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ABSTRACT

A general cubature formula with an arbitrary preassigned weight function is derived using monosplines and integration by parts. The problem of determining the best cubature is formulated in terms of monosplines of least deviation and a solution to the problem is given by theorem 3 below. This theorem may also be viewed as an optimal property of a new kind of two-dimensional spline interpolation.

1. INTRODUCTION

Let f be a member of some class of functions X , and suppose that we wish to approximate the integral $I(f)$ over a d -dimensional cube by a finite linear combination of functional (and possibly derivative) values $Q(f)$. If the elements of X satisfy certain smoothness requirements and if $Q(f)$ is a 'reasonable' approximation to $I(f)$, then it is possible to show that the remainder $E(f) = I(f) - Q(f)$ satisfies the sharp appraisal

$$|E(f)| \leq C(Q)\sigma(f), \quad (1.1)$$

where $\sigma(f)$ is some seminorm on X and where $C(Q)$ is a constant that depends on $Q(\cdot)$ but is independent of f ; see, for instance [5]. Thus arises the question of determining, among all approximate formulas for which (1.1) is sharp, that formula $Q(\cdot)$ which renders $C(Q)$ least, i.e. determining the 'best' (or 'optimal') cubature formula.

It is now well known that, in the one-dimensional case, the question of best quadrature formulas is intimately related to spline function interpolation; see, for example, [1], [2], [7]-[11].

The purpose of the present paper is to investigate the relationship between cubature formulas and two-dimensional spline interpolation. It is only recently that certain aspects of this relationship have been discussed in the literature. Handscomb [3; p. 180] is perhaps the first to have explicitly related the problem of multidimensional approximate integration to spline functions. Later, Sard [6] showed, among other things, that the optimal approximation to a given continuous linear operator on one Hilbert space to another, is obtained as the result of applying the given operator on a certain spline function. More explicit results, but in a narrower framework than that of [6], have recently been obtained by Ritter [4] whose approach extends to the two-dimensional case that of [1] and [7].

In this work, we adopt the approach of Schoenberg [8], [9], and [12]. More specifically, in Section 2 we introduce (two-dimensional) monosplines and show how they give rise to a general cubature formula with an arbitrary preassigned weight function. In Section 3, the problem of best cubature formulas is formulated in terms of monosplines of least square deviation, and a complete solution to the problem is provided by theorem 3, Section 5. The method of solution is based on a new kind of two-dimensional spline interpolation which is the content of theorem 2, Section 4. Finally, in Section 6 we reformulate theorem 3 in abstraction of cubature formulas in order to reveal an optimal property of the interpolating spline.

A few words on terminology and notations. Throughout the rest of the paper we shall use the expressions 'cubature formula' and 'cubature rule' interchangeably. We denote by π_k the class of real polynomials of degree not exceeding k , and we write $\pi_{k,\ell} = \pi_k \otimes \pi_\ell$ for the tensor product of two classes of real polynomials. Finally, we shall adopt the notation

$$g^{(i,j)}(x,y) = \frac{\partial^{i+j}}{\partial x^i \partial y^j} g(x,y)$$

$$g(t) \Big|_{t=a}^b = g(b) - g(a) \quad .$$

2. DEVELOPMENT OF A GENERAL CUBATURE RULE

We consider integration over the rectangle

$$R = [a_1, a_2] \times [b_1, b_2]$$

and assume that we are given a rectangular mesh

$$\Delta_{MN} = \Delta_M \times \Delta_N$$

where

$$\Delta_M := a_1 = x_0 < x_1 < \dots < x_{M+1} = a_2$$

$$\Delta_N := b_1 = y_0 < y_1 < \dots < y_{N+1} = b_2 .$$

Choose two positive integers m and n . Denote by $S_{m-1}(\Delta_M) = \{S(x)\}$ the class of spline functions of degree $m-1$ with simple knots $x_1, \dots, x_M$; and by $S_{n-1}(\Delta_N) = \{S(y)\}$ the class of spline functions of degree $n-1$ with simple knots $y_1, \dots, y_N$. Furthermore, let $S_{m-1, n-1}(\Delta_{MN}) = \{S(x, y)\}$ denote the class of spline functions of degree $m-1$ in x , $n-1$ in y , having the lines $x = x_\mu$ ($\mu = 1(1)M$), $y = y_\nu$ ($\nu = 1(1)N$) as simple mesh lines. Now, an elementary extension of de Boor's argument in [13, p. 56] shows that the last class is nothing else than the tensor product of the first two; specifically

$$S_{m-1, n-1}(\Delta_{MN}) = S_{m-1}(\Delta_M) \otimes S_{n-1}(\Delta_N) , \quad (2.1)$$

and we have

$$\dim S_{m-1, n-1}(\Delta_{MN}) = \dim S_{m-1}(\Delta_M) \dim S_{n-1}(\Delta_N) = (m+M)(n+N) . \quad (2.2)$$

Indeed, if $\{\phi_i(x)\}_{i=1}^{m+M}$ and $\{\psi_j(y)\}_{j=1}^{n+N}$ are bases for $S_{m-1}(\Delta_M)$ and $S_{n-1}(\Delta_N)$, respectively, then $\{\phi_i(x)\psi_j(y)\}_{i,j}$ form a basis for the class $S_{m-1, n-1}(\Delta_{MN})$. Consequently, every $S(x, y) \in S_{m-1, n-1}(\Delta_{MN})$ is uniquely represented by

$$S(x, y) = \sum_{i=1}^{m+M} \sum_{j=1}^{n+N} \tau_{ij} \phi_i(x) \psi_j(y) . \quad (2.3)$$

Now, let us be given a weight function $w(x, y) \in L(R)$ -- the class of absolutely summable functions in R -- and let

$$u(x, y) = w^{(-m, -n)}(x, y) \quad (2.4)$$

denote an, as yet, arbitrary (m,n)-fold integral of $w(x,y)$, i.e. m-fold and n-fold integrations with respect to x and y, respectively. Define the 'kernel' functions

$$\begin{cases} K_{m,n}(x,y) = u(x,y) + S(x,y) , \\ K_{m,j}(x) = u_{m,j}(x) + S(x) , \quad j = 0(1)n-1 \\ K_{i,n}(y) = u_{i,n}(y) + S(y) , \quad i = 0(1)m-1 \end{cases} \quad (2.5)$$

where $u_{m,j}(x) \in L^{(m)} [a_1, a_2]$, $u_{i,n}(y) \in L^{(n)} [b_1, b_2] - L^{(k)} [I]$ being the class of functions continuously differentiable through order $k-1$ and whose k^{th} order derivative is absolutely summable in the interval I. We shall call these three kernels monosplines of order (m,n), m and n respectively; thus extending to the two-dimensional case the terminology of Schoenberg [12]. We now propose to show how the monosplines (2.5) give rise to a general cubature rule.

Let $f(x,y) \in C^{(m,n)}(R)$. Then, successive integration by parts yields the following expressions:

$$\begin{aligned} (-1)^{m+j} \int_{a_1}^{a_2} K_{m,j}(x) f^{(m,j)}(x, b_2) dx = \\ = \sum_{\mu=1}^M \alpha_{j\mu}^0 f^{(0,j)}(x_{\mu}, b_2) + \sum_{i=0}^{m-1} \left[\alpha_{ij}^1 f^{(i,j)}(a_1, b_2) - \alpha_{ij}^2 f^{(i,j)}(a_2, b_2) \right] + \\ + (-1)^j \int_{a_1}^{a_2} K_{m,j}^{(m)}(x) f^{(0,j)}(x, b_2) dx \end{aligned} \quad (2.6)$$

where

$$\begin{cases} \alpha_{j\mu}^0 = (-1)^j K_{m,j}^{(m-1)}(x) \Big|_{x=x_{\mu}-0}^{x_{\mu}+0} \\ \alpha_{ij}^p = (-1)^{i+j} K_{m,j}^{(m-i-1)}(a_p) , \quad p = 1, 2 \end{cases} \quad (2.7)$$

$$\begin{aligned}
 (-1)^{n+i} \int_{b_1}^{b_2} K_{i,n}(y) f^{(i,n)}(a_2, y) dy = \\
 = \sum_{v=1}^N \beta_{iv}^0 f^{(i,0)}(a_2, y_v) + \sum_{j=0}^{n-1} \left[\beta_{ij}^1 f^{(i,j)}(a_2, b_1) - \beta_{ij}^2 f^{(i,j)}(a_2, b_2) \right] + \\
 + (-1)^i \int_{b_1}^{b_2} K_{i,n}^{(n)}(y) f^{(i,0)}(a_2, y) dy
 \end{aligned} \tag{2.8}$$

where

$$\begin{cases} \beta_{iv}^0 = (-1)^i K_{i,n}^{(n-1)}(y) \Big|_{y=y_v+0}^{y_v+0} \\ \beta_{ij}^q = (-1)^{j+i} K_{i,n}^{(n-j-1)}(b_q) , \quad q = 1, 2 \end{cases} \tag{2.9}$$

Finally, we have

$$\begin{aligned}
 (-1)^{m+n} \iint_R K_{m,n}(x, y) f^{(m,n)}(x, y) dx dy = \sum_{\mu=1}^M \sum_{v=1}^N \gamma_{\mu v} f(x_\mu, y_v) + \\
 + \sum_{j=0}^{n-1} \sum_{\mu=1}^M \left[\tilde{\alpha}_{j\mu}^1 f^{(0,j)}(x_\mu, b_1) - \tilde{\alpha}_j^2 f^{(0,j)}(x_\mu, b_2) \right] + \\
 + \sum_{i=1}^{m-1} \sum_{v=1}^N \left[\tilde{\beta}_{iv}^1 f^{(i,0)}(a_1, y_v) - \tilde{\beta}_{iv}^2 f^{(i,0)}(a_2, y_v) \right] + \\
 + \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \sum_{p,q=1}^2 \delta_{ij}^{pq} f^{(i,j)}(a_p, b_q) - \\
 - \sum_{i=0}^{m-1} (-1)^i \int_{b_1}^{b_2} \left[K_{m,n}^{(m-i-1,n)}(x, y) f^{(i,0)}(x, y) \right] \Big|_{x=a_1}^{a_2} dy - \\
 - \sum_{j=0}^{n-1} (-1)^j \int_{a_1}^{a_2} \left[K_{m,n}^{(m,n-j-1)}(x, y) f^{(0,j)}(x, y) \right] \Big|_{y=b_1}^{b_2} dx + \\
 + \iint_R K_{m,n}^{(m,n)}(x, y) f(x, y) dx dy
 \end{aligned} \tag{2.10}$$

where

$$\left\{ \begin{array}{l} \gamma_{\mu\nu} = K_{m,n}^{(m-1,n-1)}(x,y) \Big|_{\substack{x=x_{\mu-0} \\ y=y_{\nu-0}}}^{x_{\mu+0} \quad y_{\nu+0}} \\ \tilde{\alpha}_{j\mu}^q = (-1)^j K_{m,n}^{(m-1,n-j-1)}(x,b_q) \Big|_{x=y_{\nu-0}}^{x_{\mu+0}} \quad , \quad q = 1, 2 \\ \tilde{\beta}_{i\nu}^p = (-1)^i K_{m,n}^{(m-i-1,n-1)}(a_p,y) \Big|_{y=y_{\nu-0}}^{y_{\nu+0}} \quad , \quad p = 1, 2 \\ \delta_{ij}^{pq} = (-1)^{i+j+p+q} K_{m,n}^{(m-i-1,n-j-1)}(a_p,b_q) \quad , \quad p, q = 1, 2 \quad . \end{array} \right. \quad (2.11)$$

Now, in order to simplify (2.10) and certain subsequent expressions, we require, as we may, that $u(x,y)$ defined by (2.4) satisfies the $(m+n)$ conditions

$$\left\{ \begin{array}{l} u^{(i,n)}(a_1,y) \equiv 0 \quad , \quad i = 0(1)m-1 \\ u^{(m,j)}(x,b_1) \equiv 0 \quad , \quad j = 0(1)n-1 \end{array} \right. \quad (2.12)$$

and that $u_{m,j}(x)$ and $u_{i,n}(y)$ are such that

$$\left\{ \begin{array}{l} u_{m,j}^{(m)}(x) = u^{(m,n-j-1)}(x,b_2) \quad , \quad j = 0(1)n-1 \\ u_{i,n}^{(n)}(y) = u^{(m-i-1,n)}(a_2,y) \quad , \quad i = 0(1)m-1 \quad . \end{array} \right. \quad (2.13)$$

If we now observe that, in view of (2.4), (2.5), and (2.13), we have

$$\left\{ \begin{array}{l} K_{m,n}^{(m,n)}(x,y) = w(x,y) \\ K_{m,j}^{(m)}(x) = K_{m,n}^{(m,n-j-1)}(x,b_2) \quad , \quad j = 0(1)n-1 \\ K_{m-i-1,n}^{(n)}(y) = K_{m,n}^{(m-i-1,n)}(a_2,y) \quad , \quad i = 0(1)m-1 \end{array} \right. \quad (2.14)$$

and, in view of (2.12), we have

$$\left\{ \begin{array}{l} K_{m,n}^{(m,j)}(x, b_1) \equiv 0, \quad j = 0(1)n-1 \\ K_{m,n}^{(i,n)}(a_1, y) \equiv 0, \quad i = 0(1)m-1, \end{array} \right. \quad (2.15)$$

we readily obtain, from (2.6)-(2.11), the general cubature rule

$$\begin{aligned} \iint_R w(x,y) f(x,y) \, dx \, dy &= \sum_{\mu=1}^M \sum_{\nu=1}^N C_{\mu\nu} f(x_\mu, y_\nu) + \\ &+ \sum_{q=1}^2 \sum_{\mu=1}^M \sum_{j=0}^{n-1} A_{j\mu}^q f^{(0,j)}(x_\mu, b_q) + \\ &+ \sum_{p=1}^2 \sum_{\nu=1}^N \sum_{i=0}^{m-1} B_{i\nu}^p f^{(i,0)}(a_p, y_\nu) + \\ &+ \sum_{p,q=1}^2 \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} D_{ij}^{pq} f^{(i,j)}(a_p, b_q) + E(f) \end{aligned} \quad (2.16)$$

where

$$\begin{aligned} E(f) &= (-1)^{m+n} \iint_R K_{m,n}(x,y) f^{(m,n)}(x,y) \, dx \, dy + \\ &+ \sum_{j=0}^{n-1} (-1)^{m+j} \int_{a_1}^{a_2} K_{m,j}(x) f^{(m,j)}(x, b_2) \, dx + \\ &+ \sum_{i=0}^{m-1} (-1)^{n+i} \int_{b_1}^{b_2} K_{i,n}(y) f^{(i,n)}(a_2, y) \, dy, \end{aligned} \quad (2.17)$$

$$C_{\mu\nu} = - \left[K^{(m-1, n-1)}(x, y) \right] \Big|_{x=x_\mu-0}^{x_\mu+0} \Big|_{y=y_\nu-0}^{y_\nu+0}, \quad \mu = 1(1)M, \quad \nu = 1(1)N \quad (2.18)$$

$$\left. \begin{aligned} A_{j\mu}^1 &= -(-1)^j \left[K^{(m-1, n-j-1)}(x, b_1) \right] \Big|_{x=x_\mu-0}^{x_\mu+0} \\ A_{j\mu}^2 &= (-1)^j \left[K^{(m-1, n-j-1)}(x, b_2) - K_{m,j}^{(m-1)}(x) \right] \Big|_{x=x_\mu-0}^{x_\mu+0} \end{aligned} \right\} j = 0(1)n-1, \quad \mu = 1(1)M \quad (2.19)$$

$$\left. \begin{aligned} B_{i\nu}^1 &= -(-1)^i \left[K^{(m-i-1, n-1)}(a_1, y) \right] \Big|_{y=y_\nu-0}^{y_\nu+0} \\ B_{i\nu}^2 &= (-1)^i \left[K^{(m-i-1, n-1)}(a_2, y) - K_{i,n}^{(n-1)}(y) \right] \Big|_{y=y_\nu-0}^{y_\nu+0} \end{aligned} \right\} i = 0(1)m-1, \quad \nu = 1(1)N \quad (2.20)$$

$$\left. \begin{aligned} D_{ij}^{11} &= -(-1)^{i+j} K^{(m-i-1, n-j-1)}(a_1, b_1) \\ D_{ij}^{12} &= (-1)^{i+j} \left[K^{(m-i-1, n-j-1)}(a_1, b_2) - K_{m,j}^{(m-i-1)}(a_1) \right] \\ D_{ij}^{21} &= (-1)^{i+j} \left[K^{(m-i-1, n-j-1)}(a_2, b_1) - K_{i,n}^{(n-j-1)}(b_1) \right] \\ D_{ij}^{22} &= -(-1)^{i+j} \left[K^{(m-i-1, n-j-1)}(a_2, b_2) - K_{m,j}^{(m-i-1)}(a_2) - K_{i,n}^{(n-j-1)}(b_2) \right] \end{aligned} \right\} \begin{aligned} i &= 0(1)m-1 \\ j &= 0(1)n-1 \end{aligned} \quad (2.21)$$

Let us briefly consider the special case $w(x, y) \equiv 1$. Thus, in view of (2.12) and (2.13), we must have

$$\begin{aligned} u(x, y) &= \frac{(x-a_1)^m (y-b_1)^n}{m!n!} \\ u_{m,j}(x) &= \hat{u}_{m,j}(x) \frac{(b_2-b_1)^{j+1}}{(j+1)!}, \quad j = 0(1)n-1, \\ u_{i,n}(y) &= \hat{u}_{i,n}(y) \frac{(a_2-a_1)^{i+1}}{(i+1)!}, \quad i = 0(1)m-1, \end{aligned} \quad (2.22)$$

where $\hat{u}_{m,j}(x)$ and $\hat{u}_{i,n}(y)$ are any m -fold and n -fold integrals of the unit function, respectively. We now know that elements of the resulting classes of monosplines give rise to cubature rules (2.16), (2.17) that are evidently exact if $f(x, y) \in \pi_{m-1, n-1}$.

Conversely, let (2.16) with $w(x, y) \equiv 1$ be a cubature rule for which $E(f) = 0$ if $f \in \pi_{m-1, n-1}$. Then, by a two-dimensional analog of Peano's theorem (see [4]), the remainder $E(f)$ may be written in the form (2.17) where the kernels are monosplines of the form (2.5), (2.22).

We close this section by introducing for later use the classes $K_{m,n}(\Delta_{MN}) = \{K_{m,n}(x, y)\}$, $K_{m,j}(\Delta_M) = \{K_{m,j}(x)\}$ and $K_{i,n}(\Delta_N) = \{K_{i,n}(y)\}$ of monosplines of order (m, n) , m , and n , respectively, i.e. of the form (2.5), subject to conditions (2.12) and (2.13).

3. STATEMENT OF THE PROBLEM

Let us be given the integer subsets:

$$\begin{cases} I_{p\nu} \subset \{0, 1, \dots, m-1\}, & p = 1, 2; \quad \nu = 0(1)N+1 \\ J_{q\mu} \subset \{0, 1, \dots, n-1\}, & q = 1, 2; \quad \mu = 0(1)M+1, \end{cases} \quad (3.1)$$

where one or more of these subsets may be void. Further, for $p, q = 1, 2$ we define the subsets:

$$\tilde{I}_{pq} = \begin{cases} I_{1,0} \times J_{1,0}, & p = q = 1 \\ I_{2,0} \times J_{1,M+1}, & p = 2, q = 1 \\ I_{1,N+1} \times J_{2,0}, & p = 1, q = 2 \\ I_{2,N+1} \times J_{2,M+1}, & p = q = 2. \end{cases} \quad (3.2)$$

There are two parts to the problem that we wish to discuss:

A) Determine conditions on the subsets (3.1) and (3.2), but independent of the weight function $w(x,y)$, under which the general cubature formula (2.16) assumes the special form:

$$\begin{aligned} \iint_R w(x,y)f(x,y) dx dy &= \sum_{\mu=1}^M \sum_{\nu=1}^N C_{\mu\nu} f(x_\mu, y_\nu) + \sum_{q=1}^2 \sum_{\mu=1}^M \sum_{j \in J_{q\mu}} A_{j\mu}^q f^{(0,j)}(x_\mu, b_q) + \\ &+ \sum_{p=1}^2 \sum_{\nu=1}^N \sum_{i \in I_{p\nu}} B_{i\nu}^p f^{(i,0)}(a_p, y_\nu) + \\ &+ \sum_{p,q=1}^2 \sum_{(i,j) \in \tilde{I}_{pq}} D_{ij}^{pq} f^{(i,j)}(a_p, b_q) + E(f), \end{aligned} \quad (3.3)$$

where $E(f)$ is given by (2.17) above.

B) Assuming the reduction (3.3) possible, we wish to determine among all cubature formulas of the form (3.3) that one which renders the expression

$$\begin{aligned} \Omega_{m,n} &= \iint_R [K_{m,n}(x,y)]^2 dx dy + \sum_{j=0}^{n-1} \int_{a_1}^{a_2} [K_{m,j}(x)]^2 dx + \\ &+ \sum_{i=0}^{m-1} \int_{b_1}^{b_2} [K_{i,n}(y)]^2 dy \end{aligned} \quad (3.4)$$

a minimum. Following the terminology employed in the one-dimensional case, we will call the resulting formula the best cubature rule (in the sense of Sard [5]) relative to the subsets (3.1), (3.2).

In other words, we seek elements of the classes $K_{m,n}(\Delta_{MN})$, $K_{m,j}(\Delta_M)$, and $K_{i,n}(\Delta_N)$ that satisfy the boundary conditions:

$$\begin{cases} A_{j\mu}^q = 0, & j \notin J_{q\mu}, & \mu = 1(1)M, & q = 1, 2 \\ B_{i\nu}^p = 0, & i \notin I_{p\nu}, & \nu = 1(1)N, & p = 1, 2 \\ D_{ij}^{pq} = 0, & (i, j) \notin \tilde{I}_{pq}, & p, q = 1, 2 \end{cases} \quad (3.5)$$

and solve the problem:

$$\Omega_{m,n} = \text{minimum} . \quad (3.6)$$

4. SOME PRELIMINARY RESULTS

It is now well known that quadrature formulas are intimately related to spline interpolation; see, for example, [2] and [8]. The purpose of this section is to present a certain result on two-dimensional spline interpolation which, as we shall see in Section 5, is intimately related to cubature formulas.

We consider the interpolation problem (IP):

$$\left\{ \begin{array}{l} S(x_\mu, y_\nu) = \xi_{\mu\nu}, \quad \mu = 1(1)M, \quad \nu = 1(1)N \\ S^{(i,0)}(a_p, y_\nu) = \eta_{i\nu}^p, \quad i \in I_{p\nu}, \quad \nu = 1(1)N, \quad p = 1, 2 \\ S^{(0,j)}(x_\mu, b_q) = \zeta_{j\mu}^q, \quad j \in J_{q\mu}, \quad \mu = 1(1)M, \quad q = 1, 2 \\ S^{(i,j)}(a_p, b_q) = \xi_{ij}^{pq}, \quad (i, j) \in \tilde{I}_{pq}, \quad p, q = 1, 2. \end{array} \right. \quad (4.1)$$

DEFINITION 1

The IP (4.1) is said to be (m, n) -poised provided that if $P(x, y) \in \pi_{m-1, n-1}$ satisfies the corresponding homogeneous IP:

$$\left\{ \begin{array}{l} P(x_\mu, y_\nu) = 0, \quad \mu = 1(1)M, \quad \nu = 1(1)N \\ P^{(i,0)}(a_p, y_\nu) = 0, \quad \nu = 1(1)N, \quad i \in I_{p\nu}, \quad p = 1, 2 \\ P^{(0,j)}(x_\mu, b_q) = 0, \quad \mu = 1(1)M, \quad j \in J_{q\mu}, \quad q = 1, 2 \\ P^{(i,j)}(a_p, b_q) = 0, \quad (i, j) \in \tilde{I}_{pq}, \quad p, q = 1, 2, \end{array} \right. \quad (4.2)$$

then $P(x, y) \equiv 0$.

When does an IP possess the property of definition 1? In order to answer this question, we introduce, for $p, q = 1, 2$, the sets:

$$\left\{ \begin{array}{l} I_p^1 = \bigcap_{\nu=0}^{N+1} I_{p\nu} \\ J_q^1 = \bigcap_{\mu=0}^{M+1} J_{q\mu} \\ I_p^2 = \bigcup_{\nu=0}^{N+1} I_{p\nu} \\ J_q^2 = \bigcup_{\mu=0}^{M+1} J_{q\mu} \end{array} \right. \quad (4.3a)$$

$$\begin{cases} I_{pr}^t = I_p^t \cap \{0, 1, \dots, r\}, & r = 0(1)m-1; \quad t = 1, 2 \\ J_{qs}^t = J_q^t \cap \{0, 1, \dots, s\}, & s = 0(1)n-1; \quad t = 1, 2, \end{cases} \quad (4.3b)$$

and in terms of these are define for $t = 1, 2$ the numbers:

$$\begin{cases} M_r^t = M + |I_{1r}^t| + |I_{2r}^t|, & r = 0(1)m-1 \\ N_s^t = N + |J_{1s}^t| + |J_{2s}^t|, & s = 0(1)n-1, \end{cases} \quad (4.4)$$

where $|X|$ denotes the number of elements in the set X . We shall also need to consider, for each t , the one-dimensional quasi-Hermite problems:

$$\begin{cases} P(x_\mu) = c_\mu, & \mu = 1(1)M \\ P^{(i)}(a_p) = c_{ip}, & i \in I_p^t, \end{cases} \quad (4.5a)$$

$$\begin{cases} Q(y_\nu) = d_\nu, & \nu = 1(1)N \\ Q^{(j)}(b_q) = d_{jq}, & j \in J_q^t, \end{cases} \quad (4.5b)$$

and we shall call, together with Schoenberg [10], the inequalities:

$$M_r^t \geq r + 1, \quad r = 0(1)m-1 \quad (4.6a)$$

$$N_s^t \geq s + 1, \quad s = 0(1)n-1. \quad (4.6b)$$

Polya's conditions

For the first result of this section, we need to introduce the following concept (see [11]).

DEFINITION 2

The IP (4.5a) is said to be m -poised if $P(x) \in \pi_{m-1}$ satisfies the corresponding homogeneous system, then $P(x) \equiv 0$.

Likewise, (4.5b) is said to be n -poised if $Q(y) \in \pi_{n-1}$ satisfies the corresponding homogeneous system, then $Q(y) \equiv 0$.

The following result is known [11; Theorem 4].

LEMMA 1

The quasi-Hermite problem (4.5a) is m-poised if and only if Polya's conditions (4.6a) are satisfied. Similarly, the system (4.5b) is n-poised if and only if inequalities (4.6b) hold.

LEMMA 2

Let $P(x,y) \in \pi_{m-1,n-1}$ satisfy the homogeneous system (4.2). Then, if Polya's conditions (4.6a) for $t = 1$ hold, the functions:

$$\begin{cases} \tilde{\sigma}_v(x) = P(x, y_v) , & v = 1(1)N \\ \sigma_{jq}(x) = P^{(0,j)}(x, b_q) , & j \in J_q^1 , \quad q = 1, 2 \end{cases} \quad (4.7a)$$

vanish identically.

Similarly, if the inequalities (4.6b) for $t = 1$ are satisfied, then the functions:

$$\begin{cases} \tilde{\rho}_\mu(y) = P(x_\mu, y) , & \mu = 1(1)M \\ \rho_{ip}(y) = P^{(i,0)}(a_p, y) , & i \in I_p^1 , \quad p = 1, 2 \end{cases} \quad (4.7b)$$

vanish identically.

PROOF

Since the proof is similar for all functions, it suffices to give it for $\sigma_{jq}(x)$ only.

First, we observe that $\sigma_{jq}(x) \in \pi_{m-1}$. Secondly, since $I_p^1 \in I_{pv}$, $J_q^1 \in J_{q\mu}$ for $v = 1(1)N$, $\mu = 1(1)M$, respectively; and since $I_p^1 \times J_q^1 \in I_{pq}$, the hypothesis of the lemma give:

$$\begin{cases} \sigma_{jq}(x_\mu) = 0 , & \mu = 1(1)M \\ \sigma_{jq}^{(i)}(a_p) = 0 , & i \in I_p^1 , \quad p = 1, 2 \end{cases} \quad (4.8)$$

which defines a quasi-Hermite system of the form (4.5a). However, if (4.6a) for $t = 1$ holds, then by lemma 1 (4.8) is m-poised and on invoking definition 2 we conclude that $\sigma_{jq}(x) \equiv 0$.

Q.E.D.

COROLLARY 1

Let $P(x,y)$ be as in lemma 2. Let Polya's conditions (4.6a) and (4.6b) for $t = 1$ be satisfied. It then follows that $P(x,y) \equiv 0$.

PROOF

Let x be a fixed point of the interval $[a_1, a_2]$ and define the function:

$$\lambda_x(y) = P(x, y) .$$

Now, by (4.6a) for $t = 1$ and the first part of lemma 2, we obtain:

$$\lambda_x(y_v) = 0 , \quad v = 1(1)N$$

$$\lambda_x^{(j)}(b_q) = 0 , \quad j \in J_q^1 , \quad q = 1, 2$$

which defines a quasi-Hermite system of type (4.5b), and the result of the corollary follows by (4.6b) for $t = 1$, lemma 1 and since $\lambda_x(y) \in \pi_{n-1}$.

Q.E.D.

THEOREM 1

In order that the IP (4.1) be (m, n) -poised, it is necessary that Polya's conditions (4.6) for $t = 2$ be satisfied. Moreover, if Polya's conditions for $t = 1$ hold then the IP (4.1) is (m, n) -poised.

PROOF

To deal with the first part, let us suppose that $M_r^2 < r + 1$, for some r . Lemma 1 tells us that the quasi-Hermite problem (4.5a) for $t = 2$ is not m -poised. Thus, there is a $P(x) \in \pi_{m-1}$ [indeed, it is sufficient to take $P(x) \in \pi_r$] satisfying the conditions:

$$P(x_\mu) = 0 , \quad \mu = 1(1)M$$

$$P^{(i)}(a_p) = 0 , \quad i \in I_p^2 , \quad p = 1, 2$$

$$P(x) \neq 0 .$$

Now on taking $P(x, y) = P(x)Q(y)$, $0 \neq Q(y) \in \pi_{n-1}$, and in view of the definition of I_p^2 , we see that $P(x, y) \in \pi_{m-1, n-1}$, satisfies the homogeneous system (4.2) and moreover $P(x, y) \neq 0$. This contradicts the assumption that the IP (4.1) is (m, n) -poised. Similarly, we may deal with the case $N_s^2 < s + 1$, for some s .

The sufficiency part of the theorem is a restatement of corollary 1.

Q.E.D.

For our next basic result, we introduce the complements of the subsets (3.1) and (3.2), viz:

$$\begin{cases} I_{p\nu}^c = \{0, 1, \dots, m-1\} \setminus I_{p\nu}, & \nu = 0(1)N+1 \\ J_{q\mu}^c = \{0, 1, \dots, n-1\} \setminus J_{q\mu}, & \mu = 0(1)M+1 \\ \tilde{I}_{pq}^c = \{0, 1, \dots, m-1\} \times \{0, 1, \dots, n-1\} \setminus \tilde{I}_{pq}, & p, q = 1, 2. \end{cases} \quad (4.9)$$

We observe that:

$$\begin{cases} |I_{p\nu}^c| + |I_{p\nu}| = m \\ |J_{q\mu}^c| + |J_{q\mu}| = n \\ |\tilde{I}_{pq}^c| + |\tilde{I}_{pq}| = mn. \end{cases} \quad (4.10)$$

Now, given a sufficiently differentiable function $g(x,y)$, we associate with it the following functions:

$$g_j(x, b_q) = \begin{cases} g^{(2m-1, 2n-j-1)}(x, b_1), & q = 1 \\ g^{(2m-1, 2n-j-1)}(x, b_2) + (-1)^{n-j-1} g^{(2m-1, j)}(x, b_2), & q = 2 \end{cases} \quad (4.11)$$

$$g_i(a_p, y) = \begin{cases} g^{(2m-i-1, 2n-1)}(a_1, y), & p = 1 \\ g^{(2m-i-1, 2n-1)}(a_2, y) + (-1)^{m-i-1} g^{(i, 2n-1)}(a_2, y), & p = 2 \end{cases} \quad (4.12)$$

$$g_{ij}(a_p, b_q) = \begin{cases} g^{(2m-i-1, 2n-j-1)}(a_1, b_1), & p = q = 1 \\ g^{(2m-i-1, 2n-j-1)}(a_1, b_2) + (-1)^{n-j-1} g^{(2m-i-1, j)}(a_1, b_2), & p = 1, \quad q = 2 \\ g^{(2m-i-1, 2n-j-1)}(a_2, b_1) + (-1)^{m-i-1} g^{(i, 2n-j-1)}(a_2, b_1), & p = 2, \quad q = 1 \\ g^{(2m-i-1, 2n-j-1)}(a_2, b_2) + (-1)^{n-j-1} g^{(2m-i-1, j)}(a_2, b_2) + \\ + (-1)^{m-i-1} g^{(i, 2n-j-1)}(a_2, b_2), & p = q = 2. \end{cases} \quad (4.13)$$

THEOREM 2

If the IP (4.1) is (m,n) -poised, then there exists a unique spline function $S(x,y) \in S_{2m-1,2n-1}(\Delta_{MN})$ satisfying the following interpolatory conditions:

$$S(x_\mu, y_\nu) = z_{\mu\nu}, \quad \mu = 1(1)M, \quad \nu = 1(1)N \quad (4.14)$$

$$\left. \begin{aligned} S^{(i,0)}(a_p, y_\nu) &= \xi_{p\nu}^i, \quad i \in I_{p\nu} \\ S_i(a_p, y) \Big|_{y=y_\nu-0}^{y_\nu+0} &= \tilde{\xi}_{p\nu}^i, \quad i \in I_{p\nu}^c \end{aligned} \right\} \nu = 1(1)N, \quad p = 1, 2 \quad (4.15)$$

$$\left. \begin{aligned} S^{(0,j)}(x_\mu, b_q) &= \eta_{q\mu}^j, \quad j \in J_{q\mu} \\ S_j(x, b_q) \Big|_{x=x_\mu-0}^{x_\mu+0} &= \tilde{\eta}_{q\mu}^j, \quad j \in J_{q\mu}^c \end{aligned} \right\} \mu = 1(1)M, \quad q = 1, 2 \quad (4.16)$$

$$\left. \begin{aligned} S^{(i,j)}(a_p, b_q) &= \zeta_{pq}^{ij}, \quad (i,j) \in I_{pq} \\ S_{ij}(a_p, b_q) &= \tilde{\zeta}_{pq}^{ij}, \quad (i,j) \in I_{pq}^c \end{aligned} \right\} p, q = 1, 2, \quad (4.17)$$

where the right-hand sides have arbitrarily prescribed values.

PROOF

The basic idea behind our method of proof is that due to Greville [2].

First, in view of (2.2), we observe that the spline function $S(x,y)$ depends on $(2m+M)(2n+N)$ linear parameters and that, in view of (4.10), the number of equations (4.14)-(4.17) is indeed $(2m+M)(2n+N)$.

Next, let $\tilde{S}(x,y) \in S_{2m-1,2n-1}(\Delta_{MN})$ satisfy the homogeneous system corresponding to the system (4.14)-(4.17), i.e. with zero right-hand sides. Now, successive integration by parts yields, after some simple manipulations, the following relations:

$$\begin{aligned}
\Gamma_{m,n} &= \iint_R [\tilde{S}^{(m,n)}(x,y)]^2 dx dy = \\
&= \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} (-1)^{m+n-i-j} [\tilde{S}^{(2m-i-1, 2n-j-1)}(x,y) \tilde{S}^{(i,j)}(x,y)] \Big|_{x=a_1}^{a_2} \Big|_{y=b_1}^{b_2} + \\
&+ \sum_{\mu=1}^M \sum_{j=0}^{n-1} (-1)^{m+n-j-1} [\tilde{S}^{(2m-1, 2n-j-1)}(x,y) \tilde{S}^{(0,j)}(x,y)] \Big|_{x=x_\mu-0}^{x_\mu+0} \Big|_{y=b_1}^{b_2} + \\
&+ \sum_{v=1}^N \sum_{i=0}^{m-1} (-1)^{m+n-i-1} [\tilde{S}^{(2m-i-1, 2n-1)}(x,y) \tilde{S}^{(i,0)}(x,y)] \Big|_{x=a_1}^{a_2} \Big|_{y=y_v-0}^{y_v+0} + \\
&+ (-1)^{m+n} \sum_{\mu=1}^M \sum_{v=1}^N [\tilde{S}^{(2m-1, 2n-1)}(x,y) \tilde{S}(x,y)] \Big|_{x=x_\mu-0}^{x_\mu+0} \Big|_{y=y_v-0}^{y_v+0}. \quad (4.18)
\end{aligned}$$

$$\begin{aligned}
\Gamma_{m,j} &= \int_{a_1}^{a_2} [\tilde{S}^{(m,j)}(x,b_2)]^2 dx = \sum_{i=0}^{m-1} (-1)^{m-i-1} [\tilde{S}^{(2m-i-1,j)}(x,b_2) \tilde{S}^{(i,j)}(x,b_2)] \Big|_{x=a_1}^{a_2} + \\
&+ (-1)^m \sum_{\mu=1}^M [\tilde{S}^{(2m-1,j)}(x,b_2) \tilde{S}^{(0,j)}(x,b_2)] \Big|_{x=x_\mu-0}^{x_\mu+0}, \quad j = 0(1)n-1, \quad (4.19)
\end{aligned}$$

$$\begin{aligned}
\Gamma_{i,n} &= \int_{b_1}^{b_2} [\tilde{S}^{(i,n)}(a_2,y)]^2 dy = \sum_{j=0}^{n-1} (-1)^{n-j-1} [\tilde{S}^{(i, 2n-j-1)}(a_2,y) \tilde{S}^{(i,j)}(a_2,y)] \Big|_{y=b_1}^{b_2} + \\
&+ (-1)^n \sum_{v=1}^N [\tilde{S}^{(i, 2n-1)}(a_2,y) \tilde{S}^{(i,0)}(a_2,y)] \Big|_{y=y_v-0}^{y_v+0}, \quad i = 0(1)m-1. \quad (4.20)
\end{aligned}$$

If we now remove from each of the expressions (4.18)-(4.20) those terms that are zero in accordance with (4.14)-(4.17), we find:

$$\begin{aligned}
\Gamma_{m,n} + \sum_{j=0}^{n-1} \Gamma_{m,j} + \sum_{i=0}^{m-1} \Gamma_{i,n} &= \sum_{\substack{p,q=1 \\ p \neq q=1}}^2 \sum_{(i,j) \in \tilde{I}_{pq}^c} (-1)^{m+n-i-j+p+q} \tilde{S}_{ij}(a_p, b_q) \tilde{S}^{(i,j)}(a_p, b_q) + \\
&+ \sum_{\mu=1}^M \sum_{j \in J_2^c} (-1)^{m+n-j-1} [\tilde{S}_j(x, b_2)] \Big|_{x=x_\mu-0}^{x_\mu+0} \tilde{S}^{(0,j)}(x_\mu, b_2) + \\
&+ \sum_{v=1}^N \sum_{i \in I_2^c} (-1)^{m+n-i-1} [\tilde{S}_i(a_2, y)] \Big|_{y=y_v-0}^{y_v+0} \tilde{S}^{(i,0)}(a_2, y_v),
\end{aligned}$$

where $\tilde{S}_{i,j}(a_p, b_q)$, $\tilde{S}_j(x, b_2)$ and $\tilde{S}_i(a_2, y)$ are defined by (4.11)-(4.13), respectively; and by the second relation in each of relations (4.15)-(4.17) we obtain:

$$\Gamma_{m,n} + \sum_{j=0}^{n-1} \Gamma_{m,j} + \sum_{i=0}^{m-1} \Gamma_{i,n} = 0$$

which implies

$$\Gamma_{m,n} = \Gamma_{m,j} = \Gamma_{i,n} = 0, \quad j = 0(1)n-1, \quad i = 0(1)m-1,$$

since each is non-negative. However, $\Gamma_{m,n} = 0$ implies that $\tilde{S}(x,y)$ belongs to either (i) $S_{2m-1}(\Delta_M) \otimes \pi_{n-1}$, or (ii) $\pi_{m-1} \otimes S_{2n-1}(\Delta_N)$, or (iii) $\pi_{m-1, n-1}$. Moreover, the first and second possibilities are excluded by the fact that $\Gamma_{m,j} = 0$ for $j = 0(1)n-1$ and $\Gamma_{i,n} = 0$ for $i = 0(1)m-1$, respectively, since a polynomial of degree k can have at most k zeros. We therefore have:

$$\tilde{S}(x,y) \in \pi_{m-1, n-1}. \quad (4.21)$$

In order to complete the proof of the theorem, we now observe that $\tilde{S}(x,y)$ satisfies, in particular, the homogeneous system corresponding to (4.1) which is assumed (m,n) -poised. This observation together with (4.21) implies that $\tilde{S}(x,y) \equiv 0$ by definition 1.

Q.E.D.

5. THE SOLUTION

In this section, we shall describe the solution to the problem -- in its two parts (A) and (B) -- stated in Section 3. The reader will probably observe that our main result, theorem 3 below, extends in a natural way Schoenberg's main result in [12, p. 8].

THEOREM 3

i) In order for cubature rules of type (3.3) to exist, for any weight function $w(x,y) \in L(R)$, it is necessary and sufficient that the IP (4.1) be (m,n) -poised.

ii) If this condition is satisfied and if

$$W(x,y) = w^{(-2m,-2n)}(x,y) \quad (5.1)$$

denotes an arbitrary $(2m,2n)$ -fold integral of $w(x,y)$, then there is a unique monospline

$$H(x,y) = W(x,y) - S(x,y) \quad (5.2)$$

of order $(2m,2n)$, with $S(x,y) \in S_{2m-1,2n-1}(\Delta_{MN})$, and satisfying the conditions

$$H(x_\mu, y_\nu) = 0, \quad \mu = 1(1)M, \quad \nu = 1(1)N \quad (5.3)$$

$$\left. \begin{aligned} H^{(i,0)}(a_p, y_\nu) &= 0, & i \in I_{p\nu} \\ H_i(a_p, y) \Big|_{y=y_\nu-0}^{y_\nu+0} &= 0, & i \in I_{p\nu}^c \end{aligned} \right\} \nu = 1(1)N, \quad p = 1, 2 \quad (5.4)$$

$$\left. \begin{aligned} H^{(0,j)}(x_\mu, b_q) &= 0, & j \in J_{q\mu} \\ H_j(x, b_q) \Big|_{x=x_\mu-0}^{x_\mu+0} &= 0, & j \in J_{q\mu}^c \end{aligned} \right\} \mu = 1(1)M, \quad q = 1, 2 \quad (5.5)$$

$$\left. \begin{aligned} H^{(i,j)}(a_p, b_q) &= 0, & (i,j) \in \tilde{I}_{pq} \\ H_{ij}(a_p, b_q) &= 0, & (i,j) \in \tilde{I}_{pq}^c \end{aligned} \right\} p, q = 1, 2 \quad (5.6)$$

where $H_i(a_p, y)$, $H_j(x, b_q)$ and $H_{ij}(a_p, b_q)$ are defined by (4.12), (4.11) and (4.13), respectively.

iii) Choose $W(x,y)$ such that

$$\begin{cases} W_j^{(1,0)}(x, b_q) = 0, & j = 0(1)n-1, & q = 1, 2 \\ W_i^{(0,1)}(a_p, y) = 0, & i = 0(1)m-1, & p = 1, 2 \end{cases} \quad (5.7)$$

and define the monosplines

$$\begin{cases} K_{m,n}(x,y) = H^{(m,n)}(x,y) \\ K_{m,j}(x) = (-1)^{n-j} H^{(m,j)}(x, b_2), & j = 0(1)n-1 \\ K_{i,n}(y) = (-1)^{m-i} H^{(i,n)}(a_2, y), & i = 0(1)m-1 \end{cases} \quad (5.8)$$

of order (m,n) , m and n , respectively. Then these monosplines are, respectively, elements of $K_{m,n}(\Delta_{MN})$, $K_{m,j}(\Delta_M)$, $K_{i,n}(\Delta_N)$, and satisfy the boundary conditions

$$\left. \begin{aligned} & K_{m,n}^{(m-1, n-j-1)}(x, b_1) \Big|_{x=x_\mu^{-0}}^{x_\mu^{+0}} = 0, & j \in J_{1\mu}^c \\ & \left[K_{m,n}^{(m-1, n-j-1)}(x, b_2) - K_{m,j}^{(m-1)}(x) \right] \Big|_{x=x_\mu^{-0}}^{x_\mu^{+0}} = 0, & j \in J_{2\mu}^c \end{aligned} \right\} \mu = 1(1)M \quad (5.9)$$

$$\left. \begin{aligned} & K_{m,n}^{(m-i-1, n-1)}(a_1, y) \Big|_{y=y_\nu^{-0}}^{y_\nu^{+0}} = 0, & j \in I_{1\nu}^c \\ & \left[K_{m,n}^{(m-i-1, n-1)}(a_2, y) - K_{i,n}^{(n-1)}(y) \right] \Big|_{y=y_\nu^{-0}}^{y_\nu^{+0}} = 0, & i \in I_{2\nu}^c \end{aligned} \right\} \nu = 1(1)N \quad (5.10)$$

$$\left\{ \begin{array}{l} K_{m,n}^{(m-i-1,n-j-1)}(a_1,b_1) = 0, \quad (i,j) \in \tilde{I}_{11}^c \\ K_{m,n}^{(m-i-1,n-j-1)}(a_1,b_2) - K_{m,j}^{(m-i-1)}(a_1) = 0, \quad (i,j) \in \tilde{I}_{12}^c \\ K_{m,n}^{(m-i-1,n-j-1)}(a_2,b_1) - K_{i,n}^{(n-j-1)}(b_1) = 0, \quad (i,j) \in \tilde{I}_{21}^c \\ K_{m,n}^{(m-i-1,n-j-1)}(a_2,b_2) - K_{m,j}^{(m-i-1)}(a_2) - K_{i,n}^{(n-j-1)}(b_2) = 0, \quad (i,j) \in \tilde{I}_{22}^c \end{array} \right. \quad (5.11)$$

Observe that conditions (5.7) can always be satisfied and moreover, they completely determine $W(x,y)$.

iv) The monosplines (5.8) are among all monosplines in $K_{m,n}(\Delta_{MN})$, $K_{m,j}(\Delta_M)$, $K_{i,n}(\Delta_N)$, respectively, which satisfy the boundary conditions (5.9), (5.10) and (5.11), the unique monosplines that solve the problem (3.6), i.e. minimize the quantity

$$\Omega_{m,n} = \iint_R [K_{m,n}(x,y)]^2 dx dy + \sum_{j=0}^{n-1} \int_{a_1}^{a_2} [K_{m,j}(x)]^2 dx + \sum_{i=0}^{m-1} \int_{b_1}^{b_2} [K_{i,n}(y)]^2 dy. \quad (5.12)$$

Accordingly, and in view of (5.9)-(5.11), (2.18)-(2.21) and (3.5), the general cubature formula (2.16), (2.17), corresponding to the monosplines (5.8), reduces to the type (3.3), (2.17) and it is the best among them.

v) In terms of $H(x,y)$, the minimum value of $\Omega_{m,n}$ is given by

$$\Omega_{m,n} = (-1)^{m+n} \iint_R w(x,y) H(x,y) dx dy. \quad (5.13)$$

PROOF

We shall seal with (i) last.

ii) This part is an immediate consequence of theorem 2. For, theorem 2 tells us that there is a unique spline function $S(x,y) \in S_{2m-1,2n-1}(\Delta_{MN})$ satisfying the system (4.14) - (4.17) where

$$\begin{aligned}
 z_{\mu\nu} &= W(x_\mu, y_\nu) \\
 \xi_{p\nu}^i &= W^{(i,0)}(a_p, y_\nu) , \quad i \in I_{p\nu} \\
 \tilde{\xi}_{p\nu}^i &= 0 , \quad i \in I_{p\nu}^c \\
 \eta_{q\mu}^j &= W^{(0,j)}(x_\mu, b_q) , \quad j \in J_{q\mu} \\
 \tilde{\eta}_{q\mu}^j &= 0 , \quad j \in J_{q\mu}^c \\
 \zeta_{pq}^{ij} &= W^{(i,j)}(a_p, b_q) , \quad (i,j) \in \tilde{I}_{pq} \\
 \tilde{\zeta}_{pq}^{ij} &= W_{ij}(a_p, b_q) , \quad (i,j) \in \tilde{I}_{pq}^c
 \end{aligned}$$

and on setting

$$H(x,y) = W(x,y) - S(x,y)$$

we see that is is a unique monospline of order $(2m, 2n)$ of the type required by the theorem.

iii) (5.1) and (5.2) give

$$\begin{aligned}
 H^{(m,n)}(x,y) &= w^{(-m,-n)}(x,y) + S^*(x,y) , \\
 H^{(m,j)}(x,b_2) &= w^{(-m,-2n+j)}(x,b_2) + S_{m,j}^*(x) , \quad j = 0(1)n-1 , \\
 H^{(i,n)}(a_2,y) &= w^{(-2m+i,-n)}(a_2,y) + S_{i,n}^*(y) , \quad i = 0(1)m-1
 \end{aligned}$$

where $S^*(x,y) \in S_{m-1,n-1}(\Delta_{MN})$, $S_{m,j}^*(x) \in S_{m-1}(\Delta_M)$ and $S_{i,n}^*(y) \in S_{n-1}(\Delta_N)$. If we now observe that, in view of (5.7), (5.1), (4.11) and (4.12), we have

$$\left. \begin{aligned}
 w^{(0,-j-1)}(x,b_1) &= 0 \\
 w^{(0,-2n+j)}(x,b_2) &= (-1)^{n-j} w^{(0,-j-1)}(x,b_2)
 \end{aligned} \right\} j = 0(1)n-1$$

$$\left. \begin{aligned}
 w^{(-i-1,0)}(a_2,y) &= 0 \\
 w^{(-2m+i,0)}(a_2,y) &= (-1)^{m-i} w^{(-i-1,0)}(a_2,y)
 \end{aligned} \right\} i = 0(1)m-1$$

then it follows that the three functions (5.8) are indeed members of the classes of monosplines $K_{m,n}(\Delta_{MN})$, $K_{m,j}(\Delta_M)$ and $K_{i,n}(\Delta_N)$, respectively.

In order to establish relations (5.9)-(5.11), it is sufficient to consider typical cases. First, we have

$$\begin{aligned} K_{m,n}^{(m-1,n-j-1)}(x,b_2) &= H^{(2m-1,2n-j-1)}(x,b_1) = H_j(x,b_1) \\ K_{m,n}^{(m-1,n-j-1)}(x,b_2) - K_{m,j}^{(m-1)}(x) &= H^{(2m-1,2n-j-1)}(x,b_2) - (-1)^{n-j} H^{(2m-1,j)}(x,b_2) \\ &= H_j(x,b_2) \end{aligned}$$

and it is straightforward now to see that (5.9) follows from (5.5). Similarly, (5.10) and (5.11) are consequences of (5.4) and (5.6), respectively.

iv) Let us be given monosplines $\tilde{K}_{m,n}(x,y) \in K_{m,n}(\Delta_{MN})$, $\tilde{K}_{m,j}(x) \in K_{m,j}(\Delta_M)$ for $j = 0(1)n-1$, and $\tilde{K}_{i,n}(y) \in K_{i,n}(\Delta_N)$ for $i = 0(1)m-1$, which satisfy the boundary conditions (5.9), (5.10) and (5.11). Let

$$\tilde{H}(x,y) = \tilde{W}(x,y) + \tilde{S}(x,y)$$

denote an (m,n) -fold integral of $\tilde{K}(x,y)$ such that

$$\tilde{K}_{m,n}(x,y) = \tilde{H}^{(m,n)}(x,y) .$$

Similarly, for each i and for each j , pick an m -fold and an n -fold integral

$$\tilde{H}_{m,j}(x) = \tilde{V}_{m,j}(x) + \tilde{S}_{m,j}(x) , \quad \text{for all } j$$

$$\tilde{H}_{i,n}(y) = \tilde{V}_{i,n}(y) + \tilde{S}_{i,n}(y) , \quad \text{for all } i$$

such that

$$\tilde{K}_{m,j}(x) = (-1)^{n-j} \tilde{H}_{m,j}^{(m)}(x) ,$$

$$\tilde{K}_{i,n}(y) = (-1)^{m-i} \tilde{H}_{i,n}^{(n)}(y) ,$$

respectively. Let

$$G(x,y) = \tilde{H}(x,y) - H(x,y)$$

$$G_{m,j}(x) = \tilde{H}_{m,j}(x) - H^{(0,j)}(x,b_2) , \quad \text{for all } j$$

$$G_{i,n}(y) = \tilde{H}_{i,n}(y) - H^{(i,0)}(a_2,y) , \quad \text{for all } i$$

Then, it is clear that

$$G^{(m,n)}(x,y) = \tilde{K}_{m,n}(x,y) - K_{m,n}(x,y)$$

$$(-1)^{n-j} G_{m,j}^{(m)}(x) = \tilde{K}_{m,j}(x) - K_{m,j}(x)$$

$$(-1)^{m-i} G_{i,n}^{(n)}(y) = \tilde{K}_{i,n}(y) - K_{i,n}(y)$$

and since both sets of monosplines satisfy the boundary conditions (5.9)-(5.11), we must have

$$\left. \begin{aligned} & G^{(2m-1, 2n-j-1)}(x, b_1) \Big|_{x=x_\mu-0}^{x_\mu+0} = 0 , \quad j \in J_{1\mu}^c \\ & \left[G^{(2m-1, 2n-j-1)}(x, b_2) - (-1)^{n-j} G_{m,j}^{(2m-1)}(x) \right] \Big|_{x=x_\mu-0}^{x_\mu+0} = 0 , \quad j \in J_{2\mu}^c \end{aligned} \right\} \mu = 1(1)M \quad (5.14)$$

$$\left. \begin{aligned} & G^{(2m-i-1, 2n-1)}(a_1, y) \Big|_{y=y_\nu-0}^{y_\nu+0} = 0 , \quad i \in I_{1\nu}^c \\ & \left[G^{(2m-i-1, 2n-1)}(a_2, y) - (-1)^{m-i} G_{i,n}^{(2n-1)}(y) \right] \Big|_{y=y_\nu-0}^{y_\nu+0} = 0 , \quad i \in I_{2\nu}^c \end{aligned} \right\} \nu = 1(1)N \quad (5.15)$$

$$\left\{ \begin{aligned} & G^{(2m-i-1, 2n-j-1)}(a_1, b_1) = 0 , \quad (i,j) \in \tilde{I}_{11}^c \\ & G^{(2m-i-1, 2n-j-1)}(a_1, b_2) - (-1)^{n-j} G_{m,j}^{(2m-i-1)}(a_1) = 0 , \quad (i,j) \in \tilde{I}_{12}^c \\ & G^{(2m-i-1, 2n-j-1)}(a_2, b_1) - (-1)^{m-i} G_{i,n}^{(2n-j-1)}(b_1) = 0 , \quad (i,j) \in \tilde{I}_{21}^c \\ & G^{(2m-i-1, 2n-j-1)}(a_2, b_2) - (-1)^{n-j} G_{m,j}^{(2m-i-1)}(a_2) - (-1)^{m-i} G_{i,n}^{(2n-j-1)}(b_2) = 0 , \quad (i,j) \in \tilde{I}_{22}^c \end{aligned} \right. \quad (5.16)$$

We also have

$$\left\{ \begin{array}{l} \left[\tilde{K}_{m,n}(x,y) - K_{m,n}(x,y) \right] K_{m,n}(x,y) = G^{(m,n)}(x,y) H^{(m,n)}(x,y) \\ \left[\tilde{K}_{m,j}(x) - K_{m,j}(x) \right] K_{m,j}(x) = G_{m,j}^{(m)}(x) H^{(m,j)}(x, b_2) , \quad \text{for all } j \quad (5.17) \\ \left[\tilde{K}_{i,n}(y) - K_{i,n}(y) \right] K_{i,n}(y) = G_{i,n}^{(n)}(y) H^{(i,n)}(a_2, y) , \quad \text{for all } i . \end{array} \right.$$

Moreover, since $K_{m,n}(x,y)$ and $\tilde{K}_{m,n}(x,y)$ are both elements of $K_{m,n}(\Delta_{MN})$ we must have

$$G^{(2m,0)}(x,y) = G^{(0,2n)}(x,y) \equiv 0 , \quad (5.18)$$

and likewise we must have

$$G_{m,j}^{(2m)}(x) = G_{i,n}^{(2n)}(y) \equiv 0 , \quad \text{for all } i, j . \quad (5.19)$$

If we now write

$$\left\{ \begin{array}{l} \tilde{\Gamma}_{m,n} = \iint_R G^{(m,n)}(x,y) H^{(m,n)}(x,y) \, dx \, dy \\ \tilde{\Gamma}_{m,j} = \int_{a_1}^{a_2} G_{m,j}^{(m)}(x) H^{(m,j)}(x, b_2) \, dx \\ \tilde{\Gamma}_{i,n} = \int_{b_1}^{b_2} G_{i,n}^{(n)}(y) H^{(i,n)}(a_2, y) \, dy \end{array} \right. \quad (5.20)$$

and if we subsequently integrate by parts and derive expressions similar to (4.18)-(4.20), we obtain, in view of the boundary conditions (5.14)-(5.16) and the identities (5.18), (5.19), the result

$$\tilde{\Gamma}_{m,n} + \sum_{j=0}^{n-1} \tilde{\Gamma}_{m,j} + \sum_{i=0}^{m-1} \tilde{\Gamma}_{i,n} = 0 . \quad (5.21)$$

The important implication of (5.21) is that

$$\begin{aligned}
 & \iint_R \tilde{K}_{m,n}^2 dx dy + \sum_{j=0}^{n-1} \int_{a_1}^{a_2} \tilde{K}_{m,j}^2 dx + \sum_{i=0}^{m-1} \int_{b_1}^{b_2} \tilde{K}_{i,n}^2 dy = \\
 & = \iint_R K_{m,n}^2 dx dy + \sum_{j=0}^{n-1} \int_{a_1}^{a_2} K_{m,j}^2 dx + \sum_{i=0}^{m-1} \int_{b_1}^{b_2} K_{i,n}^2 dy + \\
 & + \iint_R (\tilde{K}_{m,n} - K_{m,n})^2 dx dy + \sum_{j=0}^{n-1} \int_{a_1}^{a_2} (\tilde{K}_{m,j} - K_{m,j})^2 dx + \\
 & + \sum_{i=0}^{m-1} \int_{b_1}^{b_2} (\tilde{K}_{i,n} - K_{i,n})^2 dy ,
 \end{aligned} \tag{5.22}$$

in view of (5.17), (5.20) and the identity

$$\tilde{T}^2 = T^2 + (\tilde{T} - T)^2 + 2T(\tilde{T} - T) .$$

Thus we have established (5.12) and hence part (iv) of the theorem.

v) Substitute $f(x,y) = H(x,y)$ in (3.3) and use (5.3)-(5.6) and the defining relations (5.8).

i) In view of (iv), we are only left with the necessity part to be proved. This is easily done by considering a polynomial $P(x,y) \in \pi_{m-1,n-1}$ satisfying the system (4.2). On substituting $f(x,y) = P(x,y)$ in (3.3) we obtain

$$\iint_R w(x,y)P(x,y) dx dy = 0$$

which is possible, for any $w(x,y) \in L(R)$, if and only if $P(x,y) \equiv 0$.

Remark

Although we have formulated our results for the space $C^{(m,n)}(R)$, but it is not difficult to see that they hold as well for Sard's wider space $B_{[m,n]}(a_2, b_2)$ (see [5], Section 4, for definition). Moreover, one may consider in the spirit of the paper the spaces $B_{[m,n]}(a_p, b_q)$ and $B_{m,n}(a_p, b_q)$.

6. AN OPTIMAL PROPERTY OF TWO-DIMENSIONAL
SPLINE INTERPOLATION

In this final section we wish to present a reformulation of theorems 2 and 3 which reveals what seems to be a new type of optimal property of two-dimensional spline interpolation.

For this purpose, we first introduce the space $\tilde{C}^{(2m,2n)}(R)$ for functions $f(x,y)$ with the following properties:

- i) $f \in C^{(m,n)}(R)$
- ii) $f^{(2m-1,j)}(x,b_q)$ is piecewise continuous in (a_1,a_2) for $j = 0(1)2n-1$,
 $q = 1, 2$
- iii) $f^{(i,2n-1)}(a_p,y)$ is piecewise continuous in (b_1,b_2) for $i = 0(1)2m-1$,
 $p = 1, 2$
- iv) $f^{(2m-i-1,j)}(a_p,b_q)$, $f^{(i,2n-j-1)}(a_p,b_q)$, $f^{(2m-i-1,2n-j-1)}(a_p,b_q)$ all exist
for $i = 0(1)m-1$, $j = 0(1)n-1$.

We continue to use the notation (4.11) - (4.13) and write

$$\begin{aligned} V_{m,n}(g) = & \iint_R \left[g^{(m,n)}(x,y) \right]^2 dx dy + \sum_{j=0}^{n-1} \int_{a_1}^{a_2} \left[g^{(m,j)}(x,b_2) \right]^2 dx + \\ & + \sum_{i=0}^{m-1} \int_{b_1}^{b_2} \left[g^{(i,n)}(a_2,y) \right]^2 dy \end{aligned} \quad (6.1)$$

THEOREM 4

Let the IP (4.1) be (m,n) -poised. Let $W(x,y) \in \tilde{C}^{(2m,2n)}(R)$.

a) There is a unique spline function $S(x,y) \in S_{2m-1,2n-1}(\Delta_{MN})$ satisfying the conditions

$$\left\{ \begin{aligned} S(s_\mu, y_\nu) &= W(x_\mu, y_\nu), \quad \mu = 1(1)M, \quad \nu = 1(1)N \\ S^{(i,0)}(a_p, y_\nu) &= W^{(i,0)}(a_p, y_\nu), \quad i \in I_{p\nu}, \quad \nu = 1(1)N, \quad p = 1, 2 \\ S^{(0,j)}(x_\mu, b_q) &= W^{(0,j)}(x_\mu, b_q), \quad j \in J_{q\mu}, \quad \mu = 1(1)M, \quad q = 1, 2 \\ S^{(i,j)}(a_p, b_q) &= W^{(i,j)}(a_p, b_q), \quad (i,j) \in \tilde{I}_{pq}, \quad p, q = 1, 2 \end{aligned} \right. \quad (6.2)$$

$$\left\{ \begin{array}{l} S_i(a_p, y) \Big|_{y=y_v-0}^{y_v+0} = W_i(a_p, y) \Big|_{y=y_v-0}^{y_v+0}, \quad i \in I_{pv}^c, \quad v = 1(1)N, \quad p = 1, 2 \\ S_j(x, b_q) \Big|_{x=x_\mu-0}^{x_\mu+0} = W_j(x, b_q) \Big|_{x=x_\mu-0}^{x_\mu+0}, \quad j \in J_{q\mu}^c, \quad \mu = 1(1)M, \quad q = 1, 2 \\ S_{ij}(a_p, b_q) = W_{ij}(a_p, b_q), \quad (i, j) \in \tilde{I}_{pq}^c, \quad p, q = 1, 2 \end{array} \right. \quad (6.3)$$

b) For any $\tilde{S}(x, y) \in S_{2m-1, 2n-1}(\Delta_{MN})$ and satisfying the boundary conditions (6.3), we have

$$V_{m,n}(W-S) \leq V_{m,n}(W-\tilde{S}) \quad (6.4)$$

with equality if and only if $\tilde{S} = S + P(x, y)$, $P(x, y) \in \pi_{m-1, n-1}$.

c) For any $f(x, y) \in \tilde{C}^{(2m, 2n)}(R)$ satisfying conditions (6.2), (6.3), we have

$$V_{m,n}(S) \leq V_{m,n}(f) \quad (6.5)$$

with equality if and only if $f(x, y) = S(x, y)$.

PROOF

a) is simply a restatement of theorem 2.

b) Set

$$H(x, y) = W(x, y) - S(x, y)$$

$$\tilde{H}(x, y) = W(x, y) - \tilde{S}(x, y)$$

and observe that

$$s(x, y) = \tilde{H}(x, y) - H(x, y) = S(x, y) - \tilde{S}(x, y)$$

is in $S_{2m-1, 2n-1}(\Delta_{MN})$ and satisfies conditions (6.3). We may now proceed as in the proof of part (iv) of theorem 3 and establish relation (6.4). Now, let $\tilde{S} = S + \hat{S}$ and suppose (6.4) is an equality. Then, it is not difficult to see that the spline function $\hat{S}$ must satisfy

$$V_{m,n}(\hat{S}) = 0$$

which, as shown in the proof of theorem 2, holds if and only if $\hat{S}(x, y) \in \pi_{m-1, n-1}$.

c) The idea of the proof is as in the proof of part (iv) of theorem 3. We define

$$F(x,y) = f(x,y) - S(x,y) \quad ;$$

note that it satisfies the homogeneous system corresponding to (6.2), (6.3); and establish

$$\begin{aligned} \iint_R S^{(m,n)} F^{(m,n)} \, dx \, dy + \sum_{j=0}^{n-1} \int_{a_1}^{a_2} S^{(m,j)}(x,b_2) F^{(m,j)}(x,b_2) \, dx + \\ + \sum_{i=0}^{m-1} \int_{b_1}^{b_2} S^{(i,n)}(a_2,y) F^{(i,n)}(a_2,y) \, dy = 0 \end{aligned}$$

which shows that

$$V_{m,n}(f) = V_{m,n}(S) + V_{m,n}(F) \quad .$$

Now, equality in (6.5) implies $F \in \pi_{m-1,n-1}$, hence $f = S$ since (4.1) is (m,n) -poised.

Q.E.D.

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