Two Photon Exchange Contributions to Elastic $ep$ Scattering in the Nonlocal Field Formalism

Pankaj Jain, Satish D. Joglekar and Subhadip Mitra
Physics Department, IIT, Kanpur - 208016, India

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Abstract

We construct a nonlocal gauge invariant Lagrangian to model the electromagnetic interaction of proton. The Lagrangian includes all allowed operators with dimension up to five. We compute the two photon exchange contribution to elastic electron-proton scattering using this effective nonlocal Lagrangian. The one loop calculation in this model includes the standard box and cross box diagram with the standard on-shell form of the hadron electromagnetic vertices. Besides this we find an extra contribution which depends on an unknown constant. We use experimentally extracted form factors for our calculation. In this paper we include only those contributions which depend on the form factor $F_1$. We find that the correction to the reduced cross section is approximately linear as a function of the photon longitudinal polarization $\epsilon$. The final result explains a substantial percentage of the difference between the form factor ratio $G_E/G_M$ extracted by Rosenbluth and polarization transfer technique. We speculate that a complete calculation including all contributions from our effective model may explain the experimental discrepancy.
1 Introduction

The electromagnetic form factors $F_1$ and $F_2$ parametrize the vertex of electromagnetic interaction of a photon with an on-shell proton,

$$
\Gamma_\mu(p, p') = \gamma_\mu F_1(q^2) + \frac{i\kappa_p}{2M_p} F_2(q^2) \sigma_{\mu\nu} q^{\nu},
$$

(1)

where $p$ and $p'$ are the proton momenta, $M_p$ is the proton mass, $\kappa_p$ its anomalous magnetic moment and $q = p' - p$ is the momentum transfer. The functions $F_1$ and $F_2$ are called the Dirac and Pauli form factors respectively. They are experimentally measured by elastic scattering of electrons on protons, assuming that the process is dominated by one photon exchange diagram (fig. 1). We also define $Q^2 = -q^2 \geq 0$. Besides the form factors $F_1$ and $F_2$, it is also convenient to define the electric and magnetic form factors (or the Sachs form factors) $G_E$ and $G_M$ which are more suitable for experimental extraction,

$$
G_E(Q^2) = F_1(Q^2) - \tau\kappa_p F_2(Q^2)
$$

$$
G_M(Q^2) = F_1(Q^2) + \kappa_p F_2(Q^2)
$$

(2)

where $\tau = Q^2/4M_p^2$. At $Q^2 = 0$, $F_1 = F_2 = 1$ and $G_E = G_M/\mu_p = 1$, where $\mu_p$ is the magnetic moment of the proton. The form factor $G_M \approx \mu_p G_D$ where $G_D$ is the dipole function,

$$
G_D = \frac{1}{(1 + \frac{Q^2}{0.71})^2}
$$

(3)

At low momenta, $G_E$ is also approximately equal to $G_D$. At large momenta, $Q^2 >> 1\text{ GeV}^2$,

$$
G_M, F_1 \propto \frac{1}{Q^4}
$$

(4)

The experimental status of $G_E$ and $F_2$ is, however, currently unclear at large momentum transfer.

A standard technique for the extraction of the proton form factors is the Rosenbluth separation [1]. Here one considers the unpolarized elastic scattering of electrons on target protons. In the one photon exchange approximation the cross section can be written as

$$
\frac{d\sigma}{d\Omega} = \frac{\sigma_{\text{Mott}}}{\epsilon(1+\tau)} \left[ \tau G_M^2(Q^2) + \epsilon G_E^2(Q^2) \right]
$$

(5)
Figure 1: The one photon exchange diagram contributing to the elastic electron proton scattering. Here $k, k'$ refer to electron momenta and $p, p'$ to the proton momenta.

where $\epsilon = 1/[1 + 2(1 + \tau)\tan^2(\theta_e/2)]$ is the longitudinal polarization of the photon and $\theta_e$ is the electron scattering angle. One finds that the reduced cross section $\sigma_R = \tau G_M^2(Q^2) + \epsilon G_E^2(Q^2)$ depends linearly on $\epsilon$. By making a linear fit to the observed $\sigma_R$ as a function $\epsilon$ at fixed $Q^2$, one can, therefore, extract both $G_M$ and $G_E$. At large $Q^2$, $G_M$ dominates at all values of $\epsilon$. Hence the uncertainty in the extraction of $G_E$ can be large at large $Q^2$. Recent results for Rosenbluth separation are available from SLAC [2] and much more precisely from JLAB [3]. The SLAC data shows that $\frac{G_E}{G_M} \approx 1$ up to momentum transfer $Q^2 \approx 6$ GeV$^2$. The JLAB data is available at $Q^2 = 2.64, 3.20$ and $4.10$ GeV$^2$ and shows a similar trend. This result also implies that the ratio $F_2/F_1 \propto 1/Q^2$.

A direct extraction of the ratio $G_E/G_M$ is possible by elastic scattering of longitudinally polarized electrons on target proton $\vec{e} + p \rightarrow e + \vec{p}$ [4]. In the one-photon exchange approximation, the recoiled proton acquires only two polarization components, $P_t$, parallel to the proton momentum and $P_{t\perp}$, perpendicular to the proton momentum in the scattering plane. The ratio,

$$\frac{G_E}{G_M} = -\frac{P_t}{P_{t\perp}} \frac{E_e + E_e'}{2M_p} \tan\left(\frac{\theta_e}{2}\right)$$

where $E_e$ and $E_e'$ are the energies of the initial and final electron. This technique, therefore, directly yields the ratio $G_E/G_M$. The results
available from JLAB, show $\mu_p G_E/G_M$ decreases with $Q^2$. A straight line fit to the data gives

$$\frac{\mu_p G_E}{G_M} = 1 - 0.13(Q^2 - 0.04)$$

in the momentum range $0.5 < Q^2 < 5.6$ GeV$^2$. The ratio, therefore, becomes as small as 0.2 at $Q^2 = 5.6$ GeV$^2$, the maximum momentum transfer in this experiment. The polarization transfer results also imply that $Q F_2/F_1 \sim 1$ for $Q^2 > 1$ GeV$^2$. The observed trend in the polarization transfer experiment is, therefore, completely different from what is measured using the Rosenbluth separation. This is clearly a serious problem and has attracted considerable attention in the literature.

2 Two Photon Exchange

An obvious source of error is the higher order corrections to the elastic scattering process. A reliable extraction of the form factors requires a careful treatment of the radiative corrections including the soft photon emission, which give a significant correction to the cross section [3, 5]. These contributions are calculated by keeping only the leading order terms in the soft photon momentum. Furthermore only the infrared divergent terms, which are required to cancel the divergences in the soft photon emission, are included in the radiative corrections. It is possible that the terms not included in these calculation may be responsible for the observed difference. Any such correction is likely to be small and hence cannot significantly change the results of the polarization transfer experiment. However a small correction to the Rosenbluth separation could imply a large correction to the extracted form factor $G_E$. A possible correction is the two photon exchange diagram which has attracted considerable attention in the literature [9, 10, 11, 12, 13]. Such a diagram is taken into account while computing the radiative corrections, but only the infrared divergent contribution is included. It is possible that the remaining contribution gives a significant correction. One may also consider next to leading order corrections in the soft photon momenta to the soft photon emission diagrams. Both of these contributions receive unknown hadronic corrections and cannot be calculated in a model independent manner.

In this paper we estimate the two photon exchange contribution using an effective non-local Lagrangian. The box and cross-box dia-
Figure 2: The two photon exchange box diagram contributing to the elastic electron proton scattering.

grams which contribute are shown in fig. 2 and 3 respectively. The two photon contribution has also been obtained by model calculations in Ref. [11,12]. The authors find that they are able to partially reconcile the discrepancy. However the results of Afanasev et al. [12] show that the predicted Rosenbluth plots are no longer linear in $\epsilon$. This is clearly a problem since the experimental results show no evidence of deviation from linearity. In Ref. [13] the authors argue, using charge conjugation and crossing symmetry, that two photon exchange contribution must necessarily be nonlinear in $\epsilon$. If this is confirmed then it will rule out two photon exchange as an explanation of the observed anomaly.

3 General electromagnetic vertex of proton

The elementary electromagnetic vertex of an on-shell proton is given in eq. 1. When the proton is off-shell, the vertex is expected to be more general. Further, it must satisfy the WT-identity, following from gauge-invariance, that implies a relation between $\Gamma_{\mu}(p,p')$ and the inverse proton propagator $S^{-1}_{F}(p)$. A local theory of interaction of a proton and a photon would have a $U(1)$ gauge-invariance, implied by
local transformations and would imply the WT-identity:

$$q^\mu \Gamma_\mu(p, p') = S_F^{-1}(p') - S_F^{-1}(p).$$

Here we are interested in formulating the theory in terms of an effective nonlocal action. Hence, instead of this local WT-identity, $\Gamma_\mu(p, p')$ satisfies a generalized non-local version of it$^1$:

$$g(q^2)q^\mu \Gamma_\mu(p, p') = S_F^{-1}(p') - S_F^{-1}(p).$$

where $g(q^2)$ is a function of $q^2$ appearing in the gauge-transformation equations, ultimately to be related to a form-factor in the next section. As we shall see in the next section, this identity follows from a non-local electromagnetic invariance, and in fact is more appropriate for an extended object like a proton. On account of the charge-conjugation invariance of the proton-photon interaction, the vertex $\Gamma_\mu(p, p')$, a $4 \times 4$ matrix, must satisfy$^2$

$$C^{-1} \Gamma_\mu(p, p') C = -\Gamma^T_\mu(-p', -p)$$

$^1$Such non-local WT-identities generally occur in non-local quantum field theories. See e.g. Ref. 14. This WT-identity reduces to the usual one as $q \to 0$, provided $g(0) = 1$.

$^2$The negative signs for momenta on the right-hand-side is a consequence of our different sign convention regarding the incoming particle (incoming momentum positive) and the outgoing particle (out-going momentum positive).
where \( C \) is the charge-conjugate matrix, with \( C\gamma_{\mu}C^{-1} = -\gamma_{\mu}^T \) [13]. We now express the vertex in its most general form, employing the 16 linearly independent Dirac matrices: \( 1, \gamma_5, \gamma_{\mu}, \gamma_\mu\gamma_5, \sigma_{\mu\nu} \) and the 4-vectors: \( P^\mu \equiv (p + p')^\mu, q^\mu \equiv (p' - p)^\mu \).

\[
\begin{align*}
\Gamma_\mu(p,p') &= aP_\mu + bq_\mu + c\gamma_\mu \\
&\quad + d\gamma_\mu P_\mu + fT^\mu P_\mu + gT^\mu q_\mu \\
&\quad + h\sigma_\mu P^\alpha + j\sigma_\mu q^\alpha \\
&\quad + k\sigma_\alpha P^\beta q^\beta P_\mu + l\sigma_\beta q^\alpha + m\gamma_\alpha\gamma_5\xi_{\mu\nu} P_{\beta}q_{\nu}
\end{align*}
\]

(11)

Here, the 12 coefficients \( a, b, ..., m \) are functions of the three Lorentz invariants \( p^2, p'^2, q^2 \). Charge conjugation requires that \( a, c, d, g, j, k \) are symmetric under \( p^2 \rightarrow p'^2 \) and \( b, e, f, h, l, m \) are antisymmetric under the same operation. To implement the WT-identity, we express,

\[
S_{F}^{-1}(p) = \alpha (p^2) \not{p} + \beta (p^2)
\]

in its most general form. We then impose the WT-identity given in eq. 9. This leads to some constraints between the coefficients. The net result of all this is to yield the following form for \( \Gamma_\mu \):

\[
\begin{align*}
\Gamma_\mu(p,p') &= a'P_\mu + c'\gamma_\mu + j\sigma_\mu q^\alpha + d'\not{T}P_\mu + 7 \text{ divergence-free terms}
\end{align*}
\]

We enumerate the divergence free (i.e. \( X_\mu \) with \( q^\mu X_\mu \equiv 0 \)) terms:

\[
\begin{align*}
b' \left[ (p^2 - p'^2)q_\mu - q^2P_\mu \right] + f \left( -P.q_{\gamma_\mu} + T^\mu P_\mu \right) + g \left( -q^2\gamma_\mu + qT^\mu \right) \\
+ k \left( -\sigma_\mu P^\alpha q^\alpha + \sigma_\alpha P^\beta q^\beta P_\mu \right) + l \left( \sigma_\mu P^\alpha q^2 - \sigma_\alpha P^\beta q^2 q_\mu \right) \\
+ e' \left( -q^2 P_{\mu} + P.q_{P_\mu} \right) + m\left( \gamma_\alpha\gamma_5\xi_{\mu\nu} \right) P_{\beta}q_{\nu}
\end{align*}
\]

We further note the relations, that arise from the WT-identity, and which restrict the form of some of the coefficients \( (a', c') \) considerably:

\[
a' = \frac{\beta (p'^2) - \beta (p^2)}{g (q^2)} \frac{p'^2 - p^2}{p^2} ; \quad c' = \frac{\alpha (p'^2) + q (p^2)}{2g (q^2)} ; \quad d' = \frac{\alpha (p'^2) - q (p^2)}{g (q^2)} \frac{p'^2 - p^2}{p^2}.
\]

whereas the coefficients of \( j, b, f, g, k, l, m \) are completely arbitrary functions of the Lorentz invariants. We make several observations:

1. We note first that power counting would associate all operators except those three with coefficients \( a', c', j \) with a local operator of dimension 6 or higher.
2. We note that the dependence on $q^2$ of both $a'$ and $c'$ are identical. Near mass-shell\(^3\), $\alpha \left(p^2\right) \sim \alpha_0 + \alpha_1 [p^2 - m^2]$, $\beta \left(p^2\right) \sim \beta_0 + \beta_1 [p^2 - m^2]$; and thus,

$$a' = \left\{ \beta_1 + O[p^2 - m^2]\right\} g^{-1} (q^2);$$

$$c' = \left\{ \alpha_0 + \frac{1}{2} \alpha_1 [p^2 + p'^2 - 2m^2]\right\} g^{-1} (q^2). \quad (13)$$

3. The on-shell expression (eq. 1) for $\Gamma_\mu(p, p')$ takes operators of dimensions 4 (electric) and 5 (magnetic) into account. It is then logical that the only other operator of dimensions 5 should also be included in the off-shell expression for the $\Gamma_\mu(p, p')$. We shall take these three terms into account in our minimal effective Lagrangian model.

4 Effective Lagrangian Model

We represent the interaction of the photon-proton system by an effective nonlocal Lagrangian model based on the discussion in the last section. We adopt the following guidelines in the process:

- The Lagrangian model should incorporate up to dimension 5 operators, for reasons partly explained in the last section. The assumption is that in the effective Lagrangian approach, the higher dimension operators will contribute much less.
- The model should incorporate the results regarding the form of the coefficients $a'$, $c'$ obtained earlier (see eq. 12), thus at least embody the form-factors on mass-shell. The resulting model is necessarily non-local.
- We assume that the model has lowest order derivatives of fermions. Our assumption about the dimensionality of operators is consistent with this.
- We require that this non-local model has an equivalent form of gauge-invariance. Such constructions of non-local versions of local symmetries are known in literature \[^{12}\][14] and we shall show explicitly that our model below has a very simple form of non-local gauge-invariance.

\(^3\)the condition that $S_F(p) \sim \frac{1}{p - m}$ near mass-shell requires that $a_0 m + \beta_0 = 0$; $a_0 + 2m^2 \alpha_1 + 2m \beta_1 = 1$
A Lagrangian model which satisfies these constraints is given by,

$$\mathcal{L} = \overline{\psi} \left( i\partial - e f_1 \left[ \frac{\partial^2}{\Lambda^2} \right] A - M_p \right) \psi + \frac{\alpha''}{2M_p} \overline{\psi} \left( \sigma_{\mu\nu} f' \left[ \frac{\partial^2}{\Lambda^2} \right] F_{\mu\nu} \right) \psi + \frac{b''}{2M_p} \overline{\psi} \tilde{D}^2 \psi$$

where $i\tilde{D} = i\partial - e f_1 \left[ \frac{\partial^2}{\Lambda^2} \right] A$ is the non-local covariant derivative. We make a number of observations regarding this effective Lagrangian:

1. $\mathcal{L}$ is invariant under the non-local form of gauge transformations:

$$\delta A_\mu = -\partial_\mu \alpha \left( x \right) ; \psi \left( x \right) \rightarrow e^{ie f_1 \left[ \frac{\partial^2}{\Lambda^2} \right] \alpha \left( x \right)} \psi \left( x \right), \overline{\psi} \left( x \right) \rightarrow \overline{\psi} \left( x \right) e^{-ie f_1 \left[ \frac{\partial^2}{\Lambda^2} \right] \alpha \left( x \right)}$$

or equivalently,

$$\delta A_\mu = -f_1 ^{-1} \left[ \frac{\partial^2}{\Lambda^2} \right] \beta \left( x \right); \psi \left( x \right) \rightarrow e^{ie \beta \left( x \right)} \psi \left( x \right), \overline{\psi} \left( x \right) \rightarrow \overline{\psi} \left( x \right) e^{-ie \beta \left( x \right)}$$

Under this transformation, $F_{\mu\nu}$ and hence the second term is gauge-invariant independent of what $f'_2$ is. Also, the non-local gauge-covariant derivative satisfies: $\tilde{D} \psi \rightarrow e^{ie \beta \left( x \right)} \tilde{D} \psi \left( x \right)$.

2. The last term generates a term proportional to $P_\mu \Gamma_\mu \left( p, p' \right)$ with a form factor proportional to $f_1$, the same one appearing in the electric term. This is consistent with the comment on the form of $\alpha'$ and $\alpha$ given earlier.

3. The (non-local) gauge-invariance of the last term requires that it is composed of the (non-local) gauge-covariant derivative: this restricts the form-factor present in this term as above.

Before we proceed, a comment on the non-local form of gauge-invariance is in order. It appears that a local form of gauge-invariance for extended particles such as a proton is inappropriate. Consider the wavefunction of an extended particle centered at $x$: viz. $\psi \left( x,t \right)$. Let $y$ be a point within the charge-radius $R$ of the proton: $|x-y| < R$. Let us imagine that a gauge-transformation on $A_\mu$ is carried out (at $t$) around $y$ with a very narrow support $\rho : \rho << |x-y|$. In the model of fundamental constituents, the quark wavefunction should be affected around $y$, which in turn should affect the proton wavefunction even though the gauge-transformation at $x$, depending on $\alpha \left( x \right)$ will be zero. Thus, the proton wavefunction should be affected by a local gauge-transformation with a support anywhere in its charge radius.
The above form of non-local version of gauge-transformations embodies this idea. Note that the Fourier transform of \( f_1 \left( \frac{\partial^2}{\Lambda^2} \right) \) has a support over a distance \( \sim 1/\Lambda \sim R \).

It proves convenient to rearrange the Lagrangian as follows: (Recall the relation \( \mathcal{D}^2 = D^2 + \frac{\epsilon}{2} \sigma_{\mu\nu} F^{\mu\nu} \))

\[
\mathcal{L} = \overline{\psi} \left( i \mathcal{D} - M_p \right) \psi + \frac{a}{2 M_p} \overline{\psi} \left( \sigma_{\mu\nu} f_2 \left[ \frac{\partial^2}{\Lambda^2} \right] F^{\mu\nu} \right) \psi + \frac{b'}{2 M_p} \overline{\psi} \left( i \mathcal{D} - M_p \right)^2 \psi
\]

Had there been no magnetic term, the last term would have formally vanished by classical equation of motion. We note that an inclusion of the last term has now modified the inverse propagator: it has non-vanishing terms at \( e = 0 \). This, in particular, gives a spurious pole in the propagator at another value of \( \mathcal{D} \). This problem can be avoided if we can write the \( \mathcal{L} \) in the following form:

\[
\mathcal{L} = \overline{\psi} \left( i \mathcal{D} - M_p \right) e x p \left\{ \frac{b'}{2 M_p} \left( i \mathcal{D} - M_p \right) \right\} \psi + \frac{a'}{2 M_p} \overline{\psi} \left( \sigma_{\mu\nu} f_2 \left[ \frac{\partial^2}{\Lambda^2} \right] F^{\mu\nu} \right) \psi
\]

which is now understood to have been consistently truncated to a given order in \( b' \). We now note that the inverse propagator is:

\[
(\mathcal{D} - M_p) e x p \left\{ \frac{b'}{2 M_p} \left( \mathcal{D} - M_p \right) \right\}
\]

and has only one zero at \( \mathcal{D} - M_p = 0 \) and the residue of the propagator at the pole is 1. [In this form of \( \mathcal{L} \), \( a' \) is the anomalous magnetic moment and \( M_p \) is the physical mass]. Since we shall consider, in the two photon exchange calculation, terms with the last operator of dimension 5 inserted in \textit{two} photon vertices, we shall do the entire calculation consistently to \( O (b'2) \) using eq. (14). In this case, the propagator for the proton is

\[
\frac{i}{\mathcal{D} - M_p} e x p \left\{ - \frac{b'}{2 M_p} \left( \mathcal{D} - M_p \right) \right\} \approx \frac{i}{\mathcal{D} - M_p} - \frac{i b'}{2 M_p} + \frac{i b'^2}{8 M_p^2} \left( \mathcal{D} - M_p \right)
\]

\footnote{Actually, the constant \( M_p \) and the normalization of KE term are also modified below. However, we shall soon modify the form of the Lagrangian further, where this proves unnecessary.}
5 Reduction of the action

In this section, we shall find an effective way to calculate the matrix elements involving insertion of the last term in the action. Since a 2-photon exchange diagram at 1-loop is at most $O[b'^2]$, we shall evaluate the effect of this term to $O[b'^2]$. What we are interested in are the tree-order matrix elements of two (possibly virtual) photon emission from an on-shell proton. The calculation of these can be simplified considerably in this context with the use of the fermion equations of motion. The result is simple: of all the terms up to $O[b'^2]$, viz. $O[b', b'^2, b'^2a', b'^2a'^2]$, only the last term of $O[b'^2a'^2]$ gives a non-zero result. While the result can be worked out, it is most effectively dealt with in the path-integral formulation.

We define,

$$W[J^\mu, K, \overline{K}] = \int D\phi \exp i \left\{ \int d^4x [L + J^\mu A_\mu + \overline{K} \psi + \bar{\psi} K] \right\}$$

where $L$ is the action of eq. (5) and $D\phi$ denotes generically the measure of the path-integral. We now perform a field transformation:

$$\psi = \exp \left\{ - \frac{b'}{2M_p} \left( i\bar{D} - M_p \right) \right\} \psi'$$

Under this transformation, the Jacobian is,

$$J = \det \exp \left\{ - \frac{b'}{2M_p} \left( i\bar{D} - M_p \right) \right\} = \exp \left\{ - \frac{b'}{2M_p} \left( i\bar{D} - M_p \right) \right\} = \exp \left\{ - \frac{b'}{2\Lambda^2} \right\} = \text{a constant.}$$

and hence can be ignored for the connected Green’s functions. This then yields,

$$W[J^\mu, K, \overline{K}]$$

$$= \int D\phi \exp i \left\{ \int d^4x \left[ \bar{\psi} \left( i\bar{D} - M_p \right) \psi' + \frac{a'}{2M_p} \bar{\psi} \sigma_{\mu\nu} f'_{\mu\nu} \left( \frac{\partial^2}{\Lambda^2} \right) F^{\mu\nu} \right] \right\}$$

$$\times \exp \left\{ - \frac{b'}{2M_p} \left[ i\bar{D} - M_p \right] \left[ \psi' \right] \right\} \times \exp i \left\{ \int d^4x \left[ J^\mu A_\mu + \bar{\psi} K \right] \right\}$$

$$+ \overline{K} \exp \left\{ - \frac{b'}{2M_p} \left[ i\bar{D} - M_p \right] \left[ \psi' \right] \right\}$$

11
We note that if we had $a' = 0$, we would have no left-over term in the action. It is easy to show that $\overline{K} \left\{ \exp \left\{ -\frac{b'}{2M_p} \left[ i\overline{D} - M_p \right] \right\} - 1 \right\} \psi'$ does not contribute to tree-level on-shell proton matrix elements. Thus all tree-level on-shell 2-proton matrix elements involving the last term in eq. [14] are at least of order $a' b'$. We now expand the action to $O(b'^2)$. We find,

$$W[J^\mu, K, \overline{K}] = \int D\phi \exp \left\{ \int d^4 x \left[ \mathcal{L}' + J^\mu A_\mu + \overline{K} \psi' + \overline{\psi} K \right] \right\} + O(b'^3)$$

with

$$\mathcal{L}' = \left\{ \overline{\psi} - \frac{a'}{2M_p} \overline{\psi} \left( \frac{\sigma_{\mu\nu} f_2}{\Lambda^2} \left[ \frac{\partial^2}{\Lambda^2} \right] F^{\mu\nu} \right) \right\} \left\{ -\frac{b'}{2M_p} \left( i\overline{D} - M_p \right) \right\} \left\{ \frac{b'^2}{8M_p^2} \left( i\overline{D} - M_p \right) \right\} \psi' + \frac{a'}{2M_p} \overline{\psi} \left( \frac{\sigma_{\mu\nu} f_2}{\Lambda^2} \left[ \frac{\partial^2}{\Lambda^2} \right] F^{\mu\nu} \right) \psi'$$

We now perform another field transformation,

$$\overline{\psi} - \frac{a'}{2M_p} \overline{\psi} \left( \frac{\sigma_{\mu\nu} f_2}{\Lambda^2} \left[ \frac{\partial^2}{\Lambda^2} \right] F^{\mu\nu} \right) \left\{ -\frac{b'}{2M_p} + \frac{b'^2}{8M_p^2} \left( i\overline{D} - M_p \right) \right\} \psi' = \overline{\psi}' \equiv \psi(1+X)$$

(17)

We can write,

$$\overline{\psi} = \overline{\psi}' \left\{ 1 + \frac{a'}{2M_p} \left( \frac{\sigma_{\mu\nu} f_2}{\Lambda^2} \left[ \frac{\partial^2}{\Lambda^2} \right] F^{\mu\nu} \right) \left\{ -\frac{b'}{2M_p} + \frac{b'^2}{8M_p^2} \left( i\overline{D} - M_p \right) \right\} \right\}^{-1}$$

$$= \overline{\psi} + \psi' \left( -\frac{a'}{2M_p} \sigma_{\mu\nu} f_2 \left[ \frac{\partial^2}{\Lambda^2} \right] F^{\mu\nu} \left\{ -\frac{b'}{2M_p} + \frac{b'^2}{8M_p^2} \left( i\overline{D} - M_p \right) \right\} \right) \psi + O(F^2)$$

The $O(F^2)$ will not matter for the present calculation of 2-photon exchange, as it will give a term having 3-photon fields. The Jacobian

\footnote{This is because one does not have a pole in at least one of the external momenta: it forbidden either by an explicit factor of $p - M_p$, or by a vertex.}
for this transformation,

\[
\frac{1}{J'} = \det[1 + X] = \det \left[ 1 + X + \frac{X^2}{2} - \frac{X^2}{2} + O[b^3] \right] \\
= \det \left[ e^X - \frac{X^2}{2} + O[b^3] \right] = \det e^X \det \left[ 1 - e^{-X} \frac{X^2}{2} + O[b^3] \right] \\
= \exp[\text{tr}X] \det \left[ 1 - e^{-X} \frac{X^2}{2} + O[b^3] \right] \\
= 1 - \text{tr} X^2 + O[b^3]
\]

The last term does not contribute to the emission of two photons from a proton line in the tree approximation. As a result of the transformation (eq. L7), the action then becomes,

\[
\mathcal{L}'' = \bar{\psi} \left( i\hat{D} - M_p \right) \psi' + \frac{a'}{2M_p} \bar{\psi}' \left( \sigma_{\mu\nu} f_2' \left[ \frac{\partial^2}{\Lambda^2} \right] F^{\mu\nu} \right) \psi \\
+ \left( \frac{a'}{2M_p} \right)^2 \left( \frac{b'}{2M_p} \right) \bar{\psi}' \left( \sigma_{\mu\nu} f_2' \left[ \frac{\partial^2}{\Lambda^2} \right] F^{\mu\nu} \right)^2 \psi' \\
- \left( \frac{a'}{2M_p} \right)^2 \left( \frac{b'^2}{8M_p^2} \right) \bar{\psi}' \left( \sigma_{\mu\nu} f_2' \left[ \frac{\partial^2}{\Lambda^2} \right] F^{\mu\nu} \right) \left( i\hat{D} - M_p \right) \\
\times \left( \sigma_{\mu\nu} f_2' \left[ \frac{\partial^2}{\Lambda^2} \right] F^{\mu\nu} \right) \psi' + O[b^3]
\]

and the source term transforms into

\[
\bar{\psi} [1 + X]^{-1} K \\
= \bar{\psi} \left\{ 1 + \left( -\frac{a'}{2M_p} \right) \sigma_{\mu\nu} f_2' \left[ \frac{\partial^2}{\Lambda^2} \right] F^{\mu\nu} \left\{ -\frac{b'}{2M_p} + \frac{b'^2}{8M_p^2} \left( i\hat{D} - M_p \right) \right\} \right\} K \\
+ \bar{\psi} \left( \frac{a'}{2M_p} \right) \left( \frac{b'^2}{8M_p^2} \right) \sigma_{\mu\nu} f_2' \left[ \frac{\partial^2}{\Lambda^2} \right] F^{\mu\nu} \sigma_{\mu\nu} f_2' \left[ \frac{\partial^2}{\Lambda^2} \right] F^{\mu\nu} K
\]

None of these terms contribute to the tree approximation 2-photon matrix element for reasons similar as before.

In conclusion, when we look at the 2-photon exchange diagrams having up to two insertions of the last term in eq. L4 each set of diagrams contain a common part, viz., two (unphysical)-photon tree
amplitude from an on-shell proton. The above discussion shows that the net effect of that comes from terms

\[
\left( \frac{a'}{2M_p} \right)^2 \left( \frac{b'}{2M_p} \right) \bar{\psi}' \left( \sigma_{\mu\nu} f_2' \left[ \frac{\partial^2}{\Lambda^2} \right] F_{\mu\nu} \right)^2 \psi' \\
- \left( \frac{a'}{2M_p} \right)^2 \left( \frac{b'^2}{8M_p^2} \right) \bar{\psi}' \left( \sigma_{\mu\nu} f_2' \left[ \frac{\partial^2}{\Lambda^2} \right] F_{\mu\nu} \right) \left(i\bar{D} - M_p\right) \psi'(18)
\]

We shall show in the appendix that the first term does not contribute in the Feynman gauge. That leaves us with the last term.

6 Results

In this section we give details of the calculation of the box and cross-box diagrams using our effective Lagrangian. The calculation turns out to be complicated due to the explicit presence of the form factors at the vertices. We also require models for the form factors both in the space-like and time-like regions. In the space-like region the form factor \( F_1(q^2) \) is known reasonably well. In the time like region experimental data exists for the form factor \( G_M(q^2) \) for \( 4M_p^2 < q^2 < 14 \) GeV\(^2\), where \( 4M_p^2 \) is the threshold energy for \( pp \) production. In Ref. [17], \( G_M(q^2) \) has been extracted in the unphysical region \( 0 < q^2 < 4M_p^2 \) by using the dispersion relations [18]. The extracted form factor shows two resonances at masses \( M \sim 770 \) MeV and \( M \sim 1600 \) MeV. The phase of the magnetic form factor also shows a large variation in the unphysical region. The electric form factor \( G_E(q^2) \), however, is not well known. In order to obtain the form factors \( F_1 \) and \( F_2 \) we require both magnetic and electric form factors. Here we assume that \( \mu_p F_1 \) is roughly the same as the magnetic form factor and ignore all contributions proportional to \( F_2 \). With this assumption the only term that contributes is

\[
\bar{\psi}' \left( -eF_1 \left[ \frac{\partial^2}{\Lambda^2} \right] A \right) \psi'(19)
\]

Motivated by the fits to dispersion relations [17] we use the following model for the proton magnetic form factor:

\[
\frac{G_M(q^2)}{\mu_p} \approx \left( \frac{1}{q^2 - m_1^2 + im_1 \Gamma_1^1} - \frac{1}{q^2 - m_2^2 + im_2 \Gamma_1^2} \right) \\
\times \frac{(m_1^2 + im_1 \Gamma_1^1)(m_2^2 + im_2 \Gamma_1^2)(q^2 - m_2^2 + im_2 \Gamma_2^2)}{im_1(\Gamma_1^1 - \Gamma_1^2)}(20)
\]

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with $\Gamma_1^1 = 0.2997$ GeV, $\Gamma_1^2 = 0.3003$ GeV, $\Gamma_2 = 0.225$ GeV, $m_1 = 0.77$ GeV and $m_2 = 1.6$ GeV. These parameters are chosen in order to agree with the position and widths of the two resonances. We point out that the two poles inside the bracket on the right hand side essentially give a dipole at $q^2 = m_1^2$. We have chosen to split the dipole in this manner since it makes the loop integrals easier to handle numerically.

The presence of the additional pole at $m_2^2$ considerably complicates our calculation. In order to make it tractable we replace the $q^2$ in this term by a constant $q^2 \sim 0.61 \text{GeV}^2$. At this momentum the magnitude of $G_M$ is maximum. This introduces a small constant phase $(\phi \sim 2.5^\circ)$ for any $q^2$. To avoid this we put an additional factor $e^{-i\phi}$. Our model for the form factor $F_1(q^2)$ may now be written as:

$$F_1(q^2) = \frac{\mathcal{F}}{im_1 \Delta \Gamma_1} \left( \frac{1}{q^2 - m_1^2 + im_1 \Gamma_1^1} - \frac{1}{q^2 - m_1^2 + im_1 \Gamma_1^2} \right)$$

with

$$\mathcal{F} = \frac{(-m_2^2 + im_1 \Gamma_1^1)(-m_2^2 + im_1 \Gamma_1^2)(-m_2^2 + im_2 \Gamma_2)}{(0.61 - m_2^2 + im_2 \Gamma_2)} \cdot e^{-i\phi},$$

$$\Delta \Gamma_1 = \Gamma_1^2 - \Gamma_1^1.$$

This model is convenient since it allows us to use the Feynman parametrization to compute the loop integrals. The model has a small imaginary part even for space like momenta. However this region contributes negligibly to the loop integrals. The dominant contribution comes from the unphysical region $0 < q^2 < 4M_p^2$ where the form factor is many orders of magnitude larger than its value in the space like region. In this region our model provides a good fit to the extracted form factor [14].

Using the model of eq. 21 one can write the amplitude of the box diagram.

$$-iM_B = \frac{e^4 \mathcal{F}^2}{(2\pi)^4(i m_1 \Delta \Gamma_1)^2} \sum_{\alpha, \beta = 1}^2 \left[ \int d^4 k' \frac{\pi(k')\gamma^\mu(k' - l + m_\nu)\gamma^\nu u(k)}{(l^2 - \mu^2 + i\xi)((q - l)^2 - \mu^2 + i\xi)} \right. \left. \times \frac{(p' + l + M_\nu)\gamma^\mu U(p)}{((k - l)^2 - m_2^2 + i\xi)((p + l)^2 - M_p^2 + i\xi)} \right] \times \frac{(-1)^{\alpha + \beta}}{(l^2 - m_1^2 + im_1 \Gamma_1^\alpha)((q - l)^2 - m_1^2 + im_1 \Gamma_1^\beta)}.$$  

\(^6\)Alternatively, one could partial-fraction the right hand side of eq. 21 and find that the dominant terms are given by eq. 24.
Here $u$ and $U$ denote the electron and proton spinors respectively, $\mu$ denotes the fictitious mass of photon and $\xi$ is an infinitesimal parameter which will be set to zero at the end of the calculation. Similarly, the amplitude of the cross-box diagram is given by:

\[
-i\mathcal{M}_{CB} = \frac{e^4 F^2}{(2\pi)^4(i m_1 \Delta \Gamma_1)^2} \sum_{\alpha,\beta=1}^{2} \left[ \int d^4l \frac{\bar{m}(k')\gamma^\mu (\not{k} - \not{l} + m_e)\gamma^\nu u(k)}{(l^2 - \mu^2 + i\xi)((q - l)^2 - \mu^2 + i\xi)} \right. \\
\times \frac{\overline{U}(p')\gamma_\nu (\not{\phi} + \not{q} - \not{l} + M_p)\gamma_\mu U(p)}{((k - l)^2 - m^2_e + i\xi)((p + q - l)^2 - M^2_p + i\xi)} \\
\left. \times \frac{(-1)^{\alpha+\beta}}{(l^2 - m^2_\gamma + im_1 \Gamma^\gamma_1)((q - l)^2 - m^2_\gamma + im_1 \Gamma^\gamma_1)} \right].
\]

These amplitudes are calculated by using the Feynman parametrization. A sample calculation is shown in Appendix 2.

A small mass of the photon $\mu$ has been introduced in order to regulate the infrared divergence in these integrals. The infrared divergent part has to be subtracted from our result since it is included in the standard radiative corrections which are applied while extracting the form factor.

The numerical calculation of the cross-box diagram is straightforward since the integral is well defined. However for the evaluation of the box diagram the numerical evaluation is facilitated by keeping a small imaginary term $\xi$ in the propagators.\(^7\) This makes the integral in the infrared limit well defined. We have verified by explicit calculations that the final result depends very little on $\xi$.

The results of the calculation for the box and cross-box diagram are given in figs. 4 and 5 respectively. Here we have taken the momentum transfer $Q^2 = 2.64, 3.20, 4.10, 5.00, 6.00$ GeV\(^2\). The first three values are same as those used in the JLAB extraction of form factors using Rosenbluth separation. The final result, adding the two contributions and subtracting the infrared $\mu^2$ dependent part, is given in fig. 6.

The reduced cross section $\sigma_R$ shows an almost linear dependence on $\epsilon$. In the calculation the parameter $\xi$ has been set equal to $10^{-4}$ for calculation of the box diagram.

In fig. 4 we show how the experimental results are modified due to two photon exchange contribution. The figure shows the ratio $\mu G_E/G_M$ extracted from the JLAB polarization transfer experiment as

\(^7\)Here we use the notation $\xi$ instead of the standard notation $\epsilon$ since $\epsilon$ is also being used to denote the photon longitudinal polarization.
Figure 4: The box diagram contribution to the elastic electron proton scattering for $Q^2 = 2.64$ GeV$^2$ (●), 3.20 GeV$^2$ (○), 4.10 GeV$^2$ (◆), 5 GeV$^2$ (△) and 6 GeV$^2$ (▲).
Figure 5: The cross-box diagram contribution to the elastic electron proton scattering for $Q^2 = 2.64$ GeV$^2$ ($\bullet$), 3.20 GeV$^2$ ($\bigcirc$), 4.10 GeV$^2$ ($\bigstar$), 5 GeV$^2$ ($\vartriangle$) and 6 GeV$^2$ ($\blacktriangle$).
well as the Rosenbluth separation experiment performed at SLAC and JLAB. The two photon exchange contribution computed in this paper should be subtracted from the Rosenbluth data in order to extract the corrected form factor ratio. In fig. 4 we instead add this contribution to the polarization transfer data. This is convenient since the Rosenbluth data from SLAC and JLAB show a slightly different trend. We point out that the two photon exchange diagrams give negligible correction to the ratio extracted by polarization transfer experiment. At \( Q^2 = 6 \text{ GeV}^2 \) the JLAB results are extrapolated using the fit given in eq. 4. We find that the two photon exchange explains a substantial part of the difference between the two experimental results. Our final result is almost in agreement with the SLAC data but falls below the JLAB Rosenbluth data. Since we have dropped many contributions this result is very encouraging. Our calculation demonstrates that the two photon exchange contributions are of the right order of magnitude to explain the discrepancy. Furthermore the predicted dependence of \( \sigma_R \) on \( \epsilon \) is very close to linear and hence may provide a satisfactory explanation of the data.

Our result may be compared to that obtained in Ref. 11 who also compute the two photon exchange contributions using monopole form factors. In our approach we have used an effective nonlocal Lagrangian model which allows us to identify all the relevant operators at a given order. We include all operators up to dimension five and find the presence of an additional operator given explicitly in eq. 13. This operator depends on the form factor \( F_2 \) and on an unknown parameter.

We have also used realistic form factor model for our calculation paying careful attention to the amplitude and phase of the form factor in the unphysical region. We find that the dependence of the reduced cross section \( \sigma_R \) on \( \epsilon \) is almost linear, in contrast to what is obtained in Ref. 11. The full calculation including the contribution of the term in eq. 13 will be presented in a separate publication.

7 Conclusions

In this paper we have constructed a nonlocal Lagrangian to model the electromagnetic interaction of proton. The model is invariant under a nonlocal form of gauge transformations and incorporates all operators up to dimension five. The model displays the standard electromagnetic vertex of an on-shell proton. The dimension five operators also
contain an operator with an unknown coefficient whose value can be extracted experimentally. We use this model to compute the two photon exchange diagrams contributing to elastic scattering of electron with proton. The calculation requires the proton form factors in the entire kinematic range. In the present paper we have focussed only on those terms which depend on the form factor $F_1$. A more complete calculation requires a realistic model for both the form factors $F_1$ and $F_2$ in the unphysical region and will be attempted in a future publication. We find that the two photon exchange diagram contribution to the reduced cross section $\sigma_R$ shows an almost linear dependence on the longitudinal polarization of the photon $\epsilon$. We also find that the contribution explains a substantial percentage of the experimental discrepancy. This is very encouraging given the fact that we have neglected many terms. It, therefore, appears that the two photon exchange may be able to explain the difference in the experimental extraction of proton electromagnetic form factor $G_E$ using the Rosenbluth separation and polarization transfer techniques.
Figure 6: The total two photon exchange contribution to elastic electron-proton scattering after removing the infrared $\mu^2$ dependent contributions for $Q^2 = 2.64 \text{ GeV}^2 (\bullet), 3.20 \text{ GeV}^2 (\varnothing), 4.10 \text{ GeV}^2 (\ast), 5 \text{ GeV}^2 (\triangle)$ and $6 \text{ GeV}^2 (\blacktriangle)$. 
Figure 7: The ratio $\mu_p G_E/G_M$, extracted by the polarization transfer technique (▼) and after adding the two photon exchange contribution to the polarization transfer result (●). For comparison the JLAB [3] (■) and SLAC [2] (○) experimental data for Rosenbluth separation are also shown. For $Q^2 = 6$ GeV$^2$, the data point shown is obtained by extrapolating the JLAB polarization transfer result, eq. 4.
However, the presence of the phase cannot introduce a large error since

\[ \phi \text{ discussed in text. The form factor should be real in this region.} \]

This will lead to a phase \( \phi \). The form factor does not change for a phase \( \phi \).

Next, our model for the effective action differs from the model

\[ \langle \phi, \Gamma \rangle \approx \mu \rho \]

We provide details of the calculation of the box diagram. We have

**Appendix 2: Sample Calculation**

In the other terms, both of these terms give zero. Similar logic applies

\[ \langle \phi, \Gamma \rangle \approx \mu \rho \]

Now, using the form factor, this becomes

\[ \langle \phi, \Gamma \rangle \approx \mu \rho \]

Thus, the result for the photon exchange diagram is the form

\[ \langle \phi, \Gamma \rangle \approx \mu \rho \]

The Feynman integral has no dependence on both \( d \) and \( d \). The Feynman integral has no dependence on both \( d \) and \( d \).

where \( a, b, c, d \) are constants and, in particular, no \( c, d \) term appears.

\[ \phi \sim \phi \phi \phi \phi \phi \phi \phi \phi \phi \]

We then note that

\[ \phi \sim \phi \phi \phi \phi \phi \phi \phi \phi \phi \]

We shall show in the appendix that the first term in each

**Appendix 1**
this region gives negligible contribution to the two photon exchange amplitude. The tree level amplitude becomes,

\[
\mathcal{M}_0' = \frac{e^2}{q^2} \left[ \overline{\tau(k')} \gamma^\delta u(k) \overline{U}(p') \left( F_1(q^2) \gamma_\delta + i\kappa F_2(q^2) \frac{1}{2M} \sigma_{\delta\rho} q^\rho \right) U(p) \right] e^{i\Phi(q^2)} \\
= \mathcal{M}_0 e^{i\Phi(q^2)} = \frac{e^2}{q^2} \mathcal{M}_0 e^{i\Phi(q^2)}
\]

(24)

It is clear that the phase will not give any contribution to the tree level cross section. The contribution to the total cross-section from the box diagram is given by,

\[2Re\{\mathcal{M}_0^* e^{-i\Phi(q^2)} \mathcal{M}_B\} + \mathcal{O}(\alpha^4)\]

Introducing Feynman parameters we can write \(\mathcal{M}_B\) as,

\[-i\mathcal{M}_B = \frac{120 e^4 F^2}{(2\pi)^4 (im_1 \Delta \Gamma_1)^2} \sum_{\alpha,\beta} \int d^4 l \int_0^1 dx_i \delta(\sum_{i=0}^6 x_i - 1) (-1)^{\alpha + \beta} \frac{N_B}{(\overline{D}_B(\Gamma_1^{\alpha}, \Gamma_1^{\beta}))^6}\]

(25)

where,

\[N_B = \overline{u}(k') \gamma^\mu (k - l + m_e) \gamma^\nu u(k) \overline{U}(p') \gamma_\mu (\phi + l + M_p) \gamma_\nu U(p),\]

\[\overline{D}_B(\Gamma_1^{\alpha}, \Gamma_1^{\beta}) = x_1(l^2 - \mu^2 + i\xi) + x_2((q - l)^2 - \mu^2 + i\xi) + x_3((k - l)^2 - m_e^2 + i\xi) + x_4((p + l)^2 - M_p^2 + i\xi) + x_5(l^2 - m_1^2 + im_1 \Gamma_1^\alpha) + x_6((q - l)^2 - m_1^2 + im_1 \Gamma_1^\beta).\]

After the Wick rotation and integration over shifted loop momentum we obtain,

\[\mathcal{M}_0^* e^{-i\Phi(q^2)} \mathcal{M}_B = \frac{e^6 F^2 e^{-i\Phi(q^2)}}{8\pi^2 (m_1 \Delta \Gamma_1)^2 q^2} \sum_{\alpha,\beta} (-1)^{\alpha + \beta} \left[ I_a(\Gamma_1^{\alpha}, \Gamma_1^{\beta}) + I_b(\Gamma_1^{\alpha}, \Gamma_1^{\beta}) \right]\]

(26)

where

\[I_a = \int_0^1 \prod_{i=0}^6 dx_i \delta(\sum_{i=0}^6 x_i - 1) \frac{2M_B^4 N_B^4}{(\Delta_B - im_1 (x_5 \Gamma_1^\alpha + x_6 \Gamma_1^\beta) - i\xi)^3},\]

\[I_b = \int_0^1 \prod_{i=0}^6 dx_i \delta(\sum_{i=0}^6 x_i - 1) \frac{3M_B^4 N_B^4}{(\Delta_B - im_1 (x_5 \Gamma_1^\alpha + x_6 \Gamma_1^\beta) - i\xi)^4},\]

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with
\[
\Delta_B = x_1^2 M_2^2 + x_2^2 m_e^2 + m_1^2 (x_5 + x_6) + \mu^2 (x_1 + x_2) - 2 x_3 x_4 E M_p
\]
\[- (x_2 + x_4)(1 - x_2 - x_3 - x_4 - x_6) q^2,
\]
\[
N_B^a = \pi (k') \gamma^\mu \gamma^\alpha \gamma^\nu u(k) U(p') \gamma_\mu \gamma_\alpha \gamma_\nu U(p),
\]
\[
N_B^b = \pi (k') \gamma^\mu \{(1 - x_3) \dot{k} + x_4 \dot{\phi} - (x_2 + x_6) \dot{\phi} \} \gamma' u(k)
\times U(p') \gamma_\mu \{(1 - x_4) \dot{\phi} + x_3 \dot{k} + (x_2 + x_6) \dot{\phi} + M_p \} \gamma_\nu U(p).
\]
Here we have approximated \((m_1 \Gamma_1^{\alpha, \beta} - \xi)\) by \(m_1 \Gamma_1^{\alpha, \beta}\) and neglected terms proportional to \(m_e\) in the numerator. We cannot neglect \(m_e\) in the denominator as that will make the integrand ill-defined in the infrared limit. To simplify the integration over \(x_3\) we replace \(x_3^2 m_e^2\) by \((0.5 m_e)^2\). The final result including both box and cross box diagrams shows a very weak dependence on electron mass.

We may now write \(I_a\) as,
\[
I_a = \int [dx_4 dx_4 dx_2] \frac{L_5^2 \overline{M_0^a N_B^b}}{B_5 (B_5 + A_3 L_5) (B_5 + A_5 L_5)}.
\]
where
\[
\int [dx_4 dx_4 dx_2] \equiv \int_0^1 dx_6 \int_0^{1-x_6} dx_4 \int_0^{1-x_6-x_4} dx_2
\]
\[
L_5 = 1 - x_6 - x_4 - x_2
\]
\[
A_3 = (x_2 + x_6) q^2 - 2 x_4 E M - \mu^2
\]
\[
A_5 = m_1^2 - \mu^2 - i m_1 \Gamma_1^\alpha
\]
\[
B_5 = x_3^2 M^2 + \delta^2 + x_6 m_1^2 + (1 - x_6 - x_4) \mu^2
\]
\[- (x_2 + x_6)(1 - x_6 - x_4 - x_2) q^2 - i x_6 m_1 \Gamma_1^\beta - i \xi.
\]

We split \(I_b\) into two parts.
\[
I_b = I_b^1 \times \text{(terms proportional to } x_3 \text{ in } \overline{M_0^a N_B^b})
\]
\[+ I_b^2 \times \text{(terms independent of } x_3 \text{ in } \overline{M_0^a N_B^b})
\]
with,
\[
I_b^1 = \int [dx_6 dx_4 dx_2] \frac{L_5^2}{2 B_5 (B_5 + A_3 L_5)^2 (B_5 + A_5 L_5)}
\]
\[
I_b^2 = \int [dx_6 dx_4 dx_2] \frac{L_5^2 (3 B_5^2 + 2 (A_3 + A_5) B_5 L_5 + A_3 A_5 L_5^2)}{2 B_5^2 (B_5 + A_3 L_5)^2 (B_5 + A_5 L_5)^2}.
\]

\(I_a, I_b^1 \) and \(I_b^2\) were calculated numerically using Gauss-Legendre integration technique \[19\] with 550 points.
References

[16] See e.g. J. W. Moffat, Phys. Rev. D 41, 1177 (1990);
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