{aligned} \quad (A.1)$$

supplemented by the assumption of canonical light cone singularities for current products. The idea is to relate $W_{\mu\nu}$, $\bar{W}_{\mu\nu}$ defined in Eqs. (2.4) and (2.8) to specific discontinuities of $T_{\mu\nu}$, $T_{\mu\nu}^{(ret)}$.

The invariant amplitudes without kinematical singularities are obtained from the decomposition in Eq. (2.5)¹⁶⁾

$$\begin{aligned} T_{\mu\nu}^{(ret)} &= (h_2)_{\mu\nu} T_2^{(ret)} + (h_1)_{\mu\nu} T_1^{(ret)} + \dots \\ h_{1,\mu\nu} &= -g_{\mu\nu} q^2 + q_\mu q_\nu \\ h_{2,\mu\nu} &= q^2 p_\mu p_\nu - \nu(p_\mu q_\nu + p_\nu q_\mu) + \nu^2 g_{\mu\nu} \end{aligned} \quad (A.2)$$

In the region of interest

$$q^2 \geq 2\nu$$

the amplitudes $T_i^{(ret)}$, $i=1,2$ feel the joint singularities corresponding to the diagrams in Figs. 1-4 in Section 2.

Treating q^2 as a parameter and neglecting eventual subtractions for the time being, we assume the following representation of $T_{1,2}$ in terms of double and single dispersion integrals (Cutkosky rules)

$$\begin{aligned}
 T_i(\nu, q^2) = & \int_{-q^2/2}^{\infty} d\nu_1 d\nu_2 \tau^i(\nu_1, \nu_2, q^2) \frac{1}{(\nu_1 - \nu - i\varepsilon)(\nu_2 + \nu - i\varepsilon)} \\
 & + \int_{-q^2/2}^{\infty} d\nu_1 \rho_1^i(\nu_1, q^2) \frac{1}{\nu_1 - \nu - i\varepsilon} \\
 & - \int_{-\infty}^{q^2/2} d\nu_2 \rho_2^i(\nu_2, q^2) \frac{1}{\nu_2 - \nu + i\varepsilon} \quad ; i=1,2
 \end{aligned} \tag{A.3}$$

In the double spectral region $q^2 > 2|\nu|$, the amplitudes T_i cannot be continued to analytic functions. Instead, one obtains

$$\begin{aligned}
 T_i(\nu, q^2) = & \int_{-q^2/2}^{\infty} d\nu' \rho_s^i(\nu', q^2) \frac{1}{\nu' - \nu - i\varepsilon} \\
 & - \int_{-\infty}^{q^2/2} d\nu' \rho_{\bar{u}}^i(\nu', q^2) \frac{1}{\nu' - \nu - i\varepsilon} \\
 & + 2\pi i \Theta(q^2/2 - \nu) \bar{\rho}_{\bar{u}}^i(\nu, q^2) \\
 \bar{\rho}_{\bar{u}}^i(\nu, q^2) = & \rho_2^i(\nu, q^2) + \int_{-q^2/2}^{\infty} d\nu_1 \frac{\tau^i(\nu_1, -\nu, q^2)}{\nu_1 - \nu - i\varepsilon} ; i=1,2
 \end{aligned} \tag{A.4}$$

In going from (A.3) to (A.4), we have assumed that

$$\begin{aligned}
 \tau^i(\nu, q^2) = & \int_{-q^2/2}^{\infty} d\nu_1 d\nu_2 \tau^i(\nu_1, \nu_2, q^2) \frac{1}{(\nu_1 - \nu - i\varepsilon)(\nu_2 + \nu + i\varepsilon)} \\
 = & \int_{-q^2/2}^{\infty} d\nu' \rho_1^i(\nu', q^2) \frac{1}{\nu' - \nu - i\varepsilon} \\
 & - \int_{-\infty}^{q^2/2} d\nu' \rho_2^i(\nu', q^2) \frac{1}{\nu' - \nu - i\varepsilon} \quad i=1,2
 \end{aligned}$$

which implies

$$\rho_s^i = \rho_1^i + \rho_1^i \alpha, \quad \rho_{\tilde{u}}^i = \rho_2^i + \rho_2^i \alpha \quad (\text{A.5})$$

In the Regge limit, q^2 fixed $\nu \rightarrow \infty$, we expect the asymptotic behaviour

$$T_1 \sim \nu \quad T_2 \sim \nu^{-1} \quad \nu \rightarrow \infty \quad (\text{A.6})$$

Taking into account the necessary subtractions for T_1 , we find, remembering the crossing relations $\rho_s^i(-\nu, q^2) = \rho_{\tilde{u}}^i(\nu, q^2)$

$$T_2 = \left\{ \begin{aligned} & \int_{-q^2/2}^{q^2/2} d\nu' \left[\rho_s^2(\nu', q^2) - \rho_{\tilde{u}}^2(\nu', q^2) \right] \frac{1}{\nu' - \nu - i\varepsilon} \\ & + \int_{q^2/2}^{\infty} d\nu' \rho_s^2(\nu', q^2) \left(\frac{1}{\nu' - \nu - i\varepsilon} + \frac{1}{\nu' + \nu + i\varepsilon} \right) \\ & + 2\pi i \theta(q^2/2 - \nu) \bar{\rho}_{\tilde{u}}^2(\nu, q^2) \end{aligned} \right\}$$

$$T_1 = \left\{ \begin{aligned} & a(\nu_0, q^2) + (\nu - \nu_0) b(\nu_0, q^2) + \\ & (\nu - \nu_0)^2 \left\{ \int_{-q^2/2}^{q^2/2} d\nu' \frac{1}{(\nu' - \nu_0)^2} \left[\rho_s^1(\nu', q^2) - \rho_{\tilde{u}}^1(\nu', q^2) \right] \frac{1}{\nu' - \nu - i\varepsilon} \right. \\ & \quad \left. + \int_{q^2/2}^{\infty} d\nu' \frac{1}{(\nu' - \nu_0)^2} \rho_s^1(\nu', q^2) \left(\frac{1}{\nu' - \nu - i\varepsilon} + \frac{1}{\nu' + \nu + i\varepsilon} \right) \right\} \\ & + 2\pi i \theta(q^2/2 - \nu) \bar{\rho}_{\tilde{u}}^1(\nu, q^2) \end{aligned} \right\} \quad (\text{A.7})$$

Analogous assumptions, as above, lead to the representation of the retarded amplitudes :

$$T_2^{\text{ret}} = \int_{-q^2/2}^{q^2/2} dv' \rho_{\text{ret}}^2(v', q^2) \frac{1}{v' - v - i\varepsilon} + \int_{q^2/2}^{\infty} dv' \rho_{\text{ret}}^2(v', q^2) \left(\frac{1}{v' - v - i\varepsilon} + \frac{1}{v' + v + i\varepsilon} \right) \quad (\text{A.8})$$

An analogous dispersion relation holds for T_1^{ret} .

We recall here the normalization relating $\rho^i(v, q^2)$ for $q^2 < 0$ to the functions V_i defined in Section 2

$$\rho^i = 2V_i$$

We consider further the function related to the current commutator

$$\rho_{\mu\nu}^{(c)} = \frac{1}{2\pi} \int dx e^{iqx} \langle P | [\bar{J}_\mu(x), J_\nu(0)] | P \rangle = (\bar{h}_1)_{\mu\nu} \rho_{(c)}^1 + (\bar{h}_2)_{\mu\nu} \rho_{(c)}^2 \quad (\text{A.9})$$

From locality and the supposed scaling properties of $\rho_{(c)}^i$ for $q^2 < 0$ it follows

$$\begin{aligned} v q^2 \rho_{(c)}^2 &\rightarrow 0 \\ q^2 \rho_{(c)}^1 &\rightarrow 0 \end{aligned} \quad \frac{q^2}{2v} = \xi > 1 \quad q^2 v \rightarrow \infty \quad (\text{A.10})$$

We consider in the following

$$\begin{aligned} T_{\mu\nu}(q, P) &= T_{\mu\nu}^{\text{ret}}(q, P) = 2\pi i \rho_{\nu\mu}^{(P)}(-q, P) \\ &= i \int dx e^{iqx} \langle P | \bar{J}_\nu(0) J_\mu(x) | P \rangle \end{aligned}$$

It follows from the light cone algebra that $\rho^{(P)}$ has to vanish in the annihilation region

$$\begin{aligned} \nu q^2 (T_2 - T_2^{\text{ret}}) &\rightarrow 0 \\ q^2 (T_1 - T_1^{\text{ret}}) &\rightarrow 0 \end{aligned} \quad \xi > 1, \nu, q^2 \rightarrow \infty \quad (\text{A.11})$$

For $q^2 > 0$ ρ_{ret}^J are in general complex and thus different from $\rho_{(C)}^J$.

We tentatively identify $\rho_{(C)}^J$ with the real part of ρ_{ret}^J and suppose that the imaginary part of ρ_{ret}^J obeys the same scaling law as the real part, or vanishes faster in the Bjorken limit. Then Eq. (A.10) implies

$$\begin{aligned} \nu q^2 \rho_{\text{ret}}^2 &\rightarrow 0 \\ q^2 \rho_{\text{ret}}^1 &\rightarrow 0 \end{aligned} \quad \xi > 1, \nu, q^2 \rightarrow \infty \quad (\text{A.12})$$

The light cone expansion for the time ordered product

$$T (\partial_\mu(x) \partial_\nu(0))$$

yields the asymptotic relations

$$\begin{aligned} \nu q^2 T_2 &\sim 2 \int_{-1}^{+1} d\alpha \frac{T_2(\alpha)}{\alpha - \xi i\epsilon\nu} \quad \xi > 1, q^2 > 0 \\ \nu q^2 V_2 &\sim T_2(\xi) \quad -1 \leq \xi \leq 1 \quad q^2 < 0 \\ q^2 T_1 &\sim 0 \quad \xi > 1 \end{aligned} \quad (\text{A.13})$$

Equation (A.13) is satisfied if we set :

$$\begin{aligned} \left. \begin{aligned} q^2 \bar{\rho}_u^1 &\sim 0 \\ q^2 \rho_s^1 &\sim 0 \end{aligned} \right\} \quad \xi > 1, \nu, q^2 \rightarrow \infty \\ \left. \begin{aligned} \nu q^2 (\rho_s^2 - \rho_u^2) &\sim 0 \\ q^2 (\rho_s^1 - \rho_u^1) &\sim 0 \end{aligned} \right\} \quad \xi > 1 \end{aligned} \quad (\text{A.14})$$

$$\nu q^2 \operatorname{Re} \rho_s^2 \sim \begin{cases} 2 F_2(\xi) & -1 \leq \xi \leq 1; q^2 < 0 \\ -2 F_2(\xi) & -1 \leq \xi \leq 1; q^2 > 0 \end{cases}$$

Comparing the full expression for T_2 in Eq. (A.7), and the asymptotic expansion

$$T_2 \sim 2\pi i \theta(q^2/2 - \nu) \bar{\rho}_2^2 + \int_{q^2/2}^{\infty} d\nu' \operatorname{Re} \rho_s^2(\nu', q^2) \left(\frac{1}{\nu' - \nu} + \frac{1}{\nu' + \nu} \right) \quad (\text{A.15})$$

we obtain the asymptotic relation

$$\bar{\rho}_2^2(\nu, q^2) \sim - \frac{1}{2\pi} \int_{q^2/2}^{\infty} d\nu' \operatorname{Im} \rho_s^2(\nu', q^2) \left(\frac{1}{\nu' - \nu} + \frac{1}{\nu' + \nu} \right) \quad (\text{A.16})$$

$q^2 \geq 2\nu, \quad \nu, q^2 \rightarrow \infty$

We emphasize at this point that the relation in (A.16) is a natural but not the unique solution of the asymptotic equations (A.10) - (A.14). Equations (A.13) and (A.14) are incompatible when we try to continue these equations to the region

$$0 \leq \xi \leq 1, \quad q^2 > 0$$

unless we drop the threshold factor $\theta(q^2/2 - \nu)$ multiplying $\bar{\rho}_u$.

Since

$$\rho_s^i(\nu, q^2) = 2\nu_2(\nu, q^2) \quad q^2 < 0 \quad (\text{A.17})$$

we assume now the existence of a dispersion relation in q^2 for ρ_s^i for fixed $s = q^2 + 2\nu$

$$\rho_s^i(\nu, q^2) = \frac{1}{\pi} \int_{s=q^2+2\nu}^{\infty} d q'^2 \frac{1}{q'^2 - q^2} \operatorname{Im} \rho_s^i(s, q'^2) \quad (\text{A.18})$$

The convergence of the integrals in (A.18) is assumed in analogy with the vanishing of $\rho_s^i(\nu, q^2)$ in the Bjorken-Johnson-Low limit ^{16), 17)}. The scaling properties of ρ_s^i are enforced if $\text{Im} \rho_s^i$ shows the following behaviour in the Bjorken limit

$$\begin{aligned} s^2 \text{Im} \rho_s^2 &\sim \tau(q^2/s) \quad q^2/s \text{ fixed}; q^2, s \rightarrow \infty \\ q^2 \text{Im} \rho_s^1 &\sim 0 \end{aligned} \quad (\text{A.19})$$

The scaling functions are obtained as limiting relations from Eqs. (A.16) - (A.19) :

$$\begin{aligned} F_2(\xi) &= \lim_{\substack{-q^2/2\nu \rightarrow \xi \\ q^2 < 0}} \frac{1}{2} \nu q^2 \rho_s^2 \quad 1 \geq \xi \geq 0 \\ &= - \frac{1}{4\pi} \frac{\xi}{1-\xi} \int_0^\infty d\tau \frac{\tau(\tau)}{(1-\xi)\tau + \xi} \\ \bar{F}_2(\xi) &= \lim_{\substack{q^2/2\nu \rightarrow \xi \\ q^2 > 0}} \frac{1}{2} \nu q^2 \bar{\rho}_s^2 = \frac{1}{2} [\phi(\xi) - \phi(-\xi)] \quad (\text{A.20}) \\ \phi(\xi) &= \frac{1}{4\pi} \int_0^\infty d\tau \frac{\tau \tau(\tau)}{(1-\xi)\tau + \xi} \quad \xi \geq 1 \end{aligned}$$

The behaviour of F_2 near the elastic threshold $\xi=1$ is given by

$$\begin{aligned} F_2(1-\varepsilon) &\sim \frac{1}{4\pi} \sum_{\nu=0}^N (-1)^\nu m_\nu \varepsilon^{\nu-1} + O(\varepsilon^N) \\ m_\nu &= \int_0^\infty d\tau \tau(\tau) \cdot \tau^\nu \end{aligned} \quad (\text{A.21})$$

provided that the moments m_ν , $\nu = 0, \dots, N$ exist.

Thus, if $F_2(1-\varepsilon)$ behaves like ε^N the first N moments of τ must vanish.

The Regge limit implies

$$F_2(\xi) \sim c \quad \xi \rightarrow 0$$

and, in turn,

$$\bar{\tau}(z) = -c \int \tau(z) + \bar{\tau}_1(z) \quad (\text{A.22})$$

If m_0 vanishes, Eq. (A.20) can be transformed to yield Eqs. (2.32) and (2.33) in Section 2 :

$$\left. \begin{aligned} \mathcal{F}_2(\xi) &= -\frac{1}{4\pi} \int_0^\infty dz \frac{\bar{\tau}(z) \cdot z}{(1-\xi)z + \xi} ; \quad 0 \leq \xi = \frac{q^2}{2\nu} \leq 1 \\ \bar{\mathcal{F}}_2(\xi) &= \frac{1}{2} \left(\phi(\xi) - \phi(-\xi) \right) \\ \phi(\xi) &= \frac{1}{4\pi} \int_0^{1/2} dz \frac{\bar{\tau}(z) \cdot z}{(1-\xi)z + \xi} \end{aligned} \right\} \quad 1 \leq \xi = \frac{q^2}{2\nu} \quad (\text{A.23})$$

$$\bar{\tau}(z) \cdot z = \tau_1(z) \cdot z$$

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FIGURE CAPTIONS

Figures 1 - 4 : Diagrams relevant to the current correlation function at timelike momenta.

Figure 5 : Diagram for the process :
 current $\rightarrow$ detected hadron state $|p\rangle$ + anything.

Figure 6 : Scattering diagrams for the processes :
 a) current + $t \rightarrow p$ + anything; and
 b) current + $t + \bar{p} \rightarrow$ anything.

Figure 7 : Typical scattering diagram for p being a fragment of t in the centre-of-mass system.

 

INCLUSIVE CURRENT INDUCED REACTIONS AND LIGHT
CONE SINGULARITIES ⁺)

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A B S T R A C T

We analyze inclusive reactions of the type

$$\text{current} + t \rightarrow p + \text{anything}$$

where t denotes either the vacuum or a hadronic target and p a produced hadron state. Conditions in momentum space are derived which enforce the scaling behaviour due to a canonical leading light cone singularity structure. These conditions imply that the scaling part of the multiplicity in e^+e^- annihilation is finite and that the corresponding multiplicity for the production of p off a hadron target grows less fast than the centre-of-mass energy. The problem of continuation of deep inelastic scattering amplitudes to the annihilation channel is investigated. Finally we outline how hadronic scaling is obtained in the limit of finite current momentum.

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1. INTRODUCTION

In this paper we consider inclusive reactions of the type :

$$\begin{aligned} \text{current} &\rightarrow \text{hadron} + \text{anything, and} \\ \text{current} + \text{hadron} &\rightarrow \text{hadron} + \text{anything} \quad 1). \end{aligned}$$

The simplest process is e^+e^- annihilation, which is considered in more detail in Section 2. The scaling behaviour as conjectured in Ref. 2) is found to be the consequence of a canonical singularity structure on the light cone, provided the scaling functions are sufficiently regular for large values of the scaling parameter $\xi = q^2/2\nu$. Thus an additional assumption is needed in order to ensure light cone dominance. This comes essentially from the fact that the matrix element for the inclusive process is not local in co-ordinate space and therefore does not have a causal representation. If the scaling functions are not well behaved for large ξ the scaling itself cannot be considered as a consequence of canonical singularities on the light cone. If this should be true in nature, we see no compelling reason why one should obtain symmetry relations derived from an $SU(3) \times SU(3)$ symmetric structure of current products, which is presumably true only at light-like distances. At the end of Section 2 we comment on the problem of continuing the electroproduction scaling functions to the annihilation region 2). Even if the light cone dominates in the annihilation region, we find no evidence for any relation between the two sets of scaling functions.

In Section 3 we generalize the light cone algebra of current commutators 3). The inclusive reactions are determined by matrix elements like

$$\sum_n \left\{ \langle t | \widehat{F}_{i\mu}(x) | p, n \rangle \langle n, p | \widehat{F}_{i\nu}(y) | t \rangle - h.c. \right\} \quad (1.1)$$

where t and p are arbitrary hadron states. The algebra of current commutators gives no direct information about matrix elements like (1.1). Nevertheless, one may abstract from field theoretic models like the free quark model or the formal quark gluon interaction model simple algebraic properties of matrix elements like (1.1). These abstractions are described in Section 3. We assume that our relations are right for the real world of hadrons, even if there are no quarks in the physical spectrum. It has to be noted that these abstractions are much more restrictive than the light cone algebra of current commutators. They might be wrong even if the latter are right. To prove or disprove the compatibility of the above abstraction with the absence of physical quarks amounts to the formidable task of constructing a non-trivial field theory model.

Our assumptions lead to the existence of operators $\phi_{k\sigma}(x, y, p)$ depending on two space-time co-ordinates x and y with $(x-y)^2 = 0$ and on the hadron state $|p\rangle$ produced in the inclusive reaction. The operators $\phi_{k\sigma}$ are generalizations of bilocal current operators and contain the latter as limiting values for $p_\mu \rightarrow 0$:

$$\phi_{k\sigma}(x, y, p) \xrightarrow{p_\mu \rightarrow 0} \widehat{f}_{k\sigma}(x, y) \quad (1.2)$$

The presence of the state $|p\rangle$ greatly influences the properties of these operators. In particular, they might be singular in the limit $x \rightarrow y$, where the bilocal currents by definition tend to local current operators. In order to ensure light cone dominance, we have to assume that these singularities are not too strong, otherwise our abstractions in co-ordinate space are still right but useless, since they cannot be applied in the scaling limit.

In Section 4 we consider current annihilation processes. They are described by vacuum expectation values of the operators $\phi_{k\sigma}$. We obtain symmetry relations and inequalities for cross-sections similar to corresponding relations derived for deep inelastic scattering. If the light cone dominates in the scaling region, we predict a finite multiplicity for the scaling contribution. Therefore an eventual (logarithmic) increase of the multiplicity with q^2 must originate from kinematical regions which shrink to 0 in the mathematical scaling limit⁴⁾. Our formalism indicates that the multiplicity of particles produced in current annihilation processes obeys a different logic as the multiplicity in deep inelastic scattering.

In Section 5 we apply the same technique to inclusive current induced reactions, which are described by one particle matrix elements of the operators $\phi_{k\sigma}$. We discuss a characteristic set of symmetry relations and inequalities for the structure functions following from our algebraic abstractions. We further outline a natural connection with hadronic scaling generalizing the exchange of scaling and Regge limits in deep inelastic scattering. The multiplicities increase logarithmically, if the scaling functions go to a finite limit at $x \rightarrow 0$. There is no indication that one should obtain SU(3) symmetric scaling functions and scaling multiplicities. The symmetry of the currents is not the symmetry of the states.

2. KINEMATICS, CONNECTION TO DEEP INELASTIC SCATTERING AND LIGHT-CONE ANALYSIS

First we consider the kinematics of the process

$$\text{current} \rightarrow \text{hadron} + \text{anything} \quad (2.1)$$

In particular, we concentrate on the reaction

$$e^+ + e^- \rightarrow \text{hadron} + \text{anything} \quad (2.2)$$

in the one-photon approximation. This process is related via crossing to virtual photon-hadron scattering

$$e^- + \text{hadron} \rightarrow \text{anything} \quad (2.3)$$

The latter is usually described by the structure functions W_1, W_2 or V_1, V_2 defined as follows ^{5),6)}.

The current correlation function is given by

$$W_{\mu\nu}(P, q) = \frac{1}{4\pi} \int dx e^{iqx} \langle P | [\bar{J}_\mu(x), J_\nu(0)] | P \rangle \quad (2.4)$$

where the connected matrix element of the current commutator is taken. We define the structure functions by the following decomposition

$$W_{\mu\nu} = \left(g_{\mu\nu} q^2 - g_{\mu\nu} q^2 \right) V_1(q^2, \nu) + \left(P_\mu P_\nu q^2 - \nu (P_\mu q_\nu + P_\nu q_\mu) + g_{\mu\nu} \nu^2 \right) \cdot V_2(q^2, \nu) \quad (2.5)$$

where $\nu = p \cdot q$. The functions V_1 and V_2 are related to the structure functions W_1 and W_2 by

$$\begin{aligned} V_1(q^2, \nu) &= \frac{1}{q^2} \left\{ W_1(q^2, \nu) + \frac{\nu^2}{q^2} W_2(q^2, \nu) \right\} \\ V_2(q^2, \nu) &= \frac{1}{q^2} W_2(q^2, \nu) \end{aligned} \quad (2.6)$$

The usual scaling assumptions in the deep inelastic region $q^2 \rightarrow -\infty, \nu \rightarrow \infty, -q^2/2\nu = \xi$ fixed, are

$$V_1 \xrightarrow{b_j} \frac{1}{2\nu} \frac{F_2(\xi)}{\xi} \quad V_2 \rightarrow -\frac{1}{2\nu^2} \frac{F_2(\xi)}{\xi} \quad (2.7)$$

or

$$W_1 \rightarrow F_1(\xi) \quad W_2 \rightarrow \frac{1}{\nu} F_2(\xi) \quad , \quad F_e(\xi) = -F_1(\xi) + \frac{F_2(\xi)}{2\xi}$$

Note also the following crossing and support properties of the scaling functions :

$$F_e(\xi) = -F_e(-\xi) \quad F_2(\xi) = F_2(-\xi)$$

$$F_e(\xi) = F_2(\xi) = 0 \quad \text{for} \quad |\xi| > 1$$

The annihilation process (2.2) is described by the matrix element ²⁾

$$\begin{aligned} \bar{W}_{\mu\nu}(p, q) &= \frac{1}{4\pi} (2\pi)^4 \sum_n \delta^4(p+n-q) \langle 0 | j_\mu(0) | p, n \rangle \times \\ &\quad \times \langle n, p | j_\nu(0) | 0 \rangle = \\ &= \frac{1}{4\pi} \int dx e^{+i q \cdot x} \sum_n \langle 0 | j_\mu(x) | p, n \rangle \langle n, p | j_\nu(0) | 0 \rangle = (2.8) \\ &= (q_\mu q_\nu - g_{\mu\nu} q^2) \bar{V}_1(q^2, \nu) + \left\{ p_\mu p_\nu q^2 - \nu (p_\mu q_\nu + p_\nu q_\mu) + g_{\mu\nu} \nu^2 \right\} \times \\ &\quad \times \bar{V}_2(q^2, \nu) \end{aligned}$$

where $|p\rangle$ is the one-particle state selected with four-momentum p_μ . The process is described by two structure functions $\bar{V}_1$ and $\bar{V}_2$ related to the previously defined functions V_1 and V_2 by crossing. It was first conjectured in Ref. 2) that the crossed functions also behave scale invariantly in the limit $q^2 \rightarrow \infty, \nu \rightarrow \infty, q^2/2\nu = \xi$ fixed

$$\begin{aligned} \bar{V}_1 &\xrightarrow[\text{by}]{\quad} \frac{1}{2\nu} \frac{\bar{F}_e(\xi)}{\xi} \quad \bar{V}_2 \xrightarrow[\text{by}]{\quad} \frac{1}{2\nu^2} \frac{\bar{F}_2(\xi)}{\xi} \\ \bar{F}_e(\xi) &= -\bar{F}_e(-\xi) \quad \text{crossing} \\ \bar{F}_2(\xi) &= \bar{F}_2(-\xi) \\ \bar{F}_1, -\bar{F}_2 &> 0 \quad \text{for} \quad \xi > 0 \end{aligned} \quad (2.9)$$

$$\bar{F}_e(\xi) = \bar{F}_2(\xi) = 0 \quad \text{for} \quad |\xi| < 1 \quad \text{support property}$$

The support property of the scaling functions is a consequence of the spectrum condition. It was also conjectured by the same authors ²⁾ that the scaling functions $\bar{F}_1$ (or $\bar{F}_l$) and $\bar{F}_2$ are simply related to the corresponding unbarred functions by analytic continuation. This is essentially a consequence of crossing symmetry in the particular parton model of Ref. 2). We shall indicate later in this section that this is not just a consequence of crossing symmetry alone, as one might suppose, but a very special assumption. In particular, the scaling itself as described by Eq. (2.9) might break down in the annihilation case but still work in the case of deep inelastic scattering without violating any principles of relativistic field theory as crossing symmetry. Before going into these details, we should like to demonstrate the connection between the scaling hypothesis (2.9) and the light-cone behaviour of the matrix element

$$\begin{aligned} \bar{W}_{\mu\nu}(x, P) &= \frac{2}{(2\pi)^3} \int d^4q e^{-iqx} \bar{W}_{\mu\nu}(q, P) \\ &= \sum_h \left\{ \langle 0 | J_\mu(x) | P, h \rangle \langle h, P | J_\nu(0) | 0 \rangle - h.c. \right\}^{(2.10)} \end{aligned}$$

In electroproduction, the scaling in the deep inelastic region may be described by the following behaviour of the functions $V_1(x^2, px)$ and $V_2(x^2, px)$ [which are just the Fourier transforms of the functions V_1 and V_2 as defined in (2.5)] near the light-cone ^{6),7)}:

$$\begin{aligned} V_1(x^2, px) &\xrightarrow{x^2 \rightarrow 0} \frac{1}{2\pi i} \varepsilon(x^0) \delta(x^2) F_1(px) \\ V_2(x^2, px) &\xrightarrow{x^2 \rightarrow 0} \frac{1}{2\pi i} \varepsilon(x^0) \theta(x^2) F_2(px) \end{aligned} \quad (2.11)$$

where the functions $F(px)$ and $F(\xi)$ are related by a simple Fourier transformation. One may suppose that a similar behaviour near the light cone also holds in the annihilation case for the corresponding functions $\bar{V}_1(x^2, px)$ and $\bar{V}_2(x^2, px)$, giving the scaling behaviour as in (2.9) ⁸⁾. Such a simple light cone analysis is not without troubles. One problem one is facing here is that the matrix element (2.10) is not local and therefore does not have to vanish at space-like distances, in contrast to the case of electroproduction. As a consequence, the structure functions in co-ordinate space $\bar{V}_1(x^2, px)$ and $\bar{V}_2(x^2, px)$ do not have to vanish at space-like x . Following the light cone assumption of (2.11), we may suppose the following behaviour of the structure functions near the light cone :

$$\begin{aligned}\bar{V}_1(x^2, px, \varepsilon(x), \dots) &\xrightarrow{x^2 \rightarrow 0} \frac{1}{2\pi i} \left\{ \varepsilon(x^0) \delta(x^2) F_1'(px) + \frac{1}{x^2} F_1''(px) \right\} \\ \bar{V}_2(x^2, px, \dots) &\xrightarrow{x^2 \rightarrow 0} \frac{1}{2\pi i} \left\{ \varepsilon(x^0) \theta(x^2) F_2'(px) + \log(x^2) F_2''(px) \right\}\end{aligned}\quad (2.12)$$

$$F_1'(px) = F_1'(-px) \quad F_1''(px) = -F_1''(-px), \dots$$

Instead of the simple singularities of the types $\varepsilon(x)\delta(x^2)$ or $\varepsilon(x)\theta(x^2)$ as in (2.11) dictated by scaling and locality all singular functions of the same dimension are allowed in (2.12). Due to this more complicated structure, the light cone analysis of the annihilation process becomes more involved. Another complication also connected with the lack of locality in the annihilation process is the following. We assume for the moment that $F_1''(px)$ in (2.12) vanishes and write :

$$\bar{V}_1(x^2, px) \rightarrow \frac{1}{2\pi i} \varepsilon(x^0) \delta(x^2) \bar{F}(px) \quad (2.13)$$

In momentum space we have :

$$\begin{aligned}\bar{V}_1(q^2, \nu) &\rightarrow \frac{1}{8\pi^2 i} \int dx e^{+i q x} \varepsilon(x^0) \delta(x^2) \bar{F}(px) = \\ &= \frac{1}{8\pi^2 i} \int_{-\infty}^{+\infty} d\alpha \bar{F}(\alpha) \int dx e^{+i(q+\alpha p)x} \varepsilon(x^0) \delta(x^2)\end{aligned}\quad (2.14)$$

where the function $\bar{F}(\alpha)$ is defined by :

$$\bar{F}(px) = \int d\alpha e^{+i\alpha(px)} \bar{F}(\alpha) \quad (2.15)$$

Finally we obtain :

$$\begin{aligned}\bar{V}_1(q^2, \nu) &\rightarrow \frac{1}{2} \int d\alpha \bar{F}(\alpha) \varepsilon(\nu + \alpha p^2) \delta(q^2 + 2\alpha\nu + \alpha^2 p^2) \\ &\sim \frac{1}{4\nu} \left[\bar{F}\left(\frac{-q^2}{2\nu}\right) - \bar{F}\left(\frac{-2\nu}{p^2}\right) \right]\end{aligned}\quad (2.16)$$

The two terms in (2.16) come from the fact that the argument of the δ function in (2.16) has two roots. It is instructive to compare the analysis above with the corresponding one for deep inelastic scattering. There one obtains just one term since the second root at $\alpha = -2\nu/p^2$ drops out because of the support properties : $F(\alpha) = 0, |\alpha| > 1$. In the annihilation case the same argument does not work and we have to keep the second root. If $\bar{F}(\omega)$ is zero, one obtains the scaling behaviour (2.9) from the light cone assumption (2.12). If $\bar{F}(\omega)$ is a constant or does not exist, say

$$\bar{F}\left(-\frac{2\nu}{p^2}\right) \xrightarrow{\nu \rightarrow \infty} \text{const.} \cdot \nu^\alpha \quad \alpha \geq 0$$

the scaling corresponding to (2.9) breaks down. This example shows clearly that in the annihilation case a simple connection between scaling and leading light cone singularity need not exist. One has to apply the second assumption that the function $\bar{F}(\alpha)$ vanishes as $\alpha \rightarrow \omega$. According to the definition (2.15), this means that the residue of the leading light cone singularity $\bar{F}(px)$ at $px \rightarrow 0$ is not too strongly divergent. Indeed, a breakdown of the scaling behaviour (2.9) is expected if $\bar{F}(px)$ diverges as $px \rightarrow 0$ stronger than $(px)^{\delta-1}$, $\delta > 0$. A stronger divergence of $\bar{F}(px)$ could mean that the decomposition (2.12) or (2.13) does not encounter the full singularity structure of the function $\bar{V}(x^2, px)$. Besides the light cone singularity described by $\varepsilon(x)\delta(x^2)$ in (2.12) there are additional singularities concentrated at the origin which may give rise to the singular behaviour of $F(px)$ at $px \rightarrow 0$. In the electroproduction case, such singularities are avoided by locality and the spectrum condition.

It is instructive to consider in this respect the light cone algebra of currents applied for deep inelastic scattering. There the scaling functions $F(px)$ are simply expectation values of a bilocal current operator $\tilde{f}_{i\mu}(x, y)$, which by definition is smooth in the limit $x \rightarrow y$, where one gets the local currents. In the annihilation case the function $\bar{F}(px)$ is not an expectation value of a bilocal operator, it might therefore be singular at $px \rightarrow 0$. In particular, the behaviour

$$\bar{F}_i(\xi) \xrightarrow{\xi \rightarrow \infty} \text{const.} \cdot \xi^{-\delta} \quad \delta > 0$$

would give

$$\bar{F}_i(p^2) \xrightarrow{p^2 \rightarrow 0} \text{const.} (p^2)^{\delta-1} \quad (2.17)$$

This is just the limiting power law where the leading light cone singularity still dominates, as one can easily see. If it should turn out by experiment that $\bar{V}_2(q^2, \nu)$ scales as in (2.9), but $\bar{F}_2(\xi)$ behaves like ξ^α , $\alpha \geq 1$ at large ξ , the light cone considerations break down. In this case we cannot hope to factorize the function $\bar{V}_2(x^2, px)$ into a singular piece in x^2 and a function depending on px , as written in (2.12). In all what follows, we assume that this is not the case and $\bar{F}_2(\xi)/\xi \rightarrow 0$, $\xi \rightarrow \infty$.

Finally we should like to comment on the conjecture that the scaling functions $F(\xi)$ and $\bar{F}(\xi)$ are pieces of one analytic function ²⁾. We first consider the current correlation function (2.4) at time-like q . Here one has to take into account besides the ordinary diagram, Fig. 1 ⁹⁾

Fig.1

the additional diagrams, Figs. 2, 3 and 4

Fig.2

Fig.3

Fig.4

The matrix element (2.4) is a sum of terms described by the various diagrams, Figs. 1-4. The Z diagram of Fig. 2 in particular describes the annihilation process, but the information one has from the matrix element (2.4) is not sufficient to determine the annihilation structure functions $\bar{V}(q^2, \nu)$, since one cannot separate the term described by Fig. 2 from the sum. But the light cone structure, as assumed in (2.11), and the fact that the scaling functions $F(\xi)$ vanish for $|\xi| > 1$ give information about the behaviour of the structure functions $V(q^2, \nu)$ in the deep inelastic limit at positive q^2 . In particular these functions have to scale to 0 at $|\xi| > 1$:

$$\left. \begin{array}{l} \nu q^2 V_2(q^2, \nu) \\ q^2 V_1(q^2, \nu) \end{array} \right\} \xrightarrow{b_2} 0 \quad \left| \frac{q^2}{2\nu} \right| = |\xi| > 1 \quad (2.18)$$

Since the annihilation process is partly determined by one of the diagrams describing the current correlation function (2.4) at positive q^2 (Fig.2), e.g.,

$$V(q^2, \nu) = \bar{V}(q^2, \nu) + \text{other terms}$$

one has to assume that in the remaining terms described by the diagrams, Figs. 1, 3 and 4, there are pieces which in the deep inelastic limit also give contributions at $|\xi| > 1$ and cancel the corresponding ones of $\bar{V}(q^2, \nu)$ in order to ensure (2.18). This non-trivial information about the behaviour of the structure functions $V(q^2, \nu)$ at time-like q^2 may be used together with dispersion relations in order to obtain information about the annihilation scaling functions $\bar{F}(\xi)$. In the Appendix we demonstrate how this could be done. The main conclusions are as follows.

If we assume the existence of dispersion relations in q^2 and ν for the virtual Compton amplitude and also that we are allowed to interchange scaling limits and dispersion integrations, we arrive at the following integral representation for $F_2(\xi)$:

$$\bar{F}_2(\xi) = - \frac{1}{4\pi} \int_0^\infty d\tau \frac{\tilde{\tau}(\tau) \cdot \tau}{(1-\xi)\tau + \xi} \quad (2.19)$$

$0 \leq \xi \leq 1$

where the function $\tau(z)$ is defined in the Appendix, (A.20). The annihilation scaling function $\bar{F}_2(\xi)$ is given by the following representation :

$$\begin{aligned}\bar{F}_2(\xi) &= \frac{1}{2} (\phi(\xi) - \phi(-\xi)) \\ \phi(\xi) &= \frac{1}{4\pi} \int_0^{\frac{1}{2}} dz \frac{\tau(z) \cdot z}{(1-\xi)z + \xi} \quad \xi \geq 1\end{aligned} \quad (2.20)$$

Corresponding representations exist for F_1 and $\bar{F}_1$. Since the integration range in (2.19) is different from the one in (2.20), one does not find any trace of the conjecture in Ref. 2) that $F(\xi)$ and $\bar{F}(\xi)$ are different pieces of one analytic function¹⁰⁾. In particular it might be that the function $\tau(z)$ vanishes in the interval $0 \leq z \leq \frac{1}{2}$, which would mean that the annihilation structure function $\bar{V}_2(q^2, \nu)$ does not scale according to (2.9). We cannot conclude from a general light cone and dispersion analysis any connection between scaling and scaling functions in electroproduction and e^+e^- annihilation.

3. LIGHT CONE RELATIONS ABSTRACTED FROM A QUARK PICTURE

The scaling behaviour observed for the electroproduction structure functions

$$\begin{aligned}V_1(q^2, \nu) &\xrightarrow{b_j} 0 \quad (\text{large errors}) \\ V_2(q^2, \nu) &\xrightarrow{b_j} - \frac{1}{2\nu^2} \frac{F_2(\xi)}{\xi}\end{aligned} \quad (3.1)$$

has given support for a simple light cone algebra of the observed electromagnetic and weak currents³⁾. The idea is to abstract from a field theoretic quark model algebraic relations for the leading light cone singularity of a current commutator. Such relations are supposed to be true in the real world of hadrons, even if quarks do not exist as real particles. The light cone relations, which one might abstract from a renormalizable quark-gluon field theory model by considering the perturbation expansion term by term, contain powers of unwanted logarithmic factors which destroy the scaling behaviour in the deep inelastic region. Therefore one abstracts the light cone algebra

of currents either from a free quark model or from a formal interacting quark-gluon theory neglecting the logarithmic terms and also unwanted additional non-canonical singularities ("anomalies") coming from renormalization¹¹⁾.

The simplest abstraction one can make is the algebraic structure of the leading light cone singularity of a current commutator, for example

$$\begin{aligned} [\widehat{T}_{i\mu}(x), \widehat{T}_{j\nu}(y)] &\cong [\widehat{T}_{i\mu}^{(5)}(x), \widehat{T}_{j\nu}^{(5)}(y)] \cong \\ &\frac{1}{4\pi} \partial^\rho (\varepsilon(z^0) \delta(z^2)) \left\{ i f_{ijk} \left[\widehat{T}_{k\sigma}(x,y) + \widehat{T}_{k\sigma}(y,x) \right] S_{\mu\nu\rho\sigma} \right. \\ &\quad \left. - i \varepsilon_{\mu\nu\rho\sigma} \left(\widehat{T}_{k\sigma}^{(5)}(x,y) - \widehat{T}_{k\sigma}^{(5)}(y,x) \right) \right] + \\ &\quad + d_{ijk} \left[S_{\mu\nu\rho\sigma} \left(\widehat{T}_{k\sigma}(x,y) - \widehat{T}_{k\sigma}(y,x) \right) - \right. \\ &\quad \left. - i \varepsilon_{\mu\nu\rho\sigma} \left(\widehat{T}_{k\sigma}^{(5)}(x,y) + \widehat{T}_{k\sigma}^{(5)}(y,x) \right) \right] \right\} \end{aligned} \quad (3.2)$$

$$S_{\mu\nu\rho\sigma} = g_{\mu\rho} g_{\nu\sigma} + g_{\mu\sigma} g_{\nu\rho} - g_{\mu\nu} g_{\rho\sigma}, \quad z = x - y$$

The bilocal current operators $\widehat{T}_{k\sigma}^{(5)}(x,y)$ as defined by (3.2) reduce to local currents in the limit $x \rightarrow y$. Their expectation values determine the corresponding scaling functions in the deep inelastic region. Because of the spectrum condition and locality, these expectation values are entire analytic functions in pz (of exponential type 1).

Let us now consider matrix elements of the following kind :

$$\begin{aligned} \overline{W}_{\mu\nu}^{ij}(p, x) &= \\ &\sum_n \left(\langle 0 | \widehat{T}_{i\mu}(x) | p, n \rangle \langle n, p | \widehat{T}_{j\nu}(y) | 0 \rangle - h.c. \right) \end{aligned} \quad (3.3)$$

or, more generally,

$$\begin{aligned} \overline{W}_{\mu\nu}^{ij}(p, x, t) &= \\ &\sum_n \left(\langle t | \widehat{T}_{i\mu}(x) | p, n \rangle \langle n, p | \widehat{T}_{j\nu}(y) | t \rangle - h.c. \right) \end{aligned} \quad (3.4)$$

where $|p\rangle$ and $|t\rangle$ are definite hadron states (for example, one-particle states) with momenta p_μ and t_μ . In the special case of the electromagnetic current, the matrix element (3.3) is related to the inclusive process

$$e^+ + e^- \rightarrow p + \text{anything}$$

In general, one might consider the operator

$$O_{\mu\nu}^{ij}(x, p) = \sum_n \left\{ \widehat{f}_{i\mu}(x) / p, n \right\rangle \langle n, p / \widehat{f}_{j\nu}(y) - \left\{ \widehat{f}_{j\nu}(y) / p, n \right\rangle \langle n, p / \widehat{f}_{i\mu}(x) \right\} \quad (3.5)$$

The question now arises if we can develop a similar abstraction framework for operators like (3.5) as the light cone algebra for ordinary current commutators. Before we consider operators like (3.5) in general, we return to the vacuum matrix element (3.3) and treat the special case when the particle detected is a vector meson or a pseudoscalar meson. Then the matrix element (3.3) is determined by a four-current function as

$$\langle 0 / T(\widehat{f}_{i\mu}(x) j(z)) \cdot T^*(\widehat{f}_{j\nu}(y) j(t)) / 0 \rangle \quad (3.6)$$

where $j(z)$ denotes a current or the divergence of an axial current, respectively ¹²⁾. In particular, one has :

$$W_{\mu\nu}^{ij}(x, p) = \int d^4z d^4t e^{ip(z-t)} \left\{ \begin{aligned} & (\square_z + p^2)(\square_t + p^2) \langle 0 / T(\widehat{f}_{i\mu}(x) j(z)) T^*(\widehat{f}_{j\nu}(y) j(t)) / 0 \rangle \\ & - \{ \mu \leftrightarrow \nu, i \leftrightarrow j, x \leftrightarrow y \} \end{aligned} \right\} \quad (3.7)$$

Note that we can, by using a field theoretic model, also treat baryons the same way, if we replace the current $j(z)$ by an interpolating field with the quantum number of the corresponding baryon, e.g., $q(z)q(z)q(z)$. The choice of this interpolating field is somehow ambiguous, but the matrix element (3.7) is independent of any ambiguities since the baryon is on its mass shell.

We are interested in the behaviour of the function (3.6) in the vicinity of the light cone $(x-y)^2 \rightarrow 0$. All other distances involved : $(x-z)$, $(x-t)$, $(y-z)$, $(y-t)$, $(z-t)$ are not constrained since we later integrate over the mass-shell as denoted by (3.7). Note that in (3.6), one is not testing a current product near the light cone since the T product in (3.6) contains terms in which the two currents, whose separation becomes light-like, are not next to each other, e.g., there are pieces like

$$\langle 0 / \widehat{f}_{i\mu}(x) j(z) j(t) \widehat{f}_{j\nu}(y) / 0 \rangle \quad (3.8)$$

Such a function is supposed to be known and identical to the one obtained from a free quark model with quark statistics if all separations are light-like (tend to a light-like line)¹³⁾. This is not the case here, since the distances involving z and t are still arbitrary. The question now arises : is it possible to obtain information about vacuum expectation values like (3.7) or, more generally, about current products like

$$\tilde{T}_{\mu}(x) j(z) j(t) \tilde{T}_{\nu}(y) \quad (3.9)$$

where $(x-y)^2 \rightarrow 0$, but all other distances are arbitrary ? We note that a Wightman function of currents like (3.7) is a boundary value of an analytic function in co-ordinate space. In particular the function (3.8) and the permuted function

$$\langle 0 / j(z) \tilde{T}_{\mu}(x) \tilde{T}_{\nu}(y) j(t) / 0 \rangle \quad (3.10)$$

are connected by analytic continuation, according to the theorem of Bargmann, Hall and Wightman¹⁴⁾. The permuted function (3.10) is given in the limit $(x-y)^2 \rightarrow 0$ as a singular function times an expectation value of a bilocal current :

$$\int^p(x-y) \langle 0 / j(z) \tilde{T}_{\mu}(x,y) j(t) / 0 \rangle \quad (3.11)$$

The question now arises, what happens if we carry out the analytic continuation from the function (3.10) to the permuted one (3.8) ? Do we still get the same type of singularity in $(x-y)^2$ for (3.8) as in (3.10) ? In other words, is it allowed just to take the asymptotic expansion (3.11) and to carry out the analytic continuation in the matrix element of the bilocal operator ? From general arguments, we cannot give an answer to these questions. Instead, we proceed similarly as in current algebra and consider first a relativistic quark field theory, abstract simple algebraic relations and then postulate these relations as being true for the real world of hadrons, even if there are no quarks or particles with triality non-zero in the physical spectrum. For simplicity, we use the free quark model for our abstraction, but we could also consider formal interacting gluon models, which lead to the same algebraic relations. In doing so, we again are faced with the question if such abstract algebraic relations already imply the existence of triality non-zero states in the particle spectrum. We assume that this is not the case. Since we always work with bilinear densities in the formal quark fields, there is no obvious contradiction to this assumption. A possible candidate for a quark field theory without quark particles would be a field theory, where any field with triality non-zero quantum number does not produce asymptotic states.

We can write formally :

$$T(\tilde{f}_{i\mu}(x) j(z)) T^*(\tilde{f}_{j\nu}(y) j(t)) \sim$$

$$T\left(\bar{q}(x) \frac{1}{2} \lambda_i \delta_{\mu} q(x) j(z)\right) T^*\left(\bar{q}(y) \frac{1}{2} \lambda_j \delta_{\nu} q(y) j(t)\right) \quad (3.12)$$

In the free theory and also in formal interacting gluon models, the leading singularity in $(x-y)^2$ is given by a single quark contraction (connected part only). We assume that we can abstract this behaviour. Our assumption makes sure that one is allowed just to take the most singular term in (3.9) as given by (3.10) and carry out the analytic continuation in co-ordinate space to arrive at the strongest light cone singularity for the function (3.8). After integrating over z and t we obtain the following algebraic relations :

$$\int d^2z d^2t e^{i p(t-z)} \left\{ (\square_z + p^2)(\square_t + p^2) T(\tilde{f}_{i\mu}(x) j(z)) \cdot T^*(\tilde{f}_{j\nu}(y) j(t)) - \right.$$

$$\left. - h.c. \right\}$$

$$= \sum_n \left(\tilde{f}_{i\mu}(x) |p, n\rangle \langle n, p| \tilde{f}_{j\nu}(y) - h.c. \right)$$

$$= \frac{1}{4\pi} \left(\partial^\rho \varepsilon_{\mu\nu\sigma} \delta(z^2) \right) \int \left\{ i f_{ijk} \left\{ \varepsilon_{\mu\nu\rho\sigma} (\phi_{k\sigma}(x, y, p) + \phi_{k\sigma}(y, x, p)) \right. \right. \quad (3.13)$$

$$\left. \left. - i \varepsilon_{\mu\nu\rho\sigma} (\phi_{k\sigma}^{(5)}(x, y, p) - \phi_{k\sigma}^{(5)}(y, x, p)) \right\} \right.$$

$$\left. + d_{ijk} \left\{ \varepsilon_{\mu\nu\rho\sigma} (\phi_{k\sigma}(x, y, p) - \phi_{k\sigma}(y, x, p)) \right. \right.$$

$$\left. \left. - i \varepsilon_{\mu\nu\rho\sigma} (\phi_{k\sigma}^{(5)}(x, y, p) + \phi_{k\sigma}^{(5)}(y, x, p)) \right\} \right\}$$

$z' = x - y$

The quantities $\phi^{(5)}(x, y, p)$ defined by relation (3.8) can be written in the model as :

$$\phi_{k\sigma}(x, y, p) = \int d^2z d^2t (\square_z + p^2)(\square_t + p^2) \cdot$$

$$\cdot \left(T(\bar{q}(x) j(z)) \frac{1}{2} \lambda_k \delta_\sigma (T^* q(y) j(t)) \right) \quad (3.14)$$

We abstract just the Lorentz - and $SU(3)$ - properties of these operators, but reject their specific structure in terms of the underlying quark fields. The algebraic relations (3.8) are very similar to the light cone relations for ordinary current commutators ³⁾, where instead of the quantities ϕ the bilocal current operators appear. But the operators ϕ have different properties than the bilocal currents :

- 1) they do not fulfil a closed algebra;
- 2) they contain information about the state p ;
- 3) especially they have additional Lorentz - $SU(3)$ - indices characterizing the transformation properties of the hadron state $|p\rangle$;
- 4) they may be singular in the limit $x \rightarrow y$.

In Section 2 we have emphasized that matrix elements like (3.3) or in general operators like (3.5) or (3.13) need not vanish at space-like distances $((x-y)^2 < 0)$, since locality does not apply to this situation. So it is astonishing that the relations (3.13) are local. This comes from our assumption that the leading singularity with respect to $(x-y)^2$ in (3.13) is described formally by a single quark contraction. This singularity is not influenced by the state $|p\rangle$, and the leading singularity in (3.13) is therefore local, but all less-leading terms, of course, will not be local.

The algebraic relations (3.13) are assumed to be true for all hadron states $|p\rangle$. If $|p\rangle$ is a baryon state, the operators ϕ have now two additional Dirac indices. One also can see easily that it is not essential that the state p detected in the final state is a one-particle state. It may also be a multiparticle state.

Note that it is essential to integrate in (3.8) over z and t . In such a way we take into account that the hadron state p is a smeared out object. If z or t happen to be identical with x or y , one gets a stronger singularity as denoted in (3.13) for the four current product (3.9). Such stronger singularities are suppressed if one integrates over z and t with the phase $e^{-ip(z-t)}$ where $|p\rangle$ is constrained on the mass shell of the corresponding particle.

We should like to note that the algebraic relations (3.13) cannot be directly related to the light cone algebra of current commutators, although the analogy is obvious. But nevertheless both abstractions are different, and the ansatz expressed in (3.13) might be wrong, even if the light cone relations for ordinary current commutators are right. Let us finally study the phenomenological structure of the relations (3.13). The tensor - and singularity structure of these relations leads, for example, to the conclusion that the scaling function $\bar{F}_\ell(\xi)$ vanishes. This, of course, is analogous to the fact that the tensor - and singularity structure of the quark light cone algebra of currents leads to the vanishing of the cross-section for longitudinal photons in the scaling limit.

The algebraic relations (3.13) are completely $SU(3) \times SU(3)$ symmetric with respect to the current indices, but one should be beware of transforming the hadron state $|p\rangle$. $SU(3) \times SU(3)$ breaking will just affect the lower lying singularities on the light cone. We emphasize again that the operators ϕ cannot be regarded as bilocal operators in the usual sense. In particular their Taylor expansion in $(x-y)$ need not exist and certainly does not start with a local current operator. Therefore there is no possibility of obtaining sum rules as in deep inelastic scattering. However, one obtains symmetry relations and inequalities for the structure functions, which arise from the $SU(3) \times SU(3)$ structure of the relations (3.13). In the following two sections we consider several immediate consequences of our assumptions.

4. CURRENT ANNIHILATION PROCESSES

Current annihilation processes are described by vacuum expectation values of the relations (3.13). They correspond to reactions of the type described in Fig. 5 :

Fig.5

In particular we concentrate on the special case where the state $|p\rangle$ is just a one-particle state. Since the formalism is analogous to the case of deep inelastic scattering, we can use the same technique as in Ref. 3) to obtain information about the structure functions. We take the usual expressions for the weak and electromagnetic currents in terms of the current-octet (in particular, we assume that the electromagnetic current transforms as an octet operator, without any charmed component). The scaling functions are given as Fourier transforms of the vacuum expectation values of the quantities ϕ_{kp} . We define :

$$\langle 0 | \phi_{kp}(x, y, p) \pm \phi_{kp}(y, x, p) / 0 \rangle = \frac{2}{2} \frac{\bar{S}_k(p^2) \cdot P_p}{\bar{A}_k(p^2) \cdot P_p} + \text{terms proportional to } 1/p^2$$

$$\bar{S}_k(p^2) = \int d\alpha e^{-i\alpha(p^2)} \tilde{S}_k(\alpha), \quad \bar{A}_k(p^2) = \int d\alpha e^{-i\alpha(p^2)} \tilde{A}_k(\alpha) \quad (4.1)$$

Using these definitions, we obtain for the electromagnetic case :

$$\begin{aligned}\bar{F}_1^e(\xi) &= \frac{1}{2} \left(\frac{2}{3} \sqrt{\frac{2}{3}} \hat{A}_0(\xi) + \frac{1}{3\sqrt{3}} \hat{A}_8(\xi) + \frac{1}{3} \hat{A}_3(\xi) \right) \\ \bar{F}_e^e(\xi) &= 0\end{aligned}\quad (4.2)$$

Similarly, one gets relations for the weak processes which are formally identical to the corresponding unbarred relations in the case of deep inelastic scattering [see Ref. 3]. In particular, we also obtain symmetry relations, for example

$$12 \left(\bar{F}_1^{eP}(\xi) - \bar{F}_1^{eN}(\xi) \right) = \left(\bar{F}_3^{\nu P}(\xi) - \bar{F}_3^{\nu N}(\xi) \right) \quad (4.3)$$

$|p\rangle$ and $|n\rangle$ denote proton and neutron states, respectively. Unfortunately, one cannot easily produce weak currents with time-like four-momenta, so these relations have no immediate practical application. We note again that it is impossible to obtain sum rules for the scaling functions. From (4.2) we also obtain relations for arbitrary isospin multiplets with total isospin $> \frac{1}{2}$:

$$\frac{\bar{F}_1^e(e) P_i^I - \bar{F}_1^e(e) P_j^I}{\bar{F}_1^e(e) P_{i'}^I - \bar{F}_1^e(e) P_{j'}^I} = \frac{i-j}{i'-j'}$$

Here the indices I and $i, j, \dots$ denote the total isospin of the multiplet and the third components, respectively. In particular, we have :

$$2 \bar{F}_1^{e\pi^0} = \bar{F}_1^{e\pi^+} + \bar{F}_1^{e\pi^-}$$

The relations (3.13) also imply, together with positivity, inequalities like the following one :

$$\frac{1}{4} \leq \frac{\bar{F}_1^{e\pi^+}(\xi)}{\bar{F}_1^{eP}(\xi)} \leq 4 \quad (4.4)$$

Note that the same inequality holds also for the (unbarred) electro-production functions. The simplest proof is the following ^{15),16)}. The electromagnetic current is written as

$$J_\mu = \widehat{f}_{3\mu} + \frac{1}{\sqrt{3}} \widehat{f}_{8\mu} = \frac{2}{3} \widehat{f}_{u\mu} - \frac{1}{3} \widehat{f}_{d\mu} - \frac{1}{3} \widehat{f}_{s\mu} \quad (4.5)$$

where the currents $\widehat{f}_{u\mu} \dots$ are certain combinations of the octet currents $\widehat{f}_{i\mu}$, with transformation properties as given by the following formal quark representation :

$$\begin{aligned} \widehat{f}_{u\mu} &\sim \bar{q} \gamma_\mu \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} q, \quad \widehat{f}_{d\mu} \sim \bar{q} \gamma_\mu \begin{pmatrix} 0 & & \\ & 1 & \\ & & 0 \end{pmatrix} q, \\ \widehat{f}_{s\mu} &\sim \bar{q} \gamma_\mu \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix} q \end{aligned} \quad (4.6)$$

The currents $\widehat{f}_{u\mu} \dots$ are Hermitean. From the current algebra at light-like distances, one finds :

$$[\partial_\mu, \partial_\nu] \cong \frac{4}{9} (\widehat{f}_{u\mu} \widehat{f}_{u\nu}) + \frac{1}{9} (\widehat{f}_{d\mu} \widehat{f}_{d\nu}) + \frac{1}{9} (\widehat{f}_{s\mu} \widehat{f}_{s\nu}) \quad (4.7)$$

Note that these relations are right only with respect to the leading singularity on the light cone. A similar decomposition as in (4.7) also holds for algebraic relations of the type (3.13). We therefore obtain :

$$\bar{F}_i^{ep} = \frac{4}{9} \bar{F}_i^{up} + \frac{1}{9} \bar{F}_i^{dp} + \frac{1}{9} \bar{F}_i^{sp} \quad (4.8)$$

and, using SU(2) symmetry :

$$\bar{F}_i^{en} = \frac{1}{9} \bar{F}_i^{up} + \frac{4}{9} \bar{F}_i^{dp} + \frac{1}{9} \bar{F}_i^{sp}$$

where the functions $\bar{F}_1^{\text{up}}(\xi) \dots$ describe the correlation functions $\langle 0 | \widehat{f}_{u\mu}(x) | p, n \rangle \langle n, p | \widehat{f}_{u\nu}(y) | 0 \rangle \dots$ in the light cone limit. Since these functions are positive definite because of the Hermiticity properties of the currents (4.6), one immediately arrives at the inequality (4.4). Similar relations one may obtain for any isospin multiplet. We denote the detected particle by $\begin{smallmatrix} I \\ i \end{smallmatrix}$ where I is the total isospin of the multiplet and i, j stand for its third component. Then we find :

$$\frac{1}{4} \leq \frac{1 - \frac{3}{5} \left| \frac{i}{I} \right|}{1 + \frac{3}{5} \left| \frac{j}{I} \right|} \leq \frac{\bar{F}_1 e^{p_1^I}}{\bar{F}_1 e^{p_2^I}} \leq \frac{1 + \frac{3}{5} \left| \frac{i}{I} \right|}{1 - \frac{3}{5} \left| \frac{j}{I} \right|} \leq 4$$

In particular, we obtain :

$$\frac{1}{4} \leq \frac{\bar{F}_1 e^{\bar{\alpha}^-}}{\bar{F}_1 e^{\bar{\alpha}^+}} \leq 4, \quad \frac{2}{5} \leq \frac{\bar{F}_1 e^{\bar{\alpha}^+}}{\bar{F}_1 e^{\bar{\alpha}^0}} \leq \frac{8}{5}$$

$$\frac{2}{3} \leq \frac{\bar{F}_1 e^{\alpha^+}}{\bar{F}_1 e^{\alpha^0}} \leq \frac{3}{2} \quad \text{etc.} \quad (4.9)$$

Let us finally consider the consequences of our assumptions for the multiplicity of the outgoing hadrons. Using the variable $\omega = 1/\xi$ we find for the scaling contribution to the multiplicity, provided the total annihilation cross-section behaves like const/q^2 17) :

$$n^{\text{scaling}}(q^2) \sim \text{const.} \int_{\sqrt{\frac{4p^2}{q^2}}}^1 d\omega \cdot \omega \bar{F}_1(\omega) \quad (4.10)$$

According to the general light cone considerations of Section 2, the asymptotic expansion of $\bar{F}_1(\omega)$ is

$$\bar{F}_1(\omega) \xrightarrow{\omega \rightarrow 0} 0 \quad (4.11)$$

and therefore $n^{\text{scaling}}(q^2)$ is finite at large q^2 18). We note that a logarithmic increase of the multiplicity $n^{\text{scaling}}(q^2)$ would imply :

$$\bar{F}_1(\omega) \xrightarrow{\omega \rightarrow 0} \frac{\text{const.}}{\omega^2} \quad (4.12)$$

Such a behaviour cannot be obtained from light cone dominance. Of course, the total multiplicity is not at all determined by the scaling contribution $n^{\text{scaling}}(q^2)$, and one can easily construct models in which the main part of the total multiplicity comes from terms which vanish in the scaling limit.

If one measures a behaviour like (4.12) then the light cone dominance breaks down and the contribution for the scaling function, given by the light cone singularity, is overwhelmed by terms like the following one :

$$\left(\frac{2\nu}{q^2}\right)^{-2} \mathcal{O}\left(\frac{2\nu}{q^2} - \sqrt{\frac{4p^2}{q^2}}\right) \quad (4.13)$$

Such terms do not come from a light cone singularity, but from singularities in (px) and from singularities outside the light cone, in particular from singularities in the variable $((px)^2 - x^2 p^2)$. Such singularities need not be local in co-ordinate space and are not described by the relations (3.13). In the latter case, the relations (3.13) are correct, but useless, since there is no connection between expectation values of the operators $\mathcal{O}$ and the scaling functions.

5. INCLUSIVE ELECTRO- AND NEUTRINO-PRODUCTION

In this section we apply the algebraic relations (3.13) to inclusive electro- and neutrino-production experiments off a hadronic target $|t\rangle$ where one detects a final particle $|p\rangle$ (Fig.6a)

Fig.6a

Fig.6b

These processes can be viewed as being the crossed ones of processes like

$$\text{current} + \text{two-particle state} \rightarrow \text{anything}$$

(see Fig.6b).

The latter is described in the scaling limit by a two-particle expectation value of a bilocal current operator while the first one is given by a one-particle expectation value of the operators $\mathcal{O}$ defined in (3.13). We describe inclusive reactions of the type (Fig.6a) by the following matrix elements :

$$W_{\mu\nu}^{ij}(q, p, t) = \frac{1}{4\pi} \int dx e^{i q \cdot x} \left(\langle \psi / \hat{f}_{i\mu}(x) / p, n \rangle \langle n p / \hat{f}_{j\nu}(0) / t \rangle - \text{h.c.} \right) \quad (5.1)$$

This tensor is described in general by 16 independent structure functions, depending on the following variables :

$$\nu = t \cdot q \quad \nu' = p \cdot q, \quad q^2, \quad u = (t - p)^2 \quad (5.2)$$

We also define four vectors R and Δ by : $R_\mu = t_\mu + p_\mu$, $\Delta_\mu = t_\mu - p_\mu$. Current conservation reduces the number of structure functions to nine. The deep inelastic limit is defined as :

$$\left. \begin{aligned} \nu, \nu' - q^2 &\rightarrow +\infty \\ \xi &= \frac{-q^2}{2\nu} \\ x &= \frac{\nu'}{\nu} \\ u &= \Delta^2 \end{aligned} \right\} \text{fixed} \quad (5.3)$$

The physical meaning of these variables can easily be seen in the centre-of-mass frame (see Fig.7), where one has :

Fig.7

$$\begin{aligned} t_\mu &= (\sqrt{m_t^2 + t_3^2}, t_3, 0, 0) & q_\mu &= (q_0, -t_3, 0, 0) \\ p_\mu &= (\sqrt{m_p^2 + p_3^2 + p_\perp^2}, p_3, p_\perp) \end{aligned} \quad (5.4)$$

We choose $t_3 > 0$. Note :

$$\begin{aligned}
 v = t q &= q_0 \sqrt{m_t^2 + t_3^2} + t_3^2 \\
 v' = p q &= q_0 \sqrt{m_p^2 + p_3^2 + p_1^2} + t_3 p_3 \\
 u = (t-p)^2 &= m_t^2 + m_p^2 + 2t_3 p_3 - 2t_3 p_1 \sqrt{\left(1 + \frac{m_t^2}{t_3^2}\right) \left(1 + \frac{m_p^2 + p_1^2}{p_3^2}\right)} \\
 \xi &= \frac{-q^2}{2v} \sim \frac{t_3 - q_0}{2t_3} \\
 x &= \frac{v'}{v} \sim \frac{p_3}{t_3}
 \end{aligned}
 \tag{5.5}$$

$$q_0 \geq 0 \rightarrow 0 \leq \xi \leq \frac{1}{2}$$

$$q_0 < 0 \rightarrow \frac{1}{2} \leq \xi \leq 1$$

The scaling limit (5.3) corresponds to :

$$\begin{aligned}
 t_3 &\rightarrow +\infty \quad x \rightarrow \frac{p_3}{t_3} \text{ fixed} \\
 u &\rightarrow (1-x) m_p^2 + (1-\frac{1}{x}) m_t^2 - \frac{1}{x} p_1^2 \text{ fixed}
 \end{aligned}$$

We emphasize that this scaling limit degenerates in the case of $-q^2$ fixed ($\xi \rightarrow 0$) into the hadronic scaling limit, where one considers the fragmentation of the target only¹⁹⁾. The range of our scaling variables is the following :

$$0 \leq \xi \leq 1, \quad 0 \leq x \leq 1-\xi \tag{5.6}$$

In the scaling limit, one tests the light cone region in co-ordinate space and we may apply the relations (3.13). Because of their singularity structure which is abstracted from a quark model, we have in the scaling limit :

$$\varepsilon_\ell^\mu \mathcal{W}_{\mu\nu} \rightarrow 0 \quad \varepsilon_\ell^\nu \mathcal{W}_{\mu\nu} \rightarrow 0 \tag{5.7}$$

where ε_ℓ^μ is the longitudinal polarization vector, normalized to one. Thus the hadronic medium is asymptotically transparent for the longitudinal current mode.

Due to the transversality conditions (5.7), the leading contributions to the tensor (5.1) can be expressed by two structure functions. In the Breit frame with

$$R_\mu = (R_0, 0, 0, 0) \quad \Delta_\mu = (\Delta_0, \Delta_1, 0, 0) \quad \Delta_1 > 0$$

$$g_\mu = (g_0, 0, 0, g_3) \quad \varepsilon_{\mu 1}^{tr} = (0, 1, 0, 0) \quad \varepsilon_{\mu 2}^{tr} = (0, 0, 1, 0)$$

we have :

$$W_{\mu\nu}^{ij} \rightarrow \varepsilon_{\mu m}^{tr} \varepsilon_{\nu n}^{tr} \bar{F}_{mn}^{ij} \quad m, n = 1, 2$$

$$\bar{F}_{mn}^{ij} = \begin{pmatrix} F_1^{ij} & i F_3^{ij} \\ -i F_3^{ij} & F_1^{ij} \end{pmatrix} \quad (5.8)$$

F_1 and F_3 are the scaling limits of structure functions W_1 and $-\frac{1}{2}W_3$ which are identical to the conventional structure functions W_1 and $-\frac{1}{2}W_3$ if $p_\mu = 0$. How are the scaling functions related to the expectation values of the operators ϕ ? We write :

$$\frac{1}{2} \langle t / \phi_{i\sigma}(\frac{z}{2}, -\frac{z}{2}, p) \pm \phi_{i\sigma}(-\frac{z}{2}, \frac{z}{2}, p) / t \rangle$$

$$= t_\sigma \left\{ \begin{matrix} \bar{S}_i^+ (t z, \frac{p^2}{t^2}, u) \\ \bar{A}_i^+ (t z, \frac{p^2}{t^2}, u) \end{matrix} \right\} + p_\sigma \left\{ \begin{matrix} \bar{S}_i^+ (t z, \frac{p^2}{t^2}, u) \\ \bar{A}_i^+ (t z, \frac{p^2}{t^2}, u) \end{matrix} \right\} \quad (5.9)$$

The four-dimensional Fourier transforms of the matrix elements (5.9), multiplied with the light cone singularity, are described by the following formula :

$$\frac{\nu + \nu'}{8\pi^2} \int d^4 z \, e^{+i q z} (\delta^4(\varepsilon \Lambda_0) \delta(\Lambda^2)) \bar{S}_i^+ (t z, \frac{p^2}{t^2}, u)$$

$$\rightarrow g^\rho \tilde{S}_i^+ (\xi, x, u), \quad \dots \quad x = \frac{p^2}{t^2} \quad (5.10)$$

The asymptotic expression (5.10) holds only if the light cone dominates in the deep inelastic limit. The correctness of (5.10) combined with the spectrum condition yields the following information on the matrix elements (5.9) in analogy to (2.17) (with $\tau = tz$, $x = \frac{p^2}{t^2}$).

$\bar{S}_i^t(\tau, x, u)$ is an entire analytic function in τ (of exponential type 1). The singularity at $x=0$ is restricted by :

$$x \cdot \bar{S}_i^t(\tau, x, u) \xrightarrow{x \rightarrow 0} 0, \dots \quad (5.11)$$

If (5.11) does not hold the leading light cone singularity will not dominate in the scaling region. The one-dimensional Fourier transforms of the matrix elements (5.9) with respect to the first variable $(tz) = \tau$ and fixed $x = \frac{pz}{tz}$ are defined by :

$$\tilde{S}_i^t(\xi, x, u) = \frac{1}{2\pi} \int d\tau e^{-i\tau\xi} \bar{S}_i^t(\tau, x, u), \dots \quad (5.12)$$

The scaling functions, as given in (5.8), depend only on the combinations :

$$\begin{aligned} \bar{F}_{S_i}(\xi, x, u) &= \tilde{S}_i^t(\xi, x, u) + x \tilde{S}_i^P(\xi, x, u) \\ \bar{F}_{A_i}(\xi, x, u) &= \tilde{A}_i^t(\xi, x, u) + x \tilde{A}_i^P(\xi, x, u) \end{aligned} \quad (5.13)$$

in the following way :

$$\begin{aligned} \bar{F}_1^{ij} &= i f_{ijk} \bar{F}_{Sk} + d_{ijk} \bar{F}_{Ak} \\ \bar{F}_3^{ij} &= i f_{ijk} \bar{F}_{Ak} + d_{ijk} \bar{F}_{Sk} \end{aligned} \quad (5.14)$$

We now define the scaling functions, which describe the scattering of a current $\tilde{T}_\mu^{(P)}(x)$ with quantum numbers $\lambda^{1/2} \cdot \rho_i$

$$\bar{T}_{13}^{(P)} = \rho^+ i \bar{T}_{13}^{ij} \rho^j \quad (5.15)$$

Later we also use the symbols ρ^+ and ρ^- referring to the chiral currents $\tilde{T}_\mu^{\rho^+} = \tilde{T}_\mu^\rho + \tilde{T}_\mu^{5\rho}$, $\tilde{T}_\mu^{\rho^-} = \tilde{T}_\mu^\rho - \tilde{T}_\mu^{5\rho}$. We denote them in the cases of electro- and neutrino-production :

$$\begin{aligned}
 \bar{F}_1^e &= \frac{2}{3} \sqrt{\frac{2}{3}} \bar{F}_{a0} + \frac{2}{3\sqrt{3}} \bar{F}_{a8} + \frac{2}{3} \bar{F}_{a3} \\
 \bar{F}_1^\nu &= 2 \sqrt{\frac{2}{3}} \bar{F}_{a0} + \frac{4}{\sqrt{3}} \bar{F}_{a8} + 4 \bar{F}_{s3} \\
 \bar{F}_3^\nu &= 2 \sqrt{\frac{2}{3}} \bar{F}_{s0} + \frac{4}{\sqrt{3}} \bar{F}_{s8} + 4 \bar{F}_{a3}
 \end{aligned} \tag{5.16}$$

The corresponding cross-sections can be obtained in the following way. Let the transverse polarization vector of the current $\hat{F}_\mu^{(\rho)}$ be given by :

$$\hat{F}_\mu^{(\rho)} = \alpha^{(\rho)} \hat{\Sigma}_{\mu 1}^{\tau\nu} + \beta^{(\rho)} \hat{\Sigma}_{\mu 2}^{\tau\nu} \tag{5.17}$$

$|\alpha|^2 + |\beta|^2 = 1$, α, β complex. The Stokes matrix can then be written as :

$$\begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix}^{(\rho)} \tag{5.18}$$

and the degree of transverse polarization is :

$$-1 \leq P_{tr}^{(\rho)} = \left(\frac{2 \operatorname{Im} \alpha \beta^*}{|\alpha|^2 + |\beta|^2} \right)^{(\rho)} \leq 1 \tag{5.19}$$

Then the inclusive cross-section of a polarized chiral current $\hat{F}_\mu^{\rho \pm}$ on a target t turns out to be :

$$E_P \frac{d\sigma(\rho^\pm, P^\rho)}{d^3p} \rightarrow \frac{1}{16\pi^2\nu} \left(\bar{F}_1^{\rho\rho^\pm} \pm \bar{F}_3^{\rho\rho^\pm} P_{tr}^{\rho^\pm} \right) / (E, \nu, 2) \tag{5.20}$$

In the following we use for the expression in the brackets the short-hand notation $\bar{F}_1 \pm \bar{F}_3 P_{tr} = F^{\rho\rho^\pm}$.

From the algebraic relations (5.16), we arrive at the following relations between structure functions :

$$6 \left(\bar{F}_1 e^{\rho \pi^+}(\xi, x, u) - \bar{F}_1 e^{\rho \pi^-}(\xi, x, u) \right) = \left(\bar{F}_3^{\nu \pi^+} - \bar{F}_3^{\nu \pi^-} \right) / (\xi, x, u)$$

$$6 \left(\bar{F}_1 e^{\rho \pi^0} - \bar{F}_1 e^{\rho \pi^0} \right) = \left(\bar{F}_3^{\nu \pi^0} - \bar{F}_3^{\nu \pi^0} \right) \quad (5.21)$$

Similar relations are obtained for the production of other isospin multiplets like Σ , etc. The functions $\bar{F}_{is}^{(5)}(\xi, x, u)$ and $\bar{F}_{ia}^{(5)}(\xi, x, u)$ defined in (5.13) transform under $SU(2)$ like

$$\langle t_p / Q_i / t_p \rangle$$

where Q_i denote the $SU(3)$ charges. Therefore, we obtain also relations like the following ones :

$$\frac{\bar{F}_1 e^{i^I} - \bar{F}_1 e^{j^I}}{\bar{F}_1 e^{i'^I} - \bar{F}_1 e^{j'^I}} = \frac{i-j}{i'-j'} = \frac{\bar{F}_1^{\nu i^I} - \bar{F}_1^{\nu j^I}}{\bar{F}_1^{\nu i'^I} - \bar{F}_1^{\nu j'^I}} =$$

$$= \frac{\bar{F}_3^{\nu i^I} - \bar{F}_3^{\nu j^I}}{\bar{F}_3^{\nu i'^I} - \bar{F}_3^{\nu j'^I}} \quad (5.22)$$

where $\begin{pmatrix} I \\ i \end{pmatrix}$ denote the total isospin and its third component of the particle produced. The symbol $\langle t \rangle$ means average over the target isospin. For example, we obtain ²⁰⁾ :

$$\bar{F}_1 e^{\pi^0} = \frac{1}{2} \left(\bar{F}_1 e^{\pi^+} + \bar{F}_1 e^{\pi^-} \right) \quad (5.23)$$

The same relation is right for neutrino-production. All these symmetry relations are analogous to the ones given in Section 2 for e^+e^- annihilation, since averaging over the targets yields the quantum numbers of the vacuum.

From positivity of the cross-sections in (5.19) and $SU(2)$ symmetry, we also get inequalities. If we average over proton and neutron targets, we obtain analogous relations as in Section 2 :

$$\frac{1}{4} \leq \frac{1 - \frac{3}{5} \left| \frac{i}{I} \right|}{1 + \frac{3}{5} \left| \frac{i}{I} \right|} \leq \frac{\bar{F}_{1, \langle t \rangle}^{e p; I}(\xi, x, u)}{\bar{F}_{1, \langle t \rangle}^{e p; I}(\xi, x, u)} \leq \frac{1 + \frac{3}{5} \left| \frac{i}{I} \right|}{1 - \frac{3}{5} \left| \frac{i}{I} \right|} \leq 4 \quad (5.24)$$

Finally, we demonstrate the connection of our scaling results with the hadronic scaling suggested by Feynman. We have already noted that the Feynman scaling region is formally obtained in the limit $\xi \rightarrow 0$, where $-q^2$ is held fixed. Since $\xi = -q^2/2\nu$ coincides with the usual scaling variable for deep inelastic scattering (where $p_\mu \rightarrow 0$), the connection between the Feynman scaling region and the deep inelastic region is just the analogue of the connection between Regge region and deep inelastic region in electro- and neutrino-production. The variable x defined in (5.3) is just the Feynman scaling variable. We note that in the limit $p_\mu \rightarrow 0$ our functions $\bar{F}(\xi, x, u)$ degenerate into the usual deep inelastic scaling functions :

$$\bar{F}(\xi, x, u) \xrightarrow{p \rightarrow 0} \bar{F}(\xi) \quad (5.25)$$

But in deep inelastic scattering, the interchange of Regge and scaling limits means that the scaling functions themselves are Regge behaved at small ξ . In particular, the Pomernanchuk trajectory gives :

$$\bar{F}_{a0}(\xi) \xrightarrow{\xi \rightarrow 0} \frac{\text{const.}}{\xi} \quad (5.26)$$

Where $\bar{F}_{a0}(\xi)$ is the Fourier transform of the expectation value of the anti-symmetrized bilocal baryon current $\hat{F}_{a0}(x, y)$ [we exclude, for simplicity, the possibility that the Pomernanchuk singularity transforms as a mixture of $\hat{F}_{a0}$ and $\hat{F}_{a8}$].

Let us now consider the function $\bar{F}_{a0}(\xi, x, u)$ at small values of ξ . According to (5.26), we assume the following behaviour

$$\bar{F}_{a0}(\xi, x, u) \xrightarrow{\xi \rightarrow 0} \frac{g(x, u)}{\xi} \quad (5.27)$$

Then the function $g(x,u) = \hat{g}(x, p_{\perp}^2)$ is our scaling function in the particular case $m^2 \ll -q^2 \ll \nu$. Especially, we get for the inclusive electroproduction:

$$E_p \frac{d\sigma^{\gamma t \rightarrow p^h}}{d^3p} \xrightarrow[\xi \rightarrow 0]{\nu, -q^2 \rightarrow \infty} \frac{\alpha}{2\pi} \frac{g(x, u)}{(-q^2)} \quad (5.28)$$

We try to extrapolate to the case $-q^2$ fixed by the substitution $-q^2 \rightarrow \bar{\beta}(q^2)$ where $\bar{\beta}(q^2)$ describes the residue of the Pommeranchuk singularity. This kind of extrapolation is suggested by the electroproduction experiments. We obtain just the usual hadronic scaling behaviour:

$$E_p \frac{d\sigma^{\gamma t \rightarrow p^h}}{d^3p} \xrightarrow[\substack{\nu \rightarrow \infty \\ -q^2 \text{ fixed}}]{\nu \rightarrow \infty} \frac{\alpha}{2\pi} \bar{\beta}(q^2) \hat{g}(x, p_{\perp}^2) \quad (5.29)$$

Further we assume that the parametrization (5.29) works even at small q^2 , as suggested by the electroproduction experiments. Then we may also study purely hadronic reactions. The residue function $\bar{\beta}(q^2)$ has double poles at $q^2 > 0$, which describe the particles produced in the current channel. The corresponding residues of the amplitude describe purely hadronic reactions. In the case of the electromagnetic current, we have for example:

$$E_p \frac{d\sigma^{\gamma t \rightarrow p^h}}{d^3p} \rightarrow \text{const.} \hat{g}(x, p_{\perp}^2) \quad (5.30)$$

Our analysis does not give an explanation why the functions $\bar{F}(\xi, x, p_{\perp}^2)$ and also the hadronic scaling functions $\hat{g}(x, p_{\perp}^2)$ fall off sharply at increasing transverse momenta. The transverse momentum cut-off comes from the specific structure of the matrix elements of the operators $\phi_{i\sigma}$.

Further, we discuss the multiplicity of the produced particles. When one scatters a current $J_{\mu}^{(S)}$ off a hadronic target t , the differential multiplicity of producing a particle p is defined by

$$\frac{dn^{pp}}{dx d^2p_{\perp}}(\xi, x, p_{\perp}^2) = \lim_{\substack{-q^2 \rightarrow \infty \\ \nu \rightarrow \infty}} \frac{1}{\sigma^{pp}(\xi^2, \nu)} \frac{d\sigma^{pp}}{dx d^2p_{\perp}} \quad (5.31)$$

Note that the total cross-sections $\sigma^{TP}(q^2, \nu)$ in the deep inelastic limit are determined by the scaling functions $F^{TP\pm}$ which stand for [see (5.20)] :

$$F^{TP\pm} = F_1 \pm F_3 P_F$$

$$\sigma^{TP}(q^2, \nu) \rightarrow \frac{\pi}{\nu} F^{TP}(\xi) \quad (5.32)$$

Therefore we get, using (5.20) :

$$\frac{d h^{PP}(\xi, x, P_\perp^2)}{dx dP_\perp^2} = \frac{1}{x} D^{PP}(\xi, x, P_\perp^2) \quad (5.33)$$

where we have defined the function D by :

$$D^{PP}(\xi, x, P_\perp^2) = \frac{1}{16\pi^3} \frac{\overline{F}^{PP}(\xi, x, P_\perp^2)}{F^P(\xi)} \quad (5.34)$$

Note that the function D goes to $1/16\pi^3$ in the limit $P_\perp \rightarrow 0$. At finite energies, we have to use non-asymptotic kinematics to evaluate (5.31) :

$$\frac{d h^{PP}}{dx dP_\perp^2} = \frac{t_3}{E_P} D^{PP} = \frac{1}{\sqrt{\frac{u_P^2 + P_\perp^2}{t_3^2} + x^2}} D^{PP} \quad (5.35)$$

We integrate over the transverse momenta and use the mean value theorem to define a formal mass parameter $\mathcal{E}$ in the following way (this definition is only possible if the integral over the transverse momenta exists, which we assume) :

$$\begin{aligned} \frac{d h^{PP}}{dx} &= \frac{1}{\sqrt{\frac{\mathcal{E}^2}{t_3^2} + x^2}} \hat{D}^{PP}(\xi, x) \\ \hat{D}^{PP}(\xi, x) &= \int_0^\infty dP_\perp^2 D^{PP}(\xi, x, P_\perp^2) \end{aligned} \quad (5.36)$$

The formal mass parameter $\mathcal{E}$ depends still on the remaining variables ξ , x and t_3 . A cut-off in the transverse momenta amounts to

$$\lim_{t_3 \rightarrow \infty} \mathcal{E}(\xi, x, t_3) = \mathcal{E}(\xi, x) < \infty$$

In that case the main contribution to the multiplicity for $t_3 \rightarrow \infty$ comes from the central region ($x \approx 0$). We evaluate the multiplicity to :

$$h^{PP}(t_3, \xi) = \int_0^{1-\xi} \frac{dx}{\sqrt{\frac{\xi^2}{t_3^2} + x^2}} \hat{D}^{PP}(\xi, x) \quad (5.37)$$

The divergent part of this integral may be evaluated replacing $\xi(\xi, x)$ by $\xi(\xi, 0)$. It seems natural that the function $\hat{D}^{PP}(\xi, x)$ approaches a finite limit for $x \rightarrow 0$. In that case we find :

$$\begin{aligned} h^{PP}(t_3, \xi) &\rightarrow \log\left(\frac{t_3}{\xi}\right) \hat{D}^{PP}(\xi, 0) \\ &= \frac{1}{2} \log\left(\frac{\nu}{\xi^2}\right) \hat{D}^{PP}(\xi, 0) \end{aligned} \quad (5.38)$$

The light cone still tolerates a behaviour like $\hat{D}^{PP}(\xi, x) \rightarrow x^{\delta-1} d^{PP}(\xi)$ with $\delta > 0$ [see (5.11)]. This amounts to a multiplicity like

$$h(\nu, \xi) \rightarrow \text{const. } d^{PP}(\xi) (\sqrt{\nu})^{1-\delta} \quad (5.39)$$

Thus the upper limit just forbidden by light cone dominance corresponds to the equipartition case of a finite characteristic energy among the particles produced.

In the particular case $m_t^2 \ll -q^2 \ll \nu$ the multiplicity is given by the limit $\xi \rightarrow 0$ in (5.38) or (5.39)

$$\begin{aligned} h^{PP}(s, \xi=0) &\rightarrow \frac{1}{2} \log\left(\frac{s}{\xi^2}\right) \hat{D}^{PP}(0, 0) \\ s = (q+t)^2 &\rightarrow 4t_3^2 \end{aligned} \quad (5.40)$$

At finite q^2 we use

$$\sigma^{TP}(q^2, \nu) \xrightarrow{\nu \rightarrow \infty} 2\pi \beta(q^2) \left(\xi F^{TP}(\xi) \right)_{\xi=0}$$

and get

$$h^{PP}(q^2, s, \xi=0) \rightarrow \frac{\bar{\beta}(q^2)}{\beta(q^2)} \frac{1}{2} \log\left(\frac{s}{\xi^2}\right) \hat{D}^{PP}(0, 0) \quad (5.41)$$

Note that all multiplicities become target independent, since the function (5.34) does not depend on the target in the limit $\xi \rightarrow 0$. It is reasonable to assume that the functions $\bar{\beta}(q^2)$ and $\beta(q^2)$ both describing the off-shell dependence of the Pomeron residue are identical. Thus (5.41) strongly suggests that the multiplicity becomes independent of q^2 as the energy goes to infinity.

6. CONCLUDING REMARKS

We have considered a light cone approach to inclusive reactions. The abstraction framework used is a speculation about the simplicity of the strong interactions at light-like distances. It is essentially based on the assumption that the algebraic structure of Green's functions in the field theory model of strong interactions resembles the one in free quark theory as far as the leading light cone singularity is concerned. The problems we encounter in describing inclusive reactions is the lack of a causal representation in contrast to the case of deep inelastic scattering. Therefore there is no immediate connection between the light cone structure in co-ordinate space and the scaling in momentum space. Nevertheless a connection can be made if one applies an additional assumption about the matrix elements [see, e.g., (2.17)]. This assumption leads in particular to a finite multiplicity in the case of e^+e^- annihilation (only the scaling part of the multiplicity is considered). However, one does not necessarily obtain a finite multiplicity for electro- and neutrino-production. Therefore in our approach the problem of multiplicity in current annihilation is quite different from the corresponding one in deep inelastic scattering. This is connected to the fact that the e^+e^- annihilation proceeds in the $L=0,1$ channels only, while in deep inelastic scattering the angular momentum distribution depends on the energy. Our algebraic abstractions are based on the quark model which has the property that leading light cone singularities are $SU(3) \times SU(3)$ invariant. This exact invariance leads to numerous symmetry relations and inequalities, as described in Sections 4 and 5. We should like to remark that these relations may also be obtained from the assumption that there are no exotic exchanges in the t channel, just as in ordinary deep inelastic scattering. In fact, it is easy to see that the algebraic relations discussed in Section 3 lead immediately to the absence of t channel exotics in the scaling limit. The advantage of our relations in co-ordinate space lies in the fact that they are supposed to be exact and therefore give us information under which conditions the symmetry relations become exact.

Finally, we should like to comment on the connection between hadronic (Feynman) scaling and deep inelastic (Bjorken) scaling. The scaling behaviour denoted in (5.8) may be considered as mixed scaling in the sense that both types of scaling are present. The scaling behaviour in the variable ξ is just the consequence of the simple light cone structure, like in deep inelastic scattering :

$$W_{\mu\nu}(t, p, z) \xrightarrow{z^2 \rightarrow 0} (\text{singular function of } z^2) \cdot F(tz, pz, u)$$

Once we have made this assumption, the scaling in the variable x arises naturally, since the residue of the leading light cone singularity $F(tz, pz, u)$ can also be written as $\hat{F}(tz, \frac{pz}{tz}, u) = \hat{F}(tz, x, u)$. The scaling parameter x just serves to describe the hadronic two-particle state consisting of the target $|t\rangle$ and the produced particle $|p\rangle$ (see Figs. 6a and 6b). Therefore the scaling in x has nothing to do with the Bjorken scaling in ξ , but only expresses the fact that one has to deal with fixed hadronic states $|t\rangle$ and $|p\rangle$ described by x and $p_\perp$ in going to the Bjorken limit. In Eqs. (5.28) - (5.30), we have indicated how to arrive at the ordinary hadronic scaling limit (q^2 fixed) by interchanging the generalized Bjorken and Regge limits. The hadronic scaling then arises as a consequence of Pomeron exchange between current and hadronic two-particle state $|tp\rangle$. This way we arrive at the description of hadronic scaling, given by Mueller²¹⁾.

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- 4) A similar conclusion was drawn by Callan and Gross [see Ref. 1] from a different, and presumably wrong, assumption.
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- 9) We should like to mention that the support properties of the structure functions $V(q^2, \nu)$ at $q^2 > 0$ are quite different from the ones at space-like q^2 . The physical region is delimited by : $q^2, \nu > 0$

$$q^2 - 2\nu \geq (m^*)^2 - p^2$$

m^* : smallest mass of a state $|X\rangle$ produced together with $|p\rangle$ in the reaction $e^+e^- \rightarrow p+X$

$$\nu^2 \geq p^2 \cdot q^2$$

- 10) Such a connection exists in specific models [see, Ref. 2), and P.V. Landshoff, J.C. Polkinghorne, DAMPT preprint 72/16 (1972); P. Menotti, Talk presented at the Informal Meeting on Electromagnetic Interactions, Frascati, May (1972)]. There one assumes that only certain diagrams survive in the scaling limit. Such models violate our assumptions about dispersion relations, made in the Appendix.

- 11) For additional references, see :
C.H. Llewellyn Smith, invited paper, presented at the Fourth International Conference on High Energy Collisions, Oxford, April (1972), SLAC-PUB 1039.
- 12) One may also consider mesons of higher spin. In this case one uses instead of $j(z)$ one of the local "generalized currents" appearing in the Taylor expansion of a bilocal current.
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A P P E N D I X

We analyze the consequences of dispersive representations for the amplitudes

$$\begin{aligned} T_{\mu\nu} &= i \int d^4x e^{+i q x} \langle p | T (j_\mu(x) j_\nu(0)) | p \rangle \\ T_{\mu\nu}^{ret} &= i \int d^4x e^{+i q x} j(x) \langle p | [j_\mu(x), j_\nu(0)] | p \rangle \end{aligned} \quad (A.1)$$

supplemented by the assumption of canonical light cone singularities for current products. The idea is to relate $W_{\mu\nu}$, $\bar{W}_{\mu\nu}$ defined in Eqs. (2.4) and (2.8) to specific discontinuities of $T_{\mu\nu}$, $T_{\mu\nu}^{(ret)}$.

The invariant amplitudes without kinematical singularities are obtained from the decomposition in Eq. (2.5) ²²⁾

$$\begin{aligned} T_{\mu\nu}^{(ret)} &= (h_2)_{\mu\nu} T_2^{(ret)} + (h_1)_{\mu\nu} T_1^{(ret)} + \dots \\ h_{1,\mu\nu} &= -g_{\mu\nu} q^2 + q_\mu q_\nu \\ h_{2,\mu\nu} &= q^2 p_\mu p_\nu - \nu (p_\mu q_\nu + p_\nu q_\mu) + \nu^2 g_{\mu\nu} \end{aligned} \quad (A.2)$$

In the region of interest

$$q^2 \gg 2\nu$$

the amplitudes $T_i^{(ret)}$, $i=1,2$ feel the joint singularities corresponding to the diagrams in Figs. 1-4 in Section 2.

Treating q^2 as a parameter and neglecting eventual subtractions for the time being, we assume the following representation of $T_{1,2}$ in terms of double and single dispersion integrals (Cutkosky rules)

$$\begin{aligned}
 T_i(v, q^2) = & \int_{-q^2/2}^{\infty} dv_1 \int_{-q^2/2}^{\infty} dv_2 \tau^i(v_1, v_2, q^2) \frac{1}{(v_1 - v - i\varepsilon)(v_2 + v - i\varepsilon)} \\
 & + \int_{-q^2/2}^{\infty} dv_1 \rho_1^i(v_1, q^2) \frac{1}{v_1 - v - i\varepsilon} \\
 & - \int_{-\infty}^{q^2/2} dv_2 \rho_2^i(v_2, q^2) \frac{1}{v_2 - v + i\varepsilon} \quad ; \quad i=1,2
 \end{aligned} \tag{A.3}$$

In the double spectral region $q^2 > 2|v|$, the amplitudes T_i cannot be continued to analytic functions. Instead, one obtains

$$\begin{aligned}
 T_i(v, q^2) = & \int_{-q^2/2}^{\infty} dv' \rho_s^i(v', q^2) \frac{1}{v' - v - i\varepsilon} \\
 & - \int_{-\infty}^{q^2/2} dv' \rho_u^i(v', q^2) \frac{1}{v' - v - i\varepsilon} \\
 & + 2\pi i \Theta(q^2/2 - v) \bar{\rho}_u^i(v, q^2) \\
 \bar{\rho}_u^i(v, q^2) = & \rho_2^i(v, q^2) + \int_{-q^2/2}^{\infty} dv_1 \frac{\tau^i(v_1, -v, q^2)}{v_1 - v - i\varepsilon} \quad ; \quad i=1,2
 \end{aligned} \tag{A.4}$$

In going from (A.3) to (A.4), we have assumed that

$$\begin{aligned}
 \tau^i(v, q^2) = & \int_{-q^2/2}^{\infty} dv_1 \int_{-q^2/2}^{\infty} dv_2 \tau^i(v_1, v_2, q^2) \frac{1}{(v_1 - v - i\varepsilon)(v_2 + v + i\varepsilon)} \\
 = & \int_{-q^2/2}^{\infty} dv' \rho_1^i(v', q^2) \frac{1}{v' - v - i\varepsilon} \\
 & - \int_{-\infty}^{q^2/2} dv' \rho_2^i(v', q^2) \frac{1}{v' - v - i\varepsilon} \quad i=1,2
 \end{aligned}$$

which implies

$$\rho_s^i = \rho_1^i + \rho_1^i \alpha, \quad \rho_{\tilde{u}}^i = \rho_2^i + \rho_2^i \alpha \quad (\text{A.5})$$

In the Regge limit, q^2 fixed $\nu \rightarrow \infty$, we expect the asymptotic behaviour

$$T_1 \sim \nu \quad T_2 \sim \nu^{-1} \quad \nu \rightarrow \infty \quad (\text{A.6})$$

Taking into account the necessary subtractions for T_1 , we find, remembering the crossing relations $\rho_s^i(-\nu, q^2) = \rho_{\tilde{u}}^i(\nu, q^2)$

$$T_2 = \left\{ \begin{aligned} & \int_{-q^2/2}^{q^2/2} d\nu' [\rho_s^2(\nu', q^2) - \rho_{\tilde{u}}^2(\nu', q^2)] \frac{1}{\nu' - \nu - i\varepsilon} \\ & + \int_{q^2/2}^{\infty} d\nu' \rho_s^2(\nu', q^2) \left(\frac{1}{\nu' - \nu - i\varepsilon} + \frac{1}{\nu' + \nu + i\varepsilon} \right) \\ & + 2\pi i \theta(q^2/2 - \nu) \bar{\rho}_{\tilde{u}}^2(\nu, q^2) \end{aligned} \right\}$$

$$T_1 = \left\{ \begin{aligned} & a(\nu_0, q^2) + (\nu - \nu_0) b(\nu_0, q^2) + \\ & (\nu - \nu_0)^2 \left\{ \int_{-q^2/2}^{q^2/2} d\nu' \frac{1}{(\nu' - \nu_0)^2} [\rho_s^1(\nu', q^2) - \rho_{\tilde{u}}^1(\nu', q^2)] \frac{1}{\nu' - \nu - i\varepsilon} \right. \\ & \quad \left. + \int_{q^2/2}^{\infty} d\nu' \frac{1}{(\nu' - \nu_0)^2} \rho_s^1(\nu', q^2) \left(\frac{1}{\nu' - \nu - i\varepsilon} + \frac{1}{\nu' + \nu + i\varepsilon} \right) \right\} \\ & + 2\pi i \theta(q^2/2 - \nu) \bar{\rho}_{\tilde{u}}^1(\nu, q^2) \end{aligned} \right\} \quad (\text{A.7})$$

Analogous assumptions, as above, lead to the representation of the retarded amplitudes :

$$T_2^{\text{ret}} = \int_{-q^2/2}^{q^2/2} dv' \rho_{\text{ret}}^2(v', q^2) \frac{1}{v' - v - i\varepsilon} + \int_{q^2/2}^{\infty} dv' \rho_{\text{ret}}^2(v', q^2) \left(\frac{1}{v' - v - i\varepsilon} + \frac{1}{v' + v + i\varepsilon} \right) \quad (\text{A.8})$$

An analogous dispersion relation holds for T_1^{ret} .

We recall here the normalization relating $\rho^i(v, q^2)$ for $q^2 < 0$ to the functions V_i defined in Section 2

$$\rho^i = 2V_i$$

We consider further the function related to the current commutator

$$\rho_{\mu\nu}^{(c)} = \frac{1}{2\pi} \int dx e^{i q x} \langle P | [\bar{j}_\mu(x), j_\nu(0)] | P \rangle = (\text{A.9})$$

$$= (h_1)_{\mu\nu} \rho_{(c)}^1 + (h_2)_{\mu\nu} \rho_{(c)}^2$$

From locality and the supposed scaling properties of $\rho_{(c)}^i$ for $q^2 < 0$ it follows

$$\begin{aligned} v q^2 \rho_{(c)}^2 &\rightarrow 0 \\ q^2 \rho_{(c)}^1 &\rightarrow 0 \end{aligned} \quad \frac{q^2}{2v} = \xi > 1 \quad q^2, v \rightarrow \infty \quad (\text{A.10})$$

We consider in the following

$$T_{\mu\nu}(q, P) = T_{\mu\nu}^{\text{ret}}(q, P) = 2\pi i \rho_{\nu\mu}^{(P)}(-q, P)$$

$$= i \int dx e^{i q x} \langle P | j_\nu(0) j_\mu(x) | P \rangle$$

It follows from the light cone algebra that $\rho^{(P)}$ has to vanish in the annihilation region

$$\begin{aligned} \nu q^2 (T_2 - T_2^{\text{ret}}) &\rightarrow 0 \\ q^2 (T_1 - T_1^{\text{ret}}) &\rightarrow 0 \end{aligned} \quad \xi > 1, \nu, q^2 \rightarrow \infty \quad (\text{A.11})$$

For $q^2 > 0$ ρ_{ret}^J are in general complex and thus different from $\rho_{(0)}^J$.

We tentatively identify $\rho_{(0)}^J$ with the real part of ρ_{ret}^J and suppose that the imaginary part of ρ_{ret}^J obeys the same scaling law as the real part, or vanishes faster in the Bjorken limit. Then Eq. (A.10) implies

$$\begin{aligned} \nu q^2 \rho_{\text{ret}}^2 &\rightarrow 0 \\ q^2 \rho_{\text{ret}}^1 &\rightarrow 0 \end{aligned} \quad \xi > 1, \nu, q^2 \rightarrow \infty \quad (\text{A.12})$$

The light cone expansion for the time ordered product

$$T(\partial_\mu(x) \partial_\nu(0))$$

yields the asymptotic relations

$$\begin{aligned} \nu q^2 T_2 &\sim 2 \int_{-1}^{+1} d\alpha \frac{T_2(\alpha)}{\alpha - \xi - i\varepsilon \nu} \quad \xi > 1, q^2 > 0 \\ \nu q^2 V_2 &\sim T_2(\xi) \quad -1 \leq \xi \leq 1 \quad q^2 < 0 \\ q^2 T_1 &\sim 0 \quad \xi > 1 \end{aligned} \quad (\text{A.13})$$

From (A.13), follows :

$$\begin{aligned} \left. \begin{aligned} q^2 \bar{\rho}_2^1 &\sim 0 \\ q^2 \rho_5^1 &\sim 0 \end{aligned} \right\} \quad \xi > 1, \nu, q^2 \rightarrow \infty \\ \left. \begin{aligned} \nu q^2 (\rho_5^2 - \rho_2^2) &\sim 0 \\ q^2 (\rho_5^1 - \rho_2^1) &\sim 0 \end{aligned} \right\} \quad \xi > 1 \end{aligned} \quad (\text{A.14})$$

$$\nu q^2 \operatorname{Re} \rho_s^2 \sim \begin{cases} 2 F_2(\xi) & -1 \leq \xi \leq 1; q^2 < 0 \\ -2 F_2(\xi) & -1 \leq \xi \leq 1; q^2 > 0 \end{cases}$$

Comparing the full expression for T_2 in Eq. (A.7), and the asymptotic expansion

$$T_2 \sim 2\pi i \theta(q^2/2 - \nu) \bar{\rho}_u^2 + \int_{q^2/2}^{\infty} d\nu' \operatorname{Re} \rho_s^2(\nu', q^2) \left(\frac{1}{\nu' - \nu} + \frac{1}{\nu' + \nu} \right) \quad (\text{A.15})$$

we obtain the asymptotic relation

$$\bar{\rho}_u^2(\nu, q^2) \sim - \frac{1}{2\pi} \int_{q^2/2}^{\infty} d\nu' \operatorname{Im} \rho_s^2(\nu', q^2) \left(\frac{1}{\nu' - \nu} + \frac{1}{\nu' + \nu} \right) \quad (\text{A.16})$$

$q^2 \geq 2\nu/1 \quad \nu, q^2 \rightarrow \infty$

We emphasize at this point that the relation in (A.16) is a natural but not the unique solution of the asymptotic equations (A.10) - (A.14). Equations (A.13) and (A.14) are incompatible when we try to continue these equations to the region

$$0 \leq \xi \leq 1, \quad q^2 > 0$$

unless we drop the threshold factor $\theta(q^2/2 - \nu)$ multiplying $\bar{\rho}_u$.

Since

$$\rho_s^i(\nu, q^2) = 2\nu_2(\nu, q^2) \quad q^2 < 0 \quad (\text{A.17})$$

we assume now the existence of a dispersion relation in q^2 for ρ_s^i for fixed $s = q^2 + 2\nu$

$$\rho_s^i(\nu, q^2) = \frac{1}{\pi} \int_{s=q^2+2\nu}^{\infty} d\eta'^2 \frac{1}{\eta'^2 - q^2} \operatorname{Im} \rho_s^i(s, \eta'^2) \quad (\text{A.18})$$

The convergence of the integrals in (A.18) is assumed in analogy with the vanishing of $\rho_s^1(\nu, q^2)$ in the Bjorken-Johnson-Low limit ^{22), 23)}. The scaling properties of ρ_s^1 are enforced if $\text{Im} \rho_s^1$ shows the following behaviour in the Bjorken limit

$$\begin{aligned} s^2 \text{Im} \rho_s^1 &\sim \tau(q^2/s) \quad q^2/s \text{ fixed}; q^2, s \rightarrow \infty \\ q^2 \text{Im} \rho_s^1 &\sim 0 \end{aligned} \quad (\text{A.19})$$

The scaling functions are obtained as limiting relations from Eqs. (A.16) - (A.19) :

$$\begin{aligned} F_2(\xi) &= \lim_{\substack{-q^2/\nu \rightarrow \xi \\ q^2 < 0}} \frac{1}{2} \nu q^2 \rho_s^1 \quad 1 \geq \xi \geq 0 \\ &= - \frac{1}{4\pi} \frac{\xi}{1-\xi} \int_0^\infty d\tau \frac{\tau(\tau)}{(1-\xi)\tau + \xi} \\ \bar{F}_2(\xi) &= \lim_{\substack{q^2/\nu \rightarrow \xi \\ q^2 > 0}} \frac{1}{2} \nu q^2 \bar{\rho}_s^1 = \frac{1}{2} [\phi(\xi) - \phi(-\xi)] \quad (\text{A.20}) \\ \phi(\xi) &= \frac{1}{4\pi} \int_0^{1/2} d\tau \frac{\tau \tau(\tau)}{(1-\xi)\tau + \xi} \quad \xi \geq 1 \end{aligned}$$

The behaviour of F_2 near the elastic threshold $\xi=1$ is given by

$$\begin{aligned} F_2(1-\varepsilon) &\sim \frac{1}{4\pi} \sum_{\nu=0}^N (-1)^\nu m_\nu \varepsilon^{\nu-1} + O(\varepsilon^N) \\ m_\nu &= \int_0^{1/2} d\tau \tau(\tau) \cdot \tau^\nu \end{aligned} \quad (\text{A.21})$$

provided that the moments m_ν , $\nu = 0, \dots, N$ exist.

Thus, if $F_2(1-\varepsilon)$ behaves like ε^N the first N moments of τ must vanish.

The Regge limit implies

$$F_2(\xi) \sim c \quad \xi \rightarrow 0$$

and, in turn,

$$\bar{\tau}(z) = -c \int(z) + \bar{\tau}_1(z) \quad (\text{A.22})$$

If m_0 vanishes, Eq. (A.20) can be transformed to yield Eqs. (2.19) and (2.20) in Section 2 :

$$\begin{aligned} \bar{T}_2(\xi) &= -\frac{1}{4\pi} \int_0^\infty dz \frac{\bar{\tau}(z) \cdot z}{(1-\xi)z + \xi} \quad ; \quad 0 \leq \xi = \frac{q^2}{2\nu} \leq 1 \\ \bar{T}_2(\xi) &= \frac{1}{2} \left(\phi(\xi) - \phi(-\xi) \right) \\ \phi(\xi) &= \frac{1}{4\pi} \int_0^{1/2} dz \frac{\bar{\tau}(z) \cdot z}{(1-\xi)z + \xi} \end{aligned} \quad \left. \vphantom{\int_0^\infty} \right\} \quad 1 \leq \xi = \frac{q^2}{2\nu} \quad (\text{A.23})$$

$$\bar{\tau}(z) \cdot z = \tau_1(z) \cdot z$$

FIGURE CAPTIONS

Figures 1 - 4 : Diagrams relevant to the current correlation function at timelike momenta.

Figure 5 : Diagram for the process :
 current $\rightarrow$ detected hadron state $|p\rangle + \text{anything}$.

Figure 6 : Scattering diagrams for the processes :
 a) current + $t \rightarrow p + \text{anything}$; and
 b) current + $t + \bar{p} \rightarrow \text{anything}$.

Figure 7 : Typical scattering diagram. for p being a fragment of t in the centre-of-mass system.