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CHIRAL LIMIT THEOREMS AND CORRECTIONS TO PCAC
IN A MODEL FOR THE SCALAR $K_{\ell 3}$ DIVERGENCE FORM FACTOR

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A B S T R A C T

Three sum rules on vertex functions are derived by applying a model on two-particle ($K\pi$) channels. $SU_3 \times SU_3$ breaking is introduced by corrections to the PCAC forms of the axial currents. Taking the $m_\pi \rightarrow 0$ limit the sum rules reduce to the Callan-Treiman theorem, a recent theorem by Dashen, Li, Pagels and Weinstein, (DLPW) and a third relation. We encounter non-analytic correction terms in the two first sum rules. In the DLPW theorem we find, contrary to a previous suggestion, non-negligible ϵ and $\epsilon^2 \ln \epsilon$ terms which are due to the large K mass. These terms are responsible for a curvature of the scalar $K_{\ell 3}$ form factor, however, negative values of the slope at the origin require also substantial $SU_2 \times SU_2$ violations. The numerical results taking into account physical masses compare favourably with $K_{\ell 3}$ decay and also with s wave ($K\pi$) scattering data.

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1. INTRODUCTION

The following paper is concerned with sum rules for vertex functions with $K_{\ell 3}$ quantum numbers. The sum rules constitute on-mass-shell versions of the Callan-Treiman relation and a recent theorem by Dashen, Li, Pagels and Weinstein ¹⁾. Furthermore, a third sum rule is obtained which is transformed into an equation for the calculation of the scalar $K_{\ell 3}$ form factor $F_0(s)$. The relations are derived from chiral $SU_3 \times SU_3$ charge algebra, and a model on the contribution of the two-particle $(K\pi)$ intermediate channels. Furthermore, corrections to PCAC for matrix elements of $\partial_\mu A_\mu^K$ and $\partial_\mu A_\mu^\pi$ in form of additional contributions are considered.

The main results are summarized in the following points:

- i) there is a dip in the form factor $F_0(s)$ in the $K_{\ell 3}$ decay region if corrections to kaon-PCAC are considered;
- ii) in the model, violations of $SU_2 \times SU_2$ symmetry through the π mass are substantial for the appearance of a negative slope at $s=0$;
- iii) threshold dominance as postulated by Li and Pagels ²⁾ as dominance of the $\epsilon \ln \epsilon$ terms does not work in practice in $K_{\ell 3}$ decay. The terms ϵ as well as $\epsilon^2 \ln \epsilon$ are essential for the suppression of the slope.

The starting points are as usual matrix elements of the chiral $SU_3 \times SU_3$ charge commutators. However, instead of including a fictitious single particle $\mathcal{K}$ in the summation over intermediate channels ³⁾ the s wave projected two-particle $(K\pi)$ channel is considered. Following a suggestion by Brene ⁴⁾ this two-particle continuum is represented as a superposition of single-particle channels with a continuous mass distribution. This "quasi-single-particle" $\mathcal{K}$ channel has little in common with the previously postulated $\mathcal{K}$ particle. The assumptions in our model are clarified by comparing them with a general result obtained from the full, local, two-particle approximation ⁵⁾. At present a model seems the only way to keep the problem tractable at all.

As compared to our previous work ⁶⁾ we have introduced PCAC corrections to K dominance in the present paper. These are essential for the new results.

The numerical calculation of the sum rules is shortly compared with experiments here. As already noticed by different authors ⁷⁾ a dip of $F_0(s)$ can be favourably connected to experimental $K_{\ell 3}$ decay data. With reasonable values for λ_+ we get $\xi(0) \cong -1$. Furthermore, λ_0 is negative. The application of the Ademollo-Gatto theorem results even in an upper bound $\lambda_0 \leq -0.013$. The calculated phase-shift of s wave ($K\pi$) scattering is in agreement with experiments up to 1050 MeV. The scattering length $a_0^{\frac{1}{2}}$ is slightly suppressed as compared to the soft meson estimate from Weinberg's formula and the $(3, \bar{3}) + (\bar{3}, 3)$ model of chiral symmetry breaking.

2. DERIVATION OF THE SUM RULES

The notation used in the following is shortly defined here at the beginning. The form factors one encounters are given by ⁶⁾

$$(2\pi)^3 \langle \pi_p^0 | \partial^\mu V_\mu^{K^-} | K_K^+ \rangle = i(m_K^2 - m_\pi^2) F_0(s); \quad s = (k-p)^2 \quad (1)$$

$$(2\pi)^3 \langle \pi_p^0 | \partial^\mu A_\mu^{K^+} | \bar{\pi}_q^- \rangle = F^{\pi K}(s, t); \quad s = q^2, \quad t = (p-q)^2 \quad (2)$$

and π replaced by K everywhere in Eq. (2) $[F_0(0) \cong \frac{1}{2}]$. Furthermore, the following decay constants are involved

$$(2\pi)^{3/2} \langle 0 | A_\mu^{K^+} | K_K^- \rangle = i f_K k_\mu \quad (3)$$

$$(2\pi)^{3/2} \langle 0 | A_\mu^{\pi^+} | \pi_p^- \rangle = i f_\pi p_\mu; \quad (f_\pi^2 \cong m_\pi^2/2) \quad (4)$$

and the "quasi-decay" constant $f_K(s)$, which should however be looked at as a form factor

$$(2\pi)^{3/2} \langle 0 | \partial^\mu A_\mu^{K^+} | \bar{\pi}_q^- \rangle = f_K(s) \cdot s; \quad s = q^2. \quad (5)$$

The $\pi K - \pi$ coupling vertex

$$g(s, t) = (2\pi)^3 \langle K_K^- | J^{\pi^0} | \bar{\pi}_q^- \rangle \quad (6)$$

is essentially the s wave projected $(K\pi)$ scattering amplitude, as is seen by replacing χ through $(K\pi)|_{s \text{ wave}}$.

The model we shall start with as it is given by Brene ⁴⁾ has already been mentioned in the Introduction. It consists of looking at the s wave projected $K\pi\chi$ $K\pi$ channel as a continuous superposition of single-particle channels ⁶⁾

$$\sum_{\alpha, \beta} \int \frac{d^3 p}{2p_0} \int \frac{d^3 k}{2k_0} K_k^\alpha \pi_p^\beta \chi_{K_k \pi_p}^\alpha \Big|_{\substack{\ell=0 \\ I=1/2}} = \sum_{\gamma} \int ds \rho(s) \int \frac{d^3 q}{2q_0} \alpha_f^\gamma \chi_{\chi_f}^\gamma \quad (7)$$

with $\rho(s)$ the mass distribution function and α, β, γ are charge and hypercharge indices. To specify the model it is only necessary to formalize it for intermediate states as in Eq. (7). In the following we refer to the χ meson as a "quasi-single-particle" channel and to $(\chi\pi)$, (χK) as "quasi-two-body" channels.

The starting point for the sum rules are the following matrix elements of the chiral $SU_3 \times SU_3$ charge commutators:

$$\langle 0 | [Q_5^{\pi^0}, Q^{K^+}] | K^- \rangle = \frac{i}{2} (2\pi)^{3/2} m_K f_K \delta^3(\underline{k}) \quad (8)$$

$$\langle 0 | [Q_5^{K^+}, Q^{K^-}] | \pi^0 \rangle = \frac{i}{2} (2\pi)^{3/2} m_\pi f_\pi \delta^3(\underline{p}) \quad (9)$$

$$\langle 0 | [Q_5^{\pi^0}, Q_5^{K^+}] | \chi_{q, \frac{1}{2}=s}^- \rangle = -\frac{i}{2} (2\pi)^{3/2} f_\chi(s) \sqrt{s} \delta^3(\underline{q}). \quad (10)$$

The commutators are saturated with single-particle, two-particle, "quasi-single-particle", and "quasi-two-particle" intermediate channels. For instance in Eq. (8) we include π^0 , $(\pi^0 K^-)$, (χ^-) and $(\chi^+ K^-)$ in the summation. According to the prescriptions to form a local approximation ⁹⁾ we take the partially disconnected parts of two-particle and quasi-two-particle channels (Z graphs) ³⁾. So we retain only terms of the type below:

$$\langle K_k^-, \pi^0 | Q_5^{\pi^0} | K_k^- \rangle = 2k_0 \delta^3(k-k') \langle \pi_p^0 | Q_5^{\pi^0} | 0 \rangle. \quad (11)$$

This means, generally speaking, keeping elastic unitarity on the level of "quasi-two-particle" channels, one has only to deal with form factors and decay constants. This is the advantage of the quasi-particle approach. The limitations of this method will become clearer after having derived a few further equations.

The equations (8)-(10) result in the three following sum rules $[s_0 = (m_K + m_\pi)^2; s_d = (m_K - m_\pi)^2]$:

$$m_K f_K = f_\pi [\sqrt{s_0} F(s_d) + \sqrt{s_d} F(s_0)] - \int_{s_0}^{\infty} ds \rho(s) f_K(s) R^-(s, m_K) \quad (12)$$

$$m_\pi f_\pi = f_K [\sqrt{s_0} F(s_d) - \sqrt{s_d} F(s_0)] + \int_{s_0}^{\infty} ds \rho(s) f_K(s) R^-(s, m_\pi) \quad (13)$$

$$\sqrt{s} f_K(s) = f_\pi R^+(s, m_\pi) - f_K R^+(s, m_K) \quad (14)$$

with the function $R^\pm(s, m)$ defined by

$$R^\pm(s, m) = (\sqrt{s} + m) F^{mK}(t = (\sqrt{s} - m)^2, s) \mp (\sqrt{s} - m) F^{mK}(t = (\sqrt{s} + m)^2, s) \quad (15)$$

In this form the sum rules contain the unknown functions $\rho(s)$, $f_K(s)$, and $g(s, t)$ which will be eliminated in the following. But, first one has to specify the form factors $F^{mK}(s, t)$. For these functions we make the ansatz

$$F^{mK}(s, t) = f_K g(s) \left[\frac{m_K^2}{t - m_K^2} + d^K(s, t) \right]; (\pi \leftrightarrow K) \quad (16)$$

which is motivated by PCAC for π and K and permitting additional correction terms $d^{K(\pi)}(s, t)$.

Speaking in terms of a dispersion relation $d^K(s)$ may result either from a subtraction or from a dispersion integral over the $(K, 2\pi)$ -cut. Implicitly it can also depend on m_K^2 . The inclusion of such a term changes the chiral symmetry content of the approach since it is well known¹⁰⁾ that strict PCAC corresponds to working in the chiral symmetry limit.

Any $d^{K(\pi)}(s)$ introduces automatically chiral symmetry breaking. Whether such terms survive even in the $m_\pi \rightarrow 0$ limit is not known ^{*}). Later we have to require only that $(d^\pi(s) - d^K(s)) \simeq O(m_\pi^2 - m_K^2)$ keeping the possibility of some residual d as $m_\pi \rightarrow 0$.

Inserting Eq. (16), Eq. (14) reduces to the following:

$$(s - m_\pi^2)(s - m_K^2) f_K(s) = -2 f_\pi f_K g(s) \left[(m_\pi^2 - m_K^2) - d^K \cdot (s - m_K^2) + d^\pi \cdot (s - m_\pi^2) \right] \quad (17)$$

which expresses a s wave projected scattering amplitude through a form factor. In many treatments of chiral $SU_3 \times SU_3$ breaking, the GOR (Gell-Mann, Oakes and Renner) and GW (Glashow, Weinberg) schemes ¹²⁾ f_π plays a rather central role ¹³⁾ but its value is always fixed, which in simple enough structures compels the π particle to act like a Goldstone boson. In our approach this need not be the case. In fact, if one looks at $f_\pi(s)$ as a scalar form factor it is difficult to avoid such a conclusion. This effect prevents the π from playing a Goldstone role ¹³⁾.

For the idea of eliminating $g(s)$, $f_\pi(s)$ and $g(s)$ from Eqs. (12)-(14) one looks at the commutator ⁶⁾:

$$\langle 0 | [\partial^\mu V_\mu^{\pi^+}(x), \partial^\mu \bar{V}_\mu^{\pi^-}(0)] | 0 \rangle = i \int_0^\infty ds s^2 \sigma_0(s) \Delta(x; s) \quad (18)$$

in which the $(K\pi)$ and $(\pi\pi)$ channels can be inserted. It is then straightforward to determine the product $g(s) |f_\pi(s)|^2$ by identifying the two contributions to Eq. (18) via Eq. (7) ^{**)}.

$$\sigma_0 = g(s) |f_\pi(s)|^2 = \frac{3}{4} \frac{(m_K^2 - m_\pi^2)}{(2\pi)^2} \frac{\sqrt{(s - s_0)(s - s_d)}}{s^3} |F_0(s)|^2 \Theta(s - s_0) \quad (19)$$

^{*}) It is a feature of $(3, \bar{3}) + (\bar{3}, 3)$ models of symmetry breaking that corrections of $d^\pi(s, t)$ type vanish in the $m_\pi \rightarrow 0$ limit. This is not true for models based on $(6, \bar{6}) + (\bar{6}, 6)$, or $(8, 8)$ ¹¹⁾.

^{**)} Our calculation of σ_0 is not affected by the discussion whether the spectral representation should be subtracted ¹⁴⁾.

Furthermore, calculating the imaginary part of $F_0(s)$ from the LSZ reduction formula for the matrix element of Eq. (1)

$$i_m \langle \pi_p^0 | \partial^\mu V_\mu^{K^-} | K_K^+ \rangle = \frac{1}{2(2\pi)^{3/2}} \int d^4x e^{ipx} \langle 0 | [j_{\pi^0}^0(x), \partial^\mu V_\mu^{K^-}] | K_K^+ \rangle \quad (20)$$

and inserting intermediate states in the commutator gives

$$i_m F_0(s) = - \frac{\pi}{(m_K^2 - m_\pi^2)} \cdot s \cdot f_K(s) \cdot g(s) \cdot \rho(s). \quad (21)$$

This relation can be combined with Eqs. (17) and (19) and results in the equation

$$i_m F_0(s) = \frac{\varphi(s)}{f_K f_\pi} |F_0(s)|^2 \quad (22)$$

with $\varphi(s)$:

$$\varphi(s) = \frac{3}{32\pi} \frac{(m_K^2 - m_\pi^2)(s - m_\pi^2)(s - m_K^2) \sqrt{(s - s_0)(s - s_d)}}{s^2 [(m_K^2 - m_\pi^2) - (s - m_K^2)d^K + (s - m_\pi^2)d^\pi]} \Theta(s - s_0). \quad (23)$$

The further sum rules Eqs. (12) and (13) take the following form, if we rescale $G(s) = F(s)/(f_K f_\pi)$:

$$-\frac{m_\pi}{f_K^2} = \left[-\sqrt{s_0} G(s_d) + \sqrt{s_d} G(s_0) \right] - m_\pi \int_{s_0}^{\infty} R(s, m_\pi) |G(s)|^2 ds \quad (24)$$

and

$$\frac{m_K}{f_\pi^2} = \left[\sqrt{s_0} G(s_d) + \sqrt{s_d} G(s_0) \right] - m_K \int_{s_0}^{\infty} R(s, m_K) |G(s)|^2 ds \quad (25)$$

with

$$R(s, m) = \frac{3}{16\pi^2} \frac{(s - m^2) [m^2(m_K^2 - m_\pi^2 - 3s) + (s - s_0)(s - s_d)d^m] (m_K^2 - m_\pi^2)^2}{s^3 \sqrt{(s - s_0)(s - s_d)} [(m_K^2 - m_\pi^2) - (s - m_K^2)d^K + (s - m_\pi^2)d^\pi]}.$$

For completeness, in terms of $G(s)$, Eq. (22) reads

$$\lim_{s \rightarrow 0} G(s) = \varphi(s) / |G(s)|^2 \quad (26)$$

with $\varphi(s)$ given by Eq. (23). It should clearly be stated that Eq. (26) is nothing but an approximation to the elastic unitarity condition for form factors which reads generally

$$\lim_{s \rightarrow 0} F_0(s) = \frac{\varphi(s)}{(s-m_\pi^2)(s-m_K^2)} F_0^*(s) T_0^{(K\pi)}(s) \quad (27)$$

where $T_0^{(K\pi)}(s)$ represents the s wave projected $(K\pi)$ scattering amplitude. In deriving Eq. (26) we made the replacement

$$T_0^{(K\pi)}(s) \rightarrow \frac{(s-m_\pi^2)(s-m_K^2)}{f_\pi f_K} F_0(s)$$

since by assumption of Eq. (11) the four-point function is replaced by the three-point function since the momenta are regarded as equal. In performing the full, two-particle saturation scheme ⁷⁾ there results only Eq. (27), which means that there are correction terms to Eq. (26) for restoring the analytic structure of $T_0^{(K\pi)}(s)$. In our model the left-hand cut of $T_0^{(K\pi)}$ has been lost ^{*)}. However, at threshold the approximation does not seem drastic, since there only the right behaviour as reflected by Eq. (22) is important. It should be noted that in the twice subtracted dispersion relation, as a consequence of Eq. (22), the asymptotic behaviour of $F_0(s)$ is determined by the threshold behaviour. Each further inelastic threshold may change the asymptotic behaviour again, on which we know nothing, however the asymptotic form of the elastic contribution should be given correctly.

Before solving (24), (25) and (26) numerically which is possible since there are no free parameters left besides $d^{K(\pi)}$, and $f_K(\pi)$ it may be more instructive to look at different chiral limits. These considerations are helpful in understanding the mechanism which suppresses the slope of $F_0(s)$ at $s=0$.

*) The higher intermediate states have been neglected too.

3. CHIRAL SYMMETRY AND NON-ANALYTIC TERMS

In this Chapter, Eqs. (24)-(26) are reduced to special limiting cases by investigating the $m_\pi \rightarrow 0$ and $m_K \rightarrow 0$ limit. Taking the view that the chiral limit can be reached in fact by letting the masses vanish, conditions on d^π , and d^K are derived first.

In the SU_3 limit we have obviously

$$m_\pi = m_K = m_\rho, \quad d^\pi = d^K$$

and we assume that $\varphi(s) \neq 0$ in that limit, which means that the quotient

$$d = \frac{d^\pi - d^K}{m_K^2 - m_\pi^2}$$

does not diverge, i.e., $(d^\pi - d^K) \simeq O(m_\pi^2 - m_K^2)$. Then $\varphi(s)$ reads

$$\varphi(s) = \frac{3}{32\pi} \frac{(s - m_\rho^2) \sqrt{(s - 4m_\rho^2)s}}{s^2(1 + s \cdot d - m_\rho^2 d)} \Theta(s - 4m_\rho^2) \quad (28)$$

It is important to note that only the difference $d^\pi - d^K$ has to vanish keeping the possibility of some residual $d^{\pi(K)} \neq 0$ in the $m_\pi \rightarrow 0$ limit.

Now in Eq. (25) the limit $m_\pi \rightarrow 0$ is performed:

$$\frac{1}{f_\pi^2} = 2 G(m_K^2) - \frac{3}{16\pi^2} m_K^2 \int_{m_K^2}^{\infty} \frac{d^\pi(s) |G(s)|^2 (s - m_K^2)}{s^2 (1 + d^K + s \frac{d^\pi - d^K}{m_K^2})} ds \quad (29)$$

It is clear that for $d^\pi = 0$, Eq. (28) represents the Callan-Treiman relation $[2 \cdot G(m_K^2) = f_0(m_K^2)/f_K f_\pi]$, where $f_0(0) = 1$. In chiral symmetry schemes in which $m_\pi \rightarrow 0$ corresponds to the $SU_2 \times SU_2$ limit the corrections to Callan-Treiman are vanishing. There one assumes a smooth extrapolation in the pion mass.

Otherwise assuming no s dependence of d^π and replacing the form factor $G(s)$ by an average value in the integral one gets the non-analytic correction terms of Li and Pagels (also assuming $d^\pi \neq 0$ for $m_\pi \rightarrow 0$)

$$m_K^2 d^\pi (d^K - d^\pi) \ln (d^\pi - d^K)$$

This term is of the order $d^\pi \epsilon^2 \ln \epsilon$ where ϵ corresponds to some strength of symmetry breaking.

The same technique applied to the other sum rule, Eq. (24) gives the theorem of Dashen, Li, Pagels and Weinstein¹⁾ (DLPW), ($\sigma = s/m_K^2$):

$$4m_K^2 \frac{d}{ds} F(s) \Big|_{s=m_K^2} = \frac{f_K}{f_\pi} - \frac{f_\pi}{f_K} - \frac{3}{16\pi^2} \frac{m_K^2}{f_\pi f_K} \int_1^{\Lambda^2/m_K^2} \frac{(1-3\sigma) + (\sigma-1)^2 d^K}{\sigma^3 (1+d^K + \sigma(d^\pi - d^K))} \times |F_0(\sigma)|^2 d\sigma \quad (30)$$

The correction term to the DLPW theorem is explicitly proportional to m_K^2 multiplied by an integral which is finite in the chiral symmetry limit [particularly with $F_0(s) \rightarrow 0$ as $s \rightarrow \infty$] and, therefore, of order $\epsilon \ln \epsilon$ and not explicitly included in threshold dominance of DLPW. These terms of order $\epsilon \ln \epsilon$ which correspond to $(f_K/f_\pi - f_\pi/f_K)$ are kept only.

The theorem of DLPW

$$4m_K^2 \frac{d}{ds} F_0(s) \Big|_{s=m_K^2} = \frac{f_K}{f_\pi} - \frac{f_\pi}{f_K} \quad (31)$$

is compatible with a linear expansion of $F_0(s)$. Using realistic values of $f_K/f_\pi = 1.23$ which should survive even for $m_\pi \rightarrow 0$ and

$$F_0(m_K^2) = F_0(0) \left(1 + \frac{s}{m_K^2} \left(\frac{f_K}{f_\pi} - 1 \right) \right) \quad (32)$$

both sides of Eq. (31) match. Thus threshold dominance does not require higher order terms in the expansion of $F_0(s)$ in powers of s . This is contrary to a suggestion by Pagels¹⁵⁾ where he proposed that mechanism to be responsible for the curvature of $F_0(s)$. A curvature of $F_0(s)$ seems to be due to the contributions which are given by the integral term in Eq. (30). The sign of this term is thus that the slope at $s=m_K^2$ is increased.

We know from general arguments ¹⁶⁾ that any enhancement of the slope at $s = m_K^2$ over the linear value suppresses the slope at $s = 0$. This is due to a constraint which is given by the attractive $(K\pi)$ scattering length. The enhancement over the linear value at $s = m_K^2$ is not negligible ¹⁶⁾, for which the contribution of the integral term in Eq. (30) was calculated to be at least 0.2.

Whether this effect produces a negative slope as it is found later cannot be said so far. This depends on the values of $F_0(s)$ at $s = 0$ and $s = m_K^2$. The actual calculation of $F_0(s)$ with $m_\pi = 0$ yields a slope $\Lambda_0 = 0.011$ at $s = 0$. This is in perfect agreement with Ref. 16). Hence, the value of $\Lambda_0 < 0$ in our model can only be due to a substantial violation of $SU_2 \times SU_2$ symmetry by the pion mass. This non-smooth extrapolation property in the mass of the pion is coming from the introduction of d^K as we know from our former investigation ⁶⁾. This point will be further discussed also in the next chapter.

After having understood the meaning of Eqs. (24)-(26) in terms of chiral symmetry and its breaking the next step is to approach these equations numerically with physical masses and to try to determine $d^K(\pi)$ from experiments.

4. NUMERICAL SOLUTIONS AND COMPARISON WITH EXPERIMENTS

The starting point for the solution of Eqs. (24)-(26) is, making a substitution $G(s) = 1/D(s)$ in Eq. (26):

$$\text{Im } D(s) = -\varphi(s)$$

which leads to the dispersion integral solution

$$D(s) = D(0) + (D(s_d) - D(0)) \frac{s}{s_d} - \frac{s(s-s_d)}{\pi} \int_{s_0}^{\infty} \frac{\varphi(s')}{s'(s'-s_d)(s'-s)} ds' \quad (33)$$

In order to get a sensible behaviour of $D(s) \sim s$ for $s \rightarrow \infty$ one needs two subtractions. The calculation of the integral can be performed analytically, but is rather lengthy, so we do not display it here. The solution depends on the parameters d^π , d^K , $D(0)$ and $D(s_d)$. For the first two we make a kind of minimal assumption $d^\pi = 0$, $d^K = \text{constant}$ which will bring us already in agreement with experiments. For $D(0)$ one has a measured value ⁷⁾

$$D(0) = \frac{2}{f_+(0)} \frac{f_K}{f_\pi} \cdot f_\pi^2 = (1.28 \pm 0.06) m_\pi^2 \quad (34)$$

Then there are still two free parameters, $D(s_d)$ and d^K . The first one is expressed by λ_0 through $D(s_d) = D(0)/(1 + \lambda_0(s_d/m_\pi^2))$ where λ_0 is now regarded as free parameter and should not be mixed with the slope of $F_0(s)$. However, experimentalists often identify λ_0 with λ_0 and even determine an "experimental value" which is clearly wrong according to the discussion in Chapter 3 on the curvature of $F_0(s)$. The selection of the values for d^K and λ_0 is constrained by the need of satisfying sum rules (24) and (25). Because the sum rule (25) is very sensitive to the precise value of d^π , we can only use (24). As seen in Fig. 1 reasonable values of $f_K/f_\pi \simeq 1.2 - 1.3$ can be obtained in a range

$$-0.02 \leq \lambda_0 \leq 0.0$$

and

$$-10^{-3} \leq d^K \leq -10^{-5} \quad (35)$$

In Fig. 2 the form factor $G(s)$ is shown for the relevant parameters. It shows the dip which was discussed in the last section. However, even using $d^\pi = 0$ does not give the Callan-Treiman point in Fig. 1. The only conclusion is that the extrapolation in the mass of the pion is not smooth. This is also the result by comparing with the previous investigation ⁶⁾ in the case $d^K = 0$ where λ_0 was predicted unambiguously positive. If there is a dip then the requirement of Callan-Treiman of a smooth off-mass-shell extrapolation

$$F'_0(m_\pi^2; s) \cong F'_0(0; s); \quad 0 \leq s \leq m_\pi^2 \quad (36)$$

is quantitatively no longer true.

In order to compare the calculated form factor with K_{l3} decay parameters at least a quadratic expansion seems to be appropriate ⁷⁾

$$G(s) = G(0) \left[1 + \lambda_1 \frac{s}{m_\pi^2} + \lambda_2 \frac{s^2}{m_\pi^4} \right] \quad (37)$$

In terms of λ_+ , λ_1 and λ_2 the usual decay parameters ⁷⁾ read

$$\begin{aligned} \xi(0) &= \frac{m_K^2 - m_\pi^2}{m_K^2} (\lambda_1 - \lambda_+) \\ \lambda_- &= \lambda_2 \frac{m_K^2}{(\lambda_1 - \lambda_+) \cdot m_\pi^2} \end{aligned} \quad (38)$$

The resulting values in dependence on the parameters d^K , λ_+ and λ_0 are given in Table 1 and Table 2 ^{*)}. The agreement with experiments is very good. $\xi(0) = -0.85 \pm 0.20$ and $\lambda_- \cong 0$, or slightly positive hint to a negative λ_0 in accordance with (36) and a large λ_+ . However, the experimental analysis should better concentrate on measuring $F_0(s)$ over-all instead of determining expansion coefficients, of which the meaning is not clear.

The evaluation of the sum rules, Eqs. (24), (25), which is contained in Fig. 1 allows some interesting conclusions. First, one notes that for $-10^{-3} \leq d^K < 0$ the decay constants get independent on d^K and are functions of λ_0 alone. This dependence is expressed in Fig. 3 and allows for an interesting bound.

With the Ademollo-Gatto theorem in the form of Quinn and Bjorken ¹⁷⁾

$$f_+(0) - 1 = \mathcal{O}(\epsilon_8) < 0 \quad (39)$$

(ϵ_8 = strength of SU_3 breaking ¹²⁾) and the experimental numbers $f_\pi^2 = 0.50 m_\pi^2$ and $f_K/f_\pi f_+(0) = (1.28 \pm 0.06)$, there follows a bound on f_K^2 :

$$f_K^2 \leq 0.89 m_\pi^2. \quad (40)$$

^{*)} λ_1 is the slope of the linear term in a quadratic expansion whereas λ_0 is regarded by experimentalists as the slope of the linear expansion.

This bound is inserted in Fig. 3 and results in

$$\lambda_0 \leq -0.013. \quad (41)$$

A mechanism for this striking result was discussed in Chapter 3. In the description of $K_{\ell 3}$ decay $SU_2 \times SU_2$ violations and the DLPW theorem play an important role.

The sum rule, Eq. (25), provides another interesting remark. In our former calculations ⁶⁾ f_K/f_π came out with its right value ($\simeq 1.28$). This means that the error of 30% in f_π^2 is due to $d^K \neq 0$. If we look at the assumption $d^\pi = 0$ we realize that this introduces an instability in Eq. (25). Thus setting $d^K \neq 0$ induces also $d^\pi \neq 0$. This $d^\pi \neq 0$ reinforces the former observation that there is no smooth extrapolation in the mass of the pion. If there is non-smooth symmetry breaking then the Callan-Treiman relation has to get correction terms of the type of the DLPW theorem as they are expressed in Eq. (28).

From the calculation of $F_0(s)$ information on s wave $(K\pi)$ scattering can be drawn. Through the use of the Fermi-Watson theorem the phase shift $\delta_{\ell=0}^{I=\frac{1}{2}}(s)$ is calculated, Fig. 4. This curve is in line with experimental points ¹⁸⁾ up to 1050 MeV. Then there the neglected inelasticities like $(K, 3\pi)$ or (K, η) channels get important. It should be stated that there is no way to reach the "up" solution in our approach. The scattering length is plotted in Fig. 5 and moves in the range

$$a_0^{1/2} \cong (0.1 \dots 0.2) m_\pi^{-1}. \quad (42)$$

This is slightly smaller than the soft meson estimate from Weinberg's formula and other approaches ¹⁹⁾.

5. DISCUSSION AND SUMMARY

In this paper a mechanism is identified which allows for a curvature of the scalar $K_{\ell 3}$ divergence form factor. This mechanism has also been proposed by Pagels ¹⁵⁾ in the light of threshold dominance. However, regarding $\epsilon \ln \epsilon$ terms only is not sufficient to exclude a linear variation. For a curvature the terms ϵ , $\epsilon^2 \ln \epsilon$, and ϵ^2 have

to be retained. Physically this means that the kaon mass as an expansion parameter is not small. This effect has to be combined with non-smooth $SU_2 \times SU_2$ violations in order to reach a negative λ_0 and not only a suppression at $s=0$.

Note that we have not committed ourselves to any explicit assumption on chiral symmetry breaking, like representation properties of H_{SB} . We have just retained physical masses and corrections to PCAC in actual calculations. These corrections can even survive for $m_\pi \rightarrow 0$ 11).

The numerical results which are calculated from the model compare favourably with presently available data. This is already true for the simple minded ansatz $d^\pi = 0$, $d^K = \text{const.} \neq 0$. We know from similar investigations 20) that such an assumption is certainly too naive. But, making the scheme more precise is a question of improved experimental knowledge of $F_0(s)$.

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λ_o \ d^K	$\lambda_+ = 0.030$		$\lambda_+ = 0.090$	
	-10^{-1}	-10^{-4}	-10^{-1}	-10^{-4}
0.005	-0.33	-0.35	-1.05	-1.07
-0.005	-0.42	-0.43	-1.13	-1.15
-0.015	-0.51	-0.53	-1.23	-1.24
-0.025	-0.62	-0.64	-1.34	-1.35

TABLE 1 : $\xi(0)$

The correlation between $\xi(0)$, λ_+ , λ_o , and d^K as derived from the linear and quadratic expansion of $F_o(s)$, Eqs. (44), (45). The dependence of $\xi(0)$ on λ_+ is linear.

λ_+ λ_- λ_0	$\lambda_+ = 0.030$		$\lambda_+ = 0.090$	
	-10^{-1}	-10^{-4}	-10^{-1}	-10^{-4}
0.005	-0.015	-0.022	-0.005	-0.007
-0.005	-0.004	-0.002	0.001	0.000
-0.015	0.012	0.004	0.005	0.003
-0.025	0.015	0.007	0.007	0.006

TABLE 2 : λ_-

The correlation between λ_- , λ_+ , λ_0 and d^K
 (see Table Caption 1). The dependence of λ_- on
 λ_+ is linear.

FIGURE CAPTIONS

- FIGURE 1 : f_K^2 and f_π^2 as functions of d^K for different fixed λ_0 .
- FIGURE 2 : The scalar $K_{\ell 3}$ divergence form factor $G(s)$ as calculated from Eq. (34).
- FIGURE 3 : f_K^2 and f_π^2 as functions of λ_0 in the linear region.
- FIGURE 4 : The $(K\pi)$ scattering phase shift $\delta_{\ell=0}^{I=\frac{1}{2}}(s)$ as compared with experimental points ¹⁸⁾. For the Padé solution see Ref. 6).
- FIGURE 5 : The scattering length $a_{\ell=0}^{I=\frac{1}{2}}$ as a function of λ_0 and d^K .

FIG.1

FIG.2

FIG. 3

FIG.4

FIG. 5