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New Criterion for Nonoscillating High-Energy Behavior. III

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The analysis of nonoscillating scattering amplitudes carried on in two previous articles is completed by proving that, as conjectured, the asymptotic relations obtained also hold, in the mean, along physical directions.

IN two recent papers, the present authors have proposed a new criterion of nonoscillation at high energy. For a two-body scattering amplitude T , it amounts to requiring that, at fixed momentum transfer, the jump $a(E)$ across the cut is such that the limit (E is the lab energy)

$$\lim_{E \rightarrow \pm\infty} \frac{a(uE)}{a(E)} \quad (1)$$

exists for fixed u in a given interval $[1-\delta, 1+\delta]$, when E goes to infinity along the cut.

It is remarkable that condition (1), which is fairly general, is still sufficient, when combined with dispersion relations, to derive the standard high-energy results for the *complete amplitude* T (such as signature, powerlike behavior at high energy, and the Pomeranchuk theorem).^{1,2}

Furthermore, condition (1) is indeed general, as it has been shown to hold if the phase of T has a limit on the cuts,³ or for a very general singularity structure of the complex l plane.^{2,4}

The analysis performed in Refs. 1 and 2 on the total amplitude T was, however, incomplete since the proofs were only carried out for $E \rightarrow \infty$ along the *complex* directions. In Ref. 1, we derived a supplementary

condition which, together with (1), allowed one to pass to the real axis, showing, by explicit construction of counterexamples, that (1) alone is not in general sufficient to obtain pointwise results on the cuts for T . It then was conjectured that the asymptotic behavior of the whole of T obtained for complex directions will also hold on the cuts but in the mean (which is, of course, what one wants physically). In the present paper, we prove this conjecture, thereby completing the analysis carried on before.

To be precise, we want to show that condition (1) is sufficient to determine the high-energy behavior on the real axis of certain *averages* of the amplitude T . Physically, these averages will correspond to the energy resolution of the experimental device, and thus one will get that the high-energy theorems (say) will be satisfied for the experimentally measured phases or cross sections, as one would desire.

To represent actual measurements, we introduce families of positive functions, $\varphi_E(x)$. For fixed E we assume that the support of $\varphi_E(x)$ lies in $E[1-\delta(E)] \leq x \leq E[1+\delta(E)]$, and we let

$$a_\varphi(E) \equiv \left(\int dE'' \varphi_E(E'') \right)^{-1} \int dE' a(E') \varphi_E(E' - E). \quad (2)$$

$\delta(E)$ gives, therefore, the energy resolution of the experimental apparatus.

We first prove that if $\delta(E)$ is suitably chosen, $a_\varphi(E)$ will also be nonoscillating. From the mean-value theorem, we obtain

$$a_\varphi(E) = a(Eg(E)), \quad 1-\delta(E) \leq g(E) \leq 1+\delta(E). \quad (3)$$

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¹ J.-L. Gervais and F. J. Yndurain, Phys. Rev. **167**, 1289 (1968).

² J.-L. Gervais and F. J. Yndurain, Phys. Rev. **169**, 1187 (1968).

³ J.-L. Gervais, Phys. Rev. **182**, 1605 (1969).

⁴ F. J. Yndurain, Phys. Letters **27B**, 581 (1968); Nuovo Cimento **60A**, 405 (1969).

On the other hand, from (1) it follows² that $a(E)$ has the representation

$$a(E) = E^\alpha C(E) \exp\left(\int^E dy \frac{\epsilon(y)}{y}\right), \quad (4)$$

$$\lim_{E \rightarrow \infty} C(E) = C, \quad \lim_{y \rightarrow \infty} \epsilon(y) = 0.$$

Therefore,

$$a_\varphi(E)/a(E) = [1 + o(1)][g(E)]^{\alpha + o(1)}, \quad (5)$$

and hence it is easy to conclude the following:

Theorem 1. If $\lim_{E \rightarrow \infty} \delta(E) = 0$, then $a_\varphi(E)/a(E) \rightarrow 1$ and, in particular, $a_\varphi(E)$ also is nonoscillating.⁵

This suggests that we admit as smearing functions those $\varphi_E(x)$ for which $\delta(E) \rightarrow 0$. For those admissible smearing functions, the results of Ref. 1 apply to $T_\varphi(E)$ defined by

$$T_\varphi(E) = \frac{1}{\pi} \int \frac{dE'}{E' - E} a_\varphi(E'). \quad (6)$$

(For definiteness we consider, from now on, only the case where $-1 < \alpha < 0$.) Namely, if $E = \rho e^{i\Phi}$ with $\rho \rightarrow \infty$ and fixed $\Phi \neq 0$, $\Phi \neq \pi$, one has, from Ref. 1,

$$T_\varphi(E) \simeq -[e^{i(\Phi-\pi)\alpha}/\sin(\pi\alpha)][a(\rho) - e^{\eta\pi\alpha}a(-\rho)], \quad (7)$$

where $\eta = +1$ (-1) if $0 \leq \Phi < \pi$ ($\pi < \Phi < 2\pi$).

Now the desired results will be expressed in the form of the following two theorems which will show that, for any smearing such that the limit exists on the real axis, this limit is obtained from (7) by simply letting $\Phi = 0$ or π , so that the analysis of Refs. 1 and 2 applies on the mean for the high-energy limits along the real axis.

Theorem 2. If $a(E)$ is nonoscillating, then there exists at least one admissible smearing such that the limit of T for $E \rightarrow \infty$ along the real axis is obtained from (7) by letting $\Phi = 0$ or π on the right-hand side.

Using the methods of Ref. 1, it is easy to see that the only piece of (6) that may cause trouble is the part

$$\frac{1}{\pi} \int_{E(1-\eta)}^{E(1+\eta)} dE' \frac{a_\varphi(E')}{E' - E} \equiv R_\varphi(E) + ia_\varphi(E), \quad (8)$$

since, when $\eta \rightarrow 0$, the remainder of (7) already gives the desired result for the real part.

Define a_φ as

$$a_\varphi(E) \equiv \frac{\ln E}{2E} \int_{E-E/\ln E}^{E+E/\ln E} dE' a(E').$$

This is clearly an admissible average. To complete the proof of Theorem 2, it only remains to show that $|R_\varphi(E)/a_\varphi(E)| \rightarrow 0$ as $\eta \rightarrow 0$. But this is so: Eq. (8)

⁵ This result was proved in Ref. 2 in the particular case $\delta(E) \sim 1/E$.

can be transformed so that

$$\frac{R_\varphi(E)}{a_\varphi(E)} = \frac{2\eta E da_\varphi(E_0)/dE_0}{\pi a_\varphi(E)}, \quad (1+\eta)E \geq E_0 \geq (1-\eta)E. \quad (9)$$

By explicit computation, however,

$$\begin{aligned} \frac{da_\varphi(E_0)}{dE_0} &= a_\varphi(E_0) o\left(\frac{1}{E_0}\right) \\ &+ \left[a\left(E_0 + \frac{E_0}{\ln E_0}\right) - a\left(E_0 - \frac{E_0}{\ln E_0}\right) \right] \frac{\ln E}{2E} \\ &+ O\left(\frac{1}{E_0}\right) \left[a\left(E_0 + \frac{E_0}{\ln E_0}\right) + a\left(E_0 - \frac{E_0}{\ln E_0}\right) \right] \\ &= O\left(\frac{1}{E_0}\right) a_\varphi(E_0) + \left[a\left(E_0 + \frac{E_0}{\ln E_0}\right) \right. \\ &\quad \left. - a\left(E_0 - \frac{E_0}{\ln E_0}\right) \right] \frac{\ln E}{2E}, \end{aligned}$$

and we have used the fact that a is nonoscillating to perform the last step, writing $a(E_0 \pm E_0/\ln E_0) = [1 + o(1)]a(E)$. Inserting this into (9), we see that

$$\left| \frac{R_\varphi(E)}{a_\varphi(E)} \right| \leq \eta O(1) + \frac{\eta \ln E}{\pi} \cdot \frac{a(E_0 + E_0/\ln E_0) - a(E_0 - E_0/\ln E_0)}{a_\varphi(E)}.$$

The first term in the right-hand side is clearly harmless. As for the second, it is of the form [writing $a_\varphi(E) = a(E_1)O(1)$, which is possible because of Theorem 1]

$$\eta O(1)[1 - a(hE_1)/a(E_1)] \ln E,$$

where

$$E_1 \equiv E_0 + \frac{E_0}{\ln E_0}, \quad h = \frac{\ln E_0 - 1}{\ln E_0 + 1} \simeq 1 - (2 \ln E_0)^{-1}.$$

Taking into account the estimates of Ref. 1, Appendix A, we see that $a(hE_1)/a(E_1) = 1 - O(1)/\ln E_0$, so that, finally,

$$|R_\varphi(E)/a_\varphi(E)| = \eta O(1). \quad \text{Q.E.D.}$$

To complete the discussion it only remains to show that the result obtained is independent of the particular smearing used. This is demonstrated by the following:

Theorem 3. If, for any admissible smearing ψ_E , the limit

$$\lim_{E \rightarrow \pm\infty} \frac{T_\psi(E)}{a_\psi(E)}$$

exists at all, then it equals the limit obtained in Theorem 2.

Proof: It is clear that, because of the estimates of Ref. 1 for complex directions, the function $[T_\varphi(z)]^{-1}$ is analytic for $\text{Im}z > 0$, $|z| > R'$, and, furthermore, is bounded by $|z|^{\alpha+\epsilon}$ for large $|z|$. Therefore, $f(z)$, given by

$$f(z) = T_\psi(z)/T_\varphi(z),$$

is analytic for $|z| > R'$, $\text{Im}z > 0$ and is polynomial-bounded at infinity. Along real directions, the limit

$$\lim_{E \rightarrow \pm\infty} f(E) = \lim_{E \rightarrow \pm\infty} \left(\frac{T_\psi(E)}{a_\psi(E)} \right) \left(\frac{T_\varphi(E)}{a_\varphi(E)} \right)^{-1}$$

exists; call it $\lambda_\pm$. Along complex directions, the estimates of Ref. 1 show that $\lim_{z \rightarrow \infty} f(z) = 1$. Therefore, applying

the Phragmén-Lindelöf theorem⁶ to $f(z)$, we obtain at once that $\lambda_\pm = 1$. Q.E.D.

As a last comment, we want to remark that, as shown in Ref. 2, in some particular cases the smearing may lead to nonoscillating functions even if $a(E)$ itself is oscillating. More generally, one could get (1) to hold after smearing even if $a(E)$ were a distribution. This indicates clearly a direction in which the analysis could be directly generalized; however, we shall not elaborate further on this subject whose interest is mainly mathematical.

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⁶ R. Nevanlinna, *Eindentliche Analytische Funktionen* (Springer-Verlag, Berlin, 1953).

Relativistic Systems of Particles in Gravitational Interaction

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An elementary heuristic discussion of relativistic systems of particles in mutual gravitational interaction is given. Both in the boson and fermion cases, the main features of rigorous treatments are very simply recovered. A conceptual dilemma touching the use of various coordinate systems in such considerations is clarified. Finally, it is shown that in any realistic boson system, the relativistic effects of gravitation are probably masked by the other fundamental interactions, in contradistinction with fermion systems.

RECENTLY, Ruffini and Bonazzola¹ have studied systems of self-gravitating particles in general relativity, discussing *ab initio* the equilibrium configurations of large systems, without relying on the approximate description given by an equation of state. I wish first to point out that an elementary discussion, non-rigorous but heuristic, reproduces the main features of their results and previous ones.²

Consider N particles in purely gravitational interaction with mass m , average momentum p , and average pair separation d . In the nonrelativistic domain, their total energy may be estimated³ as

$$E(N) \simeq Nmc^2 + N \frac{p^2}{2m} - \frac{N^2 Gm^2}{2d}, \quad (1)$$

where G is Newton's constant and $\frac{1}{2}N^2 \simeq \frac{1}{2}N(N-1)$ is the number of pairs. Quantum mechanics now implies

an inequality of the type

$$p \times d \gtrsim \hbar \quad (2)$$

up to a numerical coefficient of order unity.

Minimizing the expression (1) subject to the constraint (2), one obtains an estimate for the ground-state energy which may be confirmed by establishing rigorous upper and lower bounds.³

For real matter, consisting of nuclei and electrons, one must consider Coulomb forces in addition to gravitational forces, and take into account the effect of the Pauli principle for fermions. This can be done, both rigorously and at the heuristic level. This reasoning may be extended to the "semirelativistic" regime where the electrons become relativistic, while the nuclei, because of their much heavier masses, remain nonrelativistic. These are precisely the conditions inside white dwarf stars, and considerations such as these lead to a very simple and direct proof of existence and estimate of the Chandrasekhar mass limit for white dwarfs.³

Let us consider here a system of identical particles with purely gravitational interactions. When the non-relativistic approximation breaks down, the estimate (1) must be modified to take into account (i) the relativistic momentum-energy relationship, which gives

¹ R. Ruffini and S. Bonazzola, *Phys. Rev.* **177**, 1767 (1969).

² B. K. Harrison, K. S. Thorne, M. Wakano, and J. A. Wheeler, *Gravitation Theory and Gravitational Collapse* (The University of Chicago Press, Chicago, 1965), and additional references therein.

³ J. M. Lévy-Leblond, *J. Math. Phys.* **10**, 806 (1969); see also *J. Phys. C* **30**, 3 (1969).