Manifestly Gauge Invariant QCD

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Abstract. Building on recent work in SU(N) Yang-Mills theory, we construct a manifestly gauge invariant exact renormalization group for QCD. A gauge invariant cutoff is constructed by embedding the physical gauge theory in a spontaneously broken SU(N|N) gauge theory, regularized by covariant higher derivatives. Intriguingly, the construction is most efficient if the number of flavours is a multiple of the number of colours. The formalism is illustrated with a very compact calculation of the one-loop β function, achieving a manifestly universal result and without fixing the gauge.

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1. Introduction

One of the hallmarks of QCD is the qualitatively very different behaviours observed in the high and low energy domains. In the former case, the theory exhibits asymptotic freedom, allowing phenomena to be described in terms of weakly interacting quarks and gluons. Since the coupling is small, calculations can be performed in perturbation theory. However, as the energy scale is lowered, so the coupling strength increases, ultimately causing the quarks and gluons to be bound together into hadrons. The failure of perturbative techniques to capture this behaviour presents a stern challenge.

A promising approach to extracting information from the strongly coupled (or non-perturbative) domain of quantum field theories is the exact renormalization group (ERG) [1–3], the continuous version of Wilson’s RG. The central feature of the ERG is the implementation of a momentum cutoff, Λ, in the theory in such a way that the physics at this scale—which is encoded in the Wilsonian effective action, $S_\Lambda$—is described in terms of parameters relevant to this scale. The ERG equation determines how $S_\Lambda$ evolves with Λ, thereby linking physics at different energy scales. Consequently, an ERG for QCD has the potential to provide access to the strongly coupled regime.

In addition to providing a powerful framework for addressing a wealth of non-perturbative problems in a range of settings (see [4–12] for reviews), a particular advantage conferred by the ERG is the huge freedom inherent in its construction [13]. In the context of gauge theories, this freedom can be exploited to construct manifestly
gauge invariant ERGs [14–19] (for a comprehensive review of the alternatives, see [20]).

Whilst being of obvious novelty value, manifest gauge invariance also provides both
technical and conceptual benefits. From the technical standpoint, the gauge field is
protected from field strength renormalization and the Ward identities take a particularly
simple form since the Wilsonian effective action is built only from gauge invariant
combinations of the covariant derivative, even at the quantum level [15]. In addition,
the difficult technical issue of Gribov copies [21] is entirely avoided. Conceptually, a
strong case can be made for manifest gauge invariance being the natural language to
describe non-perturbative phenomena, not least because all conclusions drawn will be
completely gauge independent.

The majority of work into this scheme has, so far, focused on constructing [14–
19], testing [16–18, 22] and refining [23–25] the formalism. Most recently, however, real
progress has been made in understanding how to compute objects of particular interest;
specifically, the expectation values of gauge invariant operators [26]. Moreover, crucial
steps in this procedure have a non-perturbative extension [27]. Given these developments
we feel that it is timely to extend the framework to incorporate quarks, in anticipation
of application to non-perturbative QCD.

The strategy we adopt is to build directly on to the SU(N) Yang-Mills construction,
which we briefly describe. Recall that the implementation of a gauge invariant cutoff
comprises two ingredients [28]. First, we apply covariant higher derivative regularization.
However, as is well known [29], this is insufficient to completely regularize the theory,
since certain one-loop divergences slip through. The solution we employ is to instead
apply the covariant higher derivative regularization to a spontaneously broken SU(N|N)
gauge theory, into which the physical SU(N) gauge theory has been embedded. The
heavy fields arising from the symmetry breaking act as Pauli-Villars (PV) fields,
supplementing the covariant higher derivatives to furnish a complete regularization of
the physical theory.

The symmetry breaking is carried by a Higgs field, $C$. Upon acquiring a vacuum
expectation value (vev), this field breaks SU(N|N) down to its bosonic subgroup,
\[ SU(N) \times SU(N) \times U(1) \]. One of these SU(N) symmetries is identified with the symmetry
of the physical gauge field, $A^1$; the other is identified with an unphysical field, $A^2$, which
we note comes with wrong sign action. (We can effectively ignore the $U(1)$, as we will
see later.) Besides $A^1$, only $A^2$ remains massless upon spontaneous symmetry breaking,
all other fields picking up a mass of order $\Lambda$. As we send $\Lambda$ to infinity, all effective
interactions between $A^1$ and $A^2$ vanish and so the non-unitary $A^2$ sector decouples and
can be ignored [28].

The most obvious way to add quarks is to embed them in the fundamental
representation of SU(N|N). However, as we will see, the structure of the SU(N|N) group
in fact forbids this program. Instead, we embed $N$ quarks into a field which transforms
as a bifundamental under $U(N|N)$. This essentially corresponds to a gauging of not
only the physical colour symmetry but also, in an entirely unphysical way, a flavour
symmetry. The unphysical fields which accompany the physical quarks are given a mass
of order the cutoff and so act as precisely the set of PV fields we require for the theory to be regularized. In this way, we are able to incorporate quarks in multiples of $N$, such that the elements of each set have the same mass. To obtain quarks with arbitrary masses, we now modify the Higgs sector to break the unphysical $SU(N)$ symmetry completely.

Quite apart from the ERG, the inclusion of quarks in the $SU(N|N)$ regularizing scheme is of interest in itself, since it allows the construction of a real, gauge invariant cutoff in QCD. In the case of pure $\mathcal{N} = 4$ super Yang-Mills, a dual picture of this was recently constructed [30], providing a concrete understanding of how the radial direction in the AdS/CFT correspondence plays the role of a gauge invariant measure of energy scale. It will be interesting to see whether the inclusion of quarks in the $SU(N|N)$ regularizing scheme leads to a new way to introduce quarks in the dual picture.

Having incorporated massive quarks into the regularization framework, it is now a straightforward matter to generalize our manifestly gauge invariant flow equation from $SU(N)$ Yang-Mills to QCD. To understand how we go about doing this, we review the structure of general flow equations. One of the key ingredients of any flow equation is that the partition function (and hence the physics derived from it) is invariant under the flow. As a consequence of this, the family of flow equations for some generic fields, $\varphi$, follows from [13–15, 19]

$$-\Lambda \partial_\Lambda e^{-S[\varphi]} = \int_x \frac{\delta}{\delta \varphi(x)} \left( \Psi_x[\varphi] e^{-S[\varphi]} \right),$$

(1.1)

where the functional $\Psi$ parameterizes how the high energy modes are averaged over (and we have written $S_\Lambda$ as just $S$). The total derivative on the right-hand side ensures that the partition function $Z = \int \mathcal{D}\varphi \ e^{-S}$ is invariant under the flow.

Taking $\varphi$ to represent a single scalar field, we can use [32]

$$\Psi_x = \frac{1}{2} \int_y D_y \frac{\delta \Sigma_1}{\delta \varphi(y)}.$$

(1.2)

where $\partial^\lambda \equiv -\Lambda \partial_\Lambda \Delta$.

Equation (1.3) is the effective propagator relation [17] and is at the heart of the computational technique employed within our approach [17–19, 22–26, 32, 33].
Now suppose that we consider a flow equation for some set of fields, rather than a single field. It is highly desirable to insist on an effective propagator relation for each individual field which means that, in general, the number of effective propagators—and hence the number of kernels—must equal the number fields. This observation holds the key to generalizing our flow equation for SU($N$) Yang-Mills to one appropriate for QCD.

To construct an ERG for SU($N$) Yang-Mills, we use the template (1.1), covariantize the relationship (1.2) and incorporate the SU($N|N$) regularizing structure [17, 19]. By defining the covariantization appropriately (we will review this in section 4.1), we can ensure that an effective propagator exists for each of the broken phase fields. Note, however, that the form of the effective propagator relation is different to (1.3) in the gauge sector, as a consequence of the manifest gauge invariance:

$$S^{A^1A^1\alpha\beta\mu} (p) \Delta^{A^1A^1\alpha\beta\mu} (p) = \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}. \quad (1.4)$$

$S^{A^1A^1\alpha\beta\mu} (p)$ is the two-point classical vertex in the $A^1$ sector, carrying momentum $p$, and $\Delta^{A^1A^1\alpha\beta\mu} (p)$ is the associated effective propagator. Thus we see that the effective propagator is the inverse of the classical, two-point vertex only in the transverse space; equivalently, it is the inverse only up to a remainder which we call a ‘gauge remainder’.

To add quarks, we again use the template (1.1), covariantize the relationship (1.2) and incorporate the SU($N|N$) regularizing structure, but this time the covariantization is designed appropriately for the quarks. To allow independent quark masses, we update the Higgs sector of the flow equation and modify the covariantizations in the gauge and quark sectors to ensure that there are enough independent kernels to cope with the breaking of the unphysical SU($N$) symmetry.

Anticipating that the flow equation for QCD is most efficiently stated via its diagrammatic representation [17–19, 22–25], we could jump straight to this diagrammatic form, using it to hide non-universal details such as the complicated form of the covariantization. However, before doing this, we will provide an explicit example of a valid covariantization, for completeness. Nevertheless, it should be understood that this is just one of an infinite number of choices which we can make but which, in practice, we never do: in actual calculations, we implicitly work with an infinite number of flow equations. The reason we can do this is that there exists a powerful diagrammatic calculus which enables us to perform general computations, almost entirely at the diagrammatic level [17, 25, 26], in a way such that nonuniversal details cancel out. Indeed, this calculus has been employed in pure SU($N$) Yang-Mills to give a diagrammatic expression for the $\beta$-function, from which the universal answer (at least at one and two loops) can be directly extracted. This formula can be trivially adapted to QCD and we will use this to perform a very compact computation of the one-loop $\beta$-function.

The outline of this paper is as follows. In section 2, we review the regularization of SU($N$) Yang-Mills via SU($N|N$) Yang-Mills. In section 3 we add the quarks, first seeing why we cannot embed them in the fundamental of SU($N|N$) and then describing how we can instead embed them using a more elaborate scheme. We conclude this section
by showing how to give the quarks independent masses. In section 4 we review the construction of a manifestly gauge invariant flow equation for SU(N) Yang-Mills and then adapt it for QCD. In section 5 we give a diagrammatic expression for the one-loop β function which reproduces the universal result in the case that the quarks are massless. Finally, in section 6, we conclude.

2. Regularizing SU(N) Yang-Mills

2.1. Embedding in SU(N|N) Yang-Mills

Throughout this paper, we work in Euclidean dimension, D. We regularize SU(N) Yang-Mills by embedding it in spontaneously broken SU(N|N) Yang-Mills, which is itself regularized by covariant higher derivatives [28]. The supergauge field, \( A_\mu \), is valued in the Lie superalgebra and, using the defining representation, can be written as a Hermitian supertraceless supermatrix (the supertrace of a supermatrix is defined as the trace of the top block diagonal element minus the trace of the bottom block diagonal element):

\[
A_\mu = \begin{pmatrix} A^1_\mu & B_\mu \\ \overline{B}_\mu & A^2_\mu \end{pmatrix} + A^0_\mu \mathbb{I}.
\]

Here, \( A^1_\mu(x) \equiv A^{1\mu}_{\alpha\tilde{\alpha}}\tau^\alpha_1 \) is the physical SU(N) gauge field, \( \tau^\alpha_1 \) being the SU(N) generators orthonormalized to \( \text{tr}(\tau^\alpha_1 \tau^\beta_1) = \delta^{\alpha\beta}/2 \), while \( A^2_\mu(x) \equiv A^{2\mu}_{\alpha\tilde{\alpha}}\tau^\alpha_2 \) is a second, unphysical SU(N) gauge field. The \( B \) fields are fermionic gauge fields which will gain a mass of order \( \Lambda \) from the spontaneous symmetry breaking; they play the role of gauge invariant PV fields, furnishing the necessary extra regularization to supplement the covariant higher derivatives. In order to unambiguously define contributions which are finite only by virtue of the PV regularization, a preregulator must be used in \( D = 4 \) [28]. We will use dimensional regularization, emphasising that this makes sense non-perturbatively, since it is not being used to renormalize the theory, but rather as a prescription for discarding surface terms in loop integrals [14, 28].

\( A^0 \) is the gauge field for the centre of the SU(N|N) Lie superalgebra. Equivalently, one can write

\[
A_\mu = A^0_\mu \mathbb{I} + A^A_\mu T_A,
\]

where the \( T_A \) are a complete set of traceless and supertraceless generators normalised as in [28].

The theory is subject to the local invariance:

\[
\delta A_\mu = [\nabla_\mu, \Omega(x)] + \lambda_\mu(x) \mathbb{I}.
\]

The first term, in which \( \nabla_\mu = \partial_\mu - iA_\mu \), generates supergauge transformations. Note that the coupling, \( g \), has been scaled out of this definition. It is worth doing this: since we do not gauge fix, the exact preservation of (2.3) means that none of the fields suffer wavefunction renormalization, even in the broken phase [17]. The second term in (2.3) divides out the centre of the algebra. The reason for doing this is as follows.
The superfield strength is $F_{\mu\nu} = i[\nabla_\mu, \nabla_\nu]$, out of which we construct the kinetic term $\sim \text{str } F_{\mu\nu}^2$. On account of $\text{str } \mathbb{I} = 0$ and the fact that $\mathbb{I}$ commutes with everything, it is apparent that $\mathcal{A}^0$ has neither a kinetic term, nor any interactions. Consequently, if $\mathcal{A}^0$ were to appear anywhere else in the action, it would act as a Lagrange multiplier and so we forbid its presence. The resulting ‘no-$\mathcal{A}^0$ shift symmetry’ ensures that nothing depends on $\mathcal{A}^0$ and that $\mathcal{A}^0$ has no degrees of freedom.‡ This will prove important when we come to add quarks.

The spontaneous breaking is carried by a superscalar field

$$\mathcal{C} = \begin{pmatrix} C^1 & D \\ \bar{D} & C^2 \end{pmatrix}.$$  

This field is Hermitian but, unlike $\mathcal{A}_\mu$, is not supertraceless and so it is valued in the $\text{U}(N|N)$ Lie algebra. Nevertheless, the whole of $\mathcal{C}$ transforms homogeneously under local $\text{SU}(N|N)$:

$$\delta \mathcal{C} = -i [\mathcal{C}, \Omega]. \quad (2.4)$$

It can be shown that, at the classical level, the spontaneous breaking scale (effectively the mass of $B$) tracks the covariant higher derivative effective cutoff scale, $\Lambda$, if $\mathcal{C}$ is made dimensionless (by using powers of $\Lambda$) and $\hat{S}$ has the minimum of its effective potential at:

$$\langle \mathcal{C} \rangle = \sigma \equiv \begin{pmatrix} \mathbb{I}_N & 0 \\ 0 & -\mathbb{I}_N \end{pmatrix}, \quad (2.5)$$

where $\mathbb{I}_N$ is the $N \times N$ identity matrix.

In this case the classical action $S_0$ also has a minimum at (2.5). At the quantum level this can be imposed as a constraint on $S$ by taking $\langle \mathcal{C} \rangle = \sigma$ as a renormalization condition. This ensures that the Wilsonian effective action does not possess any one-point vertices, which can be translated into a constraint on $\hat{S}$ [17, 18]. In the broken phase, $D$ is a super-Goldstone mode (eaten by $B$ in the unitary gauge) whilst the $C^i$ are Higgs bosons and can be given a running mass of order $\Lambda$ [14, 17, 28]. Working in our manifestly gauge invariant formalism, $B$ and $D$ gauge transform into each other, as we will see in section 2.2.

In addition to the coupling, $g$, of the physical gauge field, the field $A_\mu^2$ carries its own coupling, $g_2$, (in the broken phase) which renormalizes separately [17–19, 22]. It is often useful not to work with $g_2$ directly but rather with

$$\alpha \equiv g_2^2/g^2. \quad (2.6)$$

The couplings $g$ and $\alpha$ are defined through their renormalization conditions:

$$S[\mathcal{A} = A^1, \mathcal{C} = \sigma] = \frac{1}{2g^2} \text{str } \int d^Dx (F_{\mu\nu}^1)^2 + \cdots, \quad (2.7)$$

‡ It is tempting to try and remove $\mathcal{A}^0$ from the algebra. However, we cannot do this directly since although $\text{SU}(N|N)$ is reducible it is indecomposable: $\mathcal{A}^0$ is generated by (fermionic) gauge transformations. It is possible to instead modify the Lie bracket [34] but this appears only to complicate matters.
where the ellipses stand for higher dimension operators and the ignored vacuum energy. The field strength tensors in the \( A^1 \) and \( A^2 \) sectors, \( F_{\mu\nu}^1 \) and \( F_{\mu\nu}^2 \), should really be embedded in the top left / bottom right entries of a supermatrix, in order for the supertraces in (2.7) and (2.8) to make sense. We will frequently employ this minor abuse of notation, for convenience.

2.2. Ward Identities

The supergauge invariant Wilsonian effective action has an expansion in terms of supertraces and products of supertraces [17]:

\[
S = \sum_{n=1}^{\infty} \frac{1}{s_n} \int d^Dx_1 \cdots d^Dx_n \, S_{a_1 \cdots a_n}^{X_1 \cdots X_n} (x_1, \ldots, x_n) \text{str} X_1^{a_1} (x_1) \cdots X_n^{a_n} (x_n) \\
+ \frac{1}{2!} \sum_{m=0}^{\infty} \frac{1}{s_n s_m} \int d^Dx_1 \cdots d^Dx_n \, d^Dy_1 \cdots d^Dy_m \, S_{a_1 \cdots a_n, b_1 \cdots b_m}^{X_1 \cdots X_n, Y_1 \cdots Y_m} (x_1, \ldots, x_n; y_1, \ldots, y_m) \text{str} X_1^{a_1} (x_1) \cdots X_n^{a_n} (x_n) \text{str} Y_1^{b_1} (y_1) \cdots Y_m^{b_m} (y_m) \\
+ \ldots \tag{2.9}
\]

where the \( X_i^{a_i} \) and \( Y_j^{b_j} \) are any of the broken phase fields, with the \( a_i \) and \( b_j \) being Lorentz indices or null, as appropriate. The vacuum energy is ignored. We take only one cyclic ordering for the lists \( X_1 \cdots X_n, Y_1 \cdots Y_m \) in the sums over \( n, m \). If any term is invariant under some nontrivial cyclic permutations of its arguments, then \( s_n (s_m) \) is the order of the cyclic subgroup, otherwise \( s_n = 1 \) (\( s_m = 1 \)).

The momentum space vertices are written

\[
S_{a_1 \cdots a_n}^{X_1 \cdots X_n} (p_1, \ldots, p_n) \, (2\pi)^D \frac{\delta}{\sum_{i=1}^{n} p_i} = \int d^Dx_1 \cdots d^Dx_n \, e^{-i \sum_{i=1}^{n} x_i p_i} \, S_{a_1 \cdots a_n}^{X_1 \cdots X_n} (x_1, \ldots, x_n),
\]

where all momenta are taken to point into the vertex. We employ the shorthand

\[
S_{a_1 a_2}^{X_1 X_2} (p) \equiv S_{a_1 a_2}^{X_1 X_2} (p, -p).
\]

Since we will ultimately be giving the flow equation for QCD via its diagrammatic representation, it is useful at this stage to introduce diagrammatics for the action (2.9). The vertex coefficient functions belonging to the action (2.9) have a simple diagrammatic representation:

\[
\begin{pmatrix} \otimes \end{pmatrix} \equiv \begin{pmatrix} S \end{pmatrix} \tag{2.10}
\]

represents all vertex coefficient functions corresponding to all cyclically independent orderings of the set of broken phase fields, \( \{ f \} \), distributed over all possible supertrace structures. For example,

\[
\begin{pmatrix} \otimes \end{pmatrix}^{c_1 c_1} \tag{2.11}
\]
represents both the coefficient functions \( S^{C^1 C^1} \) and \( S^{C^1, C^1} \) which, from (2.9), are associated with the supertrace structures \( \text{str} C^1 C^1 \) and \( \text{str} C^1 \text{str} C^1 \), respectively. (We have suppressed the momentum arguments.) Similarly,

\[
\left[ \left( S \right) \right]_{\mu \nu}^{A^1 A^1 C^1}
\]

represents \( S_{\mu \nu}^{A_1 A_1 C^1} \), \( S_{\nu \mu}^{A_1 A_1 C^1} \) and \( S_{\mu \nu}^{A_1 A_1 C^1} \). (There are no vertices which correspond to a trace of a single \( A^1 \), since \( \text{str} A^1 = 0 \).)

The (un)broken gauge transformations follow from splitting \( \Omega \) into its block components

\[
\Omega = \begin{pmatrix} \omega^1 & \tau \\ \bar{\tau} & \omega^2 \end{pmatrix} + \Omega^0 \mathbb{1}
\]

and expanding out (2.3) and (2.4) (we are not interested in the no-\( A^0 \) symmetry, here). For this purpose, it is useful to combine the fields \( A^1 \) and \( A^2 \) with the block diagonal components of \( A_0 \mathbb{1} \). We denote the resultant fields by \( \tilde{A}^1 \) and \( \tilde{A}^2 \), though note that sometimes \( A^0 \) contributions can cancel out between terms. This gives the unbroken \( SU(N) \times SU(N) \times U(1) \) transformations [17]

\[
\begin{align*}
\delta \tilde{A}^1 & = D^1 \cdot \omega^1 + \partial_\mu \Omega^0 \mathbb{1} \left[ N \right] \\
\delta \tilde{A}^2 & = D^2 \cdot \omega^2 + \partial_\mu \Omega^0 \mathbb{1} \left[ N \right] \\
\delta B_\mu & = -i (B_\mu \omega^2 - \omega^1 B_\mu) \\
\delta \bar{B}_\mu & = -i (\bar{B}_\mu \omega^1 - \omega^2 \bar{B}_\mu) \\
\delta C^1 & = -i C^1 \cdot \omega^1 \\
\delta \bar{C}^2 & = -i C^2 \cdot \omega^2 \\
\delta D & = -i (D \omega^2 - \omega^1 D) \\
\delta \bar{D} & = -i (\bar{D} \omega^1 - \omega^2 \bar{D})
\end{align*}
\]  

and the broken fermionic gauge transformations

\[
\begin{align*}
\delta \tilde{A}^1 & = -i (\tilde{B}_\mu \bar{\tau} - \tau \bar{B}_\mu) \\
\delta \tilde{A}^2 & = -i (\bar{B}_\mu \tau - \tau B_\mu) \\
\delta B_\mu & = \partial_\mu \tau - i (A_1^1 \tau - \tau A_2^1) \\
\delta \bar{B}_\mu & = \partial_\mu \bar{\tau} - i (A_2^2 \bar{\tau} - \bar{\tau} A_1^1) \\
\delta C^1 & = -i (D \bar{\tau} - \bar{\tau} \bar{D}) \\
\delta \bar{C}^2 & = -i (\bar{D} \tau - \tau D) \\
\delta D & = -i (C^1 \tau - \tau C^2) - 2i \tau \\
\delta \bar{D} & = -i (C^2 \bar{\tau} - \bar{\tau} C^1) + 2i \bar{\tau},
\end{align*}
\]

where \( D^{(1,2)} = \partial_\mu - i A_\mu^{1,2} \) are the covariant derivatives appropriate to the physical gauge field and the unphysical copy and the dot again means action by commutation.

As noted in [17], the manifest preservation of the transformations for \( \tilde{A}^1_\mu \) and \( \tilde{A}^2_\mu \) in (2.12) protects these fields from field strength renormalization. The remaining fields are similarly protected, as follows from (2.13).

The transformations (2.13) for \( B_\mu \) and \( \bar{D} \), \( \bar{B}_\mu \) and \( \bar{D} \) leads us to define [17]§

\[
\begin{align*}
F_M & = (B_\mu, D), \\
\bar{F}_N & = (\bar{B}_\nu, -\bar{D}),
\end{align*}
\]

where \( M, N \) are five-indices [18, 19]. The summation convention for these indices is that we take each product of components to contribute with unit weight.

§ Actually, these definitions differ from those of [17] by a sign in the fifth component. They are, however, consistent with [18, 19, 22–26].
Two Ward identities now follow from applying (2.12) and (2.13) to the action (2.9). The transformations (2.12) yield:

\[ q\nu S^{--X_{a\nu}^{1,2}Y_{-\nu}\ldots}(\ldots, p, q, r, \ldots) = S^{--X_{a\nu}^{1,2}Y_{-\nu}\ldots}(\ldots, p, q + r, \ldots) - S^{--X_{a\nu}^{1,2}Y_{-\nu}\ldots}(\ldots, p + q, r, \ldots). \]  

(2.15)

The effect of the transformations (2.13) are most efficiently written in the five component language of (2.14a) and (2.14b). Introducing a five momentum

\[ q_{M} = (q_{\mu}, 2), \]  

(2.16)

allows us to write

\[ q_{N}S^{--XY_{-\nu}\ldots}(\ldots, p, q, r, \ldots) = S^{--XY_{-\nu}\ldots}(\ldots, p, q + r, \ldots) - S^{--XY_{-\nu}\ldots}(\ldots, p + q, r, \ldots), \]  

(2.17)

where \( \overrightarrow{Y} \) and \( \overleftarrow{X} \) are the opposite statistics partners of the fields \( Y \) and \( X \). (For explicit expressions see \([18, 19]\).) An identical expression to (2.17) exists for when the field \( F_{N} \) is replaced by \( \overline{F}_{N} \).

The Ward identities (2.15) and (2.17) can be beautifully combined using the diagrammatics:

\[ q X \overrightarrow{Y} p = X \overrightarrow{Y} + X \overleftarrow{Y} - X \overrightarrow{Y} - X \overleftarrow{Y} + \cdots \]  

(2.18)

On the left-hand side, we contract a vertex with the momentum of the field which carries \( p \). This field—which we will call the active field—can be either \( A_{\rho}^{1}, A_{\rho}^{2}, F_{R} \) or \( \overline{F}_{R} \). In the first two cases, the triangle represents \( p_{\rho} \) whereas, in the latter two cases, it represents \( p_{R} = (p_{\rho}, 2) \). (Given that we often sum over all possible fields, we can take the Feynman rule for \( \triangleright \) in the \( C \)-sector to be null.) On the right-hand side of (2.18), we push the contracted momentum forward onto the field which directly follows the active field, in the counterclockwise sense, and pull back (with a minus sign) onto the field which directly precedes the active field. Since our diagrammatics is permutation symmetric, the struck field—which we will call the target field—can be either \( X, Y \) or any of the un-drawn fields, as represented by the ellipsis.

Allowing the active field to strike another field necessarily involves a partial specification of the supertrace structure: it must be the case that the struck field either directly followed or preceded the active field. In turn, this means that the Feynman rule for particular choices of the active and target fields can be zero. For example, as trivially follows by multiplying together supermatrices, an \( F \) can follow, but never precede an \( A_{\mu}^{1}, \) and so the pull back of an \( A_{\mu}^{1} \) onto an \( F \) should be assigned a value of zero. The momentum routing follows in an obvious manner: for example, in the first diagram on the right-hand side, momenta \( q + p \) and \( r \) now flow into the vertex. In the case that the active field is fermionic, the field pushed forward / pulled back onto is transformed into its opposite statistic partner, as above.

The half arrow which terminates the pushed forward / pulled back active field is of no significance and can go on either side of the active field line. It is necessary to keep the active field line—even though the active field is no longer part of the vertex—in order that we can unambiguously deduce flavour changes and momentum routing, without reference to the parent diagram.
We illustrate (2.18) by considering contracting $\triangleright$ into the Wilsonian the effective action two-point vertex:
\[ \rightarrow \mathcal{S} = \mathcal{S}_\mu^\nu - \mathcal{S}_\mu^\nu. \] (2.19)
Given that $\triangleright$ is null in the $C^i$ sector, the fields decorating the two-point vertex on the right-hand side can be either both $A^i$s or both fermionic. In the former case, (2.19) reads:
\[ p_\mu S_{\mu A i}^A (p) = S_{\mu A i}^A (0) - S_{\mu A i}^A (0) = 0 \]
where we note that $S_{\mu A i}^A$ is in fact zero by itself, as follows by both Lorentz invariance and gauge invariance. In the latter case, (2.19) reads:
\[ p_M S_{MN}^F (p) = \left[ S_{C^2}^F (0) - S_{C^1}^F (0) \right] \delta_{N5}, \]
where we have used (2.14a) and have discarded contributions which go like $S_{\mu A i}^A (0)$. However, the $S_{C^i}^F (0)$ must vanish. This follows from demanding that the minimum of the superhiggs potential is not shifted by quantum corrections [17]. Therefore,
\[ \rightarrow \mathcal{S} = 0. \] (2.20)

2.3. Taylor Expansion of Vertices

For the formalism to be properly defined, it must be the case that all vertices are Taylor expandable to all orders in momenta [14–16]. Consider a vertex which is part of a complete diagram, decorated by some set of internal fields and by a single external $A^1$ (or $A^2$), which we denote by a wiggly line. The diagrammatic representation for the zeroth order expansion in the momentum of the external field is all that is required for this paper [18, 19]:
\[ r \begin{array}{c} \mu \end{array} \begin{array}{c} \mu \end{array} X Y s = \begin{array}{c} \mu \end{array} + \begin{array}{c} \mu \end{array} + \begin{array}{c} \mu \end{array} + \cdots; \] (2.21)

note the similarity to (2.18).

The interpretation of the diagrammatics is as follows. In the first diagram on the right-hand side, the vertex is differentiated with respect to the momentum carried by the field $X$, whilst holding the momentum of the preceding field fixed (we assume for the time being that both $X$ and the preceding field carry non-zero momentum). Of course, using our current diagrammatic notation, this latter field can be any of those which decorate the vertex, and so we sum over all possibilities. Thus, each cyclically ordered push forward like term has a partner, cyclically ordered pull back like term, such that the pair can be interpreted as
\[ \left( \partial^r_s - \partial^s_r \right) \text{Vertex}, \] (2.22)
where $r$ and $s$ are momenta entering the vertex. In the case that $r = -s$, we can and will drop either the push forward like term or pull back like term, since the combination
can be expressed as $\partial_\mu$; we interpret the diagrammatic notation appropriately. If any of
the fields decorating the vertex carry zero momentum (besides the explicitly drawn $A^i$),
then they are transparent to this entire procedure. Thus, they are never differentiated
and, if they precede a field which is, we must look to the first field carrying non-zero
momentum to figure out which of the vertex’s momenta is held constant.

3. Adding Quarks

3.1. Massless Quarks

The simplest way to try to incorporate quarks into the setup is to embed them into the
fundamental representation of $SU(N|N)$:

$$\Psi = \begin{pmatrix} \psi \\ \varphi \end{pmatrix},$$

where $\Psi$ transforms under $SU(N|N)$, $\psi$ is a physical quark field and $\varphi$ is an unphysical,
bosonic spinor (here and henceforth, we suppress spinor indices). Immediately, we can
see that this embedding is inconsistent with (2.3): the supergauge invariant quark term,

$$\frac{1}{g^2} i \bar{\Psi} \nabla \Psi,$$

does not satisfy no-$A^0$ symmetry.

If, however, the fields $\Psi$ transform as $R \otimes \bar{R}$, for some representation, $R$, then we
can construct a no-$A^0$ invariant representation simply because $\Psi$ has zero ‘charge’ under
no-$A^0$. Thus, the strategy we pursue is to embed the quarks into fields which transform
as a bifundamental of $SU(N|N)$. To achieve this, we first embed the up-like quarks
[up, charm, top (suitably generalized for $N \neq 3$)] and down-like quarks (down, strange,
bottom) into two tensor fields $(\psi^u)_i^j$ and $(\psi^d)_i^j$, where the superscript indices carry an
SU($N$) colour symmetry and the subscript indices carry an (unphysical, gauged) SU($N$)
flavour symmetry. In turn, $\psi^u$ and $\psi^d$ are now embedded into fields $\Psi^u$ and $\Psi^d$
which are valued in complexified $U(N|N)$:

$$\Psi^u = \begin{pmatrix} \varphi^1 & \psi^u \\ \varphi^2 & \rho \end{pmatrix}, \quad \Psi^d = \begin{pmatrix} \phi^1 & \psi^d \\ \rho & \phi^2 \end{pmatrix}. \quad (3.1)$$

Notice that $\Psi^u$ and $\Psi^d$ are not Hermitian. Consequently, $\varrho$ ($\rho$) are not related to the
physical fields $\psi^u$ ($\psi^d$) and, since they will be seen to come with wrong sign action,
should be interpreted as unphysical degrees of freedom. These fields, together with $\phi^1$,
$\phi^2$, $\varphi^1$ and $\varphi^2$ (the components of which are bosonic spinors), will be given a mass
of order the cutoff. Under gauge transformations, $\Psi^u$ and $\Psi^d$ transform homogeneously:

$$\delta \Psi^u = -i[\Psi^u, \Omega], \quad \delta \Psi^d = -i[\Psi^d, \Omega]. \quad (3.2)$$

∥ If this scheme were to work, we would also have to introduce further unphysical fields to provide
sufficient PV regularization.
The SU($N|N$) invariant quark kinetic term that we include in the Lagrangian is just

$$-\frac{i}{g^2} \left( \text{str} \bar{\Psi}_u \nabla \cdot \Psi_u + \text{str} \bar{\Psi}_d \nabla \cdot \Psi_d \right), \quad (3.3)$$

where the minus sign compensates the sign buried in the supertrace, ensuring that the physical quark terms come with the correct overall sign. Notice that we have not included any covariant higher derivatives; it is straightforward to demonstrate that the supergroup structure, alone, is sufficient to provide the necessary regularization in the quark sector by repeating the analysis of [28], but this time including the fields $\Psi_i$.

To show the types of terms that we must include in the action to give the unphysical fields a mass of order the cutoff but leave the quarks massless (for the time being), it is useful to construct the following projectors:

$$\sigma_+ \equiv \frac{1}{2}(\mathbb{1} + \sigma) = \begin{pmatrix} \mathbb{1}_N & 0 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- \equiv \frac{1}{2}(\mathbb{1} - \sigma) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1}_N \end{pmatrix}. \quad (3.4)$$

With a slight abuse of notation, we can write $\varphi^1 = \sigma_+ \Psi_u \sigma_+$, $\varphi^2 = \sigma_- \Psi_u \sigma_-$, $\varrho = \sigma_- \Psi_u \sigma_+$ and $\psi_u = \sigma_+ \Psi_u \sigma_-$. We can lift these projectors to the symmetric phase by defining

$$\varsigma_\pm \equiv \frac{1}{2}(\mathbb{1} \pm C). \quad (3.5)$$

Thus, to give a mass to e.g. $\varrho$ in the broken phase, all we need to do is add to the Lagrangian the term:

$$-\frac{1}{g^2} \Lambda \text{str} \left( \bar{\Psi}_u \varsigma_- \Psi_u \varsigma_+ \right).$$

Upon spontaneous symmetry breaking, this reduces to $-\Lambda/g^2 \text{tr} \bar{\varrho} \varrho$ (plus interaction terms).

Thus, in the broken phase, the only massless fields we are left with are $\psi_i$, $A^1$ and $A^2$. In the pure gauge case, we know from [28] that $A^2$ decouples from $A^1$: integrating out the heavy fields, the lowest dimension gauge invariant effective interaction left between $A^1$ and $A^2$ is the square of the two field strengths (according to standard perturbative power counting with $g$ in the usual place)

$$\Lambda^{-D} \text{tr}(F^1)^2 \text{tr}(F^2)^2,$$

which is clearly irrelevant. Adding the quarks, however, we see immediately from (3.3) that these fields can combine with the unphysical gauge field to form a term of mass dimension $D$,

$$\alpha \text{tr}(\bar{\psi}_u \psi_u + \bar{\psi}_d \psi_d) A^2.$$
any such tuning by modifying the Higgs sector to completely break the unphysical SU(N); indeed, we will do precisely this when we adapt the formalism to give the quarks independent masses.

We conclude this section by discussing the additional (un)broken invariances which arise from the inclusion of the quarks. The up-like quarks supplement (2.12) with
\[
\begin{align*}
\delta \psi_u &= i(\omega^1 \psi_u - \psi_u \omega^2) \\
\delta \varrho &= i(\omega^2 \varrho - \varrho \omega^1) \\
\delta \varphi^1 &= i \omega^1 \cdot \varphi^1 \\
\delta \varphi^2 &= i \omega^2 \cdot \varphi^2
\end{align*}
\] (3.6)
and (2.13) with
\[
\begin{align*}
\delta \psi_u &= i(\tau \varphi^2 - \varphi^1 \tau) \\
\delta \varrho &= i(\varphi^1 \cdot \varrho^2) \\
\delta \varphi^1 &= i(\tau \varphi - \psi_u \tau) \\
\delta \varphi^2 &= i(\varphi_u \tau - \varrho \tau)
\end{align*}
\] (3.7)
(similarly for the down-like quarks). Notice that the quark field \( \psi_u \) is not protected from field strength renormalization. However, the transformations (3.7) do enforce that all components of \( \Psi_u \) have the same field strength renormalization (likewise \( \Psi_d \)). The unbroken transformation for \( \psi_u \) given by (3.6) confirms our interpretation that the physical colour symmetry is carried by \( A^1 \), whereas the unphysical flavour symmetry is carried by \( A^2 \).

3.2. Massive Quarks

If we only needed to give all the up-like quarks one mass and all the down-like quarks one mass, then we could simply add a mass term to the Lagrangian, using the fields that we have already:
\[
-\frac{1}{g^2} \left[ m_u \text{str} \bar{\psi}_u \varsigma_+ \psi_u \varsigma_- + m_d \text{str} \bar{\psi}_d \varsigma_+ \psi_d \varsigma_- \right],
\]
where we have used (3.5). Of course, to give all the quarks different masses, we will have to break the unphysical, gauged flavour symmetry. To do this, we introduce two new dimensionless superscalars, \( C_u \) and \( C_d \) which, like \( C \), lie in the adjoint of \( U(N|N) \) and transform homogeneously:
\[
\delta C_u = -i [C_u, \Omega], \quad \delta C_d = -i [C_d, \Omega].
\]
We choose the \textit{vevs} of \( C_u \) and \( C_d \) to be
\[
\langle C_u \rangle = \left( \begin{array}{cc} I_N & 0 \\ 0 & -\sigma_u \end{array} \right), \quad \langle C_d \rangle = \left( \begin{array}{cc} I_N & 0 \\ 0 & -\bar{\sigma}_d \end{array} \right).
\] (3.8)
where \( \sigma_u = \text{diag}(1, m_c/m_u, m_t/m_u) \) and, given the unitary matrix, \( U, \ U^\dagger \bar{\sigma} U = \sigma_d = \text{diag}(1, m_s/m_d, m_b/m_d) \) (with an obvious generalization for arbitrary values of \( N \)). The \textit{vev} of \( C_u \) breaks the unphysical SU(\( N \)) down to U(\( 1 \))\( ^{N-1} \). We could choose the \textit{vev} of \( C_d \) to be diagonal. However, in this case we would be forced to tune the couplings of the
residual $U(1)$'s (which we note are, on account of their wrong sign action, asymptotically free and not trivial) to zero. Consequently, we might as well choose the vev of $C_d$ such that in combination with the vev of $C_u$ the unphysical $SU(N)$ is completely broken. Thus, we implicitly assume $U$ to be such that amongst the generators broken by $\langle C_d \rangle$ are those which are not broken by $\langle C_u \rangle$. The vevs of $C_u$ and $C_d$ (unlike that of $C$) are not protected from quantum corrections, which of course corresponds to renormalization of the quark masses. The broken Ward identities (whose modification due to the breaking of the unphysical $SU(N)$ we will discuss shortly) protect the components of $C_{u,d}$ from field strength renormalization, in the broken phase.

With the introduction of $C_u$ and $C_d$, there is no requirement to retain $C$. However, the following exposition is made simpler if we keep $C$ and so we do so, noting that such considerations are anyway irrelevant from the point of view of the diagrammatic form of the flow equation. We will not give an explicit realization of the symmetry breaking potential $V(C, C_u, C_d)$ which yields (2.5) and (3.8), since we are free to work with any potential which satisfies the following requirements. First, (in unitarity gauge) all Goldstone bosons are eaten by the various components of $A_\mu$ which acquire mass \(i.e.,\) the potential must not possess any accidental symmetries: the largest continuous symmetry group is just $SU(N|N)$). Secondly, the remaining (Higgs) components of $C$, $C_u$ and $C_d$ are given a mass of order the cutoff.

For non-degenerate masses, it is useful (and always possible) to construct the set of $N$ projectors which live in the bottom right block of a supermatrix:

\[
P_1 = \text{diag}(0_N, 1, 0, 0, \ldots) \\
P_2 = \text{diag}(0_N, 0, 1, 0, \ldots) \\
\vdots \tag{3.9}
\]

To see this, simply note that e.g.

\[
\frac{(\langle C_u \rangle - \langle C \rangle)(\langle C_u \rangle - m_c/m_u \langle C \rangle)}{(1 - m_t/m_u)(m_c/m_u - m_t/m_u)} = \text{diag}(0, 0, 0, 0, 1).
\]

We can lift the $P_i$ to the symmetric phase by introducing the non-degenerate (running) parameters $a_i$ and defining:

\[
P_j = \prod_{i \neq j} \frac{C_u - a_i C}{a_i - a_j}. \tag{3.10}
\]

In the broken phase (which recall that, by construction, we are actually always in) we identify the $a_i$ with the elements of $\sigma_u$. Note that the $P_j$ gauge transform homogeneously.

The quarks' mass term can be taken to be

\[
\frac{1}{g^2} \left[ m_u \text{str} (C_u \varsigma_- \bar{\Psi}_u \varsigma_+ \Psi_u \varsigma_+) + m_d \text{str} (C_d \varsigma_- \bar{\Psi}_d \varsigma_+ \Psi_d \varsigma_-) \right], \tag{3.11}
\]

where it is understood from now on that we have rotated the down-like quark fields to the mass basis (this is exactly analogous to the introduction of the CKM matrix in the standard model). The remaining components of $\Psi_{u,d}$ are given masses of order the cutoff. Note that we have included $\varsigma_\pm$ in (3.11) purely for convenience, to ensure
that the masses of the components of the fields $\phi^2$ and $\varphi^2$ (see (3.1)) do not pick up contributions from the quarks’ mass matrices.

Neglecting the covariant higher derivative regularization, the kinetic terms for $C_u$ and $C_d$ take the form

$$\frac{1}{2g^2} \text{str} \left[ (\nabla_\mu \cdot C_u)^2 + (\nabla_\mu \cdot C_d)^2 \right].$$

Notice that this term provides differing contributions to the masses of the various components of $B$. Specifically, $B$ decomposes into columns, with each column receiving a different mass. This is precisely what we would expect from the unbroken gauge transformations, as we now discuss.

An immediate effect of breaking the unphysical SU($N$) is that the relationships (2.12) and (2.13), (3.6) and (3.7) (which we supplement by those appropriate for $C_{u,d}$) decompose. The only relationships which are completely unaffected are those involving just $\omega_1$, i.e. the unbroken relationships for $A^1_{\mu}, C^1, C^1_{u,d}, \varphi^1$ and $\phi^1$.

The relationships involving just $\omega_2$ are completely broken. This means that the independent components of each of the bottom right block fields are no longer related by unbroken gauge transformations and so can be expected to propagate separately.

In the fermionic sectors, the previously unbroken gauge transformations involve both $\omega_1$ and $\omega_2$ e.g. $\delta B_\mu = -i(B_\mu \omega^2 - \omega^1 B_\mu)$. Now, however, the $\omega_2$ part is completely broken. In matrix language, the surviving unbroken transformation involving $\omega_1$ mixes up elements of each column with elements of the same column. Consequently, upon the breaking of the unphysical SU($N$), $B_\mu$ decomposes into $N$ ‘flavours’, $B_{\mu a}$, corresponding to the $N$ columns, with unbroken transformation law

$$(\delta B_{\mu a})^i = i \left( \omega^1 \right)^i_j (B_{\mu a})^j.$$  

This is precisely what we want: in the quark sector, colour remains a good symmetry and the unphysical, gauged flavour symmetry is completely broken.

With the above decomposition of many of our fields, we must adapt our expansion of the action in terms of fields (2.9) by appropriately expanding the set of fields represented by $X$ and $Y$. To maintain the supermatrix structure, we should ensure that the new fields are still embedded in supermatrices. For example, the field $B_{\mu a}$ should be in the appropriate column of the top-right block of a supermatrix, with all other elements set to zero. Equivalently, we can project the field $B_{\mu a}$ out by using the $P_i$ of (3.9).

At first sight, the breaking of the unphysical SU($N$) considerably complicates the Ward identities. However, we can anticipate from [23–25] that we can and should hide these complications in the diagrammatics. This will be made particularly straightforward if we now sum over the flavours of the target fields in (2.18). This helps for the following reason. Consider (2.15) where the target fields are fermionic. We know that the fermionic fields decompose by column and so the right-hand side of (2.15) will contain a sum of terms such that, if the unphysical SU($N$) were restored, these terms could be combined back into block supermatrix components.
We conclude this section by giving the renormalization conditions for the quarks:

\[
S = \frac{1}{g^2} \int d^D x \, \text{tr} \left[ \psi_u \left( i \nabla^1 + \sigma_u \right) \psi_u + \psi_d \left( i \nabla^1 + \sigma_d \right) \psi_d \right] + \cdots,
\]

where the ellipsis represents all other operators contributing to the effective action.

4. A flow Equation for QCD

4.1. Review of SU(N) Yang-Mills

4.1.1. Setup

We begin by describing the flow equation used for pure Yang-Mills [19], the basic form of which is:

\[
- \Lambda \partial_{\Lambda} S = a_0[S, \Sigma_g] - a_1[\Sigma_g],
\]

(4.1)

where \( \Sigma_g \equiv g^2 S - 2 \hat{S} \) (\( \hat{S} \), we recall, being the seed action). On the right hand side of the flow equation is the bilinear functional, \( a_0[S, \Sigma_g] \), which generates classical corrections and the functional \( a_1[\Sigma_g] \) which generates quantum corrections. These terms are given by

\[
a_0[S, \Sigma_g] = \frac{1}{2} \frac{\delta S}{\delta A_\mu} \left\{ \hat{\Delta} A A \right\} \frac{\delta \Sigma_g}{\delta A_\mu} + \frac{1}{2} \frac{\delta S}{\delta C} \frac{\delta \Sigma_g}{\delta C},
\]

(4.2)

\[
a_1[\Sigma_g] = \frac{1}{2} \frac{\delta}{\delta A_\mu} \left\{ \hat{\Delta} A A \right\} \frac{\delta \Sigma_g}{\delta A_\mu} + \frac{1}{2} \frac{\delta}{\delta C} \frac{\delta \Sigma_g}{\delta C},
\]

(4.3)

where the \( \hat{\Delta} \) represent the ERG kernels and the notation \( \{ \hat{\Delta} \} \) denotes their covariantization [14, 15].

The natural definitions of functional derivatives of SU\((N|N)\) matrices are used [16, 17, 28]:

\[
\frac{\delta}{\delta C} \equiv \begin{pmatrix} \delta/\delta C^1 & -\delta/\delta D \\ \delta/\delta D & -\delta/\delta C^2 \end{pmatrix},
\]

(4.4)

and from (2.2) [17, 28]:

\[
\frac{\delta}{\delta A_\mu} \equiv 2T_A \frac{\delta}{\delta A_\mu} + \frac{\sigma}{2N} \frac{\delta}{\delta A_\mu}.
\]

(4.5)

The wonderful simplicity of (4.2) and (4.3) arises from the realization that the fine detail of the flow equation (which, as we will see, does not affect universal quantities anyway) can be buried in the definition of the covariantization. Nonetheless, for the purpose of transparently generalizing to QCD, we now discuss the covariantization in some detail. The primary ingredient is the supercovariantization [17] of the kernel \( W \), \( \{ W \}_{\hat{A}} \). This is defined according to

\[
u \{ W \}_A = \sum_{m,n=0}^\infty \int_{x_1, \ldots, x_n; y_1, \ldots, y_m} \left( x_1, \ldots, x_n; y_1, \ldots, y_m; x, y \right) W_{\mu_1 \cdots \mu_n, \nu_1 \cdots \nu_m}(x_1, \ldots, x_n, y_1, \ldots, y_m) \text{str} \left[ u(x)A_{\mu_1}(x_1) \cdots A_{\mu_n}(x_n)v(y)A_{\nu_1}(y_1) \cdots A_{\nu_m}(y_m) \right],
\]

(4.6)

Notice that we use \( \Sigma_g \) instead of the \( \Sigma_1 \) of (1.2), on account of \( g \) being scaled out of the covariant derivative.
where \( u \) and \( v \) are supermatrix representations transforming homogeneously as in (2.4) and where, without loss of generality, we may insist that \( \{W\}_A \) satisfies \( u \{W\}_A v \equiv v \{W\}_A u \). For simplicity’s sake, we have chosen (4.6) to contain only a single supertrace. (In the diagrammatic form of the flow equation, such details make no difference.) The \( m = n = 0 \) term is just the original kernel, i.e.

\[
W_{(\hat{\tau}; x, y)} \equiv W_{xy}. \tag{4.7}
\]

The requirement that (4.6) is supergauge invariant enforces a set of Ward identities on the vertices \( W_{\mu_1 \cdots \mu_n, \nu_1 \cdots \nu_m} \) which we describe later. The no-\(\mathcal{A}^0\) symmetry is obeyed by requiring the coincident line identities [15]. These identities are equivalent to the requirement that the gauge fields all act by commutation [16], ensuring that the no-\(\mathcal{A}^0\) part of (2.3) is satisfied. A consequence of the coincident line identities, which also trivially follows from the representation of (4.6) in terms of commutators, is that if \( v(y) = 1\!l_g(y) \) for all \( y \), i.e. is in the scalar representation of the gauge group, then the covariantization collapses to

\[
u \{W\}_A v = (\text{str} u) \cdot W \cdot g, \tag{4.8}
\]

where we define

\[
f \cdot W \cdot g = \int_{x,y} f(x) W_{xy} g(y) = \int_x f(x) W(-\partial^2/\Lambda^2) g(x)
\]

which holds for any momentum space kernel \( W(p^2/\Lambda^2) \) and functions of spacetime \( f, g \), using

\[
W_{xy} = W(-\partial^2/\Lambda^2) \delta(x-y) = \int \frac{d^D p}{(2\pi)^D} W(p^2/\Lambda^2) e^{ip(x-y)}.
\]

At this point, it is instructive to recall the demonstration of the SU(\(N|N\)) invariance of the flow equation, assuming that the covariantizations \( \{\hat{\Delta}\} \) are just of the type (4.6) [17].

Under (2.4), the \( C \) functional derivative transforms homogeneously:

\[
\delta \left( \frac{\delta}{\delta C} \right) = -i \left[ \frac{\delta}{\delta C}, \Omega \right], \tag{4.9}
\]

and thus by (4.6), the corresponding terms in (4.2) and (4.3) are invariant. The \( \mathcal{A} \) functional derivative, however, transforms as [17]:

\[
\delta \left( \frac{\delta}{\delta \mathcal{A}_\mu} \right) = -i \left[ \frac{\delta}{\delta \mathcal{A}_\mu}, \Omega \right] + \frac{i \mathbb{I} \!l}{2N} \text{tr} \left[ \frac{\delta}{\delta \mathcal{A}_\mu}, \Omega \right]. \tag{4.10}
\]

The correction is there because (4.5) is traceless, which in turn is a consequence of the supertracelessness of (2.1). The fact that \( \delta/\delta \mathcal{A} \) does not transform homogeneously means that supergauge invariance is destroyed unless the correction term vanishes for other reasons.

Here, no-\(\mathcal{A}^0\) symmetry comes to the rescue. Using the invariance of (4.6) for homogeneously transforming \( u \) and \( v \), and the invariance of \( S \) and \( \hat{S} \), we have by (4.10) and (4.8), that the \( \mathcal{A} \) term in (4.2) transforms to

\[
\delta \left( \frac{\delta S}{\delta \mathcal{A}_\mu} \right) \delta \hat{\Delta} \approx \delta \left( \frac{\delta S}{\delta \mathcal{A}_\mu} \right) = \frac{i \mathbb{I} \!l}{2N} \text{tr} \left[ \frac{\delta S}{\delta \mathcal{A}_\mu}, \Omega \right] \cdot \hat{\Delta} \approx \delta \left( \frac{\delta S}{\delta \mathcal{A}_\mu} \right) \delta \hat{\Delta} \approx \delta \left( \frac{\delta S}{\delta \mathcal{A}_\mu} \right) \delta \hat{\Delta} = \delta \left( \frac{\delta S}{\delta \mathcal{A}_\mu} \right) \delta \hat{\Delta} \approx \delta \left( \frac{\delta S}{\delta \mathcal{A}_\mu} \right) \delta \hat{\Delta} \approx \delta \left( \frac{\delta S}{\delta \mathcal{A}_\mu} \right) \delta \hat{\Delta} \approx \delta \left( \frac{\delta S}{\delta \mathcal{A}_\mu} \right) \delta \hat{\Delta} \approx \delta \left( \frac{\delta S}{\delta \mathcal{A}_\mu} \right) \delta \hat{\Delta}, \tag{4.11}
\]

This completes the demonstration of the invariance of the flow equation under SU(\(N|N\)) transformations.
where $S \leftrightarrow \Sigma_g$ stands for the same term with $S$ and $\Sigma_g$ interchanged. But by (4.5) and no-$\mathcal{A}^0$ symmetry,
\[
\delta \frac{\delta \Sigma_g}{\delta \mathcal{A}_\mu} = 0
\]
(similarly for $S$), and thus the tree level terms are invariant under (2.3) and (2.4). Likewise, the quantum terms in (4.3) are invariant and this completes the proof that, for covariantizations of the form (4.6), the flow equation is both supergauge and no-$\mathcal{A}^0$ invariant.

As it stands, the covariantization (4.6) is not general enough for our purposes: we require the broken phase fields to come with their own kernels. The first part of the solution to this [17] is to define a new covariantization
\[
u \{ W \}_{AC} v = \nu \{ W \}_{A} v - \frac{1}{4} \{ \mathcal{C}, \nu \} \{ W \}_{m} \mathcal{C}, v \].
\] (4.12)
The $\mathcal{C}$ commutator terms are introduced to allow a difference between $A$ and $B$ kernels, and $C$ and $D$ kernels, in the broken phase. They do this because at the level of two-point flow equations $\mathcal{C}$ is replaced by $\sigma$ in (4.12), and $\sigma$ (anti)commutes with the (fermionic) bosonic elements of the algebra. Thus, extracting the broken phase two-point, classical flow equations from (4.2), we find that the $A^i$ kernels are given by $\hat{\Delta}^{AA}$, the $C^i$ kernels by $\hat{\Delta}^{CC}$, but the $B$ kernel is $\hat{\Delta}^{AA} + \hat{\Delta}_m^{AA}$ and the $D$ kernel is $\hat{\Delta}^{CC} + \hat{\Delta}_m^{CC}$ [17]. The $B$ and $D$ kernels can be combined (cf. (2.14a) and (2.14b)):
\[
\hat{\Delta}_{MN}(p) = \begin{pmatrix} \hat{\Delta}_{BB} & 0 \\ 0 & -\hat{\Delta}_{DD} \end{pmatrix}.
\]

As must be the case, the extra term in (4.12) is consistent with both supergauge and no-$\mathcal{A}^0$ invariance. If $\nu$ and $v$ transform homogeneously, then so do $[\mathcal{C}, \nu]$ and $[\mathcal{C}, v]$. Correction terms proportional to the identity (cf. (4.10)) are killed by the commutator structure; this structure also ensures no-$\mathcal{A}^0$ invariance. Notice that (4.8) holds for the extended covariantization.

We are still not quite done: it is convenient to generalize the covariantization yet further. In particular, since the physical coupling, $g$, and the unphysical coupling, $g_2$, renormalize differently [17, 19], it is useful to furnish $A^1$ and $A^2$ with different kernels (there is no need to do this for $C^1$ and $C^2$ [19]). To construct a term which does this, we cannot use a commutator, since $[\sigma, \delta/\delta A^i] = 0$. The simplest solution is to define in (4.2) and (4.3)
\[
u \{ \Delta^{AA} \}_{AC} v = \nu \{ \Delta^{AA} \}_{A} v + \nu \{ \Delta^{AA} \}_{A} \mathcal{P}(v) + \mathcal{P}(\nu) \{ \Delta^{AA} \}_{A} v,
\]
where
\[
8N \mathcal{P}(X) = \{ \mathcal{C}, X \} \str \mathcal{C} - 2 \mathcal{C} \str \mathcal{X}.
\] (4.14)
$\mathcal{P}(X)$ has the following properties which ensure both no-$\mathcal{A}^0$ invariance and supergauge invariance:
\[
\begin{align*}
\str \mathcal{P}(X) &= 0, \quad (4.15a) \\
\mathcal{P}(\mathbb{I}) &= 0. \quad (4.15b)
\end{align*}
\]
In the broken phase, we see that
\[ \mathcal{P} \left( \frac{\delta}{\delta A_{\mu}} \right) = \frac{1}{2} \sigma \frac{\delta}{\delta A_{\mu}} + \cdots, \]
where we have noted from (4.5) that \( \text{str} \sigma \delta / \delta A_{\mu} = 0 \) and the ellipsis includes terms with additional fields. Thus, a minus sign is introduced in the \( A^2 \) sector, compared to the \( A^1 \) sector, and we find that \( \hat{\Delta}^{AA} = \hat{\Delta}^{AA} + \hat{\Delta}^{AA}, \hat{\Delta}^{A^2A^2} = \hat{\Delta}^{AA} - \hat{\Delta}^{AA} \) [19].

We conclude this section by commenting on the Ward identities satisfied by the vertices of the covariantized kernels. Throughout this section, we have used the fact that the \( u \) and \( v \) in e.g. (4.13) are functional derivatives with respect to, say, \( Z^1 \) and \( Z^2 \) to label the kernels \( \hat{\Delta}^{Z^1Z^2} \). The diagrammatic form of the Ward identities (2.18) holds for the vertices of the kernels, also, so long as two of the target fields are identified with the ends of the kernels \( i.e. \) with \( Z^1 \) and \( Z^2 \) [18, 19].

4.1.2. Diagrammatics
As mentioned in the introduction, the most useful representation of the flow equation is a diagrammatic one, which is shown in figure 1 [18, 19, 23, 25].

- The term on the left-hand side generates the flow of all cyclically independent Wilsonian effective action vertex coefficient functions which correspond to the set of broken phase fields \( \{f\} \).
- The objects on the right-hand side of figure 1 have two different types of component. The lobes represent vertices of action functionals. The object attaching to the various lobes, \( \bullet \), is the sum over vertices of the covariantized ERG kernels [15, 17] and, like the action vertices, can be decorated by fields belonging to \( \{f\} \). The fields of the action vertex (vertices) to which the vertices of the kernels attach act as labels for the ERG kernels. We henceforth loosely refer to both individual and summed over vertices of the kernels simply as a kernel. The dumbbell-like term corresponds to the classical term, \( a_0 \), whereas the padlock-like diagram corresponds to the quantum term, \( a_1 \). The rule for decorating the classical and quantum terms is simple: the set of fields, \( \{f\} \), are distributed in all independent ways between the component objects of each diagram.
- Embedded within the diagrammatic rules is a prescription for evaluating the group theory factors. Suppose that we wish to focus on the flow of a particular vertex coefficient function which, necessarily, has a unique supertrace structure. For example, we might \[ \hat{\Delta}^{AA} = \hat{\Delta}^{AA} + \hat{\Delta}^{AA}, \hat{\Delta}^{A^2A^2} = \hat{\Delta}^{AA} - \hat{\Delta}^{AA} \]
be interested in just the $S_{C_1}C_1$ component of (2.11). On the right-hand side of the flow equation, we must focus on the components of each diagram with precisely the same supertrace structure as the left-hand side, noting that the kernel, like the vertices, has multi-supertrace contributions. In this more explicit diagrammatic picture, the kernel is to be considered a double sided object (for more details see [18, 19]). Thus, whilst the dumbbell like term of figure 1 has at least one associated supertrace, the next diagram has at least two, on account of the loop (this is strictly true only in the case that kernel attaches to fields on the same supertrace). If a closed circuit formed by a kernel is devoid of fields then it contributes a group theory factor, depending on the flavours of the fields to which the kernel forming the loop attaches. This is most easily appreciated by noting that $\text{str} \sigma_\pm = \pm N$ (see (3.4)). In the counterclockwise sense, a $\sigma_+$ can always be inserted for free after an $A^1$, $C^1$ or $\bar{F}$, whereas a $\sigma_-$ can always be inserted for free after an $A^2$, $C^2$ or $F$.

The above prescription for evaluating the group theory factors receives $1/N$ corrections in the $A^1$ and $A^2$ sectors, as a consequence of the SU($N$) completeness relation [15]. If a kernel attaches to an $A^1$ or $A^2$, it comprises a direct attachment and an indirect attachment. In the former case, one supertrace associated with some vertex coefficient function is ‘broken open’ by an end of a kernel: the fields on this supertrace and the single supertrace component of the kernel are on the same circuit. In the latter case, the kernel does not break anything open and so the two sides of the kernel pinch together at the end associated with the indirect attachment. This is illustrated in figure 2; for more detail, see [18, 19, 26].

![Figure 2](image-url)

**Figure 2.** The $1/N$ corrections to the group theory factors.

We can thus consider the diagram on the left-hand side as having been unpackaged, to give the terms on the right-hand side. The dotted lines in the diagrams with indirect attachments serve to remind us where the loose end of the kernel attaches in the parent diagram.

4.2. Adding Quarks

The game now is easy. We add to the flow equation classical and quantum terms for the fields $\Psi_u$, $\Psi_d$ and modify the superhiggs sector. For all fields, we ensure that there is sufficient freedom in the covariantization to allow enough different kernels for the broken phase fields.

Though the details will ultimately be hidden in the diagrammatic form of the flow equation, we will give examples of choices we can make for the covariantization. Just
as in the pure Yang-Mills case, the kernels of the propagating fields may turn out to be linear combinations of those which appear in the flow equation. Though we will not explicitly perform this change of basis, it is instructive to see how we can, if we so desire, construct the covariantization so as to make this procedure as easy as possible.

As in the pure Yang-Mills case, the starting point for constructing the covariantization is (4.6). To this we now add additional terms which reflect the complete breaking of the unphysical $\text{SU}(N)$ gauge symmetry. Knowing that $B_\mu$ decomposes into $N$ flavours, it makes sense to add to the covariantization a term of the form

$$-\frac{1}{4} \sum_{j=1}^{N} \left( [\mathcal{P}_j, [\mathcal{C}, u]] \{ \hat{\Delta}_{j+}\}_{\mu} [\mathcal{P}_j, [\mathcal{C}, v]] \right),$$

where we have used (3.10). (As usual, the overall factor is merely a matter of convention.) The presence of the $\mathcal{C}$ commutators is purely for convenience. In the broken phase, they project on to the block off-diagonal components of $u$ and $v$, which ensures that, at the two-point level, the above term does not interfere with the flow of the components of the field $A_\mu^2$. This makes it easier to extract the kernels of the propagating fields in terms of the kernels in the flow equation.

The gauge field $A_\mu^2$ has $N^2 - 1$ independent components. Since we can use the same kernel for a field and its Hermitian conjugate, we require a total of $N(N + 1)/2 - 1$ kernels. So, for the components of $A_\mu^2$, we add to the covariantization a term of the form:

$$\sum_{j=1}^{N(N+1)/2-1} \left[ u \{ \hat{\Delta}_{j+N}^{\mu} \mathcal{P}_j'(\mathcal{P}(v)) + \mathcal{P}_j'(\mathcal{P}(u)) \{ \hat{\Delta}_{j+N}^{\mu} \mathcal{P}_j v \} \right],$$

(4.16)

where

$$\mathcal{P}_j'(X) = (Y_j X Z_j + Z_j X Y_j)_{\text{str}} Y_j Z_j - Y_j Z_j_{\text{str}} Y_j X Z_j - Z_j Y_j_{\text{str}} Z_j X Y_j,$$

and the $Y_j$ and $Z_j$ contain linear combinations of the symmetric phase versions of the projectors defined by (3.10) in their bottom right block (all other elements being zero). There are clearly many different choices we can take for $Y_j$ and $Z_j$. An example for $N > 2$ would be to set $Y_j = Z_j$ and choose the first $N(N - 1)/2 Y_j$ to be such that, in the the broken phase, they reduce to the $N C_2$ independent combinations of projectors of the form $Y_j = P_k + P_{t \neq k}$. Then we can take the remaining $Y_j$ to be any $N - 1$ of the $N P_1$. Notice that $\mathcal{P}'$ satisfies the conditions (4.15a) and (4.15b) and thus does not spoil either the supergauge or no-$\mathcal{A}^0$ invariance of the flow equation. The appearance of $\mathcal{P}$ (see (4.14)) in (4.16) is again for convenience ensuring that, at the two-point level, the kernels of (4.16) do not appear in the flow of the components of $B_\mu$.

Finally, then, a suitable choice for the covariantization of $\hat{\Delta}^{AA}$ is:

$$u \{ \hat{\Delta}^{AA} \}_{\mu} v \equiv u \{ \hat{\Delta}^{AA} \}_{\mu} v - \frac{1}{4} \sum_{j=1}^{N} \left( [\mathcal{P}_j, [\mathcal{C}, u]] \{ \hat{\Delta}_{j+}^{AA} \}_{\mu} [\mathcal{P}_j, [\mathcal{C}, v]] \right)$$

$$+ \sum_{j=1}^{N(N+1)/2-1} \left[ u \{ \hat{\Delta}_{j+N}^{AA} \mathcal{P}_j'(\mathcal{P}(v)) + \mathcal{P}_j'(\mathcal{P}(u)) \{ \hat{\Delta}_{j+N}^{AA} \mathcal{P}_j v \} \right].$$
The modifications to the covariantization in the superhiggs sector are almost identical; the only real difference is that, since the superscalars are not supertraceless, there are more propagating degrees of freedom in than in the $A_\mu$ sector, and so we must introduce additional kernels to take account of this.

The inclusion of quarks follows a similar pattern. The contributions of the up-like quarks to the flow equation are the standard ones for spinor fields [33], with the contribution to the classical term given by

$$\frac{1}{2} \left( \frac{\delta S}{\delta \Psi_u} \left\{ \Delta \bar{\Psi}_u \Psi_u \right\} + \frac{\delta S}{\delta \bar{\Psi}_u} \left\{ \Delta \bar{\Psi}_u \Psi_u \right\} \right),$$

and the contribution to quantum term given by

$$\frac{1}{2} \left( \frac{\delta}{\delta \Psi_u} \left\{ \Delta \bar{\Psi}_u \Psi_u \right\} + \frac{\delta}{\delta \bar{\Psi}_u} \left\{ \Delta \bar{\Psi}_u \Psi_u \right\} \right),$$

where we have suppressed spinor indices and functional derivatives with respect to $\Psi$ are defined as for any other unconstrained (i.e. not supertraceless) superfield (see (4.4)).

The covariantization is chosen to be:

$$u \left\{ \Delta \bar{\Psi}_u \Psi_u \right\} v = u \left\{ \Delta \bar{\Psi}_u \Psi_u \right\}_A v + \sum_{j=1}^N [P_j, \varpi_-(u)] \left\{ \Delta \bar{\Psi}_u \Psi_u \right\}_A [P_j, \varpi_+(v)] + \ldots \tag{4.17}$$

The first term on the right-hand side is the usual contribution involving just the supercovariantization. In the second term we have introduced, for convenience, the objects

$$\varpi_\pm(X) = \frac{1}{8} ([C, [C, X]] \pm 2[C, X])$$

defined so that, in the broken phase, they reduce to $\sigma_+ X \sigma_-$ and $\sigma_- X \sigma_+$ (plus interaction terms). Consequently, at the two-point level, the second term on the right-hand side of (4.17) contributes only to the flow of the physical quarks. The ellipsis represents additional terms which provide kernels for the rest of the propagating fields embedded in $\Psi_u$.

4.3. Diagrammatics for QCD

4.3.1. The Exact Flow Equation The beauty of the diagrammatic form of the flow equation given in figure 1 is that it can be directly generalized from SU($N$) Yang-Mills to QCD: all we need to do is to extend the set of broken phase fields which can decorate the diagrams both as internal and external fields. A consequence of this is that the prescription for extracting the group theory factors receives further corrections, which follow from inserting the appropriate projectors to go from derivatives with respect to supermatrix block fields to derivatives with respect to their appropriate components.

In fact, it is useful to employ a slightly different flow equation, in which we work directly with renormalized fields [31–33] (recall that the components of all fields bar
Ψ_u and Ψ_d are protected from field strength renormalization). This flow equation is given in figure 3. It is not the result of scaling the wavefunction renormalizations out of the version of (4.1) appropriate to QCD but is a perfectly valid flow equation, nonetheless [31, 32]. This is a manifestation of the tremendous freedom we have in constructing flow equations, encapsulated in (1.1).

\[
\left(-\Lambda \partial_\Lambda + \sum_{\chi \in \{f\}} \gamma^{(\chi)}\right) \begin{bmatrix} S \end{bmatrix}^{\{f\}} = \frac{1}{2} \begin{bmatrix} \Sigma_g \\
\cdot \\
S \\
\Sigma_g \end{bmatrix}^{\{f\}}
\]

**Figure 3.** The diagrammatic form of the flow equation.

The term \(\sum_{\chi \in \{f\}} \gamma^{(\chi)}\) explicitly takes account of the anomalous dimensions of the fields which suffer field strength renormalization. The field \(\chi\) belongs to the set of fields \(\{f\}\) and the notation \(\gamma^{(\chi)}\) just stands for the anomalous dimension of the field \(\chi\) (which is zero for all but the components of \(\Psi_{u,d}\)).

### 4.3.2. Perturbative Diagrammatics

In the perturbative domain, we have the following weak coupling expansions [15, 17–19]. The Wilsonian effective action is given by

\[
S = \sum_{i=0}^{\infty} g^{2(i-1)} S_i = \frac{1}{g^2} S_0 + S_1 + \cdots, \tag{4.18}
\]

where \(S_0\) is the classical effective action and the \(S_{i>0}\) the \(i\)th-loop corrections. The seed action has a similar expansion:

\[
\hat{S} = \sum_{i=0}^{\infty} g^{2i} \hat{S}_i, \tag{4.19}
\]

and the \(\beta\) function and anomalous dimensions are defined as usual:

\[
\beta \equiv \Lambda \partial_\Lambda g = \sum_{i=1}^{\infty} g^{2i+1} \beta_i, \tag{4.20}
\]

\[
\gamma^{(\chi)} \equiv \Lambda \partial_\Lambda \ln Z^{(\chi)} = \sum_{i=1}^{\infty} g^{2i} \gamma_i^{(\chi)}, \tag{4.21}
\]

where \(Z^{(\chi)}\) is the field strength renormalization of the field of species \(\chi\).

We also introduce \(\beta\) functions for the dimensionless mass parameters, \(\overline{m} \equiv m^2/\Lambda:\)

\[
\beta^j \equiv \Lambda \partial_\Lambda \overline{m}^j = \sum_{i=1}^{\infty} g^{2i} \beta^j_i. \tag{4.22}
\]

Defining \(\Sigma_i = S_i - 2\hat{S}_i\), the weak coupling flow equations follow from substituting (4.18)–(4.22) into the flow equation, as shown in figure 4.
Figure 4. The weak coupling flow equations.

The symbol $\bullet$ means $-\Lambda \partial_\Lambda |_{m_u,m_c,...}$. We will see shortly why the notation for the ERG kernels, $\bullet$, includes this symbol. A vertex whose argument is an unadorned letter, say $n$, represents $S_n$. We define $n_r \equiv n - r$ and $n_\pm \equiv n \pm 1$. The bar notation of the dumbbell term is defined as follows:

$$a_0[S_{n-r}, \bar{S}_r] \equiv a_0[S_{n-r}, S_r] - a_0[S_{n-r}, \hat{S}_r] - a_0[\hat{S}_{n-r}, S_r].$$

We illustrate the use of the flow equation by considering the flow of all two-point, classical vertices. This is done by setting $n = 0$ in figure 4 and specializing $\{f\}$ to contain two fields, as shown in figure 5. We note that we can and do choose all such vertices to be single supertrace terms [18, 19].

Figure 5. Flow of all possible two-point, classical vertices.

Following [14–19, 32], we use the freedom inherent in $\hat{S}$ by choosing the two-point, classical seed action vertices equal to the corresponding Wilsonian effective action vertices. Equation (4.23) now simplifies. Rearranging, integrating with respect to $\Lambda$ and choosing the appropriate integration constants [18, 19], we arrive at the following relationship between the integrated ERG kernels and the two-point, classical vertices.

$$M_{\bullet} = M_{\bullet} - M_{\bullet} \equiv \hat{M}_{\bullet}$$

(4.24)

We have attached the integrated ERG kernel, denoted by a solid line, to an arbitrary structure since it only ever appears as an internal line. The field labelled by $M$ can be any of the broken phase fields. The object $\bullet \equiv \hat{\bullet}$ is a gauge remainder (cf. (1.4)). The gauge remainder components are non-null only in the sectors corresponding to (components of) $A_\mu^i$ and $F_R$ and, in these sectors, $\hat{\bullet}$ and $\hat{\bullet}$ are related as a consequence of gauge invariance, as we will see shortly. Note that, in the case that a full gauge remainder bites a vertex, as opposed to just a $\hat{\bullet}$, we can replace the half arrows on the
right-hand side of (2.18) (which we recall just indicate to former presence of a $\triangleright$) with $a > [18, 19]$.

We have been able to construct the effective propagator for each and every independent classical, two-point vertex because we ensured that, for each such vertex, there exists an independent (integrated) kernel.

From the effective propagator relation and (2.20) follows a series of diagrammatic identities. In QCD, as opposed to pure Yang-Mills, the renormalization of $\langle C \rangle_{u,d}$—equivalently the renormalization of the quark masses—means that one-point $C_{u,d}^{1,2}$ vertices exist beyond tree level, spoiling (2.20) at the loop level. The first of the diagrammatic identities is, then, the classical part of (2.20):

$$\triangleright \begin{array}{c} 0 \end{array} \leftarrow = 0 = 0 = \triangleright \triangleright + \triangleright \triangleright ,$$

which implies the following diagrammatic identity:

$$\triangleright \triangleright = 1.$$ (4.25)

From the effective propagator relation and (4.25), two further diagrammatic identities follow. First, consider attaching an effective propagator to the right-hand field in (4.25) and applying the effective propagator before $\triangleright$ has acted. Diagrammatically, this gives

$$\begin{array}{c} \triangleright \end{array} \bigtriangledown \begin{array}{c} 0 \end{array} = 0 = \begin{array}{c} \triangleright \end{array} - \begin{array}{c} \triangleright \triangleright \triangleright \end{array} ,$$

which implies the following diagrammatic identity:

$$\begin{array}{c} \triangleright \triangleright \triangleright = 1.$$ (4.26)

The effective propagator relation, together with (4.26), implies that

$$\begin{array}{c} 0 \end{array} \rightarrow = \begin{array}{c} \triangleright \end{array} - \begin{array}{c} \triangleright \triangleright \triangleright \end{array} ,$$

In other words, the (non-zero) structure $\rightarrow$ kills a classical, two-point vertex. But, by (4.25), this suggests that the structure $\rightarrow$ must be equal, up to some factor, to $\leftarrow$. Hence,

$$\rightarrow \equiv \leftarrow ,$$ (4.27)

where the dot-dash line represents the pseudo effective propagators of [18, 19].

In practice, pseudo effective propagators only ever appear in a very specific way [25], which we now describe. Consider a three-point vertex attached to two arbitrary structures, $A$ and $B$, by two effective propagators. The third field on the vertex is taken to be an $A^1$ carrying momentum $p$ and we suppose that we now Taylor expand the vertex to zeroth order in $p$. Using (2.21) we have:

$$0 \overline{\overline{\langle A \rangle B}} = - \overline{\overline{\langle A \rangle B}} - \overline{\overline{\langle A \rangle B}} - \overline{\overline{\langle A \rangle B}},$$ (4.28)

where the arrow on the momentum derivative symbol indicates in which sense the momentum derivative acts. This is unnecessary in the parent diagram on the left-hand side of (4.28), since it is obvious that the momentum derivative has ‘pushed forward’ and
so corresponds to a derivative with respect to the momentum flowing into the vertex, from the structure $B$. However, once the vertex has been removed via the effective propagator relation, this information is lost, unless explicitly indicated.

Allowing the active gauge remainders in (4.28) to act (according to (2.18)) leads us to define

$$\equiv - \frac{1}{2} \phi \phi$$  \hspace{2cm} (4.29)

which we will require shortly. Notice that the second term of the right-hand side contains a pseudo effective propagator (differentiated with respect to the momentum flowing into its bottom end).

5. The One-loop $\beta$ function

As an illustration of the formalism, we would like to reproduce a standard result, namely the one-loop $\beta$ function for massless QCD (we take the massless case since, in our mass-dependent scheme, massive quarks will spoil universality, rendering any comparison with other methods of limited use). Fortunately, due to the developments in [24, 25], this calculation is extremely easy. In [25] a diagrammatic expression has been derived for the $n$-loop $\beta$ function in SU($N$) Yang-Mills from which the universal answer (at least at one and two loops) can be directly extracted. The key to deriving this expression is the effective propagator relation, which we have ensured holds for QCD. Indeed, at the one loop level, the pure SU($N$) Yang-Mills expression is exactly the same as in QCD, modulo the changes to the Feynman rules, and it is given in figure 6. (Beyond one loop, the expression of [25] is only slightly modified.) We define $\Box_{\mu\nu}(p) \equiv p^2 \delta_{\mu\nu} - p_\mu p_\nu$ and take wiggly lines to denote physical gauge fields (with Lorentz indices suppressed). Note that in $D = 4$ only the second, third and final diagrams contribute, so the expression for $\beta_1$ is really very simple.

![Diagrammatic expression for $\beta_1$.](image)

\[4\beta_1 \Box_{\mu\nu}(p) + O(p^4) = -\frac{1}{2} \begin{bmatrix} 0 & - & 0 & +4 & - & 0 \\ 0 & +4 & 0 & -8 & 0 & \end{bmatrix} \]
From comparison with the Yang-Mills (YM) expression [19, 22] and the QED expression [33], it is immediately clear that, as in conventional approaches (though with the number of flavours set equal to twice the number of colours),

$$\beta_{\text{QCD}}^1 = \beta_{\text{YM}}^1 + 2N \beta_{\text{QED}}^1.$$

Setting the quark masses to zero yields

$$\beta_{\text{QCD}}^1 = -\frac{N}{(4\pi)^2} \left( \frac{11}{3} - \frac{4}{3} \right).$$

6. Conclusion

We have constructed a manifestly gauge invariant ERG for QCD and have used it to compute the one-loop $\beta$ function for SU($N$) Yang-Mills coupled to $2N$ quarks. In the massless limit, we recovered the universal result.

The formalism is a direct extension of the one developed for SU($N$) Yang-Mills in [19]. The incorporation of the quarks comprised three steps. First, the quarks had to be added in a way that respected the SU($N|N$) regularization scheme. The symmetry associated with the centre of this algebra in fact prevented simply embedding the quarks into the fundamental representation of SU($N|N$). Instead, we first embedded sets of $N$ quarks into $N \times N$ matrices, whose rows (columns) were labelled by colour (flavour). Each of these matrices was then embedded into a separate supermatrix valued in complexified U($N|N$). In this way, we were able to include multiples of $N$ quarks, with each set of $N$ having degenerate masses.

The second step was to give each of the quarks independent masses, and this required that we broke an unphysical, gauged SU($N$) flavour symmetry which is carried by one of the fields belonging to the SU($N|N$) regularizing structure. To do this, we introduced Higgs fields for each of the sets of $N$ quarks, whose vevs are essentially mass matrices. The introduction of non-degenerate masses lifts the restriction that the number of flavours must be a multiple of the number of colours, since we are at liberty to remove quarks from the spectrum by tuning their masses to infinity. Clearly, though, the construction is most efficient when, suggestively, the number of flavours is a multiple of the number of colours.

The third and final step was to adapt the ERG equation. This involved not just including additional terms for the additional fields but also defining the covariantization of the ERG kernels appropriately. The key point is that we require the number of independent kernels to be equal to the number of independent, propagating fields. Having given algebraic examples of the covariantizations needed, we wrote down the full QCD flow equation in its diagrammatic form. Besides explicitly including the anomalous dimensions of the renormalizing fields, this expression has exactly the same form as that used for pure SU($N$) Yang-Mills. Indeed, this similarity allowed us to directly write down a diagrammatic expression for the one-loop $\beta$ function in QCD. By setting the quark masses to zero we were able to recover the standard, universal result.
The development of a manifestly gauge invariant ERG for QCD is timely. In SU($N$) Yang-Mills, methods exist for computing the expectation values of gauge invariant operators without fixing the gauge [26] and this work can now be directly generalized to include quarks. We find this particularly exciting in view of the fact that important progress is being made in understanding the structure of non-perturbative contributions both to these expectation values and the $\beta$ function [27].

Acknowledgments

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