Variational methods in supersymmetric lattice field theory: The vacuum sector

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The application of variational methods to the computation of the spectrum in supersymmetric lattice theories is considered, with special attention to $O(N)$ supersymmetric σ models. Substantial cancellations are found between bosonic and fermionic contributions even in approximate *Ansätze* for the vacuum wave function. The nonlinear limit of the linear σ model is studied in detail, and it is shown how to construct an appropriate non-Gaussian vacuum wave function for the nonlinear model. The vacuum energy is shown to be of order unity in lattice units in the latter case, after infinite cancellations.

I. INTRODUCTION

String theory and lattice gauge theory are two of the most active fields of research in particle physics. Lattice simulations are one of the most powerful ways of investigating nonperturbative phenomena in physical theories. String theories will have physical importance only if we can understand their nonperturbative predictions. But there is as yet very little work in lattice studies of string theories. One reason is that supersymmetry is an essential ingredient of string theory, and lattice formalisms respecting supersymmetry are hard to find.

A lattice formalism which preserves some supersymmetry has been introduced in a Hamiltonian formalism by Elitzur, Rabinovici, and Schwimmer.¹ The basic idea is that one can find a formalism which preserves one supersymmetric charge, while violating the others. The Hamiltonian is the square of this particular supersymmetric charge, so that the true ground state of the lattice theory has exactly zero energy.

We have previously studied bosonic² and fermionic³ lattice theories in a Hamiltonian variational formalism. To obtain estimates of the masses in such theories, we gave *Ansätze* for the ground-state and excited-state wave functions. The mass estimate is the difference between the expectation value of the Hamiltonian in the trial excited state and in the trial ground state. One recurring problem in this approach was that the ground-state energy has both ultraviolet and infrared divergences. Although these are regulated in the lattice formalism, the ground-state energy estimates necessarily grow as one proceeds towards the continuum and/or infinite-volume limit. The mass estimates become a small difference between two large numbers, which requires that each of the large numbers be known to greater and greater precision. Moreover, even though the energy estimates for both the ground state and first excited states are strict

upper bounds, the estimates for the masses are not. On the other hand, if supersymmetry is incorporated in the formalism, the ground-state energy must exactly vanish. The mass estimate is then a strict upper bound to the exact mass, and does not have to be obtained by subtracting two large numbers. This is our primary motivation for studying lattice versions of supersymmetric theories in a Hamiltonian formalism.

To gain experience that we might later apply to realistic four-dimensional theories, we studied the linear and nonlinear supersymmetric σ models in two space-time dimensions.⁴ We obtained the nonlinear models as the limit of the linear model as the coupling tends to infinity. This limit is quite tricky, and we explain the differences between the naive (physically uninteresting) limit and the correct (physically interesting) limit. This paper studies trial wave functions for the vacuum sector of both linear and nonlinear supersymmetric σ models. The one-particle sector together with mass estimates will be undertaken in another publication.

In Sec. II we study the lattice formalism for the supersymmetric linear σ model. The formalism for dealing with the Grassmann variables is developed further in Sec. III. In Sec. IV we show that a Gaussian *Ansatz* for the trial wave function is exact in both the $N \rightarrow \infty$ limit and formally in the $f_0 \rightarrow \infty$ limit, where f_0 is the coupling constant that appears in the linear σ model. The latter result is very surprising since the nonlinear σ model is a nontrivial theory, and we should not be able to guess its exact ground-state wave function. We show in Sec. V that the physically interesting nonlinear σ model is not simply the formal limit of the linear model as $f_0 \rightarrow \infty$. The appropriate modifications are made in Sec. VI, where a trial *Ansatz* for the true nonlinear theory is presented. The numerics of this *Ansatz* are also presented in Sec. VI, where it is shown that the energy density associated with this *Ansatz* is much smaller than

the energy associated with the fermionic or bosonic sectors separately, so that much of the supersymmetric cancellation between fermionic and bosonic zero-point energies is still maintained. Indeed, we obtain, even in the highly nontrivial nonlinear limit, total vacuum energies of order unity in lattice units, even after an effectively infinite cancellation (for $f_0 \rightarrow \infty$) between the bosonic and fermionic sectors. This suggests that our *Ansatz* has a strong overlap with the exact ground state of the theory. It may even be the case that the continuum-region scaling of the mass gap in this asymptotically-free theory may be obtained from a purely variational calculation, without the necessity for improvement by including states generated in the manner of Lanczos. This possibility will be explored in a future publication, where we consider the one-particle sector.

One may wonder whether models with exact supersymmetry, which appears to be crucial for the vacuum energy cancellation, can be at all relevant to the real world. However, at least one feature of some such models, the absence of a cosmological constant, does appear to be a phenomenological fact. It is, perhaps, not unreasonable to hope that an understanding of this remarkable and mysterious phenomenon can lead at the same time to technical advances in our control of Hamiltonian field theory.

II. SUPERSYMMETRIC σ MODELS: CONTINUUM AND LATTICE FORMULATION

The basic features of lattice supersymmetry in a Hamiltonian formulation were studied some time ago by Elitzur, Rabinovici, and Schwimmer.¹ In this section we shall review the formalism, with particular emphasis on $N=1$ supersymmetric σ models in two space-time dimensions.

The supersymmetry (SUSY) algebra in two space-time dimensions takes the form

$$\{Q_\alpha, Q_\beta\} = P^0 \delta_{\alpha\beta} + (\gamma^0 \gamma^1)_{\alpha\beta} P^1. \quad (2.1)$$

(We use the Majorana representation $\gamma^0 = \sigma_2$, $\gamma^1 = i\sigma_1$ throughout.) As Elitzur, Rabinovici, and Schwimmer point out, if we define the linear combinations $Q_\pm = Q_1 \pm Q_2$, the SUSY algebra (2.1) becomes

$$Q_+^2 = Q_-^2 = P^0, \quad \{Q_+, Q_-\} = 2P^1. \quad (2.2)$$

On the lattice, nonlinear (higher than quadratic) terms in the action cause a failure in (2.2), specifically $Q_+^2 \neq Q_-^2$, so we are forced to select a single element of the algebra by defining (for example) the Hamiltonian $H = Q_+^2$. Since Q_+ is Hermitian, this gives a positive spectrum with a ground-state energy of zero, even on the lattice. The failure of the other elements of the supersymmetry is directly related to the loss of the Leibniz principle for supercovariant derivatives on the lattice, as we shall see more explicitly below. This problem really arises because the lattice (discrete) structure is introduced only in the bosonic (ordinary space-time) sector of superspace—the Grassmannian sector is not touched and continues to have the same formal properties as in a fully continuum theory.⁵

The construction of supersymmetric σ models is most easily accomplished using superspace techniques. First, parametrize superspace by the pair (x^μ, θ^α) , $\mu=0,1$, $\alpha=1,2$, with θ^α a Majorana spinor taking values in a Grassmann algebra. The supersymmetry generators may be represented as superspace derivatives

$$Q_\alpha = \frac{\partial}{\partial \theta^\alpha} + i(\gamma^\mu \theta)_\alpha \frac{\partial}{\partial x^\mu}. \quad (2.3)$$

One easily verifies that the covariant derivatives

$$D_\alpha \equiv \frac{\partial}{\partial \theta^\alpha} - i(\gamma^\mu \theta)_\alpha \frac{\partial}{\partial x^\mu} \quad (2.4)$$

anticommute with the generators (2.3),

$$\{Q_\alpha, D_\beta\} = 0, \quad (2.5)$$

thus allowing the construction of super-Casimir operators, e.g.,

$$[Q_\alpha, \epsilon_{\beta\gamma} D_\beta D_\gamma] = 0. \quad (2.6)$$

Consequently, if $\Phi(x, \theta)$ is a general superfield

$$\Phi(x, \theta) = A(x) + \bar{\theta} \psi(x) + \frac{1}{2} \bar{\theta} \theta F(x), \quad (2.7)$$

a supersymmetric kinetic term is easily constructed,

$$I_{\text{kin}} = - \int d^2x d^2\theta \frac{1}{4} \Phi \epsilon_{\alpha\beta} D_\alpha D_\beta \Phi, \quad (2.8)$$

while interactions may be added via a superpotential W

$$I_{\text{pot}} = \int d^2x d^2\theta W(\Phi^2). \quad (2.9)$$

Under a supersymmetry transformation, $\delta\Phi = Q_\alpha \Phi$, (2.9) will be invariant either if the derivatives $\partial/\partial x^\mu$, $\partial/\partial \theta^\alpha$ in Q_α satisfy the chain rule or, if W is at most quadratic, they satisfy the rule for integration by parts. The latter property can always be assured on the lattice by defining derivatives via symmetric differences, thus

$$\Delta\Phi_m = \frac{1}{2}(\Phi_{m+1} - \Phi_{m-1}). \quad (2.10)$$

Thus, for free theories with quadratic actions, we can indeed preserve the full SUSY algebra on the lattice. For interacting theories, however, δW does not equal the total lattice derivative so it will be impossible to preserve all of (2.2).

To construct the Hamiltonian for explicit calculations it will be necessary to expand the action in terms of component fields. We shall be dealing with $O(N)$ σ models where the basic superfield Φ is an N -multiplet. Defining $V(\Phi^2) = (\partial/\partial\Phi^2)W(\Phi^2)$, one finds for the total action

$$\begin{aligned} I = \int d^2x & \left[\frac{1}{2} (\partial_\mu \mathbf{A})^2 + \frac{1}{2} \bar{\Psi} i \cdot \partial \psi + \frac{1}{2} \mathbf{F}^2 \right. \\ & + 2V(\mathbf{A}^2) \mathbf{A} \cdot \mathbf{F} - V(\mathbf{A}^2) \bar{\psi} \cdot \psi \\ & \left. - 2V'(\mathbf{A}^2) \bar{\mathbf{A}} \cdot \bar{\psi} \mathbf{A} \cdot \psi \right]. \end{aligned} \quad (2.11)$$

The field $\mathbf{F}$ is nondynamical and may be eliminated by using its equations of motion

$$\begin{aligned} I = \int d^2x & \left[\frac{1}{2} (\partial_\mu \mathbf{A})^2 + \frac{1}{2} \bar{\Psi} i \cdot \partial \psi \right. \\ & \left. - 2\mathbf{A}^2 V^2 - V \bar{\psi} \cdot \psi - 2V' \bar{\mathbf{A}} \cdot \bar{\psi} \mathbf{A} \cdot \psi \right]. \end{aligned} \quad (2.12)$$

Finally, we specialize to a “double-well” model in which

$$W = \frac{f}{4}(\Phi^2 - \beta)^2, \quad V = \frac{f}{2}(\Phi^2 - \beta). \quad (2.13)$$

The Hamiltonian corresponding to (2.12) is readily constructed (some caution is required as the reality of the ψ implies the presence of second-class constraints). One finds

$$H = \int dx \left[\frac{1}{2} \pi(x)^2 + \frac{1}{2} |\partial_1 \mathbf{A}|^2 - \frac{i}{2} \psi \gamma_5 \partial_1 \psi \right. \\ \left. + \frac{1}{2} f^2 \mathbf{A}^2 (\mathbf{A}^2 - \beta)^2 + \frac{1}{2} f [\bar{\psi} \cdot \psi (\mathbf{A}^2 - \beta) \right. \\ \left. + 2 \bar{\mathbf{A}} \cdot \bar{\psi} \mathbf{A} \cdot \psi \right] \quad (2.14)$$

with the canonical commutation relations (CCR's)

$$[\pi^A(x), A^B(y)] = -i \delta^{AB} \delta(x-y), \quad (2.15) \\ \{\psi_\alpha^A(x), \psi_\beta^B(y)\} = \delta^{AB} \delta_{\alpha\beta} \delta(x-y).$$

The supersymmetry generator $\bar{\epsilon}Q$ induces on the component fields the transformations

$$\delta \mathbf{A} = \bar{\epsilon} \psi, \quad (2.16) \\ \delta \psi = [-i \not{\partial} \mathbf{A} - 2V(\mathbf{A}^2) \mathbf{A}] \epsilon.$$

To go over to a Hamiltonian lattice formulation of this theory, we discretize the spatial direction $x^1 \equiv x = ma$ ($m = 1, 2, \dots, L$) and note that the discrete version of (2.16) is implemented by

$$Q_\alpha = \frac{1}{\sqrt{2a}} \sum_m [\Pi_m - \gamma_5 \Delta \mathbf{A}_m \\ + 2ia \gamma^0 \mathbf{A}_m V(\mathbf{A}_m^2)] \cdot \psi_{\alpha m}. \quad (2.17)$$

As mentioned previously, the symmetric difference operator Δ is defined by

$$\Delta \mathbf{A}_m \equiv \frac{1}{2} (\mathbf{A}_{m+1} - \mathbf{A}_{m-1}) \quad (2.18)$$

with the option of either symmetric ($\mathbf{A}_{m+L} = \mathbf{A}_m$) or antisymmetric ($\mathbf{A}_{m+L} = -\mathbf{A}_m$) boundary conditions. The discrete version of the CCR's (2.15) is simply

$$[\Pi_m^A, A_n^B] = -i \delta^{AB} \delta_{mn}, \quad (2.19) \\ \{\psi_{\alpha m}^A, \psi_{\beta n}^B\} = \delta^{AB} \delta_{\alpha\beta} \delta_{mn}.$$

We may define a lattice Hamiltonian for this theory in terms of either Q_+^2 or Q_-^2 [cf. (2.2)]; a straightforward calculation, using (2.19), gives

$$aH_\pm \equiv aQ_\pm^2 = \frac{1}{2} \sum_m \Pi_m^2 + \frac{1}{2} \sum_m |\Delta \mathbf{A}_m|^2 \\ - \frac{i}{2} \sum_m \psi_m \gamma_5 \Delta \psi_m \\ + \frac{(af)^2}{2} \sum_m \mathbf{A}_m^2 (\mathbf{A}_m^2 - \beta)^2 \\ + \frac{af}{2} \sum_m [\bar{\psi}_m \cdot \psi_m (\mathbf{A}_m^2 - \beta) \\ + 2 \bar{\mathbf{A}}_m \cdot \bar{\psi}_m \mathbf{A}_m \cdot \psi_m] \\ \pm af \sum_m (\mathbf{A}_m^2 - \beta) \mathbf{A}_m \cdot \Delta \mathbf{A}_m. \quad (2.20)$$

The supersymmetry-violating term is clearly visible as the last entry on the right-hand side of the above expression. We shall simply choose the positive sign ($H \equiv Q_+^2$) throughout. In fact, in the class of *Ansätze* considered in this paper, the expectation value of this final term will always vanish, so the sign chosen here is actually irrelevant.

As far as the bosonic $\mathbf{A}_m$ fields are concerned, the discrete CCR's (2.19) are immediately realizable in a Schrödinger representation

$$\Pi_m = -i \frac{\partial}{\partial \mathbf{A}_m}. \quad (2.21)$$

The CCR's for the fermionic fields are more usually realized in a creation and annihilation operator formalism. However, for the calculations to be performed below, it turns out to be much more convenient to realize the fermionic part of H also in terms of differential operators, this time with respect to Grassmann variables. This preserves a certain formal symmetry between the bosonic and fermionic parts of the wave function, but also simplifies considerably the calculation of the needed expectation values. In the following section, we shall develop the necessary formalism for carrying out variational calculations in “Grassmann wave mechanics.”

III. GRASSMANN WAVE MECHANICS

As a very simple but nonetheless illustrative example of the formalism we are aiming for, consider a fermionic system with one degree of freedom. The creation and annihilation algebra is just

$$\{a, a^\dagger\} = 1, \quad (3.1)$$

which is clearly realized by setting

$$a \rightarrow \frac{\partial}{\partial \theta}, \quad a^\dagger \rightarrow \theta, \quad (3.2)$$

where θ is a single anticommuting Grassmann variable.⁶ The Hilbert space for this system is spanned by two Fock states $|0\rangle$ and $|1\rangle$, with $a^\dagger|0\rangle = |1\rangle$, etc. Another representation, however, is the “ θ representation” in which the wave function of the state $|0\rangle$ is represented by the constant function and that of the

state $|1\rangle$ by the linear function θ . Thus

$$\langle\theta|0\rangle=1, \quad \langle\theta|1\rangle=\theta, \quad (3.3)$$

which implies that the θ basis is not orthonormal. Rather

$$\begin{aligned} \langle\theta'|\theta\rangle &= \langle\theta'|0\rangle\langle 0|\theta\rangle + \langle\theta'|1\rangle\langle 1|\theta\rangle \\ &= 1 + \theta'\theta = e^{\theta'\theta}. \end{aligned} \quad (3.4)$$

From (3.4) we derive directly the completeness relation

$$1 = \int d\theta d\theta' |\theta'\rangle e^{\theta'\theta} \langle\theta|, \quad (3.5)$$

and from (3.5) the formula for overlap of two Grassmann wave functions

$$\begin{aligned} \langle\psi'|\psi\rangle &= \int d\theta d\theta' \langle\psi'|\theta'\rangle e^{\theta'\theta} \langle\theta|\psi\rangle \\ &= \int d\theta d\theta' \psi'^*(\theta') e^{\theta'\theta} \psi(\theta). \end{aligned} \quad (3.6)$$

The extension to several degrees of freedom is essentially straightforward. There is only one tricky point: in $\psi^*(\theta'_i)$, appearing in (3.6), the order of the θ'_i must be inverted relative to their order of appearance in the unconjugated wave function $\psi(\theta_i)$. Thus, if

$$|1,1\rangle \rightarrow \theta_1\theta_2, \quad (3.7)$$

then the conjugate wave function

$$\langle 1,1| \rightarrow \theta'_2\theta'_1 \quad (3.8)$$

(this guarantees that $\langle 1,1|1,1\rangle = +1$).

We shall be particularly interested in overlaps of Gaussian fermionic wave functions of the form

$$\psi(\theta_i) = e^{\theta_i M_{ij} \theta_j}, \quad (3.9)$$

where M_{ij} is a Hermitian, purely imaginary matrix. After application of the Hamiltonian, multinomials in θ_i appear multiplying the exponential in (3.9). Equivalently, such products may be generated by introducing sources for θ_i, θ'_i and computing the following generating functional once and for all:

$$\begin{aligned} Z(\eta_i, \eta'_i) &\equiv \int d\theta_i d\theta'_i \exp(\theta'_i M_{ij} \theta'_j + \theta'_i \theta_i + \theta_i M_{ij} \theta_j \\ &\quad + \eta_i \theta_i + \eta'_i \theta'_i) \\ &= \exp \left[-\frac{1}{2} (\eta, \eta')_i \mathbb{D}_{ij} \begin{pmatrix} \eta \\ \eta' \end{pmatrix}_j \right]. \end{aligned} \quad (3.10)$$

The propagator $\mathbb{D}_{ij}$ is given by

$$\mathbb{D} = \begin{pmatrix} iD & -\tilde{D} \\ +\tilde{D} & iD \end{pmatrix} \quad (3.11)$$

with

$$D \equiv 2i \frac{M}{1+4M^2}, \quad \tilde{D} \equiv \frac{1}{1+4M^2}. \quad (3.12)$$

We may immediately evaluate expectations of θ quantities using this result, for example,

$$E_0(\Lambda) \equiv \frac{\int d\mathbf{A}_m d\boldsymbol{\theta}_m d\boldsymbol{\theta}'_m \psi_0^*(\mathbf{A}_m, \boldsymbol{\theta}'_m) \exp \left[\sum \boldsymbol{\theta}'_m \cdot \boldsymbol{\theta}_m \right] H \psi_0(\mathbf{A}_m, \boldsymbol{\theta}_m)}{\int d\mathbf{A}_m d\boldsymbol{\theta}_m d\boldsymbol{\theta}'_m \psi_0(\mathbf{A}_m, \boldsymbol{\theta}'_m) \exp \left[\sum \boldsymbol{\theta}'_m \cdot \boldsymbol{\theta}_m \right] \psi_0(\mathbf{A}_m, \boldsymbol{\theta}_m)}, \quad (3.19)$$

over the range of permissible Λ .

$$\begin{aligned} \langle\theta_i \theta_j\rangle &\equiv \frac{\int d\theta d\theta' \theta_i \theta_j \exp(\theta' M \theta' + \theta' \theta + \theta M \theta)}{\int d\theta d\theta' \exp(\theta' M \theta' + \theta' \theta + \theta M \theta)} \\ &= \frac{\partial}{\partial \eta_i} \frac{\partial}{\partial \eta_j} Z \bigg|_{\eta_i=0} = iD_{ij}. \end{aligned} \quad (3.13)$$

With this formal machinery in hand, we can return to the task of reexpressing the lattice Hamiltonian (2.20) in completely differential form. The CCR's (2.19) may be realized via the replacement

$$\begin{aligned} \frac{\psi_{1m} + i\psi_{2m}}{\sqrt{2}} &= \mathbf{a}_m^\dagger \rightarrow \boldsymbol{\theta}_m, \\ \frac{\psi_{1m} - i\psi_{2m}}{\sqrt{2}} &= \mathbf{a}_m \rightarrow \frac{\partial}{\partial \boldsymbol{\theta}_m}. \end{aligned} \quad (3.14)$$

We substitute (3.14) into (2.20) and introduce (to facilitate the large- N limit discussed below) the dimensionless variables β_0, f_0 by

$$af \equiv \frac{f_0}{N}, \quad \beta \equiv \beta_0 N. \quad (3.15)$$

After some algebra, one finds for the total Hamiltonian, in lattice units,

$$aH = H_{\text{bos}} + H_{\text{ferm}}, \quad (3.16)$$

where

$$\begin{aligned} H_{\text{bos}} &\equiv -\frac{1}{2} \sum_m \frac{\partial^2}{\partial \mathbf{A}_m^2} - \frac{1}{2} \sum_{mn} (\Delta^2)_{mn} \mathbf{A}_m \cdot \mathbf{A}_n \\ &\quad + \frac{f_0^2}{2N^2} \sum_m \mathbf{A}_m^2 (\mathbf{A}_m^2 - \beta)^2 \\ &\quad + \frac{f_0}{N} \sum_m \mathbf{A}_m^2 \mathbf{A}_m \cdot \Delta_{mn} \mathbf{A}_n, \end{aligned} \quad (3.17)$$

$$\begin{aligned} H_{\text{ferm}} &\equiv -\frac{i}{2} \sum_{mn} \Delta_{mn} \left[\frac{\partial}{\partial \boldsymbol{\theta}_m} \cdot \frac{\partial}{\partial \boldsymbol{\theta}_n} + \boldsymbol{\theta}_m \cdot \boldsymbol{\theta}_n \right] \\ &\quad + \frac{f_0}{N} \sum_m (\mathbf{A}_m^2 - \beta) \boldsymbol{\theta}_m \cdot \frac{\partial}{\partial \boldsymbol{\theta}_m} \\ &\quad + 2 \frac{f_0}{N} \sum_m \mathbf{A}_m \cdot \boldsymbol{\theta}_m \mathbf{A}_m \cdot \frac{\partial}{\partial \boldsymbol{\theta}_m} \\ &\quad - \frac{f_0}{2} \sum_m \left[\left(1 + \frac{2}{N} \right) \mathbf{A}_m^2 - \beta \right]. \end{aligned} \quad (3.18)$$

Although the last term in (3.18) appears to be completely bosonic in character, it is included here with the fermionic Hamiltonian as it originates as a reordering term of fermion operators.

A variational calculation in this model will involve the choice of an appropriate *Ansatz* vacuum wave function $\psi_0(\mathbf{A}_m, \boldsymbol{\theta}_m; \Lambda)$ depending on some set of parameters Λ . Then our task is to minimize the expectation $E_0(\Lambda)$,

IV. THE GAUSSIAN ANSATZ

Bosonic and supersymmetric σ models are well known^{7,8} to become Gaussian in the large- N limit: namely, the limit in which the number of field components in (3.16)–(3.18) tends to infinity while the bare parameters f_0 and β_0 remain fixed. To establish this at the lattice level is quite straightforward given the machinery developed in preceding sections. It turns out that an obvious fermionic generalization of the free bosonic ground-state wave function does the trick. Consider the direct-product wave function

$$\psi_0(\mathbf{A}_m, \boldsymbol{\theta}_m) = \exp\left(-\frac{1}{2} \mathbf{A}_m G_{mn} \cdot \mathbf{A}_n\right) \times \exp\left(\frac{1}{2} \boldsymbol{\theta}_m H_{mn} \cdot \boldsymbol{\theta}_n\right) \quad (4.1)$$

which is manifestly $O(N)$ invariant, and translation-invariant provided

$$G_{mn} = \frac{1}{L} \sum_p G_p e^{2\pi i p(m-n)/L}, \quad (4.2)$$

$$H_{mn} = \frac{1}{L} \sum_p h_p e^{2\pi i p(m-n)/L}.$$

In (4.2), p labels lattice momenta and will be taken to run over *half-integral* values $-(L-1)/2, -(L-3)/2, (L-1)/2$ throughout this section (corresponding to antiperiodic boundary conditions). We take G_{mn} real and H_{mn} pure imaginary. All the integrals relevant to the evaluation of (3.19) are Gaussian, and one finds, for the energy density,

$$E_0 = E_{\text{bos}} + E_{\text{ferm}}, \quad (4.3)$$

$$\begin{aligned} \frac{1}{NL} E_{\text{bos}} = & \frac{1}{4L} \sum_p \left[G_p + \frac{\delta_p^2}{G_p} \right] \\ & + \frac{f_0^2}{4L} \sum_p \left[\frac{1}{G_p} \right] \left[\frac{1}{2} \left[1 + \frac{2}{N} \right] \frac{\sum \frac{1}{G_p}}{L} - \beta_0 \right]^2 \\ & + \frac{f_0^2}{8N} \left[1 + \frac{2}{N} \right] \left[\frac{1}{L} \sum \frac{1}{G_p} \right]^3, \end{aligned} \quad (4.4)$$

$$\begin{aligned} \frac{1}{NL} E_{\text{ferm}} = & \frac{1}{L} \sum_p \frac{\delta_p h_p}{1 + h_p^2} \\ & + f_0 \left[\frac{1}{2} \left[1 + \frac{2}{N} \right] \frac{\sum \frac{1}{G_p}}{L} - \beta_0 \right] \\ & \times \left[\frac{1}{2} - \frac{1}{L} \sum \frac{1}{1 + h_p^2} \right], \end{aligned} \quad (4.5)$$

with $\delta_p \equiv \sin(2\pi p/L)$ the eigenvalues of the lattice difference operator Δ . Minimizing with respect to h_p ,

$$\frac{\partial E_{\text{ferm}}}{\partial h_p} = 0, \quad h_p = \frac{\nu f_0}{\delta_p} - \text{sgn}(\delta_p) \left[1 + \nu^2 \frac{f_0^2}{\delta_p^2} \right]^{1/2} \quad (4.6)$$

with

$$\nu \equiv \frac{1}{2} \left[1 + \frac{2}{N} \right] \frac{\sum \frac{1}{G_p}}{L} - \beta_0. \quad (4.7)$$

Another useful combination is

$$\mu \equiv \frac{1}{2} - \frac{1}{L} \sum_p \frac{1}{1 + h_p^2}. \quad (4.8)$$

In virtue of the extremal condition (4.6), μ may also be written

$$\mu = -\frac{\nu f_0}{2L} \sum_p (\delta_p^2 + \nu^2 f_0^2)^{-1/2}. \quad (4.9)$$

The extremal condition with respect to G_p leads to a second constraint

$$\begin{aligned} \frac{\partial E_0}{\partial G_p} = & 0, \\ G_p = & \left[\delta_p^2 + \nu^2 f_0^2 + 2\mu f_0 \left[1 + \frac{2}{N} \right] \right. \\ & \left. + 2\nu f_0^2(\nu + \beta_0) + \frac{6f_0^2}{N+2}(\nu + \beta_0)^2 \right]^{1/2}. \end{aligned} \quad (4.10)$$

Practitioners of the Hartree-Fock art⁸ will recognize at this point the appearance of a pair of self-consistent gap equations for μ, ν (directly related to the fermion and boson masses), namely,

$$\begin{aligned} \nu + \beta_0 = & \frac{1}{2L} \left[1 + \frac{2}{N} \right] \sum_p \left[\delta_p^2 + \nu^2 f_0^2 + 2\mu f_0 \left[1 + \frac{2}{N} \right] \right. \\ & \left. + 2\nu f_0^2(\nu + \beta_0) \right. \\ & \left. + \frac{6f_0^2(\nu + \beta_0)^2}{N+2} \right]^{-1/2} \end{aligned} \quad (4.11)$$

together with (4.9).

The gap equations (4.9) and (4.11) may be solved and shown to lead to zero ground-state energy in two limits.

(a) Large N ($N \rightarrow \infty$; f_0, β_0 fixed). Eliminate μ from (4.11) using (4.9) and drop terms of order $1/N$. Then (4.11) becomes

$$\nu + \beta_0 = \frac{1}{2L} \sum_p \left[\delta_p^2 + \nu^2 f_0^2 + 2\nu f_0^2 \left[\nu + \beta_0 - \frac{1}{2L} \sum_q (\delta_q^2 + \nu^2 f_0^2)^{-1/2} \right] \right]^{-1/2}. \quad (4.12)$$

Inspection of (4.12) reveals that the quantity in large parentheses repeats the structure of the outer part of the equation. Accordingly, a sufficient condition for a solution of (4.12) is

$$\nu + \beta_0 = \frac{1}{2L} \sum_p (\delta_p^2 + \nu^2 f_0^2)^{-1/2}, \quad (4.13)$$

or, equivalently [recall (4.10)],

$$G_p = (\delta_p^2 + \nu^2 f_0^2)^{1/2}. \quad (4.14)$$

(4.9) and (4.11) have, in general, *more* solutions than (4.13): However, these additional solutions may be studied numerically and shown to lead to nonzero energy, at least for large enough L . Substituting (4.14) into (4.3) one finds an exact cancellation of the bosonic and fermionic self-energies

$$\begin{aligned} E_{\text{bos}} &= \frac{N}{2} \sum_p G_p, \\ E_{\text{ferm}} &= -\frac{N}{2} \sum_p (\delta_p^2 + \nu^2 f_0^2)^{1/2} = -E_{\text{bos}}. \end{aligned} \quad (4.15)$$

It should be emphasized that even for finite N , the cancellation is substantial (for example, for $N=4$, $L=8$, $E_{\text{bos}}=23.5$, $E_{\text{ferm}}=-21.6$, $E_0=1.9$ in lattice units).

(b) Much more surprising is the finding that E_0 also vanishes for fixed, finite N in the limit $f_0 \rightarrow \infty$. Indeed, in this limit

$$\begin{aligned} \mu &= \frac{1}{2} + O\left[\frac{1}{f_0^2}\right], \\ \nu &= -\beta_0 + \frac{1}{2\beta_0 f_0} \left[1 + \frac{2}{N}\right] + O\left[\frac{1}{f_0^2}\right], \end{aligned} \quad (4.16)$$

whence

$$\begin{aligned} E_{\text{bos}} &\rightarrow \frac{1}{2} f_0 \beta_0 - \frac{1}{4\beta_0} \left[1 + \frac{2}{N}\right] + O\left[\frac{1}{f_0}\right], \\ E_{\text{ferm}} &\rightarrow -\frac{1}{2} f_0 \beta_0 + \frac{1}{4\beta_0} \left[1 + \frac{2}{N}\right] + O\left[\frac{1}{f_0}\right], \end{aligned}$$

so

$$E_0 = O\left[\frac{1}{f_0}\right], \quad f_0 \rightarrow \infty.$$

For example, for $N=3$, $L=16$, $f_0=8$, one finds $E_{\text{bos}}=171.2$, $E_{\text{ferm}}=-170.6$, $E_0=0.6$. This result is, at first glance, shocking as we are accustomed to regard this limit as leading to the nonlinear regime of the σ model, which is certainly not a free Gaussian theory. Nevertheless, we seem to have obtained essentially the

exact ground state using a simple Gaussian wave function in which, moreover, the bosonic and fermionic degrees are totally decoupled. The reason for this paradox will be explained in Sec. V, where it will be seen that the nonlinear limit of supersymmetric σ models exhibits a peculiarity not present in purely bosonic σ models of the standard double-well type—namely, the unwanted persistence of a contribution from the region at the origin of field space unless active steps are taken to suppress this region as we take $f_0 \rightarrow \infty$.

V. THE NONLINEAR LIMIT

To understand the precise nature of the nonlinear limit in a supersymmetric σ model, it is convenient to return temporarily to a Euclidean formulation. The superpotential part of the action (2.9) and (2.13) corresponds to a factor

$$\exp\left[i\frac{f}{4} \int d^2x \, d^2\theta (\Phi^2 - \beta)^2\right] \quad (5.1)$$

in the functional integral, where Φ is the superfield (2.7). Naively one would expect this factor to generate a delta function $\delta(\Phi^2 - \beta)$ in the limit $f \rightarrow \infty$, which would precisely correspond to the nonlinear supersymmetric σ model. Indeed, the constraints defining the latter model are⁴

$$\mathbf{A}^2 = \beta, \quad \mathbf{A} \cdot \psi = 0, \quad \mathbf{A} \cdot \mathbf{F} = \frac{1}{2} \bar{\psi} \cdot \psi, \quad (5.2)$$

which arise from examining the individual components of the superfield constraint $\Phi^2 = \beta$.

In fact, the limit of (5.1) as $f \rightarrow \infty$ leads to an additional contribution with support at the origin of field space $\Phi=0$. The possibility for such a contribution is evident in the component expansion of the superpotential (2.13), which gives a bosonic potential $\mathbf{A}^2(\mathbf{A}^2 - \beta)^2$ with a central minimum at $\mathbf{A}=0$, as well as the spherically symmetric well at $\mathbf{A}^2 = \beta$. Indeed, there will be such a minimum for any (field-analytic) choice of the super potential, as a consequence of the $\int d^2\theta$ present in the definition of the superaction. In components, (5.1) becomes

$$\exp\{if[(\mathbf{A}^2 - \beta)(\mathbf{A} \cdot \mathbf{F} - \frac{1}{2} \bar{\psi} \cdot \psi) - \overline{\mathbf{A} \cdot \psi} \mathbf{A} \cdot \psi]\}. \quad (5.3)$$

Inspection of (5.3) reveals that at large f we are forced into two distinct regions of field space: (a) the conventional nonlinear region, where $\mathbf{A}^2 = \beta$, $\mathbf{A} \cdot \mathbf{F} = \frac{1}{2} \bar{\psi} \cdot \psi$, and $\mathbf{A} \cdot \psi = 0$; (b) the origin in field space. If $\mathbf{A}=0$, (5.3) reduces to $e^{+(if\beta/2)\bar{\psi}\psi}$, which forces $\psi=0$ at large f .

With some trial and error, one determines that the correct distributional limit takes the form

$$\exp\{if[(\mathbf{A}^2 - \beta)(\mathbf{A} \cdot \mathbf{A} - \frac{1}{2} \bar{\psi} \cdot \psi) - (\overline{\mathbf{A} \cdot \psi} \mathbf{A} \cdot \psi)]\} \xrightarrow{f \rightarrow \infty} c_1 \delta(\mathbf{A} \cdot \hat{\mathbf{A}} \cdot \mathbf{F}) \delta(\psi) + c_2 \delta(\mathbf{A}^2 - \beta) \delta(\mathbf{A} \cdot \psi) \delta(\mathbf{A} \cdot \mathbf{F} - \frac{1}{2} \bar{\psi} \cdot \psi). \quad (5.4)$$

By integrating both sides of (5.4) with suitable test functions, one easily finds

$$c_1 = 2\pi(f\beta)^{N-1}, \quad c_2 = 4\pi. \quad (5.5)$$

Consequently, in the large- f limit, the contribution from the region $\mathbf{A}=\psi=0$ is enhanced relative to that from the nontrivial, nonlinear region projected by the second term in (5.4), provided $N > 1$. This accounts for our

finding in Sec. IV that a decoupled Gaussian wave function appears to give zero energy in the large- f limit. Such a wave function is simply constructing the appropriate ground state for the degenerate and trivial theory defined by the first term in (5.4). In order to build wave functions which accurately describe the physics of the true nonlinear limit, we shall be forced to introduce non-Gaussian wave functions with explicit coupling between bosonic and fermionic sectors.

VI. VARIATIONAL VACUUM WAVE FUNCTIONS FOR THE NONLINEAR LIMIT

The discussion in Sec. V underscores the necessity for properly building into the vacuum *Ansatz* the crucial features of the nonlinear physics, which in path-integral language amount to the constraints (5.2). At the Hamiltonian level, where the nondynamical field has already been eliminated, these constraints imply

(1) The wave function should be concentrated around the region $\mathbf{A}_m^2 = \beta$ (for each lattice site m), with a width that vanishes as $f_0 \rightarrow \infty$. Otherwise, the positive-definite bosonic contribution $f_0^2 \sum_m \mathbf{A}_m^2 (\mathbf{A}_m^2 - \beta)^2$ to (3.17) will lead to infinite energy.

(2) The wave function should be annihilated by the operator $\mathbf{A}_m \cdot \partial / \partial \theta_m$. In fact, θ_m and $\partial / \partial \theta_m$ are Hermitian adjoints of another [cf. (3.14)], so the term $f_0 \sum_m \mathbf{A}_m \cdot \theta_m \mathbf{A}_m \cdot \partial / \partial \theta_m$ in (3.18) is positive definite, and must annihilate the wave function to avoid a large positive energy as $f_0 \rightarrow \infty$. One might think that it is not necessary to have the wave function annihilated by $\mathbf{A}_m \cdot \partial / \partial \theta_m$, but just that the expectation value of $(2f_0/N) \sum_m \mathbf{A}_m \cdot \theta_m \mathbf{A}_m \cdot \partial / \partial \theta_m$ be of order 1. A simple argument in the Appendix shows that it is no loss of generality to have the wave function annihilated by $\mathbf{A}_m \cdot \partial / \partial \theta_m$. This condition implies that the wave function $\psi_0(\mathbf{A}_m, \theta_m^{\text{tr}})$ be a function of the Grassmann variables θ_m^{tr} projected transverse to the bosonic field at each site:

$$\theta_m^{\text{tr}} = G_m \theta_m = (1 - \hat{A}_m \hat{A}_m) \cdot \theta_m. \quad (6.1)$$

The simplest *Ansatz* building in these requirements is

$$\begin{aligned} \psi_0(\mathbf{A}_m, \theta_m^{\text{tr}}) = & \exp \left[-\frac{f_0}{4N} \sum_m (\mathbf{A}_m^2 - \beta)^2 \right] \\ & \times \exp \left[\lambda \sum_m \hat{A}_m \cdot \hat{A}_{m+1} \right] \\ & \times \exp \left[i\rho \sum_m \theta_m^{\text{tr}} \cdot \theta_{m+1}^{\text{tr}} \right], \end{aligned} \quad (6.2)$$

where for the time being we have restricted ourselves to nearest-neighbor interactions for the angular bosonic and fermionic terms. The radial term is chosen such that the dependence on f_0 generated by applying $\sum_m \partial^2 / \partial \mathbf{A}_m^2$ to ψ_0 exactly cancels all other terms in the Hamiltonian involving f_0 , with the exception of the term

$$\frac{f_0}{N} \sum_m (\mathbf{A}_m^2 - \beta) \theta_m \cdot \frac{\partial}{\partial \theta_m} \xrightarrow{f_0 \rightarrow \infty} \frac{N-2}{2\beta} \sum_m \theta_m \cdot \frac{\partial}{\partial \theta_m}. \quad (6.3)$$

The above limit is obtained by explicitly integrating out the radial degrees of freedom in the large- f_0 limit, so that the factor $f_0(\mathbf{A}_m^2 - \beta)$ at each site is replaced by a fixed constant.

The radial dependence in (6.2), although related in a natural way to the potential, is clearly very restrictive. In fact, we shall now consider more general choices of radial wave function—the final result will be a modified version of (6.2).

The important point is that ψ_0 be concentrated around $A_m = \sqrt{\beta}$. Writing

$$\psi_0 = \left[\prod_m \psi_R(A_m) \right] \psi_T(\hat{A}_m, \theta) \quad (6.4)$$

we can choose

$$\psi_R = e^{-(f_0\beta/N)(A_m - \sqrt{\beta})^2}. \quad (6.5)$$

A more general wave function is

$$\psi_R = e^{-\sigma^2 \left[1 + \frac{\sigma E_1(\sigma)}{\sqrt{f_0}} + \frac{E_2(\sigma)}{f_0} \right]}, \quad (6.6)$$

where

$$\sigma = \left[\frac{\beta f_0}{N} \right]^{1/2} (A_m - \sqrt{\beta}) \quad (6.7)$$

and E_1, E_2 are even functions of σ . It is straightforward to compute the expectation value of the terms in the Hamiltonian which depend on E_1 and E_2 in the limit $f_0 \rightarrow \infty$. The result is independent of E_2 and is given by

$$\frac{\beta}{2N} \frac{\int d\sigma e^{-2\sigma^2} \left[4\sigma^2 E_1^2 + \sigma^2 \left[\frac{\partial E_1}{\partial \sigma} \right]^2 + \left[\frac{N}{\beta} \right]^{1/2} (24\sigma^4 - 12\sigma^2) E_1 + \frac{8\eta\sqrt{N}}{\beta} \sigma^2 E_1 \right]}{\int d\sigma e^{-2\sigma^2}}, \quad (6.8)$$

where

$$\eta = \frac{\langle \psi_T | \frac{1}{L} \sum_m \theta_m \cdot \frac{\partial}{\partial \theta_m} | \psi_T \rangle}{\langle \psi_T | \psi_T \rangle} . \quad (6.9)$$

This functional can easily be minimized. The only solution is

$$E_1 = - \left[\frac{N}{\beta^2} \right]^{1/2} (\sigma^2 + \eta) . \quad (6.10)$$

The energy of this wave function is lower than the energy of wave function (6.2) by

$$\frac{1}{2\beta} L \eta^2 . \quad (6.11)$$

We do our calculations by using (6.2) and subtracting (6.11) from the energy at the end. The integrals in this case are not Gaussian, so we cannot perform the evaluation of E_0 analytically as in Sec. IV. Instead, the quantities obtained by applying the Hamiltonian to (6.2) are averaged in a Monte Carlo simulation (with the fermion determinant properly included). We calculate the fermion determinant by keeping the exact current value of the fermion propagator, as in Ref. 9.

To get some analytic idea of the λ values required to minimize the expectation of the Hamiltonian, one can ignore the fermion degrees of freedom completely, setting $\rho=0$ and ignoring the determinant. We choose $N=3$. The problem then reduces to minimizing the energy density

$$E_0/L = \left[\left(\frac{\beta}{4} - \frac{\lambda^2}{\beta} \right) (1-x^2) + \frac{2\lambda x}{\beta} \right] , \quad (6.12)$$

where

$$x = \frac{\cosh 2\lambda}{\sinh 2\lambda} - \frac{1}{2\lambda} . \quad (6.13)$$

For $\beta < \sqrt{6}$, the minimum occurs at $\lambda=0$ and $E_0/L = \beta/4$. As β grows beyond $\sqrt{6}$, the best value of λ rises rapidly.

For the complete treatment including the fermions we also see a rapid rise in λ for β near $\sqrt{6}$. The results for $N=3$, $L=8$ are shown in Table I. They were obtained by averaging over 2000 sweeps and errors in the energies are at the level of a few percent. After an effectively infinite cancellation between bosonic and fermionic energies, the total vacuum energy is reduced to the range 3–4 (in lattice units) in the crossover region. It should be possible to further reduce this by a more ingenious choice of fermionic wave function [although at present the indications are that the energy is not greatly reduced simply by including next-nearest terms, etc., in (6.2)].

Ansätze of the form (6.2) therefore seem to be adequate starting points for attacking the continuum region of the theory. The Lanczos algorithm should converge very rapidly in this case and indeed it may be possible to obtain mass gap scaling at the purely variational level. This possibility will be examined in a future publication.

TABLE I. Ground-state energies for values of the variational parameters in (6.2).

$\sqrt{\beta}$	$\lambda_{\min}$	$\rho_{\min}$	$E_{0 \min}$
0.8	0.02	0.08	0.95
1.0	0.02	0.10	1.50
1.2	0.02	0.16	2.17
1.4	0.04	0.24	2.98
1.5	0.15	0.26	3.45
1.6	0.8	0.3	3.90
1.7	1.2	0.4	4.02

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APPENDIX

We show that it is no loss of generality to assume that ψ_T is annihilated by $\mathbf{A}_m \cdot \partial / \partial \theta_m$. This must surely be true to leading order in f_0 , but what if we wrote

$$\psi_T = \psi_1(\hat{A}, \theta) + \frac{1}{f_0} \psi_2(\hat{A}, \theta) , \quad (A1)$$

where

$$\mathbf{A}_m \cdot \frac{\partial}{\partial \theta_m} \psi_1 = 0 ? \quad (A2)$$

We can assume that

$$\langle \psi_1 | \psi_2 \rangle = 0 . \quad (A3)$$

Then

$$\frac{\langle \psi_T | H | \psi_T \rangle}{\langle \psi_T | \psi_T \rangle} = \frac{\langle \psi_1 | H | \psi_1 \rangle}{\langle \psi_1 | \psi_1 \rangle} + \frac{1}{f_0} \frac{\langle \psi_2 | H | \psi_1 \rangle + \text{c.c.}}{\langle \psi_1 | \psi_1 \rangle} . \quad (A4)$$

We now show that the second term on the right-hand side of (A4) does not contribute as $f_0 \rightarrow \infty$. Clearly only the pieces of H proportional to f_0 survive. But we have already shown that with our choice of radial wave function, all these pieces cancel to give a contribution of order 1 except for the term $(2f_0/N) \sum \mathbf{A}_m \cdot \theta_m \mathbf{A}_m \cdot \partial / \partial \theta_m$ which we had omitted. But the contribution of this term to $\langle \psi_2 | H | \psi_1 \rangle$ vanishes because $\mathbf{A}_m \cdot \partial / \partial \theta_m$ annihilates ψ_1 .

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