This crude approach to the concept of canonical coordinates is important to the practicing physicist, who often likes to work in a large box with periodic boundary conditions, but we obtain precise definition of the coordinate operator only in the $L \rightarrow \infty$ limit, where $\mathbf{q} \equiv \lim_{k \rightarrow 0} \boldsymbol{\sigma}(k)$ is equivalent to defining $\mathbf{q}$ as the generator of the unitary group of operators $\boldsymbol{\sigma}(k)$. However, while $\boldsymbol{\sigma}(k)$ is a bounded operator on the Hilbert space, $\mathbf{q}$ has a restricted domain. Consequently, it seems advisable to treat the operators $\boldsymbol{\sigma}(k)$ and $\mathbf{c}(k)$ as basic observables whether L is finite or infinite, and to recognize that $\mathbf{q}$ appears formally only as a limit operator or generator.

There is of course the alternate approach of Dirac, which is to generalize the concept of Hilbert space by reinterpreting the metric in terms of distributions.⁶ This enables one to include nonintegrable functions as members of the vector space, but it may also exclude functions of physical significance, such as eigenfunctions

⁶ P. A. M. Dirac, *Principles of Quantum Mechanics* (Oxford U. P., Oxford, England, 1958), p. 40.

of nonanalytic Hamiltonians.⁷ In this formalism one interprets Eq. (2) as a relationship between distributions, obtained by taking the $k \rightarrow 0$ limit of Eq. (5), i.e.,

$$(p-p') \frac{\partial \langle p | p' \rangle}{\partial p'} = \langle p | p' \rangle.$$

We hope that it is clear from the above remarks that coordinate operators do not play an essential role in quantum mechanics and that the theory of periodic systems can be formulated quite satisfactorily without them. Since they arise fundamentally as generators of continuous transformation groups, their occurrence is dictated by no deeper consideration than whether such transformation groups exist in the problem at hand.

We are indebted to Herman Munczek for several helpful discussions of these topics.

⁷ M. J. Lighthill, *Fourier Analysis and Generalized Functions* (Cambridge U. P., Cambridge, England, 1960), Chap. 2.

Studies in the Yang-Mills Field*

ALFRED C. T. WU†

Theoretical Study Division, CERN, Geneva, Switzerland‡
and

Department of Physics, University of Michigan, Ann Arbor, Michigan 48104

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By explicit calculation, the cancellation between longitudinal and scalar components is verified in the lowest-order scattering of massless Yang-Mills fields. The problem of stability of the subsidiary condition under a local gauge transformation is briefly discussed. The question is raised as to whether the (infinitesimal) non-Abelian gauge function should be quantized. A possible connection between the fictitious ghost of Feynman and DeWitt and the fictitious gauge quanta is mentioned.

THE peculiarities in the quantized treatment of massless spin-one particles are well known. One is faced with either a manifestly covariant formulation by the use of the artifice of the indefinite metric (such as in the Gupta-Bleuler method), or, alternatively, one chooses to work only with the transverse components (such as in Coulomb gauge) and hope to be able to justify the covariance afterwards. With the Gupta-Bleuler method, it is crucial in quantum electrodynamics to verify that the longitudinal and the scalar components always conspire to cancel each other exactly.

With the incorporation of local internal symmetry in the form of the massless non-Abelian isotopic gauge

field (the Yang-Mills field¹), it is by no means obvious that the cancellation between the scalar and longitudinal components will always take place.

The purpose of this note is twofold. (1) We verify by an explicit calculation that the above mentioned cancellation holds in the lowest (g^2) order of the $\mathbf{b}_\mu$ -field scattering. Explicit calculations for higher-order processes or when closed loops are involved lie outside the scope of the present paper. (2) We discuss the problem of stability of the subsidiary condition under a local gauge transformation.

I. CANCELLATION BETWEEN LONGITUDINAL AND SCALAR COMPONENTS

Consider the $\mathbf{b}_\mu$ - $\mathbf{b}_\mu$ scattering to order g^2 . The scattering amplitude $M(k)$ consists of contributions from a sum of four graphs. The first three are the one-particle-

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† John Simon Guggenheim Memorial Foundation Fellow.

‡ On leave of absence from University of Michigan, Ann Arbor, Mich. 48104.

¹ C. N. Yang and R. Mills, Phys. Rev. 96, 191 (1954). The notation of this paper is followed.

exchange graphs with trilinear self-couplings; the fourth is the so-called seagull graph with quadrilinear self-coupling. Let $k_{\lambda l}^{i_l}$ ($l=0, 1, 2, 3$) denote the momentum of the l th line having the polarization index λ_l and isospin index i_l . Consider $k_{\lambda_0} + k_{\lambda_1} \rightarrow k_{\lambda_2} + k_{\lambda_3}$. Without loss of generality, we can take the incoming lines (labeled 0 and 1) to be in the transverse states TT and discuss the following classes of polarization states for the outgoing particles (labeled 2 and 3): (a) TT ; (b) TL, TS ; (c) LL, LS, SL, SS , where T, L, S , stand for the transverse, longitudinal, and scalar components, respectively. There is nothing to be proved for class (a). For the remainder, a straightforward calculation shows that

$$k_{\lambda_2}^{i_2} e_{\lambda_0} e_{\lambda_1} e_{\lambda_2} M_{\lambda_0 \lambda_1 \lambda_2 \lambda_3}(k) = (e_{\lambda_2} \cdot k_{\lambda_2}) f(k; e_{\lambda_0}, e_{\lambda_1}), \quad (1)$$

where $f(k)$ is a rational function whose explicit form does not concern us here. Thus, for class (b), the choice $\lambda_2 = T$ makes the right-hand side of Eq. (1) vanish and the left-hand side gives the desired cancellation, namely,

$$|M_{TTTL}|^2 - |M_{TTTS}|^2 = 0. \quad (2)$$

Finally, for the class (c) states, we get from (1), after summing over the λ_2 states of $|M|^2$,

$$|M_{LL}|^2 + |M_{SS}|^2 - |M_{LS}|^2 - |M_{SL}|^2 = 0, \quad (3)$$

which is the desired cancellation.

II. STABILITY OF SUBSIDIARY CONDITION

We turn next to the stability problem of the subsidiary condition under a local gauge transformation in the quantization of the Yang-Mills field. The motivation is as follows: In recent years, the question of deriving Feynman-rule prescriptions for the massless Yang-Mills $\mathbf{b}_\mu$ field has attracted considerable interest.²⁻⁴ It has been asserted by several authors^{3,4} that, in addition to the usual trilinear and quadrilinear self-couplings among the $\mathbf{b}_\mu$ fields, there is, for graphs involving closed loops, an extra coupling of the form

$$\phi \cdot (\mathbf{b}_\mu \times \partial_\mu \phi), \quad (4)$$

where ϕ is a massless scalar field with isospin one. The arguments given by the authors in Ref. 3 are not, to say the least, transparent. We observe that an intuitive way to interpret the origin of such a coupling (4) is as a result of ensuring the stability of the subsidiary

condition $\partial_\mu B_\mu = 0$ under a local gauge transformation

$$B_\mu \rightarrow B'_\mu = S^{-1} B_\mu S + (i/\epsilon) S^{-1} \partial_\mu S, \quad (5)$$

$$\partial_\mu B'_\mu = 0. \quad (6)$$

As is well known, the implication of (6) under an *infinitesimal* gauge transformation S in the unquantized version was first pointed out by Yang and Mills in their original paper. Our starting point is their Eqs. (10) and (18), namely,

$$S = 1 - 2i\boldsymbol{\tau} \cdot \delta\boldsymbol{\omega}, \quad (7)$$

$$\mathbf{b}'_\mu = \mathbf{b}_\mu + 2\mathbf{b}_\mu \times \delta\boldsymbol{\omega} + (1/\epsilon) \partial_\mu \delta\boldsymbol{\omega} \quad (B_\mu = 2\boldsymbol{\tau} \cdot \mathbf{b}_\mu), \quad (8)$$

$$\partial^2 \delta\boldsymbol{\omega} = -2\epsilon \mathbf{b}_\mu \times \partial_\mu \delta\boldsymbol{\omega}. \quad (9)$$

Thus, in a consistent scheme of quantization of the $\mathbf{b}_\mu$ field, a new role for the infinitesimal gauge function $\delta\boldsymbol{\omega}$ seems to be indicated. The question is, in treating both sides of Eq. (8) on an equal footing, whether one has to quantize the gauge function $\delta\boldsymbol{\omega}$. As is well known, the corresponding Abelian gauge function in quantum electrodynamics never has to be quantized, since it is not coupled to the electromagnetic field. The coupling of $\delta\boldsymbol{\omega}$ with the $\mathbf{b}_\mu$ field makes $\delta\boldsymbol{\omega}$ look very much like a fictitious massless scalar field with isospin one. It would be very tempting to correlate $\delta\boldsymbol{\omega}$ in (9) with the ϕ field in (4).

As far as the *infinitesimal* non-Abelian gauge transformation is concerned, there is no apparent contradiction in entertaining the notion of a quantized gauge function $\delta\boldsymbol{\omega}$, with its intrinsic coupling to $\mathbf{b}_\mu$, apart from giving up the instinct associated with the freedom of making arbitrary choices of c -number gauge functions, which situations are so much more familiar in the case of Abelian gauge transformations. Even if the need to quantize the gauge function may not be particularly compelling, it might serve its purpose in pointing a way, albeit heuristically, to the origin of the mysterious coupling of the virtual ghost ϕ states. The existence of fictitious ghosts in the theory should be no more palatable than the notion of dealing with quantized gauge functions.

Note added in manuscript. The author's attention has been called to the existence of the work of Remler⁵ on Yang-Mills scattering amplitudes in a different context.

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⁵ E. A. Remler, Phys. Rev. **133**, B1267 (1964).

² C. N. Yang (private communication).

³ Feynman rules for the massless Yang-Mills field are discussed by the following authors: R. P. Feynman, Acta Phys. Polon. **24**, 697 (1963); B. S. DeWitt, Phys. Rev. **162**, 1195 (1967); **162**, 1239 (1967); L. D. Faddeev and V. N. Popov, Phys. Rev. Letters **25B**, 29 (1967); S. Mandelstam, Phys. Rev. **175**, 1580 (1968).

⁴ For a discussion on massive Yang-Mills field, see M. Veltman, Nucl. Phys. **B7**, 637 (1968).