MOSAICKING WITH COSMIC MICROWAVE BACKGROUND INTERFEROMETERS

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ABSTRACT

Measurements of cosmic microwave background (CMB) anisotropies by interferometers offer several advantages over single-dish observations. The formalism for analyzing interferometer CMB data is well developed in the flat-sky approximation, valid for small fields of view. As the area of sky is increased to obtain finer spectral resolution, this approximation needs to be relaxed. We extend the formalism for CMB interferometry, including both temperature and polarization, to mosaics of observations covering arbitrarily large areas of the sky, with each individual pointing lying within the flat-sky approximation. We present a method for computing the correlation between visibilities with arbitrary pointing centers and baselines and illustrate the effects of sky curvature on the $\ell$-space resolution that can be obtained from a mosaic.

Subject headings: cosmic microwave background — techniques: interferometric

1. INTRODUCTION

The study of anisotropies in the cosmic microwave background (CMB) radiation has revolutionized cosmology. Key to this revolution have been coupled advances in theory, data analysis, and instrumentation. In particular, the design of experiments with exquisite systematic error control has been crucial for progress in the field. Interferometers offer several advantages in this respect, with simple optics, instantaneous differentiating of sky signals without scanning and no differentiating of detectors. The shape of the beam can be well understood and the measurement is done directly in Fourier space where the theory most naturally lives.

Pioneering attempts to detect CMB anisotropy with interferometers were made by Martin & Partridge (1988) and Subrahmanyan et al. (1993). Several groups have successfully detected primary CMB anisotropies (O’Sullivan et al. 1993; Raker et al. 1994; Halverson et al. 2002; Pearson et al. 2003; Taylor et al. 2003) and polarization (Readhead et al. 2004; Leitch et al. 2007), using interferometers. The formalism for analyzing CMB data from interferometers has been developed by Hobson & Magueijo (1996); Hobson & Magueijo (1996); White et al. (1999); Hobson & Maisinger (2002); and Myers et al. (2003), as well as in the experimental papers cited above. Park et al. (2003) and Park & Ng (2004) examined interferometric polarimetry.

In the Fraunhofer limit an interferometer measures the Fourier transform of the sky, multiplied by the primary beam. The primary beam determines the instantaneous field of view of the instrument and its Fourier transform is simply the autocorrelation of the Fourier transform of the point response of the receiver to an electric field. The angular scale probed by any pair of telescopes being correlated is determined by their spacing in units of the observational wavelength. The range of scales probed by the interferometer is then determined by the spacing of the elements, while the resolution in spatial wavenumber is determined by the area of sky surveyed. By “mosaicking” several smaller patches together, the resolution in spatial wavenumber can be increased, although the range of spatial scales remains fixed by the geometry of the interferometer elements.

In most cases it has been assumed that the field of view is small, so that one can use the “small-angle” or “flat-sky” approximation. However, if we want fine resolution in spatial wavenumber – which future experiments are driving towards – we need to survey large areas of sky (Hobson & Magueijo 1996) and thus relax this assumption. The purpose of this paper is to extend the formalism presented in the above papers to the case where each individual pointing of the interferometer is still within the flat-sky approximation but by mosaicking many pointings together a significant area of sky is surveyed. Our extension allows one to see how large an error is being made in assuming the flat-sky approximation and shows how corrections can be systematically incorporated.

The central idea of this paper is the following. The key ingredient in analyzing a mosaic of interferometer pointings is the set of two-point visibility correlations. For each pair of pointings, we can calculate the correlations in a spherical coordinate system that places both pointing centers on the equator. If each pointing has a small field of view, then we can approximate the sphere by a cylinder in the vicinity of the equator, allowing the use of Fourier analysis rather than a more cumbersome expansion in spherical harmonics.

The outline of this paper is as follows. We begin in §2 by reminding the reader of some basic results in the flat-sky limit. We then show how this can be extended using a cylindrical projection in §3 and make contact with the exact spherical harmonic treatment in §4. Section 5 ex-
tends our results to include polarization, and we conclude in

2. FLAT-SKY LIMIT

We begin by considering the flat-sky limit and focusing on temperature anisotropies. Thus our interferometer is measuring a scalar field, \( T(x) \), defined on the 2D plane. In this limit the fundamental observable, a visibility, can be written

\[
V(u) = \frac{\partial B_\nu}{\partial T} \int d^2x \Delta T(x) A(x) e^{2\pi i u \cdot x},
\]

where \( \partial B_\nu / \partial T \) converts from temperature to intensity units and \( A(x) \) is the primary beam (typically normalized to unity at peak). From now on we will neglect the flux-temperature conversion factors and write \( T \) for \( \Delta T \).

For Gaussian fluctuations, such as the primary CMB anisotropies, we need to compute the visibility correlation matrix

\[
\mathcal{V}_{ij} \equiv \langle V(u_i)V^*(u_j) \rangle,
\]

where \( u_i \) and \( u_j \) represent the baselines to be correlated and \( \langle \cdot \rangle \) represents an ensemble average. This can be related to the usual temperature correlation function

\[
\left\langle T(n_i)T(n_j) \right\rangle = \frac{1}{4\pi} \sum_{\ell=0}^{\infty} (2\ell+1)C_\ell P_\ell(n_i \cdot n_j)
\]

for temperatures measured in directions \( n_i \) and \( n_j \) where \( C_\ell \) is the multipole moments. In our flat-sky limit, for a single patch (e.g., White et al. 1999)

\[
\mathcal{V}_{ij} \propto \int d^2w S(w) \tilde{A}^*(2\pi[w+u_i])\tilde{A}(2\pi[w+u_j]),
\]

where the angular power spectrum \( S(u) \) is defined by

\[
(2\pi)^2 u^2 S(u) \simeq \ell(\ell+1)C_\ell \quad \text{for} \\ \ell = 2\pi u,
\]

and \( \tilde{A} \) is the Fourier transform of the antenna pattern,

\[
\tilde{A}(k) = (2\pi)^{-2} \int d^2x A(x)e^{-ik \cdot x}.
\]

The extension to multiple different patches, each with its own pointing center \( \mathcal{P} \), merely inserts a phase factor

\[
\mathcal{V}_{ij}^{\mathcal{P}_1\mathcal{P}_2} \propto \int d^2w S(w) \tilde{A}^*(2\pi[w+u_i])\tilde{A}(2\pi[w+u_j])e^{2\pi i w \cdot D},
\]

where \( D \) is the separation between the pointing centers \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \). From now on we will drop the superscripts on \( \mathcal{V}_{ij} \).

3. CYLINDRICAL METHOD

The flat-sky approximation above is valid only if both the field of view of an individual pointing and the separation \( D \) between pointings are small. We will now assume a mosaic of pointings that cover a large area, although each individual pointing observes only a small patch of sky. We therefore relax the second assumption while retaining the first.

For a statistically isotropic temperature field we are free to use any coordinate system we like to compute the visibility correlation \( \mathcal{V}_{12} \). In particular, we can arrange to have the two pointing centers lie on the equator of a spherical coordinate system \( (\theta = \pi/2) \) and be separated by an angle \( \beta \). We introduce a cylindrical coordinate system, with the cylinder tangent to the sphere at the equator, denoted by \( \xi = (\phi, z) \). Since both observations sample only regions near the equator, we can pretend that the data live on the cylinder rather than on the sphere. In this approximation, it is natural to expand the temperature \( T(x) \) in a discrete Fourier series in \( \phi \) and a continuous Fourier transform in \( z \):

\[
T(\xi) = \sum_m \int dn \tilde{T}_m(n)e^{im\phi+nz}
\]

with

\[
\left\langle \tilde{T}_m(n)\tilde{T}_m^*(n') \right\rangle = \mathcal{P}\left(\frac{m^2+n^2}{(2\pi)^2}\right)\delta_{mm'}\delta(n-n').
\]

The power spectrum is \( \mathcal{P}(k) = (2\pi)^2 S(k/2\pi) \) with \( S \) as defined in Eq. (5). It is related to the spherical harmonic angular power spectrum by \( \mathcal{P}(k) \simeq C_k \) for large \( k \).

The visibility becomes

\[
V(\mathcal{P}_1, u_1) = (2\pi)^2 \int dn \sum_m \tilde{T}_m(n)\tilde{A}^*(2\pi[u_1+w]),
\]

where the vector \( w \) has coordinates \( (w_\phi, w_z) = (m,n)/(2\pi) \) and

\[
\tilde{A}(k) = \int \frac{d^2\xi}{(2\pi)^2} A(\xi)e^{-ik \cdot \xi}
\]

is the usual Fourier transform of the primary beam. Since we imagine \( A \) is non-zero only over a small region we can extend the integral over the entire plane.

The visibility for the second pointing center is analogous, except for a phase factor \( e^{im\beta} \), so the correlation between two pointings becomes

\[
\mathcal{V}_{12} \propto \int dn \sum_m S(w)W_{12}(u_1, u_2, w)e^{im\beta},
\]

FIG. 1.— Window functions for the covariance between two interferometer pointings in both the flat and cylindrical approximations. The antenna pattern is Gaussian with beam width \( \sigma = 5' \). The two pointings are separated by 120°. Each visibility has a baseline of magnitude \( u = 20 \) pointing in the \( \phi \) direction. The plus signs are the cylindrical approximation, and the stars are the flat approximation. For clarity, only the real parts of the window functions are shown.
where we have defined the window function

\[ W_{12}(u_1, u_2, w) \equiv \bar{A}^* (2\pi|u_1 + w|) \bar{A} (2\pi|u_2 + w|) \]  

(13)

It is convenient to define a window function that is averaged over direction:

\[ V_{12} = \sum_\ell W^{(12)}_\ell \mathcal{C}_\ell. \]  

(14)

To compute \( W^{(12)}_\ell \), we divide the integral and sum in equation (12) into bands with \( \ell - \frac{1}{2} < 2\pi w < \ell + \frac{1}{2} \). Within each band we assume the power spectrum is constant and remove it from the integral. We can calculate this window function in the flat-sky approximation instead of the cylindrical approximation if we like, simply by replacing the sum over \( m \) by an integral.

Figure 2 illustrates the difference between the flat and cylindrical approximations for large \( \beta \). The difference is most significant when the baseline vectors \( u_i \) are nearly equal in magnitude and parallel to the separation direction \( \phi \); otherwise the correlation \( V_{12} \) is always small whenever \( \beta \) is large.

We can use this prescription to calculate the full visibility covariance matrix for a mosaic of many pointings. For each pair of pointings, we must transform to a coordinate system in which both pointings lie on the equator. In performing this rotation, the components of the baseline vectors \( u_i \) will naturally be transformed. The Appendix contains an explicit recipe for performing this rotation.

Figures 2 and 3 present a simple illustration of how mosaicking increases the \( \ell \)-space resolution of an experiment. In each pointing, the beam pattern is a Gaussian with beam width \( \sigma = 5^\circ \). We assume a 10 \( \times \) 10 grid of pointings, separated by 5\(^\circ\) in both \( \theta \) and \( \phi \) in a spherical coordinate system, with the center of the grid on the equator (\( \theta = \pi/2 \)). For each pointing, we consider only a single baseline \( u = 22\phi \). Figure 2 shows the locations of the pointing centers and the baselines in Aitoff projection. Note that although all the baselines have identical components in the spherical coordinate system, they do not when rotated to the appropriate coordinate system for computing the covariances. As an example, to compute the covariance between the two pointings in the upper corners of the grid, we must use a coordinate system in which the great circle represented by the dashed line becomes the equator. In this coordinate system, the two baseline vectors have \( \phi \) components of opposite sign.

The visibility obtained from any single pointing provides an estimate of the power spectrum with a fairly wide window function (solid curve in Fig. 3). We can obtain an estimate of the power spectrum with a narrower window function by simply adding together all 100 visibilities. To find the window function for the sum, we write down the absolute square of the sum of all 100 visibilities:

\[ \left\langle \left| \sum_{i=1}^{100} V_i \right|^2 \right\rangle = \sum_{i,j=1}^{100} \langle V_i V_j^\ast \rangle = \sum_{i,j=1}^{100} V_{ij}. \]  

(15)

The window function for the sum of all the visibilities is therefore the sum of \( W^{(i,j)}_\ell \) over all visibility pairs \( i, j \). We can compute each \( W^{(i,j)}_\ell \) using the recipe described above. Specifically, for each pair of pointings, we find a rotation that brings both pointing centers to the equator, apply that rotation to the vectors \( u_1, u_2 \) using the recipe in Appendix, and apply equation (13).

The dashed curve in Figure 3 is the window function for the sum of all pointings. As expected, the \( \ell \)-space resolution has improved. The dotted curve shows the window function obtained by incorrectly assuming the sky is flat over the entire mosaic – that is, assuming that all of the baseline vectors illustrated in Fig. 2 lie in the...
same plane and are parallel.

The difference between the dashed and dotted curves is almost entirely due to “baseline rotation” – the fact that, e.g., the baseline vectors at the corners of Fig. 2 are not in fact parallel to each other when viewed in a coordinate system in which both lie on the equator. It makes virtually no difference whether we use the flat method (integral over m), the cylindrical method (sum over m), or an exact spherical harmonic calculation as described below in calculating each \( W_{\ell}^{(ij)} \), as long as we get the baseline rotation right.

Of course, other linear combinations could be used instead of a simple sum of all 100 pointings, resulting in window functions with peaks in different places (within the envelope set by the single-pointing window function).

4. HARMONIC METHOD

We can also make direct contact with usual spherical harmonic treatment in which

\[
T(\hat{r}) = \sum_{\ell,m} a_{\ell m} Y_{\ell m}(\hat{r})
\]

and

\[
(a_{\ell m}, a_{\ell m}') = C \delta_{\ell \ell'} \delta_{m m'}.
\]

The visibility for a single pointing is

\[
V(u) = \int d^2 \hat{r} A(\hat{r}) T(\hat{r}) e^{2\pi i u \cdot \hat{r}} = \sum_{\ell,m} a_{\ell m} F_{\ell m}(u),
\]

where

\[
F_{\ell m}(u) = \int A(\hat{r}) Y_{\ell m}(\hat{r}) e^{2\pi i u \cdot \hat{r}}.
\]

It is of course possible to perform these integrals numerically and calculate the visibility covariance matrix without any approximations at all. In this section we will see how to obtain the cylindrical approximation from this exact expression.

In previous treatments (e.g., [White et al. 1999]), the flat-sky limit for a single pointing was taken by approximating the spherical harmonics near the pole of the spherical coordinate system (\( \theta = 0 \)). To obtain the visibility covariance for two different pointings it is more convenient to place the pointing centers on the equator as in the previous section.

Near the equator \( z \equiv \cos \theta \approx \pi/2 - \theta \). Using the recurrence relations for the associated Legendre polynomials [Abramowitz & Stegun 1972, Gradshteyn & Ryzhik 1980], one can show

\[
Y_{\ell m}(\phi, z) \rightarrow N_{\ell m} e^{i m \phi} \left\{ \begin{array}{ll}
\cos n_{\ell m} z & \text{if } \ell - m \text{ even} \\
-i \sin n_{\ell m} z & \text{if } \ell - m \text{ odd}
\end{array} \right.
\]

with

\[
n_{\ell m} = \sqrt{\ell(\ell + 1) - m^2}
\]

and

\[
N_{\ell m} = (-1)^{(\ell + m)/2} \frac{2^m}{\sqrt{\pi}} \frac{\sqrt{2\ell + 1}}{4\pi} \sqrt{\frac{(\ell - m)!}{(\ell + m)!}} \frac{(\ell + m - 1)/2)!}{(\ell - m)/2)!},
\]

which can also be written

\[
N_{\ell m} = \left( -1 \right)^{(\ell + m)/2} \frac{2^m}{\sqrt{\pi}} \frac{\sqrt{2\ell + 1}}{4\pi} \sqrt{\frac{(\ell - m)!}{(\ell + m)!}} \frac{(\ell + m - 1)/2)!}{(\ell - m)/2)!}.
\]

by using [Gradshteyn & Ryzhik 1980]

\[
\left( n + \frac{1}{2} \right)! = \sqrt{\pi} \frac{(2n + 1)!}{2^{2n + 1} n!}
\]

In all of these expressions the factorials should be interpreted as \( \Gamma \) functions for non-integer arguments. In the limit when all of the factorial moments are large we can use the approximation [Abramowitz & Stegun 1972]

\[
\ln N \approx \left( \frac{N + 1}{2} \right) \ln \frac{N - N + \text{const}}{2}
\]

to write the normalization factor as

\[
N_{\ell m} \approx \left( -1 \right)^{(\ell + m)/2} \frac{2^m}{\sqrt{\pi}} \frac{\sqrt{2\ell + 1}}{4\pi} \frac{1}{\ell(\ell + 1)}
\]

Note that the \( Y_{\ell m} \) are eigenfunctions of the 2D Laplacian with eigenvalues \(-\ell(\ell + 1)\), and the form of \( n_{\ell m} \) guarantees that this is preserved in the cylindrical coordinates:

\[
\nabla^2 Y_{\ell m} \rightarrow \left( \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} \right) Y_{\ell m}
\]

\[
= -m^2 Y_{\ell m}
\]

\[
= -\ell(\ell + 1) Y_{\ell m},
\]

where the arrow indicates the cylindrical coordinate limit.

If we define

\[
\alpha_{\ell m} = \left( -1 \right)^{(\ell + m)/2} \frac{N_{\ell m}}{2} a_{\ell m}, \quad \bar{\alpha}_{\ell m} = \frac{N_{\ell m}}{2} \bar{a}_{\ell m},
\]

then we can rewrite Eq. 16 as

\[
T(\xi) = \sum_{\ell,m} \alpha_{\ell m} e^{i k \xi} + \bar{\alpha}_{\ell m} e^{i k \xi}
\]

with the definitions \( k = (m, n_{\ell m}) \) and \( \bar{k} = (m, -n_{\ell m}) \).

From now on we will take the sum over both \( \pm n_{\ell m} \) as implicit and write

\[
T(\xi) = \sum_{k} \alpha_{k} e^{i k \xi},
\]

This way of writing the spherical harmonic expansion makes the correspondence with the Fourier representation explicit.

The visibility becomes

\[
V(u) = \left( 2\pi \right)^2 \sum_{k} \alpha(k) \tilde{A}^{*}(k + 2\pi u),
\]

and the correlation matrix is

\[
\mathcal{V}_{12} = \frac{\left( 2\pi \right)^4}{4} \sum_{k} \left| N_{\ell m} \right|^2 C_{\ell} W_{12}(u_1, u_2, k/2\pi) e^{i k \beta},
\]

plus oscillatory terms that average to zero in the sum over \( \ell \) and \( m \).
must consider observations of the two Stokes parameters

\[ P_{\pm} = \frac{1}{\sqrt{2}}(Q \pm iU), \]

because these quantities transform in a relatively simple way under rotations: under a rotation by an angle \( \psi \) about a given point \( \hat{r} \), \( P_{\pm}(\hat{r}) \to P_{\pm}(\hat{r})e^{\pm 2i\psi} \). In other words, \( P_+ \) is a quantity of spin weight \(-2\) and \( P_- \) has spin weight \(+2\). These transformation properties make \((P_+,P_-)\) a more convenient basis of observables to work with than \((Q,U)\). The two bases are related by a unitary transformation,

\[
\begin{pmatrix} P_+ \\ P_- \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} Q \\ U \end{pmatrix}, \tag{37}
\]

so any results derived in one basis can easily be transformed to the other.

An interferometer that works by interfering circularly polarized radiation from the two antennas measures the visibilities

\[
V_{\pm}(u) = \int d^2\hat{r} P_{\pm}(\hat{r}) A(\hat{r}) e^{2\pi iu \cdot \hat{r}}. \tag{38}
\]

Specifically, interfering left-circularly polarized radiation from antenna 1 with right-circularly polarized radiation from antenna 2 yields \( V_+ \), and reversing the senses of both circular polarizations yields \( V_- \). (Interfering right with left and left with left yields visibilities that probe total intensity and circular polarization.) On the other hand, an interferometer that works by combining linear polarization states would measure visibilities \( V_Q \) and \( V_U \) for the individual Stokes parameters. (For instance, interfering \( E_x \) from antenna 1 with \( E_y \) from antenna 2 yields \( V_U \).) We will assume that the measured\(^2\) quantities are \( V_{\pm} \) rather than \( V_Q,V_U \); but all results are easily transformed to the \( Q,U \) basis using Eq. (37).

As in the case of temperature anisotropy, the key ingredient in analyzing CMB interferometric observations of polarization is the visibility covariance matrix:

\[
V_{\pm \pm}^{\pm \pm} = \langle V_{\pm}(u_1,P_1) V_{\pm}(u_2,P_2)^* \rangle, \tag{39}
\]

where \( u_i \) and \( P_i \) represent baselines and pointing centers for a pair of visibilities. Note that in this equation the signs of \( \pm \) and \( \pm \) can be varied independently — that is, there are in general four distinct covariances, \( V_{12}^{++}, V_{12}^{-+}, V_{12}^{+-}, V_{12}^{--} \).

Our primary interest will continue to be the case where the flat-sky approximation is appropriate for each individual pointing but the separation between pointings is not necessarily small. We will present exact expressions for the visibility covariances in terms of spherical harmonics first, then show that they reduce in the this limit to cylindrical-sky expressions similar to the anisotropy results above.

5.1 Flat sky

\(^2\) We are ignoring some sources of systematic error in this expression. For instance, in an instrument with cross-polar beam response, each measured visibility would contain contributions from both \( P_+ \) and \( P_- \), with different effective antenna patterns.
Assume our observations cover a small enough patch of sky that we can replace spherical harmonic expansions with Fourier transforms:

$$P_{\pm}(x) = \int d^2k \tilde{P}_{\pm}(k)e^{i\mathbf{k}\cdot\mathbf{x}}.$$  \hspace{1cm} (40)

The two Fourier transforms are related like this:

$$\tilde{P}_{\pm}(k) = P_{\pm}(-k).$$  \hspace{1cm} (41)

A key insight into the analysis of CMB polarization was the observation [Kamionkowski et al. 1997; Zaldarriaga & Seljak 1997] that any polarization field can be decomposed into a scalar part (conventionally denoted $E$) and a pseudoscalar part (denoted $B$). The $E$-$B$ separation is particularly simple in Fourier space: modes with polarization parallel or perpendicular to $\mathbf{k}$ are $E$ modes, while modes polarized at 45° are $B$ modes. In terms of $P_{\pm}$, this means that

$$\tilde{P}_{\pm}(k) = (\tilde{E}(k) \pm i\tilde{B}(k))e^{\pm 2i\psi_k},$$  \hspace{1cm} (42)

where $\psi_k$ is the angle made by the wavevector $\mathbf{k}$ with respect to the $x$ axis.

Assuming that the polarization was generated by a homogeneous, isotropic, parity-respecting process, the two-point correlations between $E$ and $B$ are determined by two power spectra $P_{E,B}$:

$$\langle \tilde{E}(\mathbf{k})\tilde{E}^*(\mathbf{k}') \rangle = (2\pi)^{-2}P_{E,E}(k)\delta(k - k'),$$  \hspace{1cm} (43)

$$\langle \tilde{B}(\mathbf{k})\tilde{B}^*(\mathbf{k}') \rangle = (2\pi)^{-2}P_{B,B}(k)\delta(k - k'),$$  \hspace{1cm} (44)

$$\langle \tilde{E}(\mathbf{k})\tilde{B}^*(\mathbf{k}') \rangle = 0.$$  \hspace{1cm} (45)

This means that the covariances of the polarization $P_{\pm}$ are

$$\langle \tilde{P}_{\pm}(\mathbf{k})\tilde{P}_{\pm}^*(\mathbf{k}') \rangle = (2\pi)^{-2}(P_{E,E}(k) + P_{B,B}(k))\times \delta(k - k'),$$  \hspace{1cm} (46)

$$\langle \tilde{P}_{\pm}(\mathbf{k})\tilde{P}_{\pm}^*(\mathbf{k}') \rangle = (2\pi)^{-2}(P_{E,E}(k) - P_{B,B}(k))\times e^{\pm 4i\psi_k}\delta(k - k').$$  \hspace{1cm} (47)

Just as in the scalar case, the visibility associated with a pointing center $\mathbf{b}$ and a baseline $\mathbf{u}$ can be expressed in terms of the Fourier transform of the antenna pattern:

$$V_{\pm}(\mathbf{u}, \mathbf{b}) = (2\pi)^{2}\int d^2k \tilde{A}^*(\mathbf{k} + 2\pi\mathbf{u})\tilde{P}_{\pm}(k)e^{i\mathbf{k}\cdot\mathbf{b}}.$$  \hspace{1cm} (48)

The correlation between two visibilities is

$$V_{12}^{\pm} = (2\pi)^{2}\int d^2k \tilde{A}^*(\mathbf{k} + 2\pi\mathbf{u}_1)\tilde{A}(\mathbf{k} + 2\pi\mathbf{u}_2)$$

$$e^{i\mathbf{k}\cdot(\mathbf{b}_1 - \mathbf{b}_2)}(P_{E,E}(k) + P_{B,B}(k)), \hspace{1cm} (49)$$

$$V_{12}^{\mp} = (2\pi)^{2}\int d^2k \tilde{A}^*(\mathbf{k} + 2\pi\mathbf{u}_1)\tilde{A}(\mathbf{k} + 2\pi\mathbf{u}_2)$$

$$e^{i\mathbf{k}\cdot(\mathbf{b}_1 - \mathbf{b}_2)}(P_{E,E}(k) - P_{B,B}(k))e^{\pm 4i\psi_k}. \hspace{1cm} (50)$$

5.2. Spherical harmonics

Since the quantities $P_{\pm}$ are quantities of spin weight $\mp 2$, it is natural to expand them in spin-$(\pm 2)$ spherical harmonics:

$$P_{\pm}(\hat{r}) = \sum_{\ell,m} a_{\mp 2,\ell m} Y_{\ell m}(\hat{r}).$$  \hspace{1cm} (51)

The decomposition into $E$ and $B$ components is particularly simple in terms of the spherical harmonic coefficients:

$$a_{\pm 2,\ell m} = E_{\ell m} \mp iB_{\ell m}. \hspace{1cm} (52)$$

The two-point statistics are completely described by two power spectra $C_{\ell E}^{EE}, C_{\ell B}^{BB}$:

$$\langle E_{\ell m}E_{\ell' m'}^* \rangle = C_{\ell E}^{EE}\delta_{\ell\ell'}\delta_{mm'}, \hspace{1cm} (53)$$

$$\langle B_{\ell m}B_{\ell' m'}^* \rangle = C_{\ell B}^{BB}\delta_{\ell\ell'}\delta_{mm'}, \hspace{1cm} (54)$$

$$\langle E_{\ell m}B_{\ell' m'}^* \rangle = 0. \hspace{1cm} (55)$$

As in the case of temperature anisotropy, the spherical and flat-sky power spectra are related via $C_{\ell} \sim \mathcal{P}(u)$ with $l = 2\pi u$.

The visibilities can be expressed in terms of the spherical harmonic coefficients as

$$V_{\pm}(\mathbf{u}) = \sum_{\ell,m} a_{\mp 2,\ell m} F_{\mp 2,\ell m}(\mathbf{u}), \hspace{1cm} (56)$$

where

$$F_{\mp 2,\ell m}(\mathbf{u}) = \int d^2\hat{r} A(\hat{r}) Y_{\ell m}(\hat{r})e^{2\pi i\mathbf{u}\cdot\hat{r}}. \hspace{1cm} (57)$$

Consider first the covariance between two visibilities with identical pointing centers. Combining equations 39 and 42 through 50, the visibility covariances can be shown to be

$$\langle V_{12}^{\pm} \rangle = \sum_{\ell} (C_{\ell E}^{EE} + C_{\ell B}^{BB})W_{\ell}^{\pm}, \hspace{1cm} (58)$$

$$\langle V_{12}^{\mp} \rangle = \sum_{\ell} (C_{\ell E}^{EE} - C_{\ell B}^{BB})W_{\ell}^{\mp}, \hspace{1cm} (59)$$

where

$$W_{\ell}^{\pm} = \sum_{m} F_{\mp 2,\ell m}(\mathbf{u}_1)F_{\mp 2,\ell m}(\mathbf{u}_2). \hspace{1cm} (60)$$

In the case where the two observations have different pointing centers, we once again transform to a coordinate
system with both pointing centers on the equator, separated by an angle \( \beta \). Because the spin-weighted spherical harmonics have azimuthal dependence \( e^{im\phi} \), the only change is an additional factor of \( e^{im\beta} \):

\[
W_{\ell}^{\pm \pm} = \sum_{m} e^{im\beta} F_{\mp 2, \ell m}(u_1) F_{\mp 2, \ell m}^*(u_2). \tag{61}
\]

In order to calculate the correlation between a pair of observations with arbitrary pointing centers, we simply rotate to a new coordinate system that places both centers on the equator before applying the above results. In performing this rotation, it is important to remember to transform \( P_{\ell} \) (and hence \( V_{\ell} \)) by \( e^{2i\delta} \) where \( \delta \) is the angle through which the polarization basis directions are rotated by the transformation. To be specific, if the change of coordinates results in a rotation of the \( \theta, \phi \) directions at each of the pointing centers by \( \delta_1, \delta_2 \), then

\[
V_{12}^{\pm \pm} \rightarrow V_{12}^{\pm \pm} e^{2i(\pm \delta_1 \mp \delta_2)}. \]

See the Appendix for an explicit recipe for finding these angles.

5.3. Connecting flat-sky to spherical.

As in the case of temperature anisotropy, we can see the connection between the spherical and flat-sky calculations of polarization by considering observations that lie near the equator of our spherical coordinate system and approximating the sphere by a cylinder. By applying the spin-raising operator (e.g., Lewis et al. 2002) to the plane-wave approximation to the spherical harmonics \( \{Y_{\ell m}(\hat{n})\} \), one can show that in this limit

\[
2Y_{\ell m}(\hat{r}) \approx N_{\ell m}^{(2)} e^{im\phi} \left\{ \cos(n_{\ell m}z + \delta_{\ell m}) \right. \text{ if } l - m \text{ even } \left. -i \sin(n_{\ell m}z + \delta_{\ell m}) \right\} \text{ if } l - m \text{ odd}, \tag{62}
\]

with

\[
N_{\ell m}^{(2)} = N_{\ell m}(l + 1) \sqrt{(l - 2)! (l + 2)!} = N_{\ell m} \sqrt{\frac{(l)(l+1)}{(l+2)(l-1)}} \tag{63}
\]

and

\[
\delta_{\ell m} = 2 \cos^{-1} \left( \frac{m}{l(l+1)} \right). \tag{64}
\]

By reasoning similar to the previous section we can use this to connect the spherical harmonic formalism to the flat-sky limit.

5.4. Example

Consider a \( 10 \times 10 \) mosaic of pointings of an interferometer, with the same parameters as in the example of Section 3, the Gaussian beam width is \( \sigma = 5^\circ \), and the pointings are centered on the equator and separated by \( 5^\circ \) in both \( \theta \) and \( \phi \). We consider only one baseline per pointing, with \( u = 22.5^\circ \). Assume that both visibilities \( V_Q \) and \( V_U \) are measured (either directly or by measuring \( V_{\pm} \)).

For any pair of pointings we can define \( E \) and \( B \) window functions such that

\[
\langle V_{Q_1} V_{Q_2}^* \rangle = \sum_{\ell} (W_{\ell}^E C_{\ell}^{EE} + W_{\ell}^B C_{\ell}^{BB}) \tag{65}
\]

where \( V_{Q_i} \) is the visibility for Stokes \( Q \) corresponding to baseline \( i \). In the limit of infinitely sharp \( \ell \)-space resolution, we would expect \( W_{\ell}^B \) to vanish, since the polarization would be parallel to the baseline \( u \). Inevitably, however, when only part of the sky is covered (leading to imperfect Fourier space resolution), there is some mixing of \( E \) and \( B \) modes (Lewis et al. 2002; Bunn 2002; Bunn et al. 2003).

For the case of the two visibilities at the upper corners of the grid, these window functions are shown in Fig. 6. As in the scalar case, the correlations are strongly affected by the rotation of the coordinate basis. If we incorrectly model the entire mosaic as flat, treating the basis vectors \( \theta, \phi \) at each point to be parallel, the correlation between these two pointings would be dramatically overestimated. In fact, by treating the sky as flat,
we would be making two separate errors: treating the two baseline vectors as parallel (just as in the case of temperature anisotropy in Sec. 3) and failing to apply the appropriate transformation to the Stokes parameters \((Q, U)\).

Fig. 7 illustrates the improvement of resolution due to mosaicking in this example. Like Fig. 3 this figure shows the autocorrelation window function for a single pointing as well as that of the sum of all 100 pointings in the grid. If we neglect sky curvature, we overestimate the correlation between distant baselines and hence also overestimate the improvement in \(\ell\)-space resolution.

6. CONCLUSIONS

Interferometers have been used to great effect in measuring CMB temperature and polarization anisotropies. The formalism for analyzing interferometer data, however, has only been fully developed in the small-field-of-view or flat-sky limit. Future experiments which aim for exquisite \(\ell\)-space resolution will need to survey large areas of sky – outside the realm of validity of the existing formalism.

In this paper we have extended the formalism to the situation where we can approximate the sky as flat for each individual pointing of the instrument, but we relax the assumption that the angle between pointings is also small. We have connected the full-sky spherical harmonic approach to the flat-sky Fourier approach in two distinct ways and derived approximations for the visibility covariance matrix in each. We find that the cylindrical method of \([8]\) and \([12]\) works in all cases better than the harmonic method of \([4]\) and provides accurate approximations to the full-sky expressions for individual pointings smaller than 20° FWHM. Mosaicking together many pointings increases the \(\ell\)-space resolution, but in the cases considered here the improvement is less than would be predicted from the flat-sky formalism, in large part due to the effects of baseline rotation. If we neglect sky curvature we overestimate the correlation between distant baselines and hence also overestimate the improvement in \(\ell\)-space resolution.

APPENDIX

In calculating the covariance between visibilities at two different pointing centers \(\hat{r}_1, \hat{r}_2\), we must transform to a coordinate system that places both pointing centers on the equator. This affects both the components of the baseline vectors \(\mathbf{u}_1, \mathbf{u}_2\) and (in the case of polarization) the Stokes parameters \((Q, U)\). We present here an explicit recipe for performing this transformation.

Throughout this appendix, unprimed symbols will refer to the original coordinate system, and primed symbols will refer to a coordinate system \((x', y', z')\) such that both \(\hat{r}_1\) lies on the \(x'\) axis and \(\hat{r}_2\) is in the \(x'y'\) plane. First, choose the \(z'\) axis to be perpendicular to both vectors:

\[
\mathbf{z}' = \frac{\hat{r}_1 \times \hat{r}_2}{|\hat{r}_1 \times \hat{r}_2|}.
\]  

Next, choose the \(y'\) axis to be perpendicular to both \(\mathbf{z}'\) and \(\hat{r}_1\):

\[
\mathbf{y}' = \hat{r}_1 \times \mathbf{z}'.
\]

Finally, choose \(\mathbf{x}' = \mathbf{y}' \times \mathbf{z}'\). In spherical coordinates \((\theta', \phi')\) defined with respect to the primed coordinate system, we have \(\hat{r}_1 = (\pi/2, 0)\) and \(\hat{r}_2 = (\pi/2, \beta)\) with \(\beta\) such that \((\cos \beta, \sin \beta) = (\hat{r}_2 \cdot \mathbf{x}', \hat{r}_2 \cdot \mathbf{y}')\).

Say that the baseline vector \(\mathbf{u}_i\) \((i = 1, 2)\) is expressed in the original (unrotated) spherical coordinate system as

\[
\mathbf{u}_i = u_{i\theta}\mathbf{\hat{\theta}} + u_{i\phi}\mathbf{\hat{\phi}}.
\]

We need to know the corresponding components in the rotated coordinate system. The components of the basis vectors \(\mathbf{\hat{\theta}}, \mathbf{\hat{\phi}}\) in the rotated coordinate system are

\[
\mathbf{\hat{\theta}} = (\cos \delta_1)\mathbf{\hat{\theta}}' - (\sin \delta_1)\mathbf{\hat{\phi}}',
\]

\[
\mathbf{\hat{\phi}} = (\sin \delta_1)\mathbf{\hat{\theta}}' + (\cos \delta_1)\mathbf{\hat{\phi}}'.
\]
The easiest way to find the rotation angle $\delta_i$ is to compute the components of $\hat{\phi} = (\hat{z} \times \hat{r}_i)/|\hat{z} \times \hat{r}_i|$, and take the dot product $\sin \delta_i = \hat{\phi} \cdot \hat{\theta}' = \hat{\phi} \cdot (-\hat{z}')$ since $\hat{r}_i$ is on the equator in the primed coordinate system.

Once the rotation angles $\delta_1, \delta_2$ are known, the components of the baseline vectors are

$$
\begin{pmatrix}
u_{i\theta'} \\
u_{i\phi'}
\end{pmatrix} =
\begin{pmatrix}
\cos \delta_i & \sin \delta_i \\
-\sin \delta_i & \cos \delta_i
\end{pmatrix}
\begin{pmatrix}
u_{i\theta} \\
u_{i\phi}
\end{pmatrix}.
$$

In calculating the polarization visibilities we replace $(Q, U)$ with

$$
\begin{pmatrix}
Q' \\
U'
\end{pmatrix} =
\begin{pmatrix}
\cos 2\delta_i & \sin 2\delta_i \\
-\sin 2\delta_i & \cos 2\delta_i
\end{pmatrix}
\begin{pmatrix}
Q \\
U
\end{pmatrix}.
$$

REFERENCES


