Exact analytical solution to the relativistic Klein-Gordon equation with non-central equal scalar and vector potentials

Fevziye Yasuk, Aysen Durmus and Ismail Boztosun

Department of Physics, Erciyes University, 38039 Kayseri Turkey

Abstract

We present an alternative and simple method for the exact solution of the Klein-Gordon equation in the presence of the non-central equal scalar and vector potentials by using Nikiforov-Uvarov (NU) method. The exact bound state energy eigenvalues and corresponding eigenfunctions are obtained for a particle bound in a potential of $V(r, \theta) = \frac{\alpha}{r} + \frac{\beta}{r^2 \sin^2 \theta} + \gamma \frac{\cos \theta}{r^2 \sin^2 \theta}$ type.

PACS numbers: 03.65.Pm; 03.65.Ge

Keywords: Klein-Gordon equation, relativistic wave equations, non-central potentials, Nikiforov-Uvarov Method, exact solutions.
I. INTRODUCTION

In nuclear and high energy physics, one of the interesting problems is to obtain exact solutions of the Klein-Gordon, Duffin-Kemmer-Petiau and Dirac equations for mixed vector and scalar potentials. The Klein-Gordon, Dirac and Duffin-Kemmer-Petiau wave equations are frequently used to describe the particle dynamics in relativistic quantum mechanics. In recent years, a great deal of efforts have been spent to solve these relativistic wave equations for various potentials by using different methods. These relativistic equations contain two objects: the four-vector linear momentum operator and the scalar rest mass. They allow us to introduce two types of potential coupling which are the four vector potential ($V\left(\mathbf{r}\right)$) and the space-time scalar potential ($S\left(\mathbf{r}\right)$).

The Klein-Gordon equation with the vector and scalar potentials can be written as follows:

$$
\left[-\left(i\frac{\partial}{\partial t} - V\left(\mathbf{r}\right)\right)^2 - \nabla^2 + \left(S\left(\mathbf{r}\right) + M\right)^2\right] \psi(\mathbf{r}) = 0 \tag{1}
$$

For the case $S(\mathbf{r}) = \pm V(\mathbf{r})$, the solution of the Klein-Gordon equation has been studied recently [1, 2]. The exact solutions of these equations are possible only for certain potentials such as Coulomb, Morse, Pöschl-Teller, Hulthen and harmonic oscillator etc. by using different methods [3]. The other exactly solvable ones are the ring-shaped potentials introduced by Hartmann [4] and Quesne [5]. These potentials involve an attractive Coulomb potential with a repulsive inverse square potential one. In particular, the Coulombic ring-shaped potential [6] revived in quantum chemistry by Hartmann and coworkers [7] and the oscillatory ring-shaped potential, systematically studied by Quesne [5], have been investigated from a quantum mechanical viewpoint by using various approaches. The special case of the potential in spherical coordinates is

$$
V(\mathbf{r}, \theta) = \frac{\alpha}{r} + \frac{\beta}{r^2 \sin^2 \theta} + \frac{\gamma \cos \theta}{r^2 \sin^2 \theta} \quad \tag{2}
$$

introduced by Makarov et al. [8]. This potential can be used in quantum chemistry and nuclear physics to describe the ring-shaped molecules like benzene and the interactions between the deformed pairs of the nuclei.
In this paper, we introduce an alternative and simple method for the exact solution of the Klein-Gordon equation for the case where $S(\vec{r}) = \pm V(\vec{r})$, considering a general angle dependent (non-central) potential by using Nikiforov-Uvarov (NU) method [9]. This method is based on solving the second-order linear differential equations by reducing to a generalized equation of hypergeometric type.

NU-method is used to solve Schrödinger, Dirac, Klein-Gordon and Duffin-Kemmer-Petiau wave equations in the presence of the exponential type potentials such as Woods-Saxon, Pöschl-Teller [10] and Hulthen [11, 12] and non-central potential [13]. The aim of this study is to show that the Nikiforov-Uvarov method can be used to obtain exact solutions of non-central potentials for Klein-Gordon equation. Thus, radial and angular parts of the Klein-Gordon equation with non-central potential are solved by NU-method and it is seen that this method is applicable to non-central type potential for relativistic wave equations.

In the following section, the Klein-Gordon equation with equal scalar and vector potentials. In section III the Klein-Gordon equation in spherical coordinates for a particle in the presence of non-central potential is separated into radial and angular parts. Section IV is devoted to a brief description of the Nikiforov-Uvarov method. The solutions of the radial and angular parts of the Klein-Gordon equation by using the Nikiforov-Uvarov method in section V. Finally, concluding remarks are given in section VI.

II. KLEIN-GORDON EQUATION WITH EQUAL SCALAR AND VECTOR POTENTIALS

For the time-independent potentials we can write the total wave function as $\psi(\vec{r}, t) = e^{-i\varepsilon t}\psi(\vec{r})$, where $\varepsilon$ is the relativistic energy. The three-dimensional Klein-Gordon equation with the mixed vector and scalar potentials can be written as follows:

$$\left[\vec{\nabla}^2 + (V(\vec{r}) - \varepsilon)^2 - (S(\vec{r}) + M)^2 \right] \psi(\vec{r}) = 0, \quad (3)$$
where \( M \) is the mass, \( \varepsilon \) is the energy and \( S(\vec{r}) \) and \( V(\vec{r}) \) is the scalar and vectorial potentials respectively. Now, if we take \( S(\vec{r}) = \pm V(\vec{r}) \), the Klein-Gordon equation becomes:

\[
\left[ \nabla^2 - 2 (\varepsilon \pm M) V(\vec{r}) + \varepsilon^2 - M^2 \right] \psi(\vec{r}) = 0 \quad (4)
\]

This equation describes a scalar particle, i.e., spin-0 particle. It is the Schrödinger equation for the potential \( 2V \) in the non-relativistic limit. Thus, Alhaidari et al. concludes that only the choice \( S = +V \) produces a nontrivial nonrelativistic limit with a potential function \( 2V \), and not \( V \). Accordingly, it would be natural to scale the potential terms in Eq.\((3)\) so that in the relativistic limit the interaction potential becomes \( V \), not \( 2V \). Therefore, they modify Eq.\((3)\) to read as follows [1]:

\[
\left[ \nabla^2 + \left( \frac{1}{2} V(\vec{r}) - \varepsilon \right)^2 - \left( \frac{1}{2} S(\vec{r}) + M \right)^2 \right] \psi(\vec{r}) = 0 \quad (5)
\]

Thus, Eq.\((4)\) is acquired as,

\[
\left[ \nabla^2 - (\varepsilon \pm M) V(\vec{r}) + \varepsilon^2 - M^2 \right] \psi(\vec{r}) = 0 \quad (6)
\]

In the following section, for \( S(\vec{r}) = +V(\vec{r}) \), if we take \( V(\vec{r}) \) as a general non-central potential, three dimensional Klein-Gordon equation is separated into variables and these equation can be solved by using Nikiforov-Uvarov method.

### III. SEPARATING VARIABLES OF THE KLEIN-GORDON EQUATION WITH NON-CENTRAL POTENTIAL

In the spherical coordinates, the Klein-Gordon equation for a particle in the existence of a general non-central potential \( V(r, \theta) \) becomes

\[
\left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} - (\varepsilon + M) V(r, \theta) + \varepsilon^2 - M^2 \right] \psi(r, \theta, \varphi) = 0 \quad (7)
\]

where \( V(r, \theta) \) is a general non-central potential as given by Eq.\((2)\). If one assigns the corresponding spherical total wave function as \( \psi(r, \theta, \varphi) = \frac{1}{r} R(r) Y(\theta, \varphi) \), then by selecting
\( Y(\theta, \varphi) = \Theta(\theta)\Phi(\varphi) \), the wave equation (7) for a general non-central potential is separated into variables and the following equations are obtained:

\[
\left[ \frac{d^2}{dr^2} - \frac{\lambda}{r} - (\varepsilon + M) \frac{\alpha}{r} + \varepsilon^2 - M^2 \right] R(r) = 0 ,
\]

(8)

\[
\frac{d^2\Theta(\theta)}{d\theta^2} + \cot \theta \frac{d\Theta(\theta)}{d\theta} + \left[ \lambda - \frac{m^2}{\sin^2 \theta} - (\varepsilon + M) \left( \frac{\beta + \gamma \cos \theta}{\sin^2 \theta} \right) \right] \Theta(\theta) = 0
\]

(9)

\[
\frac{d^2\Phi(\varphi)}{d\varphi^2} + m^2\Phi(\varphi) = 0
\]

(10)

where \( m^2 \) and \( \lambda \) are the separation constants. The solution of equation (10) is well-known and it is the azimuthal angle solution,

\[
\Phi_m = A e^{im\varphi} , \quad (m = 0, \pm 1, \pm 2 \ldots)
\]

(11)

Eqs. (8) and (9) are radial and polar-angle equations and they will be solved by using Nikiforov-Uvarov method [9], given briefly in the following section.

**IV. NIKIFOROV-UVAROV METHOD**

The non-relativistic Schrödinger equation or similar time-independent second-order differential equations can be solved by using Nikiforov-Uvarov method which is based on the solutions of a general second-order linear differential equation with special orthogonal functions. In this method, for a given real or complex potential, the Schrödinger equation is transformed into a generalized equation of hypergeometric type with an appropriate coordinate transformation and it can be written in the following form

\[
\psi''(s) + \frac{\tau(s)}{\sigma(s)} \psi'(s) + \frac{\sigma'(s)}{\sigma^2(s)} \psi(s) = 0
\]

(12)

where \( \sigma(s) \) and \( \tilde{\sigma}(s) \) are polynomials, at most second-degree, and \( \tau(s) \) is a first-degree polynomial. Hence, from Eq. (12), the Schrödinger equation or the Schrödinger-like equations can be solved by means of this method for potentials we consider. In order to find a particular solution of Eq. (12), we use the separation of variables with the transformation
\[ \psi(s) = \phi(s)y(s) \]  
(13)

it reduces Eq. (12) to an equation of hypergeometric type,

\[ \sigma(s)y'' + \tau(s)y' + \lambda y = 0 \]  
(14)

and \( \phi(s) \) is defined as a logarithmic derivative in the following form and its solutions can be obtained from

\[ \frac{\phi'(s)}{\phi(s)} = \frac{\pi(s)}{\sigma(s)} \]  
(15)

The other part \( y(s) \) is the hypergeometric type function whose polynomial solutions are given by Rodrigues relation

\[ y_n(s) = \frac{B_n}{\rho(s)} \frac{d^n}{ds^n} [\sigma^n(s)\rho(s)] \]  
(16)

where \( B_n \) is a normalizing constant and the weight function \( \rho(s) \) must satisfy the condition

\[ (\sigma \rho)' = \tau \rho \]  
(17)

The function \( \pi \) and the parameter \( \lambda \) required for this method are defined as follows

\[ \pi(s) = \frac{\sigma' - \tilde{\tau}}{2} \pm \sqrt{\left( \frac{\sigma' - \tilde{\tau}}{2} \right)^2 - \tilde{\sigma} + k\sigma} \]  
(18)

\[ \lambda = k + \pi' \]  
(19)

On the other hand, in order to find the value of \( k \), the expression under the square root must be square of a polynomial. Thus, a new eigenvalue equation for the Schrödinger equation becomes

\[ \lambda = \lambda_n = -n\tau' - \frac{n(n-1)}{2}\sigma'' \]  
(20)

where

\[ \tau(s) = \tilde{\tau}(s) + 2\pi(s) \]  
(21)

and its derivative is negative. By comparison of Eqs. (19) and (20), we obtain the energy eigenvalues.
V. SOLUTIONS OF THE RADIAL AND ANGLE-DEPENDENT EQUATIONS

A. Solutions of the Radial Equation and Energy Eigenvalues

The radial part of the Klein Gordon equation is given as

\[
\left[ \frac{d^2}{dr^2} - \frac{\lambda}{r^2} - \frac{\varepsilon + M}{r} + \varepsilon^2 - M^2 \right] R(r) = 0 \tag{22}
\]

This equation can be further arranged as

\[
R''(r) + \left( -\eta^2 r^2 - \xi^2 r - \lambda \right) \frac{1}{r^2} R(r) = 0 \tag{23}
\]

with

\[
\varepsilon^2 - M^2 = -\eta^2, \quad (\varepsilon + M) = \xi^2, \quad \lambda = \ell(\ell + 1), \quad \alpha = -Ze^2 \tag{24}
\]

which is now amenable to a NU solution. In order to find the solution of this equation, it is necessary to compare Eq. (23) with Eq. (12). By comparison, we obtain the following polynomials:

\[
\tilde{\tau} = 0, \quad \sigma = r, \quad \tilde{\sigma} = -\eta^2 r^2 - \xi^2 r - \lambda \tag{25}
\]

Substituting these polynomials in Eq. (18), we obtain \( \pi \) function as

\[
\pi = \frac{1}{2} \pm \frac{1}{2} \sqrt{4\eta^2 r^2 + 4r(k + \xi^2) + 4\lambda + 1} \tag{26}
\]

The expression in the square root must be square of polynomial in respect of the NU method.

Therefore, we can determine the constant \( k \) by using the condition that the discriminant of the square root is zero, that is

\[
k = -\xi^2 \pm 2\sqrt{\eta^2} \left( \ell + \frac{1}{2} \right) \tag{27}
\]

In view of that, one can find new possible functions for each \( k \) as

\[
\pi = \begin{cases} 
\frac{1}{2} \pm \left[ \sqrt{\eta^2 r} + \left( \ell + \frac{1}{2} \right) \right], & \text{for } k = -\xi^2 + 2\sqrt{\eta^2} \left( \ell + \frac{1}{2} \right) \\
\frac{1}{2} \pm \left[ \sqrt{\eta^2 r} - \left( \ell + \frac{1}{2} \right) \right], & \text{for } k = -\xi^2 - 2\sqrt{\eta^2} \left( \ell + \frac{1}{2} \right)
\end{cases} \tag{28}
\]
For the polynomial of $\tau = \tilde{\tau} + 2\pi$ which has a negative derivative, we get

$$k = -\xi^2 - 2\sqrt{\eta^2 \left( \ell + \frac{1}{2} \right)} \quad \text{and} \quad \pi = \frac{1}{2} - \left[ \sqrt{\eta^2 r} - \left( \ell + \frac{1}{2} \right) \right]$$

(29)

Using $\lambda = k + \pi'$ together with the values $k$ and $\pi$, $\tau$ and $\lambda$ can be respectively obtained as

$$\tau = 2 \left( \ell + 1 - \sqrt{\eta^2 r} \right)$$

(30)

$$\lambda = -\xi^2 - \sqrt{\eta^2} (2\ell + 2)$$

(31)

Another definition of $\lambda_N$ is given at Eq.(20),

$$\lambda_N = 2N \sqrt{\eta^2}$$

(32)

comparing this with Eq.(31) and inserting the values of $\eta$ and $\xi$, the exact energy eigenvalues of radial part of Klein-Gordon equation with non-central potential are derived as

$$\varepsilon_{N\ell} = M \frac{(N + \ell + 1)^2 - \frac{\alpha^2}{4}}{(N + \ell + 1)^2 + \frac{\alpha^2}{4}}$$

(33)

where $N$ denotes the radial quantum number. This is not equal to the well-known positive energy spectrum of the relativistic Klein-Gordon-Coulomb problem but gives the correct non-relativistic limit in the case of weak coupling.

Using $\sigma$ and $\pi$ in Eqs.(14) to (16), we can find the wave functions $y(r) = y_{N\ell}(r)$ and $\phi(r)$:

$$R_{N\ell}(z) = C_{N\ell} r^{\ell+1} \exp\left( -\frac{z}{2} \right) L_{N\ell}^{2\ell+1}(z)$$

(34)

where $L_{N\ell}^{2\ell+1}(z)$ stands for the associated Laguerre functions whose argument is equal to $z = \frac{(\varepsilon + M)Z\alpha^2}{(N+\ell+1)^2}$ and $C_{N\ell}$ is normalization constant determined by $\int_0^\infty R_{N\ell}^2(r)dr = 1$. The corresponding normalized wave functions are finally obtained as

$$R_{n'\ell}(r) = \left( \frac{2(\varepsilon + M)Z\alpha^2}{n'} \right)^{1/2} \left( \frac{(n'-\ell-1)!}{n'!} \right)^{1/2} \left( \frac{2(\varepsilon + M)Z\alpha^2}{n'} \right)^{\ell+1} \times$$

$$\times r^{\ell+1} \exp\left( -\frac{(\varepsilon + M)Z\alpha^2}{n'} r \right) L_{n'\ell}^{2\ell+1} \left( \frac{(\varepsilon + M)Z\alpha^2}{n'} r \right)$$

(35)

where $n' = N + \ell + 1$. This equation is also stands for solution of the radial Klein-Gordon equation with Coulomb potential, since radial Klein-Gordon equation with non-central potential contains only Coulombic potential terms.
B. Eigenvalues and Eigenfunctions of the Angle-dependent Equation

As for the solutions of angle-dependent part of the Klein-Gordon equation, we may also derive eigenvalues and eigenfunctions of polar angle part of the Klein-Gordon equation similar to the method as given in section IV.

Eq. (9) can be written in the following form by introducing a new variable, \( x = \cos \theta \),

\[
\frac{d^2 \Theta(x)}{dx^2} - \frac{2x}{1-x^2} \frac{d\Theta(x)}{dx} + \left( \frac{\lambda(1-x^2) - m^2 - (\varepsilon + M) (\beta + \gamma x)}{(1-x^2)^2} \right) \Theta(x) = 0
\]  

(36)

To apply the Nikiforov-Uvarov method, we compare Eq.(36) with Eq.(12). By comparison, we obtain the following polynomials

\[ \tilde{\tau} = -2x, \quad \sigma = 1 - x^2, \quad \tilde{\sigma} = -\lambda x^2 - \gamma x + (\lambda - m^2 - \beta) \]  

(37)

The function \( \pi \) is obtained by putting the above-expression in Eq.(18),

\[ \pi = \pm \sqrt{x^2(\lambda - k) + \gamma x - (\lambda - m^2 - \beta - k)} \]  

(38)

The expression in the square root must be square of a polynomial. Then, one can find new possible functions for each \( k \) as

\[
\pi = \pm \begin{cases} 
  x \sqrt{\frac{m^2 + \beta + u}{2}} + \sqrt{\frac{m^2 + \beta - u}{2}}, & \text{for } k = \frac{2\lambda - m^2 - \beta - 1}{2} \frac{1}{2} u \\
  x \sqrt{\frac{m^2 + \beta - u}{2}} + \sqrt{\frac{m^2 + \beta + u}{2}}, & \text{for } k = \frac{2\lambda - m^2 - \beta + 1}{2} u 
\end{cases}
\]  

(39)

where \( u = \sqrt{(m^2 + \beta)^2 - \gamma^2} \). For the polynomial of \( \tau = \tilde{\tau} + 2\pi \) which has a negative derivative,

\[ \tau = -2 \sqrt{\frac{m^2 + \beta - u}{2}} - 2x \left( 1 + \sqrt{\frac{m^2 + \beta + u}{2}} \right) \]  

(40)

Using \( \lambda = k + \pi' \) and its other definition \( \lambda_n = -n\tau' - \frac{n(n-1)}{2} \sigma'' \) given by Eqs. (19) and (20), following expressions for the \( \lambda \) are obtained respectively

\[
\lambda = \frac{2\lambda - (m^2 + \beta)}{2} - \frac{1}{2} u - \sqrt{\frac{m^2 + \beta + u}{2}} 
\]  

(41)

\[
\lambda_n = 2n \left( 1 + \sqrt{\frac{m^2 + \beta + u}{2}} \right) + n(n - 1) 
\]  

(42)
Equating Eqs. (41) and (42) and using the definition of \( \lambda = \ell(\ell + 1) \), we obtain the \( \ell \) values as

\[
\ell = \sqrt{\frac{m^2 + \beta + \sqrt{(m^2 + \beta)^2 - \gamma^2}}{2}} + n
\]

(43)

If we insert \( \ell \) values obtained by Eq. (43) into eigenvalues of radial part of the Klein-Gordon equation with non-central potential given by Eq. (33), we finally find the energy eigenvalues for a bound electron in the presence of a non-central potential by Eq. (2)

\[
E_{Nm} = M \left[ \frac{\left( N + \sqrt{\frac{m^2 + \beta + \sqrt{(m^2 + \beta)^2 - \gamma^2}}{2} n + 1} \right)^2 - \frac{\alpha^2}{4}}{\left( N + \sqrt{\frac{m^2 + \beta + \sqrt{(m^2 + \beta)^2 - \gamma^2}}{2} n + 1} \right)^2 + \frac{\alpha^2}{4}} \right]
\]

(44)

where \( \beta = (\varepsilon + M)\beta \) and \( \gamma = (\varepsilon + M)\gamma \). The non-relativistic limit \( (\alpha \ll 1) \) of the energy spectrum for the Hartmann problem where \( \beta \neq 0 \) and \( \gamma = 0 \) is,

\[
E_{Nm} = -\frac{M\alpha^2}{2} \left( N + n + 1 + \sqrt{m^2 + \beta (\varepsilon + M)} \right)
\]

(45)

Then, the wave functions of polar-angle part of the Klein-Gordon equation, using \( \sigma \) and \( \pi \) in Eqs. (14) to (16), are obtained:

\[
\phi = (1 - x)^{B+C/2} (1 + x)^{B-C/2}
\]

(46)

\[
\rho = \left( 1 - x^2 \right)^B \left( \frac{1 + x}{1 - x} \right)^{-C}
\]

(47)

\[
y_n = B_n \left( 1 - x \right)^{-B+C} \left( 1 + x \right)^{-B-C} \frac{d^n}{dx^n} \left[ (1 + x)^{n+B-C} (1 - x)^{n+B+C} \right]
\]

(48)

where \( B = \sqrt{\frac{m^2 + \beta + u}{2}} \) and \( C = \sqrt{\frac{m^2 + \beta - u}{2}} \). The polynomial solution of \( y_n \) is expressed in terms of Jacobi polynomials which are one of the orthogonal polynomials, giving \( P_n^{(B+C, B-C)}(x) \). Substituting Eqs. (46) to (48) into Eq. (13), the corresponding wave functions are found to be

\[
\Theta_n(x) = N_n \left( 1 - x \right)^{(B+C)/2} \left( 1 + x \right)^{(B-C)/2} P_n^{(B+C, B-C)}(x)
\]

(49)
where $N_n$ is normalization constant determined by $\int_{-1}^{+1} [\Theta_n(x)]^2 \, dx = 1$ and using the orthogonality relation of Jacobi polynomials \[14, 15\], the normalization constant becomes

\[N_n = \sqrt{\frac{(2n + 2B + 1)\Gamma(n + 1)\Gamma(n + 2B + 1)}{2^{2B+1}\Gamma(n + B + C + 1)\Gamma(n + B - C + 1)}} \quad (50)\]

\noindent \textbf{VI. CONCLUSIONS}

This paper presented a different approach, the Nikiforov-Uvarov (NU) method, to the calculation of the non-zero angular momentum solutions of the relativistic Klein-Gordon equation. Exact eigenvalues and eigenfunction for the Klein-Gordon equation in the presence of the non-central equal scalar and vector potentials are derived easily. In the non-relativistic limit, the energy eigenvalue spectrum is shown to be equivalent to Hartmann one and the radial and polar angle wave functions are found in terms of Laguerre and Jacobi polynomials respectively. The method presented in this study is general and worth extending to the solution of other interaction problems.

\textbf{Acknowledgments}

This paper is an output of the project supported by the Scientific and Technical Research Council of Turkey (TÜBİTAK), under the project number TBAG-2398 and Erciyes University (FBA-03-27, FBT-04-15, FBT-04-16). The authors wish to thank Professor Coşkun Öнем for many helpful discussions and suggested improvements to the paper.


