On electromagnetics of an isotropic chiral medium moving at constant velocity

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Abstract

A medium which is an isotropic chiral medium from the perspective of a co–moving observer is a Faraday chiral medium (FCM) from the perspective of a non–co–moving observer. The Tellegen constitutive relations for this FCM are established. By an extension of the Beltrami field concept, these constitutive relations are exploited to show that planewave propagation is characterized by four generally independent wavenumbers. This FCM can support negative phase velocity at certain translational velocities and with certain wavevectors, even though the corresponding isotropic chiral medium does not. The constitutive relations and Beltrami–like fields are also used to develop a convenient spectral representation of the dyadic Green functions for the FCM.

Keywords: Beltrami field, Bohren transform, dyadic Green function, Faraday chiral medium, isotropic chiral medium, Lorentz transformation, negative phase velocity
1 Introduction

A fundamental issue in electromagnetics is the variation in the perceived properties of a linear medium according to the observer’s inertial frame of reference. Interest in this topic dates from the earliest days of the special theory of relativity and it remains an active area of research. Electromagnetic fields in mediums which are isotropic dielectric–magnetic from the perspective of a co–moving observer have been widely studied (Chen 1983; Kong 1986; Pappas 1965). Recent studies involving an isotropic dielectric–magnetic medium have demonstrated that planewave propagation with positive, negative or orthogonal phase velocity can arise depending upon the observer’s inertial frame of reference (Mackay & Lakhtakia 2004a; Mackay et al. 2006). The electromagnetics of simply moving plasmas have also been extensively investigated (Chawla & Unz 1971).

In this paper we consider electromagnetic fields in linear isotropic chiral mediums moving at constant velocities. A natural formalism for investigating the electromagnetic properties of an isotropic chiral medium, as observed from a co–moving inertial frame of reference, is provided by Beltrami fields. The defining characteristic of a Beltrami field is that the curl of the field is a scalar multiple of the field itself (Lakhtakia 1994a,b). These fields are useful for analysis of a wide range physical phenomenons, as in astrophysics (Chandrasekhar 1956, 1957), hydrodynamics and magnetohydrodynamics (Dritschel 1991), thermoacoustics (Ceperley 1992), chaotic flows (McLaughlin and Pironneau 1991), plasma physics (Yoshida 1991) and magnetostatics (Marcinkowski 1992), for example. In the following sections, an extension of the Beltrami field concept is developed to investigate the electromagnetic properties of an isotropic chiral medium as observed from a non–co–moving inertial frame of reference.

In earlier studies involving isotropic chiral mediums moving at a constant velocity, the Lorentz–transformed wavevector and Lorentz–transformed angular frequency have been utilized to explore Doppler shift and aberration (Engheta et al. 1989); the scattering response of an electrically small sphere made of an isotropic chiral medium has been formulated (Lakhtakia et al. 1991); and planewave propagation has been investigated for relatively low translational speeds (Hillion 1993; Ben–Shimol & Censor 1995, 1997). Reflection and transmission coefficients for a uniformly moving isotropic chiral slab have also been calculated using the Lorentz–transformed electromagnetic fields (Hinders et al. 1991).

In contrast to these earlier works, the analysis presented in the following sections begins with a derivation of the Tellegen constitutive relations, from the perspective of a non–co–moving observer, for a medium which is an isotropic chiral medium for a co–moving observer. By means of the Bohren transform and the consequent introduction of Beltrami–like fields, these constitutive relations are exploited to consider planewave propagation — specifically, the propensity for negative phase velocity — from the perspective of a non–co–moving observer. The constitutive relations, combined with Beltrami–like fields, are also used to develop a convenient spectral representation of the dyadic Green functions for the isotropic chiral medium moving at constant velocity.

As regards notational matters, 3–vectors are underlined whereas 3×3 dyadics are double underlined. The identity 3×3 dyadic is written as $\mathbf{I}$. Vectors with the $\hat{\cdot}$ symbol overhead are unit vectors. The operators $\text{Re} \{ \cdot \}$ and $\text{Im} \{ \cdot \}$ deliver the real and imaginary parts, respectively, of complex–valued quantities; the superscript * denotes a complex conjugate; and $i = \sqrt{-1}$. The
permittivity and permeability of free space are denoted $\varepsilon_0$ and $\mu_0$, respectively; $c_0 = 1/\sqrt{\varepsilon_0\mu_0}$ is the speed of light in free space.

2 Constitutive relations

We contrast the electromagnetic properties in two different inertial frames of reference, denoted as $\Sigma'$ and $\Sigma$, where $\Sigma'$ moves with constant velocity $\mathbf{v} = v\hat{\mathbf{v}}$ relative to $\Sigma$. The spacetime coordinates $(r', t')$ in frame $\Sigma'$ are related to the spacetime coordinates $(r, t)$ in frame $\Sigma$ by the Lorentz transformation

$$ r' = Y \cdot r - \gamma vt, \quad t' = \gamma \left( t - \frac{r \cdot \mathbf{v}}{c_0^2} \right), $$

wherein

$$ Y = I + (\gamma - 1) \frac{\hat{\mathbf{v}} \hat{\mathbf{v}}}{c_0^2}, \quad \gamma = \frac{1}{\sqrt{1 - \beta^2}}, $$

and the relative speed $\beta = v/c_0$.

In the absence of sources, the spatiotemporal variations of the (time–domain) electromagnetic fields in frame $\Sigma'$ are related as

$$ \begin{align*}
\nabla' \times \mathbf{E}'(r', t') + \frac{\partial}{\partial t'} \mathbf{B}'(r', t') &= 0, \\
\nabla' \times \mathbf{H}'(r', t') - \frac{\partial}{\partial t'} \mathbf{D}'(r', t') &= 0,
\end{align*} $$

whereas those in frame $\Sigma$ are related as

$$ \begin{align*}
\nabla \times \mathbf{E}(r, t) + \frac{\partial}{\partial t} \mathbf{B}(r, t) &= 0, \\
\nabla \times \mathbf{H}(r, t) - \frac{\partial}{\partial t} \mathbf{D}(r, t) &= 0,
\end{align*} $$

as dictated by the Lorentz covariance of the Maxwell curl postulates. The primed and unprimed fields in (3) and (4) are connected via the Lorentz transformation as (Chen 1983)

$$ \begin{align*}
\mathbf{E}'(r', t') &= \gamma \left[ Y^{-1} \cdot \mathbf{E}(r, t) + \mathbf{v} \times \mathbf{D}(r, t) \right], \\
\mathbf{B}'(r', t') &= \gamma \left[ Y^{-1} \cdot \mathbf{B}(r, t) - \frac{1}{c_0^2} \mathbf{v} \times \mathbf{E}(r, t) \right], \\
\mathbf{H}'(r', t') &= \gamma \left[ Y^{-1} \cdot \mathbf{H}(r, t) - \mathbf{v} \times \mathbf{D}(r, t) \right], \\
\mathbf{D}'(r', t') &= \gamma \left[ Y^{-1} \cdot \mathbf{D}(r, t) + \frac{1}{c_0^2} \mathbf{v} \times \mathbf{H}(r, t) \right].
\end{align*} $$

Let us consider a homogeneous medium, which is an isotropic chiral medium from the perspective of an observer co–moving relative to the frame $\Sigma'$. In the most general scenario, the medium is spatiotemporally nonlocal. From the perspective of the observer co–moving relative to $\Sigma'$, the constitutive relations of the medium may be expressed in the Tellegen form as
(Lakhtakia 1994b)

\[
\vec{D}'(r', t') = \epsilon_0 \int_{\mathcal{V}'} \int_{\mathcal{V}'} \vec{\varepsilon}'(s', u') \vec{E}'(r' - s', t' - u') \, du' \, ds'
+ i \sqrt{\epsilon_0 \mu_0} \int_{\mathcal{V}'} \int_{\mathcal{V}'} \vec{\zeta}'(s', u') \vec{H}'(r' - s', t' - u') \, du' \, ds',
\]

(9)

\[
\vec{B}'(r', t') = -i \sqrt{\epsilon_0 \mu_0} \int_{\mathcal{V}'} \int_{\mathcal{V}'} \vec{\xi}'(s', u') \vec{E}'(r' - s', t' - u') \, du' \, ds'
+ \mu_0 \int_{\mathcal{V}'} \int_{\mathcal{V}'} \vec{\mu}'(s', u') \vec{H}'(r' - s', t' - u') \, du' \, ds',
\]

(10)

with the real–valued (time–domain) constitutive parameters \(\vec{\varepsilon}'(r', t'), \vec{\xi}'(r', t')\) and \(\vec{\mu}'(r', t')\). By implementing the spatiotemporal Fourier transforms (Lakhtakia & Weighhofer 1996)

\[
\begin{bmatrix}
E'_x(k', \omega') \\
H'_x(k', \omega') \\
D'_x(k', \omega') \\
E'_y(k', \omega')
\end{bmatrix}
= \int_{\mathcal{V}'} \int_{\mathcal{V}'} \begin{bmatrix}
\vec{E}'(r', t') \\
\vec{H}'(r', t') \\
\vec{D}'(r', t') \\
\vec{B}'(r', t')
\end{bmatrix}
\exp \left[ i \left( \omega' t' - \vec{k}' \cdot \vec{r}' \right) \right] \, dt' \, dr',
\]

(11)

and

\[
\chi'_x(k', \omega') = \int_{\mathcal{V}'} \int_{\mathcal{V}'} \tilde{\chi}'(r', t') \exp \left[ i \left( \omega' t' - \vec{k}' \cdot \vec{r}' \right) \right] \, dt' \, dr', \quad (\chi = \epsilon, \xi, \mu),
\]

(12)

along with the convolution theorem (Walker 1988), the frequency–domain constitutive relations emerge as

\[
\begin{bmatrix}
E'_x(k', \omega') \\
H'_x(k', \omega') \\
D'_x(k', \omega') \\
E'_y(k', \omega')
\end{bmatrix}
= \epsilon_0 \epsilon'_x(k', \omega') E'_x(k', \omega') + i \sqrt{\epsilon_0 \mu_0} \xi'_x(k', \omega') H'_x(k', \omega')
\]

(13)

\[
B'_y(k', \omega') = -i \sqrt{\epsilon_0 \mu_0} \xi'_y(k', \omega') E'_y(k', \omega') + \mu_0 \mu'_y(k', \omega') H'_y(k', \omega')
\]

In many applications the effects of spatial nonlocality are negligible in comparison to those of temporal nonlocality. The constitutive relations (13) may then be approximated as

\[
\begin{bmatrix}
E'_x(r', \omega') \\
H'_x(r', \omega') \\
D'_x(r', \omega') \\
E'_y(r', \omega')
\end{bmatrix}
= \int_{\mathcal{V}'} \begin{bmatrix}
E'_x(r', t') \\
H'_x(r', t') \\
D'_x(r', t') \\
B'_y(r', t')
\end{bmatrix}
\exp \left( i \omega' t' \right) \, dt',
\]

(15)

and

\[
\chi'(\omega') = \int_{\mathcal{V}'} \tilde{\chi}'(t') \exp \left( i \omega' t' \right) \, dt', \quad (\chi = \epsilon, \xi, \mu),
\]

(16)

with \(\tilde{\chi}'(t') \equiv \tilde{\chi}'(r', t')\) for \(\chi = \epsilon, \xi\) and \(\mu\).
Let us now proceed to develop the frequency–domain constitutive relations in the reference frame $\Sigma$. After using (5), (7) and (8) to substitute for $\vec{E}'(r',t')$, $\vec{H}'(r',t')$ and $\vec{D}'(r',t')$, respectively, the constitutive relation (9) may be expressed in terms of $\Sigma$ fields as

$$
\vec{D}(r,t) = \epsilon_0 \int_{s'} \int_{u'} \left( \varepsilon(s',u') \vec{E}(r - Y \cdot s' - \gamma y u',t - u' \gamma - \frac{\gamma s' \cdot v}{c_0^2})
+ Y \cdot \left\{ v \times \left[ \varepsilon(s',u') \vec{E}(r - Y \cdot s' - \gamma y u',t - u' \gamma - \frac{\gamma s' \cdot v}{c_0^2}) \right] \right\} \right) \, du' \, ds'
+i \sqrt{\epsilon_0 \mu_0} \int_{s'} \int_{u'} \left( \varepsilon(s',u') \vec{H}(r - Y \cdot s' - \gamma y u',t - u' \gamma - \frac{\gamma s' \cdot v}{c_0^2})
- Y \cdot \left\{ v \times \left[ \varepsilon(s',u') \vec{D}(r - Y \cdot s' - \gamma y u',t - u' \gamma - \frac{\gamma s' \cdot v}{c_0^2}) \right] \right\} \right) \, du' \, ds'
- Y \cdot \left[ \frac{1}{c_0^2} v \times \vec{H}(r,t) \right].
$$

Similarly, the constitutive relation (10) may be expressed in terms of $\Sigma$ fields as

$$
\vec{B}(r,t) = -i \sqrt{\epsilon_0 \mu_0} \int_{s'} \int_{u'} \left( \varepsilon(s',u') \vec{E}(r - Y \cdot s' - \gamma y u',t - u' \gamma - \frac{\gamma s' \cdot v}{c_0^2})
+ \gamma s' \times (\varepsilon(s',u') \vec{E}(r - Y \cdot s' - \gamma y u',t - u' \gamma - \frac{\gamma s' \cdot v}{c_0^2})) \right) \, du' \, ds'
+i \mu_0 \int_{s'} \int_{u'} \left( \varepsilon(s',u') \vec{H}(r - Y \cdot s' - \gamma y u',t - u' \gamma - \frac{\gamma s' \cdot v}{c_0^2})
- Y \cdot \left\{ v \times \left[ \varepsilon(s',u') \vec{D}(r - Y \cdot s' - \gamma y u',t - u' \gamma - \frac{\gamma s' \cdot v}{c_0^2}) \right] \right\} \right) \, du' \, ds'
+ Y \left[ \frac{1}{c_0^2} v \times \vec{E}(r,t) \right],
$$

by using (5), (7) and (6) to substitute for $\vec{E}'(r',t')$, $\vec{H}'(r',t')$ and $\vec{D}'(r',t')$, respectively. Implementation of the spatiotemporal Fourier transforms

$$
\begin{align*}
\left\{ \begin{array}{l}
E_x(k,\omega) \\
H_y(k,\omega) \\
D_z(k,\omega) \\
D_z(k,\omega)
\end{array} \right\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ \begin{array}{l}
\vec{E}(r,t) \\
\vec{H}(r,t) \\
\vec{D}(r,t) \\
\vec{B}(r,t)
\end{array} \right\} \exp \left[ i (\omega t - k \cdot r) \right] \, dt \, dr
\end{align*}
$$

and (12) with (17) delivers

$$
\begin{align*}
D_z(k,\omega) &= \epsilon_0 \mu_0 \{ E_x(k,\omega) + Y \cdot [v \times D_z(k,\omega)] \}
+i \sqrt{\epsilon_0 \mu_0} \{ H_y(k,\omega) - Y \cdot [v \times D_z(k,\omega)] \}
- Y \left[ \frac{1}{c_0^2} v \times H_y(k,\omega) \right],
\end{align*}
$$

5
and with (18) yields
\[
B_4(k,\omega) = -i\sqrt{\frac{\epsilon_0}{\mu_0}}\xi_4(k',\omega') \left\{ F_4(k,\omega) + \frac{Y}{c_0} \times B_4(k,\omega) \right\} \\
+ \mu_0\mu_r^2(k',\omega') \left\{ H_4(k,\omega) + \frac{Y}{c_0} \times D_4(k,\omega) \right\} \\
+ \frac{Y}{c_0} \times B_4(k,\omega) .
\] (21)

In the derivation of (20) and (21), the principle of phase invariance (Pappas 1965; Kong 1986) has been invoked to obtain the relations
\[
k' = \frac{Y}{c_0} \times k - \frac{\omega_0}{c_0} v, \quad \omega' = \gamma (\omega - k \cdot v) .
\] (22)

The two independent expressions (20) and (21) which relate \(D_4(k,\omega)\) and \(B_4(k,\omega)\) to \(E_4(k,\omega)\) and \(H_4(k,\omega)\) can be manipulated to deliver the \(\Sigma\) frequency–domain constitutive relations
\[
\begin{align*}
D_4(k,\omega) &= \epsilon_0 \xi_4(k',\omega') \times E_4(k,\omega) + i\sqrt{\frac{\epsilon_0}{\mu_0}}\xi_4(k',\omega') \times H_4(k,\omega) \\
B_4(k,\omega) &= -i\sqrt{\frac{\epsilon_0}{\mu_0}}\xi_4(k',\omega') \times E_4(k,\omega) + \mu_0 \mu_r^2(k',\omega') \times H_4(k,\omega)
\end{align*}
\] (23)

Herein the \(3\times3\) constitutive dyads all have the form
\[
\chi_4(k',\omega') = \chi_4^1(k',\omega') I - \chi_4^{02}(k',\omega') \hat{\nu} \times I + \left[ \chi_4^{0}(k',\omega') - \chi_4^{1}(k',\omega') \right] \hat{\nu} \hat{\nu}, \quad (\chi = \epsilon, \xi, \mu) .
\] (24)

The unprimed constitutive parameters emerge as
\[
\begin{align*}
\epsilon_4^1(k',\omega') &= \epsilon_4^1(k',\omega') \left\{ \beta^2 \left[ \epsilon_4^1(k',\omega') \mu_2^1(k',\omega') - \xi_4^2(k',\omega') \right] - 1 \right\} (\beta^2 - 1) \Delta_4(k',\omega') \\
\epsilon_4^2(k',\omega') &= 2\epsilon_4^1(k',\omega') \xi_4^2(k',\omega') \beta (1 - \beta^2) \Delta_2(k',\omega') \\
\epsilon_4^3(k',\omega') &= \epsilon_4^3(k',\omega') \\
\xi_4^1(k',\omega') &= \xi_4^1(k',\omega') \left\{ \beta^2 \left[ \epsilon_4^1(k',\omega') \mu_2^1(k',\omega') - \xi_4^2(k',\omega') \right] + 1 \right\} (1 - \beta^2) \Delta_4(k',\omega') \\
\xi_4^2(k',\omega') &= \beta \left( \left[ \epsilon_4^1(k',\omega') \mu_2^1(k',\omega') - 1 \right] \left[ 1 - \beta^2 \epsilon_4^1(k',\omega') \mu_2^1(k',\omega') \right] - \xi_4^2^2(k',\omega') \right) \Delta_4^2(k',\omega') \\
\xi_4^3(k',\omega') &= \xi_4^3(k',\omega') \\
\mu_4^1(k',\omega') &= \mu_4^1(k',\omega') \left\{ \beta^2 \left[ \epsilon_4^1(k',\omega') \mu_2^1(k',\omega') - \xi_4^2(k',\omega') \right] - 1 \right\} (\beta^2 - 1) \Delta_4(k',\omega') \\
\mu_4^2(k',\omega') &= 2\mu_4^1(k',\omega') \xi_4^2(k',\omega') \beta (1 - \beta^2) \Delta_2(k',\omega') \\
\mu_4^3(k',\omega') &= \mu_4^3(k',\omega')
\end{align*}
\] (25-27)

with
\[
\frac{1}{\Delta_4(k',\omega')} = 1 - 2\beta^2 \left[ \epsilon_4^1(k',\omega') \mu_2^1(k',\omega') + \xi_4^2(k',\omega') \right] + \beta^4 \left[ \epsilon_4^1(k',\omega') \mu_2^1(k',\omega') - \xi_4^2(k',\omega') \right]^2 .
\] (28)
The constitutive relations (23) in Σ reduce to those in Σ′ specified by (13) in the limit \( v \to 0 \). We note the similarity of these constitutive relations (23) to the tensor formulation that is used in plasma physics (Melrose & McPhedran, 1991). Interestingly, the constitutive dyadics (24) have the same form as that ascribed to a Faraday chiral medium (FCM) (Weiglhofer & Lakhtakia 1998). Hitherto, FCMs have been conceptualized as homogenized composite mediums (Weiglhofer & Lakhtakia 1998; Engheta et al. 1992), arising from blending together an isotropic chiral medium with either a magnetically biased ferrite (Weiglhofer et al. 1998) or a magnetically biased plasma (Weiglhofer & Mackay 2000). Through the homogenization process, the natural optical activity of isotropic chiral mediums (Lakhtakia 1994b) is combined with the Faraday rotation exhibited by gyrotropic mediums (Lax & Button 1962).

For the spatially local medium represented by the Σ′ constitutive relations (14), the corresponding constitutive relations in Σ are derived from (23) as

\[
\begin{align*}
D(r, \omega) &= \varepsilon_0 \xi (\omega') \cdot E(r, \omega) + i\sqrt{\varepsilon_0\mu_0} \xi (\omega') \cdot H(r, \omega) \\
B(r, \omega) &= -i\sqrt{\varepsilon_0\mu_0} \xi (\omega') \cdot E(r, \omega) + \mu_0 \mu (\omega') \cdot H(r, \omega)
\end{align*}
\]

with the 3×3 constitutive dyadics defined as in (24), but with no dependency on \( k' \).

As an illustrative example, let us consider the case of the medium specified by the Σ′ constitutive parameters \( \varepsilon' = 6.5 + i1.5, \mu' = 1 + i0.2, \) and \( \mu' = 3.0 + i0.5 \). The corresponding constitutive parameters in Σ, as specified in (25)-(27), are plotted in Figure 1 against the relative speed \( \beta \in [0, 1] \). The parameters are constrained such that \( \text{Re} \{ \chi^t, \chi^z \} \to \text{Re} \{ \chi' \}, \text{Im} \{ \chi^t, \chi^z \} \to \text{Im} \{ \chi' \} \) and \( |\chi^g| \to 0 \) as \( \beta \to 0 \) for \( \chi = \varepsilon, \mu \) and \( \xi \). In Figure 1, whereas \( \text{Re} \{ \chi^t, \chi^g \} \) and \( \text{Im} \{ \chi^t, \chi^g \} \) become vanishingly small as \( \beta \) approaches unity for \( \chi = \varepsilon \) and \( \mu \), as do \( \text{Re} \{ \xi^t \} \) and \( \text{Im} \{ \xi^t, \xi^g \} \), this is not the case for \( \text{Re} \{ \xi^g \} \). The parameters \( \chi^z \) are independent of \( \beta \) for \( \chi = \varepsilon, \mu \) and \( \xi \).

### 3 Planewave propagation

Let us now consider the propagation of plane waves in spatially local mediums of the chosen kind. A plane wave characterized in frame Σ by the wavevector \( k \) and angular frequency \( \omega \) is related to a plane wave characterized by the wavevector \( k' \) and angular frequency \( \omega' \) in frame Σ′ by the relations (22). The Doppler shift and aberration arising from the transformation from Σ′ to Σ have been explored previously (Engheta et al. 1989). In the remainder of this section, we exploit the frequency–domain constitutive relations (14) and (29) for the two frames to consider planewave propagation and, in particular, investigate the phenomenon of negative phase velocity for the isotropic chiral medium moving at constant velocity. A central element in the analysis is the introduction of Beltrami and Beltrami–like fields.

Planewave propagation in reference frame Σ′, with field phasors of the form

\[
\begin{align*}
E'(r', \omega') &= E_0' (\omega') \exp (i k' \cdot r') \\
H'(r', \omega') &= H_0' (\omega') \exp (i k' \cdot r')
\end{align*}
\]

is a well–documented matter (Lakhtakia 1994b). It is both mathematically expedient and physically insightful to implement the Bohren transform and introduce the Beltrami field phasors...
(Bohren 1974)
\[
\mathcal{Q}_1'(\mathbf{r}', \omega') = \frac{1}{2} \left\{ \mathbf{E}'(\mathbf{r}', \omega') + i\eta'(\omega') \mathbf{H}'(\mathbf{r}', \omega') \right\},
\]
\[
\mathcal{Q}_2'(\mathbf{r}', \omega') = \frac{1}{2} \left\{ \mathbf{H}'(\mathbf{r}', \omega') + \frac{i}{\eta'(\omega')} \mathbf{E}'(\mathbf{r}', \omega') \right\},
\]
with the intrinsic impedance
\[
\eta'(\omega') = \left[ \frac{\mu_0\mu'(\omega')}{\epsilon_0\epsilon'(\omega')} \right]^{1/2}.
\]
Thereby, the frequency-domain Maxwell curl postulates in reference frame Σ′, namely
\[
\begin{align*}
\nabla' \times \mathbf{E}'(\mathbf{r}', \omega') &- i\omega' \mathbf{B}'(\mathbf{r}', \omega') = \mathbf{0}, \\
\nabla' \times \mathbf{H}'(\mathbf{r}', \omega') &+ i\omega' \mathbf{D}'(\mathbf{r}', \omega') = \mathbf{0},
\end{align*}
\]
may be recast as two uncoupled first-order differential equations, which yield
\[
i\mathbf{k}' \times \mathcal{Q}_\ell'(\mathbf{r}', \omega') + (-1)^\ell \kappa_0' \mathbf{k}' \mathcal{Q}_\ell'(\mathbf{r}', \omega') = \mathbf{0}, \quad (\ell = 1, 2),
\]
for plane waves (30). Regardless of the direction of propagation, two wavevectors \( \mathbf{k}' \in \{ \mathbf{k}' \} \) with \( \mathbf{k}'_0 = \mathbf{k}'_0 \mathbf{e}' \mathbf{g} \) and corresponding wavenumbers \( k'_\ell = k'_0 k'_\ell \) are supported, where (Lakhtakia 1994b)
\[
\begin{align*}
k'_1 &= \sqrt{\epsilon'(\omega')} \mu'(\omega') - \xi'(\omega'), \\
k'_2 &= \sqrt{\epsilon'(\omega')} \mu'(\omega') + \xi'(\omega'),
\end{align*}
\]
and \( k'_0 = \omega' \sqrt{\epsilon_0 \mu_0} \).

The nonreciprocal bianisotropic nature (Krowne 1984) of the medium specified by (29) in reference frame Σ leads to more complicated planewave characteristics than in Σ′. Following the strategy used for frame Σ′, it is helpful to utilize the field phasors
\[
\begin{align*}
\mathcal{Q}_1(\mathbf{r}, \omega) &= \frac{1}{2} \left\{ \mathbf{E}(\mathbf{r}, \omega) + i\eta(\omega') \mathbf{H}(\mathbf{r}, \omega) \right\}, \\
\mathcal{Q}_2(\mathbf{r}, \omega) &= \frac{1}{2} \left\{ \mathbf{H}(\mathbf{r}, \omega) + \frac{i}{\eta(\omega')} \mathbf{E}(\mathbf{r}, \omega) \right\},
\end{align*}
\]
This enables the frequency-domain Maxwell curl postulates in frame Σ, namely
\[
\begin{align*}
\nabla \times \mathbf{E}(\mathbf{r}, \omega) &- i\omega \mathbf{B}(\mathbf{r}, \omega) = \mathbf{0}, \\
\nabla \times \mathbf{H}(\mathbf{r}, \omega) &+ i\omega \mathbf{D}(\mathbf{r}, \omega) = \mathbf{0},
\end{align*}
\]
to be decoupled as
\[
i\mathbf{k} \times \mathcal{Q}_\ell(\mathbf{r}, \omega') + (-1)^\ell \kappa_0 \hat{\mathbf{k}}_\ell(\omega') \cdot \mathcal{Q}_\ell(\mathbf{r}, \omega') = \mathbf{0}, \quad (\ell = 1, 2),
\]
for plane waves
\[
\begin{align*}
\mathbf{E}(\mathbf{r}) &= \mathbf{E}_0 \exp(i \mathbf{k} \cdot \mathbf{r}), \\
\mathbf{H}(\mathbf{r}) &= \mathbf{H}_0 \exp(i \mathbf{k} \cdot \mathbf{r}),
\end{align*}
\]
with wavevector \( \mathbf{k} = \mathbf{k}_0 \) and \( \kappa_0 = \omega \sqrt{\epsilon_0 \mu_0} \). The 3×3 dyadics in (38) are given as
\[
\mathbf{k}_\ell(\omega') = \kappa_\ell(\omega') \mathbf{I} - i\kappa_\ell(\omega') \hat{\mathbf{k}}_\ell \times \mathbf{I} + \left[ \kappa_\ell(\omega') - \kappa_\ell(\omega') \right] \hat{\mathbf{k}}_\ell, \quad (\ell = 1, 2),
\]
with
\[
\tilde{k}_1 = \left[ \frac{\mu'(\omega')}{\varepsilon'(\omega')} \right]^{1/2} \varepsilon''(\omega') - \xi''(\omega') \equiv \left[ \frac{\varepsilon'(\omega')}{\mu'(\omega')} \right]^{1/2} \mu''(\omega') - \xi''(\omega') \], \quad \nu = t, g, z. \tag{41}
\]
\[
\tilde{k}_2 = \left[ \frac{\mu'(\omega')}{\varepsilon'(\omega')} \right]^{1/2} \varepsilon''(\omega') + \xi''(\omega') \equiv \left[ \frac{\varepsilon'(\omega')}{\mu'(\omega')} \right]^{1/2} \mu''(\omega') + \xi''(\omega') \]

The unprimed constitutive parameters in (41) are defined as in (25)–(27), but with no dependency on \( k' \). Whereas \( \Omega'_\nu(t', \omega') \) are Beltrami field phasors (Lakhtakia 1994a), \( \Omega_\nu(t, \omega) \) should be called Beltrami–like field phasors.

The dispersion relations
\[
\det \left[ i\tilde{k} \times I + (-1)^\ell k_0 \tilde{k}_\nu(\omega') \right] = 0, \quad (\ell = 1, 2), \tag{42}
\]
arise immediately from (38). For an arbitrary direction of propagation specified by the relative orientation angle \( \theta = \cos^{-1} \hat{\nu} \cdot \hat{k} \), the dispersion relations (42) may be expressed as the pair of quadratic equations
\[
a_\ell \tilde{k}^2 + b_\ell \tilde{k} + c_\ell = 0, \quad (\ell = 1, 2), \tag{43}
\]
wherein the relative wavenumber \( \tilde{k} = k/k_0 \). The coefficients in these equations are given as
\[
a_1 = 1 + \beta^2 \left\{ \sqrt{\varepsilon'(\omega') \mu'(\omega')} + \xi'(\omega') \right\}^2 \cos^2 \theta + \sin^2 \theta
\]
\[
\times \left\{ \beta^2 \left[ \sqrt{\varepsilon'(\omega') \mu'(\omega')} - \xi'(\omega') \right]^2 - 1 \right\} - \left[ \sqrt{\varepsilon'(\omega') \mu'(\omega')} - \xi'(\omega') \right]^2
\]
\[
b_1 = 2\beta \cos \theta \left\{ 1 - \left[ \sqrt{\varepsilon'(\omega') \mu'(\omega')} + \xi'(\omega') \right]^2 \right\}
\]
\[
c_1 = \beta^2 \left\{ \left[ \varepsilon'(\omega') \mu'(\omega') - \xi^2(\omega') \right]^2 + 1 \right\} - \left[ \sqrt{\varepsilon'(\omega') \mu'(\omega')} - \xi'(\omega') \right]^2 (\beta^4 + 1)
\]
and
\[
a_2 = -1 + \beta^2 \left\{ \sqrt{\varepsilon'(\omega') \mu'(\omega')} - \xi'(\omega') \right\}^2 \cos^2 \theta + \sin^2 \theta
\]
\[
\times \left\{ 1 - \beta^2 \left[ \sqrt{\varepsilon'(\omega') \mu'(\omega')} + \xi'(\omega') \right]^2 \right\} + \left[ \sqrt{\varepsilon'(\omega') \mu'(\omega')} + \xi'(\omega') \right]^2
\]
\[
b_2 = 2\beta \cos \theta \left[ \sqrt{\varepsilon'(\omega') \mu'(\omega')} + \xi'(\omega') \right]^2
\]
\[
\times \left( \beta^2 \left\{ \left[ \sqrt{\varepsilon'(\omega') \mu'(\omega')} - \xi'(\omega') \right]^2 - 1 \right\} - 1 \right) + 1
\]
\[
c_2 = \left[ \sqrt{\varepsilon'(\omega') \mu'(\omega')} + \xi'(\omega') \right]^2 (\beta^4 + 1) - \beta^2 \left\{ \left[ \varepsilon'(\omega') \mu'(\omega') - \xi^2(\omega') \right]^2 + 1 \right\}
\]

Thus, four relative wavenumbers \( \tilde{k} \in \{ \tilde{k}_1, \tilde{k}_2, \tilde{k}_3, \tilde{k}_4 \} \) emerge as the roots of (43). While these may be straightforwardly extracted from (43), explicit algebraic representations of the wavenumbers are generally cumbersome. The following two special cases are noteworthy exceptions. For
propagation parallel to the direction of translation (i.e., \( \hat{k} = \hat{v} \)) we have

\[
\begin{align*}
\tilde{k}_1 &= \frac{\sqrt{\epsilon'(\omega')\mu'(\omega') \left(1 - \beta^2\right) + \beta \left[1 - \epsilon'(\omega')\mu'(\omega') + \xi^2(\omega')\right] - \xi'(\omega') \left(1 + \beta^2\right)}}{1 - \beta \left[\beta \epsilon'(\omega')\mu'(\omega') - \xi^2(\omega')\right] - 2\xi'(\omega')} \\
\tilde{k}_2 &= \frac{\sqrt{\epsilon'(\omega')\mu'(\omega') \left(1 - \beta^2\right) + \beta \left[1 - \epsilon'(\omega')\mu'(\omega') + \xi^2(\omega')\right] + \xi'(\omega') \left(1 + \beta^2\right)}}{1 - \beta \left[\beta \epsilon'(\omega')\mu'(\omega') - \xi^2(\omega')\right] - 2\xi'(\omega')} \\
\tilde{k}_3 &= -\frac{\sqrt{\epsilon'(\omega')\mu'(\omega') \left(1 - \beta^2\right) + \beta \left[1 - \epsilon'(\omega')\mu'(\omega') + \xi^2(\omega')\right] + \xi'(\omega') \left(1 + \beta^2\right)}}{1 - \beta \left[\beta \epsilon'(\omega')\mu'(\omega') - \xi^2(\omega')\right] - 2\xi'(\omega')} \\
\tilde{k}_4 &= -\frac{\sqrt{\epsilon'(\omega')\mu'(\omega') \left(1 - \beta^2\right) + \beta \left[1 - \epsilon'(\omega')\mu'(\omega') + \xi^2(\omega')\right] - \xi'(\omega') \left(1 + \beta^2\right)}}{1 - \beta \left[\beta \epsilon'(\omega')\mu'(\omega') - \xi^2(\omega')\right] - 2\xi'(\omega')}
\end{align*}
\]

whereas the relative wavenumbers are delivered as

\[
\begin{align*}
\tilde{k}_1 &= \left\lfloor \frac{\sqrt{\epsilon'(\omega')\mu'(\omega') - \xi'(\omega')}}{1 - \beta} \right\rfloor ^{1/2} \\
\tilde{k}_2 &= \left\lfloor \frac{\sqrt{\epsilon'(\omega')\mu'(\omega') + \xi'(\omega')}}{1 - \beta} \right\rfloor ^{1/2} \\
\tilde{k}_3 &= -\tilde{k}_1 \\
\tilde{k}_4 &= -\tilde{k}_2
\end{align*}
\]

for propagation perpendicular to the direction of translation (i.e., \( \hat{k} \cdot \hat{v} = 0 \)).

By way of numerical illustration, let us return to the constitutive parameters used for Figure 1. The corresponding relative wavenumbers in \( \Sigma \), computed as the roots of (43), are plotted in Figure 2 against relative speed \( \beta \in [0, 1] \) and wavevector orientation angle \( \theta \in [0, \pi] \). For clarity, the wavenumbers in Figure 2 are ordered such that \( \text{Im} \{\tilde{k}_1\} > \text{Im} \{\tilde{k}_2\} > \text{Im} \{\tilde{k}_3\} > \text{Im} \{\tilde{k}_4\} \).

Notice that \( \text{Im} \{\tilde{k}_{1,2}\} > 0 \), whereas \( \text{Im} \{\tilde{k}_{3,4}\} < 0 \). At \( \beta = 0 \), the wavenumbers are independent of \( \theta \). As \( \beta \) increases from zero, the dependencies of the wavenumbers upon \( \theta \) are observed to be highly asymmetric with respect to \( \theta = \pi/2 \). In the limit \( \beta \rightarrow 1 \), the \( \theta \)-dependencies of the real parts of the wavenumbers become antisymmetric relative to \( \theta = \pi/2 \), whereas the \( \theta \)-dependencies of the imaginary parts of the wavenumbers become symmetric relative to \( \theta = \pi/2 \).

In relation to planewave propagation, a topic of considerable current interest is whether the phase velocity is negative or positive (Lakhtakia et al. 2003). Negative phase velocity (NPV) is closely related to the phenomenon of negative refraction (Ramakrishna 2005). Planewave propagation with NPV in \( \Sigma \) is signified by (Mackay & Lakhtakia 2004b)

\[
\text{Re} \{\tilde{k}\} \cdot \mathbf{P}(\Omega, \omega) < 0,
\]

where \( \mathbf{P}(\Omega, \omega) \) is the time-averaged Poynting vector; conversely, positive phase velocity (PPV) in \( \Sigma \) is signified by

\[
\text{Re} \{\tilde{k}\} \cdot \mathbf{P}(\Omega, \omega) > 0.
\]
In \( \Sigma' \), NPV is signified by \( \text{Re} \{ k' \} \cdot \mathbf{P}'(r', \omega') < 0 \) and PPV by \( \text{Re} \{ k' \} \cdot \mathbf{P}'(r', \omega') > 0 \). Issues concerning NPV propagation for isotropic chiral mediums (Mackay 2005) and FCMs arising as homogenized composite mediums (Mackay & Lakhtakia 2004b) have been reported previously.

For the medium of interest here, NPV propagation occurs in \( \Sigma' \) provided that (Mackay 2005)

\[
\text{Re} \left\{ \sqrt{\varepsilon' \mu'(\omega')} - \xi'(\omega') \right\} \times \text{Re} \left\{ \sqrt{\varepsilon' \mu'(\omega')} + \xi'(\omega') \right\} < 0 \quad \text{for} \quad k' = k'_0 \tilde{k}'_{1,3}
\]

\[
\text{Re} \left\{ \sqrt{\varepsilon' \mu'(\omega')} - \xi'(\omega') \right\} \times \text{Re} \left\{ \sqrt{\varepsilon' \mu'(\omega')} + \xi'(\omega') \right\} < 0 \quad \text{for} \quad k' = k'_0 \tilde{k}'_{2,4}
\]

For the same medium, by virtue of (48), NPV propagation occurs in \( \Sigma \) provided that

\[
\Omega(k) < 0, \tag{51}
\]

the form of the real–valued NPV parameter \( \Omega(k) \) being provided in Appendix 1.

Let us return to the numerical example considered in Figures 1 and 2. The \( \beta \theta \)–regimes of NPV and PPV, as determined by evaluating \( \Omega(k) \) for \( k \in \{ k_1, k_2, k_3, k_4 \} \), are mapped in Figure 3 with respect to the relative speed \( \beta \in [0, 1) \) and wavevector orientation angle \( \theta \in [0, \pi) \). The medium clearly does not support NPV propagation when \( \beta = 0 \); i.e., all plane waves in \( \Sigma' \) must be of the PPV kind. As \( \beta \) increases, the \( \beta \theta \)–regimes supporting NPV propagation in \( \Sigma \) emerge in the range \( \pi/2 < \theta < \pi \) for the relative wavenumbers \( \tilde{k}_{1,2} \), and in the range \( 0 < \theta < \pi/2 \) for the relative wavenumbers \( \tilde{k}_{3,4} \).

Finally in this section, we note that an alternative derivation of the dispersion relations (42) in \( \Sigma \) may be developed via the Lorentz transformation of the corresponding dispersion relations in \( \Sigma' \). NPV arises when this Lorentz transformation brings about a change of sign in the angular frequency.

### 4 Dyadic Green functions

The problem of finding the (frequency–domain) field phasors generated by a specified distribution of sources within a linear medium is conveniently tackled by means of dyadic Green functions (DGFs) (Tai 1994). For the isotropic chiral medium in reference frame \( \Sigma' \), the DGFs are well–known (Lakhtakia 1994b). However, explicit representations of DGFs are generally unavailable for anisotropic and bianisotropic mediums (Mackay & Lakhtakia 2006). In this section we exploit the constitutive relations derived in §2, together with the Beltrami–like fields introduced in §3, to establish a convenient spectral representation of the DGFs for the FCM described by (29).

Let a source electric current density phasor \( \mathbf{J}_e(r, \omega) \) and a source magnetic current density phasor \( \mathbf{J}_m(r, \omega) \) exist, from the perspective of the non–co–moving observer. Extending the approach adopted in §3 wherein Beltrami–like fields are introduced to aid the planewave analysis...
in $\Sigma$, we recast $\mathcal{J}_{e,m}(\mathbf{r},\omega)$ as the Beltrami current density phasors

$$
\mathcal{W}_1(\mathbf{r},\omega) = \frac{1}{2} \left[ i \eta'(\omega') \mathcal{J}_e(\mathbf{r},\omega) - \mathcal{J}_m(\mathbf{r},\omega) \right],
$$

$$
\mathcal{W}_2(\mathbf{r},\omega) = \frac{1}{2} \left[ \mathcal{J}_e(\mathbf{r}',\omega) - \frac{i}{\eta'(\omega')} \mathcal{J}_m(\mathbf{r},\omega) \right],
$$

The Beltrami–like field phasors generated by the source term $s$ (52) may then be expressed in terms of the DGFs $\mathcal{G}_\ell(\mathbf{r} - \mathbf{s},\omega')$ as

$$
\mathcal{Q}_\ell(\mathbf{r},\omega) = \mathcal{\hat{Q}}_\ell(\mathbf{r},\omega) + \int_V \mathcal{G}_\ell(\mathbf{r} - \mathbf{s},\omega') \cdot \mathcal{W}_\ell(\mathbf{r},\omega) \, d^3\mathbf{s}, \quad (\ell = 1, 2),
$$

where $V$ is the region containing the source current density phasors. The complementary functions $\mathcal{\hat{Q}}_{1,2}(\mathbf{r},\omega)$ are given by

$$
\mathcal{\hat{Q}}_1(\mathbf{r},\omega) = \frac{1}{2} \left[ \mathcal{\hat{E}}(\mathbf{r},\omega) + i \eta'(\omega') \mathcal{\hat{H}}(\mathbf{r},\omega) \right],
$$

$$
\mathcal{\hat{Q}}_2(\mathbf{r},\omega) = \frac{1}{2} \left[ \mathcal{\hat{H}}(\mathbf{r},\omega) + \frac{i}{\eta'(\omega')} \mathcal{\hat{E}}(\mathbf{r},\omega) \right],
$$

wherein $\mathcal{\hat{E}}(\mathbf{r},\omega)$ and $\mathcal{\hat{H}}(\mathbf{r},\omega)$ satisfy the relations

$$
\nabla \times \mathcal{\hat{E}}(\mathbf{r},\omega) - i\omega \mathcal{\hat{B}}(\mathbf{r},\omega) \equiv \mathbf{0},
$$

$$
\nabla \times \mathcal{\hat{H}}(\mathbf{r},\omega) + i\omega \mathcal{\hat{D}}(\mathbf{r},\omega) \equiv \mathbf{0},
$$

along with

$$
\mathcal{\hat{B}}(\mathbf{r},\omega) = \varepsilon_0 \epsilon(\omega') \cdot \mathcal{\hat{E}}(\mathbf{r},\omega) + i \sqrt{\varepsilon_0 \mu_0} \xi(\omega') \cdot \mathcal{\hat{H}}(\mathbf{r},\omega)
$$

$$
\mathcal{\hat{D}}(\mathbf{r},\omega) = -i \sqrt{\varepsilon_0 \mu_0} \xi(\omega') \cdot \mathcal{\hat{E}}(\mathbf{r},\omega) + \mu_0 \mu(\omega') \cdot \mathcal{\hat{H}}(\mathbf{r},\omega)
$$

The DGFs in (53) are provided as the solutions of the differential equations

$$
\nabla \times \mathcal{G}_\ell(\mathbf{r} - \mathbf{s},\omega') + (-1)^\ell k_0 \mathcal{G}_\ell(\omega') \cdot \mathcal{G}_\ell(\mathbf{r} - \mathbf{s},\omega') = \delta(\mathbf{r} - \mathbf{s}) \mathcal{I}_\ell, \quad (\ell = 1, 2),
$$

with $\delta(\mathbf{\cdot})$ being the Dirac delta function. By implementing the spatial Fourier transforms

$$
\mathcal{G}_\ell^\sharp(\mathbf{q},\omega') = \int \mathcal{G}_\ell(\mathbf{r},\omega') \exp(-i \mathbf{q} \cdot \mathbf{r}) \, d\mathbf{r}, \quad (\ell = 1, 2)
$$

with (57), the components of the spectral DGFs $\mathcal{G}_\ell^\sharp(\mathbf{q},\omega')$ with respect to the Cartesian basis
vectors \( \{ \tilde{x}, \tilde{y}, \hat{z} \} \) emerge as

\[
\begin{align*}
\left[ G^1_1 \left( q, \omega' \right) \right]_{11} &= \left[ q \cdot \hat{x} \right]^2 - \tilde{\kappa}^1_1 \tilde{\kappa}^1_1 \Lambda_1 \\
\left[ G^1_1 \left( q, \omega' \right) \right]_{12} &= \left[ q \cdot \hat{x} \left( q \cdot \hat{y} \right) + i\tilde{\kappa}^1_1 \left( \tilde{\kappa}^1_1 + q \cdot \hat{z} \right) \right] \Lambda_1 \\
\left[ G^1_1 \left( q, \omega' \right) \right]_{13} &= \left[ q \cdot \tilde{x} \left( q \cdot \hat{y} + \tilde{\kappa}^1_2 \right) - i\tilde{\kappa}^1_1 q \cdot \hat{y} \right] \Lambda_1 \\
\left[ G^1_1 \left( q, \omega' \right) \right]_{21} &= \left[ q \cdot \hat{x} \left( q \cdot \hat{y} \right) - i\tilde{\kappa}^1_1 \left( \tilde{\kappa}^1_1 + q \cdot \hat{z} \right) \right] \Lambda_1 \\
\left[ G^1_1 \left( q, \omega' \right) \right]_{22} &= \left[ q \cdot \hat{y} \right]^2 - \tilde{\kappa}^1_1 \tilde{\kappa}^1_1 \Lambda_1 \\
\left[ G^1_1 \left( q, \omega' \right) \right]_{23} &= \left[ q \cdot \hat{y} \left( q \cdot \hat{z} + \tilde{\kappa}^1_2 \right) + i\tilde{\kappa}^1_1 q \cdot \hat{z} \right] \Lambda_1 \\
\left[ G^1_1 \left( q, \omega' \right) \right]_{31} &= \left[ q \cdot \tilde{x} \left( q \cdot \hat{z} + \tilde{\kappa}^1_2 \right) + i\tilde{\kappa}^1_1 q \cdot \hat{y} \right] \Lambda_1 \\
\left[ G^1_1 \left( q, \omega' \right) \right]_{32} &= \left[ q \cdot \tilde{y} \left( q \cdot \hat{z} + \tilde{\kappa}^1_2 \right) - i\tilde{\kappa}^1_1 q \cdot \hat{z} \right] \Lambda_1 \\
\left[ G^1_1 \left( q, \omega' \right) \right]_{33} &= \left[ \left( \tilde{\kappa}^1_1 + \tilde{\kappa}^1_2 + q \cdot \hat{z} \right) \left( \tilde{\kappa}^1_1 + \tilde{\kappa}^1_2 - q \cdot \hat{z} \right) \right] \Lambda_1 \\
\end{align*}
\]

with

\[
\frac{1}{\Lambda_1} = \frac{1}{2} \left\{ \left[ \left( q \cdot \hat{x} \right)^2 - \left( q \cdot \hat{x} \right)^2 - \left( q \cdot \hat{y} \right)^2 \right] \left( \tilde{\kappa}^1_1 - \tilde{\kappa}^1_1 \right) + q \cdot q \left( \tilde{\kappa}^1_1 + \tilde{\kappa}^1_1 \right) \right\} + \left( \tilde{\kappa}^1_1 \right)^2 + 2\tilde{\kappa}^1_1 \tilde{\kappa}^1_1 q \cdot \hat{z} - \left( \tilde{\kappa}^1_1 \right)^2,
\]

and

\[
\begin{align*}
\left[ G^2_2 \left( q, \omega' \right) \right]_{11} &= \left[ - q \cdot \hat{x} \right]^2 + \tilde{\kappa}^1_1 \tilde{\kappa}^1_1 \Lambda_2 \\
\left[ G^2_2 \left( q, \omega' \right) \right]_{12} &= \left[ - q \cdot \hat{x} \left( q \cdot \hat{y} \right) + i\tilde{\kappa}^1_1 \left( -\tilde{\kappa}^1_2 + q \cdot \hat{z} \right) \right] \Lambda_2 \\
\left[ G^2_2 \left( q, \omega' \right) \right]_{13} &= \left[ q \cdot \hat{x} \left( q \cdot \hat{y} + \tilde{\kappa}^1_2 \right) + i\tilde{\kappa}^1_1 q \cdot \hat{y} \right] \Lambda_2 \\
\left[ G^2_2 \left( q, \omega' \right) \right]_{21} &= \left[ - q \cdot \hat{x} \left( q \cdot \hat{y} \right) - i\tilde{\kappa}^1_1 \left( -\tilde{\kappa}^1_2 + q \cdot \hat{z} \right) \right] \Lambda_2 \\
\left[ G^2_2 \left( q, \omega' \right) \right]_{22} &= \left[ - q \cdot \hat{y} \right]^2 + \tilde{\kappa}^1_1 \tilde{\kappa}^1_1 \Lambda_2 \\
\left[ G^2_2 \left( q, \omega' \right) \right]_{23} &= \left[ q \cdot \hat{y} \left( q \cdot \hat{z} - \tilde{\kappa}^1_2 \right) - i\tilde{\kappa}^1_1 q \cdot \hat{z} \right] \Lambda_2 \\
\left[ G^2_2 \left( q, \omega' \right) \right]_{31} &= \left[ q \cdot \hat{x} \left( q \cdot \hat{z} - \tilde{\kappa}^1_2 \right) - i\tilde{\kappa}^1_1 q \cdot \hat{y} \right] \Lambda_2 \\
\left[ G^2_2 \left( q, \omega' \right) \right]_{32} &= \left[ q \cdot \hat{y} \left( q \cdot \hat{z} - \tilde{\kappa}^1_2 \right) + i\tilde{\kappa}^1_1 q \cdot \hat{z} \right] \Lambda_2 \\
\left[ G^2_2 \left( q, \omega' \right) \right]_{33} &= \left[ \left( \tilde{\kappa}^1_2 - \tilde{\kappa}^1_2 + q \cdot \hat{z} \right) \left( \tilde{\kappa}^1_2 + \tilde{\kappa}^1_2 - q \cdot \hat{z} \right) \right] \Lambda_2 \\
\end{align*}
\]

with

\[
\frac{1}{\Lambda_2} = \frac{1}{2} \left\{ \left[ \left( q \cdot \hat{x} \right)^2 - \left( q \cdot \hat{x} \right)^2 - \left( q \cdot \hat{y} \right)^2 \right] \left( \tilde{\kappa}^1_2 - \tilde{\kappa}^1_2 \right) - q \cdot q \left( \tilde{\kappa}^1_2 + \tilde{\kappa}^1_2 \right) \right\} + \left( \tilde{\kappa}^1_2 \right)^2 + 2\tilde{\kappa}^1_1 \tilde{\kappa}^1_1 q \cdot \hat{z} + \left( \tilde{\kappa}^1_2 \right)^2.
\]
Having established the spectral DGFs $G_{1,2}^\sharp (q, \omega')$, we obtain the $\Sigma$ field phasors generated by the source phasors $J_{e,m}(r, \omega)$ as

$$
E(r, \omega) = \hat{E}(r, \omega) + \frac{1}{4\pi^3} \int_V \left( \left\{ \int q G_{1,2}^\sharp (q, \omega') \exp \left[ i q \cdot (r - s) \right] dq \right\} \cdot \left[ \eta'(\omega') J_e(r, \omega) - J_m(r, \omega) \right] \right) ds
$$

and

$$
H(r, \omega) = \hat{H}(r, \omega) + \frac{1}{4\pi^3} \int_V \left( \left\{ \int q G_{1,2}^\sharp (q, \omega') \exp \left[ i q \cdot (r - s) \right] dq \right\} \cdot \left[ J_e(r, \omega) - \frac{i}{\eta'(\omega')} J_m(r, \omega) \right] \right) ds.
$$

5 Discussion

The Tellegen constitutive relations for an isotropic chiral medium moving at constant velocity are presented in (23) and (29). The availability of these constitutive relations facilitates a full analysis of the planewave characteristics of the medium, and also enables the spectral DGFs to be derived in a convenient form. In contrast to earlier studies, the analysis presented herein is not restricted to low relative speeds (Hillion 1993; Ben–Shimol & Censor 1995, 1997). Furthermore, the constitutive relations (23) and (29) establish that the uniformly moving isotropic chiral medium in fact belongs to the category of FCMs.

In §3, the analysis of planewave propagation in reference frames $\Sigma'$ and $\Sigma$, is aided by the introduction of the Beltrami field phasors $Q_{1,2}' (r', \omega')$ in (31) and the Beltrami–like field phasors $Q_{1,2} (r, \omega)$ in (36), respectively. They facilitate a decoupling of the Maxwell curl postulates. A key property of $Q_{1,2}' (r', \omega')$ is that the curl of $Q_{1,2}' (r', \omega')$ is a scalar multiple of $Q_{1,2}' (r', \omega')$, as demonstrated in (34). Such fields are known as Beltrami fields and their properties are firmly established (Hillion & Lakhtakia 1993; Lakhtakia 1994b,b). In contrast, the curl of $Q_{1,2} (r, \omega)$ is not generally parallel to $Q_{1,2} (r, \omega)$, as may be observed from (38). The Beltrami–like field phasors $Q_{1,2} (r, \omega)$ therefore represent an important extension of the usual Beltrami field concept which can be traced back to at least the late 1880s (Beltrami 1889; Silberstein 1907; Trkal 1919).

Owing to their relatively large parameter space, linear bianisotropic mediums support a richer palette of planewave properties than do anisotropic and isotropic mediums, as has been highlighted lately by investigations of NPV propagation (Mackay & Lakhtakia 2004b) and optical singularities (Berry 2005). The planewave study presented in §3 reveals that the bianisotropic
FCM described by the constitutive relations (29) generally supports four independent wavenumbers for each direction of propagation, from the perspective of a non–co–moving observer (the exception being propagation perpendicular to the direction of translation for which only two independent wavenumbers are supported). This contrasts with the two independent wavenumbers supported by the isotropic chiral medium from the perspective of the co–moving observer. We see in Figure 3 that an isotropic chiral medium, which does not support NPV propagation from the perspective of a co–moving observer, does support NPV propagation from the perspective of a certain class of non–co–moving observers, provided that the relative speed is sufficiently high. This finding is consistent with results presented for an isotropic dielectric–magnetic medium moving at constant velocity (Mackay & Lakhtakia 2004a). This is also consistent with a study which showed that a FCM arising as a homogenized composite medium can support NPV propagation provided that the gyrotropy parameter of the gyrotropic constituent medium is sufficiently large (Mackay & Lakhtakia 2004b).

While explicit representations of DGFs are available for isotropic mediums, these are generally not available for anisotropic and bianisotropic mediums (Mackay & Lakhtakia 2006). However, as shown in §4, the field phasors for the FCM described by the constitutive relations (29) may be formulated in terms of spectral DGFs. By exploiting the constitutive relations (29) and the Beltrami–like field phasors $Q_{1,2} (\boldsymbol{\zeta}, \omega)$, a convenient representation of the spectral DGFs is established in (58)–(62).

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Appendix 1

As indicated by (51), whether or not plane waves propagate with negative phase velocity in a Faraday chiral medium is determined by the sign of the parameter $\Omega (\hat{k})$. The form of $\Omega (\hat{k})$ is provided here; a complete description of the derivation of $\Omega (\hat{k})$ is available elsewhere (Mackay & Lakhtakia 2004a).

As a function of the relative wavenumber $\hat{k}$, $\Omega (\hat{k})$ may be expressed in terms of the constitutive parameters (25)–(27) and the wavevector orientation angle $\theta$ as

$$
\Omega (\hat{k}) = \text{Re} \left\{ \hat{k} \right\} \times \text{Re} \left\{ \frac{1}{\mu^z^*(\omega')} \left[ \hat{k}^* \sin \theta - i \tau^* \xi^z^*(\omega') \right] \sin \theta \right. \\
+ \frac{1}{[\mu^t^*(\omega')]^2 - [\mu^g^*(\omega')]^2} \left[ \hat{k}^* \left( [\mu^t^*(\omega') (|\alpha|^2 + 1) + i \mu^g^*(\omega') (\alpha - \alpha^*)] \cos^2 \theta \\
+ |\tau|^2 \mu^t^*(\omega') \sin^2 \theta - [\mu^t^*(\omega') (\alpha^* \tau + \alpha \tau^*) + i \mu^g^*(\omega') (\tau - \tau^*)] \sin \theta \cos \theta \right) \\
+ [\mu^t^*(\omega') \xi^g^*(\omega') - \mu^g^*(\omega') \xi^t^*(\omega')] \left( [|\alpha|^2 + 1] \cos \theta - \alpha^* \tau \sin \theta \right) \\
- i [\mu^t^*(\omega') \xi^t^*(\omega') - \mu^g^*(\omega') \xi^g^*(\omega')] \left( (\alpha - \alpha^*) \cos \theta - \tau \sin \theta \right) \right\}.
$$

(65)
Herein,

\[
\alpha = \frac{L_{12}L_{33} + L_{13}L_{23}}{L_{13}^2 - L_{11}L_{33}}
\]

\[
\tau = \frac{L_{12}L_{23} - L_{13}L_{22}}{L_{13}L_{23} + L_{12}L_{33}}
\]

(66)

with

\[
L_{11} = \epsilon^t(\omega') - \frac{2\Gamma \mu^g(\omega') \xi^t(\omega') - \mu^t(\omega') \left\{ \left[ \xi^t(\omega') \right]^2 + \Gamma^2 \right\}}{[\mu^t(\omega')]^2 - [\mu^g(\omega')]^2}
\]

\[
L_{22} = \epsilon^t(\omega') - \frac{2\Gamma \mu^g(\omega') \xi^t(\omega') - \mu^t(\omega') \left\{ \left[ \xi^t(\omega') \right]^2 + \Gamma^2 \right\}}{[\mu^t(\omega')]^2 - [\mu^g(\omega')]^2}
\]

\[
L_{33} = \epsilon^z(\omega') - \frac{\left[ \xi^z(\omega') \right]^2 - \mu^t(\omega') \kappa^2 \sin^2 \theta}{[\mu^t(\omega')]^2 - [\mu^g(\omega')]^2}
\]

\[
L_{12} = i \left( \epsilon^g(\omega') + \frac{\mu^g(\omega') \left\{ \left[ \xi^t(\omega') \right]^2 + \Gamma^2 \right\} - 2\Gamma \mu^t(\omega') \xi^t(\omega')}{[\mu^t(\omega')]^2 - [\mu^g(\omega')]^2} \right)
\]

\[
L_{13} = \frac{\Gamma \mu^t(\omega') - \mu^g(\omega') \xi^t(\omega')}{[\mu^t(\omega')]^2 - [\mu^g(\omega')]^2} L \sin \theta
\]

\[
L_{23} = i \left[ \frac{\Gamma \mu^g(\omega') - \mu^t(\omega') \xi^t(\omega')}{[\mu^t(\omega')]^2 - [\mu^g(\omega')]^2} - \frac{\left. \xi^z(\omega') \right|}{\mu^z(\omega')}, \frac{\sin \theta}{L} \right]
\]

and \( \Gamma = \xi^g(\omega') + \kappa \cos \theta \).

References


Figure 1: Real (left) and imaginary (right) parts of the constitutive parameters $\chi^t$, $\chi^g$ and $\chi^z$, where $\chi \in \{\epsilon, \xi, \mu\}$, plotted against the relative speed $\beta = v/c_0$. Solid curves represent $\chi_t$, dashed curves represent $\chi_z$, and broken dashed curves represent $\chi_g$. Constitutive parameters in $\Sigma'$: $\epsilon' = 6.5 + i1.5$, $\xi' = 1 + i0.2$, and $\mu' = 3.0 + i0.5$. 
Figure 2: Real (left) and imaginary (right) parts of the four relative wavenumbers $\tilde{k}_{1,2,3,4}$ plotted against the relative speed $\beta = v/c_0$ and the propagation angle $\theta = \cos^{-1} \hat{v} \cdot \hat{k}$ (in degree). Constitutive parameters in $\Sigma'$: $\epsilon' = 6.5 + i1.5$, $\xi' = 1 + i0.2$, and $\mu' = 3.0 + i0.5$. 
Figure 2: continued
Figure 3: $\beta\theta$-regimes of NPV and PPV, as viewed from the inertial frame $\Sigma$, with respect to the relative speed $\beta = v/c_0$ and the propagation angle $\theta = \cos^{-1} \hat{\nu} \cdot \hat{k}$ (in degree), as determined by the parameter $\Omega(\tilde{k})$ for the four relative wavenumbers $\tilde{k} \in \{\tilde{k}_1, \tilde{k}_2, \tilde{k}_3, \tilde{k}_4\}$. Constitutive parameters in $\Sigma'$: $\epsilon' = 6.5 + i1.5$, $\xi' = 1 + i0.2$, and $\mu' = 3.0 + i0.5$. 