The quantum capacity with symmetric side channels

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We present an upper bound for the quantum channel capacity that is both additive and convex. Our bound can be interpreted as the capacity of a channel for high-fidelity communication when assisted by the family of all channels mapping symmetrically to their output and environment. The bound seems to be quite tight, and for degradable quantum channels it coincides with the unassisted channel capacity. Using this symmetric side channel capacity, we find new upper bounds on the capacity of the depolarizing channel. We also briefly indicate an analogous notion for distilling entanglement using the same class of (one-way) channels, yielding one of the few genuinely 1-LOCC monotonic entanglement measures.

I. INTRODUCTION

The archetypal problem in information theory is finding the capacity of a noisy channel to transmit messages with high fidelity. Already in [1], Shannon provided a simple formula for the capacity of a discrete memoryless channel, with single-letter capacity formulas of more general channels to follow later (see e.g. [2]).

The status of the quantum channel capacity question is not nearly as nice. While there has recently been significant progress in finding the quantum capacity of a quantum channel [3, 4, 5], the resulting expressions cannot be evaluated in any tractable way, with the exception of some very special channels (e.g., the capacity of the amplitude-damping [6], dephasing [7] and erasure [8] channels are known, most others are not). In fact, there are several capacities that can be defined for a quantum channel, and so far only two of them seem to admit single-letter formulas: the entanglement-assisted capacity [9, 10] and the environment-assisted quantum capacity [11, 12]. The multi-letter formulas available for the other capacities, including the quantum capacity, provide, at best, partial characterizations.

For instance, it was shown in [3, 4, 5, 13] that the capacity of a quantum channel $N$ is given by

$$Q(N) = \lim_{n \to \infty} \frac{1}{n} \max_{\lambda = A^n A^n'} I(A^n B^n) \omega_{A^n B^n},$$

where $\omega_{A^n B^n} = \text{id} \otimes N \otimes^n (\langle \phi | \phi \rangle_{A^n A^n'})$, and $I(A^n B^n) \omega_{A^n B^n} = S(\omega_{B^n}) - S(\omega_{A^n B^n})$ is known as the coherent information [13]. In order to evaluate this regularized formula one would have to perform an optimization over an infinite number of variables, making a numerical approach essentially impossible. Furthermore, it is known that the limit on the right is in general strictly larger than the corresponding single-letter expression [14, 15, 16]: there are channels, $N$, for which

$$Q^{(1)}(N) := \max_{\lambda = A B} I(A) \omega_{AB} < Q(N).$$

In the absence of an explicit formula for the quantum capacity, it is desirable to find upper and lower bounds for Eq. 1. Unfortunately, most known bounds are as difficult to evaluate in general as Eq. 1. Examples of upper bounds that can be easily evaluated, at least in some special cases, are given by the no-cloning based arguments of [17, 18], the semi-definite programming bounds of Rains [7, 19] and the closely related relative entropy of entanglement [20]. None of these is expected to be particularly tight—the last two are also upper bounds for the capacity assisted by two-way classical communication (which can be much larger than one-way), whereas the first is based solely on reasoning about where the channel’s capacity must be zero. As such, it would be useful to find new upper bounds for the quantum capacity that are both free of regularization and fundamentally one-way. In the following we present just such a bound.

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Inspired by the fact that allowing free forward classical communication does not increase the quantum channel capacity \[21\], we will consider the capacity of a quantum channel assisted by the use of a quantum channel that maps symmetrically to the receiver (Bob) and the environment (Eve). Such assistance channels, which we call symmetric side channels, can be used for forward classical communication but are apparently somewhat stronger. They can, however, immediately be seen to have zero quantum capacity, so that while the assisted capacity we find may in general be larger than the usual quantum capacity, one expects that it will provide a fairly tight upper bound. In particular, the symmetric side channel capacity (ss-capacity) we find will not be an upper bound for the capacity assisted by two-way classical communication.

The expression we find for the assisted capacity, which we’ll call \(Q_{ss}\), turns out to be much easier to deal with than Eq. (1) and has several nice properties. Most importantly, our expression is free of the regularization present in so many quantum channel capacities. We will also see that \(Q_{ss}\) is convex, additive, and that it is equal to \(Q\) for the family of degradable channels \[22\]. We will use these properties to find upper bounds on \(Q_{ss}\) of the depolarizing channel which, in turn, will give a significant improvement over known bounds for its unassisted capacity.

It should be emphasized that we have not found an upper bound on the dimension of the side channel needed to attain the assisted capacity, which in general prevents us from evaluating \(Q_{ss}\) explicitly or even numerically. While we cannot rule out such a bound, the arguments we use to establish several of \(Q_{ss}\)'s nice properties rely explicitly on the availability of an unbounded dimension. This suggests that dealing with an assistance channel of unbounded dimension may be the price we pay for such desirable properties as additivity and convexity, which is reminiscent of the findings of \[23, 24\].

II. DEFINITION AND PROPERTIES OF \(Q_{ss}\)

Before studying the symmetric side channel capacity, we must first make explicit some definitions. We will be concerned, as outlined above, with noisy channels \(\mathcal{N} : A \rightarrow B\) (i.e., completely positive and trace preserving linear maps). The Stinespring dilation theorem \[25\] tells us that we may always think of \(\mathcal{N}\) as an isometric embedding \(U_{\mathcal{N}} : A \hookrightarrow B \otimes E\) of \(A\) into a combined system of Bob and the environment (‘Eve’), followed by tracing over \(E\): \(\mathcal{N}(\rho) = \text{Tr}_E U_{\mathcal{N}\rho} U_{\mathcal{N}}^\dagger\). This dilation, of which we shall make free use, is unique up to equivalences of \(E\), ensuring that entropies will be well-defined.

We employ a more flexible notation for the coherent information than mentioned above \[26\]: Any \(|\phi\rangle\) living on a bipartite system \(A \otimes A\) determines a density operator on \(A \otimes B\) according to \(\omega_{AB} = (\text{id} \otimes \mathcal{N})|\phi\rangle \langle \phi|\). For any such state, we define the coherent information of \(A\) given \(B\) as

\[
I(A|B) := S(B) - S(AB) = -S(A|B),
\]

(3)

where we have introduced the shorthand \(S(A) = S(\omega_A)\) with the reduced state \(\omega_A = \text{Tr}_B \omega_{AB}\), and likewise \(S(AB) = S(\omega_{AB})\). If there is possible ambiguity as to which state we refer, it is added as a subscript index to the entropies.

Letting \(\mathcal{A}_d = \mathbb{C}^{(d^2+\delta)/2} \rightarrow S \subset \mathbb{T} \otimes \bot \rightarrow \mathcal{T} \simeq \mathbb{C}^d\),

(4)

the \(d\)-dimensional symmetric side channel.

We say that a rate \(R\) is ss-achievable if for all \(\epsilon > 0\) and all sufficiently large \(n\), there is a dimension \(d_n\), a code \(C_n \subset A^{\otimes n} \otimes S_{d_n}\) with \(\text{log dim} C_n \geq Rn\), and a decoding operation \(D_n\) such that for all states \(|\psi\rangle \in C_n\), the reconstructed state \(D_n\left([A^{\otimes n} \otimes \mathcal{A}_{d_n}]|\psi\rangle\langle \psi|\right)\) has a fidelity of at least \(1 - \epsilon\) with the original state \(|\psi\rangle\). The ss-capacity, which we will denote by \(Q_{ss}(\mathcal{N})\), is defined as the supremum of all ca-achievable rates.

Note that assistance by the symmetric channels includes free use of classical communication, as the dephasing operation \(|x\rangle \rightarrow |x\rangle|x\rangle\) is obtained by restricting \(\mathcal{A}_d\) to a subspace.

We are now in a position to introduce a quantity that will play a central role in our study of the ss-capacity. Letting \(\mathcal{N} : \hat{A} \rightarrow \hat{B}\) be a completely positive map, we define \(Q_{ss}^{(1)}(\mathcal{N})\) to be the supremum over all states \(|\phi\rangle\langle \phi|_{\mathcal{A}\bot\bot}\) that are invariant under the permutation of \(\mathcal{T}\) and \(\bot\), of the coherent information of \(A\) given \(B\mathcal{\hat{T}}\), evaluated after the \(\hat{A}\) register of \(\phi\) is acted on by \(\mathcal{N}\). That is, we let

\[
\omega_{A\mathcal{T}\bot} = (\text{id}_{A\bot} \otimes \mathcal{N})\phi_{A\mathcal{A}\mathcal{T}\bot},
\]

\[
Q_{ss}^{(1)}(\mathcal{N}) = \sup_{\phi_{A\mathcal{A}\mathcal{T}\bot}} I(A|B\mathcal{\hat{T}})_{\omega} = \sup_d Q^{(1)}(\mathcal{N} \otimes \mathcal{A}_d),
\]

(5)

(6)
where the supremum is over all pure states $\phi_{AA\top\bot}$ invariant under the swap $\top\leftrightarrow\bot$ of $\top$ and $\bot$. The alternative expression for $Q_{ss}^{(1)}(\mathcal{N})$ is seen as follows. On the one hand, for every state $|\phi\rangle \in A\mathcal{A}$, $(\mathbb{1}_{AA} \otimes C)|\phi\rangle$ is a state on $A\mathcal{A}\top\bot$ that is symmetric in $\top\bot$, so that the coherent information of $(\id \otimes \mathcal{N} \otimes \mathcal{A}_d)\phi_{A\mathcal{A}\mathcal{S}}$ is exactly $I(A)B\top$. On the other hand, if we have a pure state $\phi_{AA\top\bot}$ that is invariant under the exchange of $\top$ and $\bot$, it must be an eigenvector of the swap operator with eigenvalue 1 or $-1$. In the latter case we can extend $\top$ and $\bot$ with a qubit and tensor a singlet onto $|\phi\rangle$—this doesn’t change the coherent information but results in a vector $|\phi\rangle$ which is invariant under swapping $\top$ and $\bot$. As a result, $Tr_{AA}\phi$ is supported on the symmetric subspace of $\top\bot$ and we can present $|\phi\rangle$ as the image of a pure state under some $\mathbb{1}_{AA} \otimes C$.

For later use, we start by deriving a different formula for $Q_{ss}^{(1)}$.

**Lemma 1** For any channel $\mathcal{N}$ with Stinespring dilation $U_\mathcal{N} : A \rightarrow BE$,

$$Q_{ss}^{(1)}(\mathcal{N}) = \sup_{\rho_{AAF}} \frac{1}{2} \left[ I(A)BF - I(A)EF \right]$$

with respect to the state $\omega_{ABEF} = (\mathbb{1}_{AF} \otimes U_\mathcal{N})\rho(\mathbb{1}_{AF} \otimes U_\mathcal{N})^\dagger$.

**Proof** We may think of $\rho_{AAF}$ as the reduced state $Tr_{F'}\phi_{AAF'F'}$ of a pure state $|\phi\rangle$, and look at the information quantities in the lemma w.r.t. the state $(\mathbb{1}_{AF'} \otimes U_\mathcal{N})|\phi\rangle$. Then, it is an elementary identity that $I(A)EF = -I(A)BF'$, and in the r.h.s. of Eq. (7) the expression becomes

$$\frac{1}{2} \left[ I(A)BF + I(A)BF' \right].$$

Notice that if $\phi$ is symmetric under swapping $F$ and $F'$, this is equal to $I(A)BF$.

In general, we can, with $\top = FG$ and $\bot = F'G'$ (where $G$ and $G'$ label qubit registers), define

$$|\varphi\rangle_{AA\top\bot} = \left[ \left( \phi_{AAFF'} \right)_{G} |01\rangle_{GG'} + \left( \mathbb{1}_{AA} \otimes \SWAP_{BF'} \right) \phi_{AAFF'} |10\rangle_{GG'} \right],$$

and with respect to the state $\Omega_{AB\top\bot} = (\id_{AB\top\bot} \otimes \mathcal{N})|\varphi\rangle$,

$$\frac{1}{2} \left[ I(A)BF + I(A)BF' \right]_{\omega} = I(A)B\top\bot_{\omega},$$

and we are done. \qed

It will turn out that $Q_{ss}^{(1)}(\mathcal{N})$ is exactly the ss-capacity of $\mathcal{N}$.

**Theorem 2** For all channels $\mathcal{N}$, $Q_{ss}(\mathcal{N}) = Q_{ss}^{(1)}(\mathcal{N})$, where $Q_{ss}^{(1)} = \sup_{\phi_{AA\top\bot}} I(A)B\top \omega$ with $\omega_{AB\top\bot} = (\id_{AB\top\bot} \otimes \mathcal{N})\phi_{AA\top\bot}$ and the optimization is over all $\phi_{AA\top\bot}$ invariant under permuting $\top$ and $\bot$.

We will prove this with the following two lemmas.

**Lemma 3** $Q_{ss}^{(1)}$ is additive; that is, $Q_{ss}^{(1)}(\mathcal{N}_1 \otimes \mathcal{N}_2) = Q_{ss}^{(1)}(\mathcal{N}_1) + Q_{ss}^{(1)}(\mathcal{N}_2)$ for arbitrary channels $\mathcal{N}_1$ and $\mathcal{N}_2$.

**Proof** We use the previous lemma, and observe, for a state $\rho_{A\hat{A}_1\hat{A}_2\hat{F}}$, and

$$\omega_{AB_iE_1E_2F_2} = (\mathbb{1}_{AF} \otimes U_{\mathcal{N}_1} \otimes U_{\mathcal{N}_2})\rho(\mathbb{1}_{AF} \otimes U_{\mathcal{N}} \otimes U_{\mathcal{N}_2})^\dagger,$$

the identity (w.r.t. $\omega$)

$$I(A)B_1B_2F - I(A)E_1E_2F = (I(A)B_1B_2F - I(A)E_1B_2F) + (I(A)E_1B_2F - I(A)E_1E_2F).$$

If we introduce new auxiliary systems $F_1 := B_2F$ and $F_2 := E_1F$, the above right hand side becomes

$$(I(A)B_1F_1 - I(A)E_1F_1) + (I(A)B_2F_2 - I(A)E_2F_2),$$

which is evidently upper bounded by $Q_{ss}^{(1)}(\mathcal{N}_1) + Q_{ss}^{(1)}(\mathcal{N}_2)$. This shows $Q_{ss}^{(1)}(\mathcal{N}_1 \otimes \mathcal{N}_2) \leq Q_{ss}^{(1)}(\mathcal{N}_1) + Q_{ss}^{(1)}(\mathcal{N}_2)$.

Furthermore, by restricting the optimization in Eq. (8) to states of the form $\phi_{A_1\hat{A}_1U_{V_1}} \otimes \phi_{A_2\hat{A}_2U_{V_2}}$ we see that $Q_{ss}^{(1)}(\mathcal{N}_1 \otimes \mathcal{N}_2) \geq Q_{ss}^{(1)}(\mathcal{N}_1) + Q_{ss}^{(1)}(\mathcal{N}_2)$. \qed

The other ingredient we need is the following expression for the ss-capacity, which follows by standard arguments (see, e.g., [5]).
Lemma 4 The ss-capacity $Q_{ss}$ is given by the regularization of $Q_{ss}^{(1)}$: for any channel $N$,

$$Q_{ss}(N) = \lim_{n \to \infty} \frac{1}{n} Q_{ss}^{(1)}(N^\otimes n).$$

Proof To see that the ss-capacity is no less than the right hand side, note that for any $\phi_A \otimes B^n \otimes I$ symmetric under the interchange of $\top$ and $\perp$, the rate $\frac{1}{n} I(A^n)B^n \top$ is achievable by the quantum noisy channel coding theorem applied to the channel $N^\otimes n \otimes A_{d\tau}$. Furthermore, we will now show that the ss-capacity of a degradable channel is given

$$D \circ N$$

words, dilation is unique up to isometric equivalence of $E$ that results by tracing out system $B$ such that $D$ degradable if there exists a completely positive map, $\epsilon$. As mentioned above, every channel, degradable $Q$ is a convex function of the channel $Q$. Letting $\rho = (\id \otimes N \otimes A_{d\tau}) \phi$, $I(A^n)B^n \top \geq I(A^n)C_{(\id \otimes D_{B^n \top})\omega}$

$$\geq Rn - \frac{2}{e} - 8 \log(d_C) \sqrt{\epsilon} = Rn - \frac{2}{e} - 8Rn \sqrt{\epsilon},$$

so that $R \leq (1 - \sqrt{\epsilon})^{-1} \left( \frac{1}{n} Q_{ss}^{(1)}(N^\otimes n) + \frac{2}{ne} \right)$.

Lemma 4 and 5 immediately imply the expression for $Q_{ss}(N)$ quoted in Theorem 2. From Theorem 2 we can easily show

Proposition 5 $Q_{ss}$ is a convex function of the channel $N$.

Proof Letting $N_1$ and $N_2$ be channels and $\omega_i = (\id \otimes N_i \otimes A_d) \phi$, the convexity of $I(A)B^\top \omega_{AB^\top}$ gives us

$$I(A)B^\top \omega_{1} + \left(1 - p\right)\omega_2 \leq p I(A)B^\top \omega_1 + \left(1 - p\right) I(A)B^\top \omega_2,$$

where $p\omega_1 + \left(1 - p\right)\omega_2 = [\id \otimes (pN_1 + \left(1 - p\right)N_2) \otimes A_d] \phi$. This implies

$$\max_\phi I(A)B^\top \omega \leq p \max_\phi I(A)B^\top \omega_1 + \left(1 - p\right) \max_\phi I(A)B^\top \omega_2,$$

which tells us exactly that $Q_{ss}(pN_1 + \left(1 - p\right)N_2) \leq pQ_{ss}(N_1) + \left(1 - p\right)Q_{ss}(N_2)$.

III. IMPLICATIONS FOR THE UNASSISTED QUANTUM CAPACITY

In this section we explore some of the limitations that the ss-capacity places on the standard capacity of a quantum channel. As noted in the introduction, by simply not using the cloning channel provided, it is possible to communicate over a channel at the unassisted rate. In other words,

$$Q(N) \leq Q_{ss}(N).$$

Furthermore, as we will now see, this upper bound is actually an equality for the class of channels known as degradable. As mentioned above, every channel, $N$, can be expressed as an isometry $U_N : A \to BE$ followed by a partial trace, such that $N(\rho) = \Tr_B U_N^\dagger \rho U_N^\dagger$. The complementary channel of $N$, which we call $\hat{N}$, is the channel that results by tracing out system $B$ rather than the environment: $\hat{N}(\rho) = \Tr_B U_N^\dagger \rho U_N^\dagger$. Since the Stinespring dilation is unique up to isometric equivalence of $E$, $\hat{N}$ is well-defined up to isometries on the output. A channel is degradable if there exists a completely positive map, $D : B \to E$, which “degrades” the channel $N$ to $\hat{N}$. In other words, $D \circ N = \hat{N}$. The capacity of a degradable channel is given by the single letter maximization of the coherent information, as shown in [22]. Furthermore, we will now show that the ss-capacity of a degradable channel is given by the same formula. That is, the assistance channels we have been considering are of no use at all for a degradable channel.

Theorem 6 If $N$ is degradable, then $Q_{ss}(N) = Q(N)$.

Proof Fix $|\phi\rangle_{A\hat{A}S}$. Then, with respect to the state $\omega_{AB^\top} = (\id_A \otimes N \otimes A) \phi$,

$$I(A)B^\top \leq I(A^\top \perp)B + I(ABE^\top)$$

(14)
exactly when \( I(E; \perp) \leq I(B; \top) \), which is true if \( \mathcal{N} \) is degradable by the monotonicity of mutual information under local operations. This implies that the maximum value of the left hand side of Eq. (14) is no larger than the maximum of the right hand side. The maximum of the first term on the right is exactly the single-shot maximization of the coherent information, \( Q^{(1)}(\mathcal{N}) \), whereas the maximum of the second is zero (because of the no-cloning argument), so that \( I(\mathcal{A})B\top \leq Q(\mathcal{N}) \). Furthermore, by choosing a trivial assistance channel, the left hand side can attain the right hand side.

As an aside, we note that the definition of \( Q^{(1)}_\text{ss} \) can be reformulated in terms of degradable channels. In particular, we call a channel \( \mathcal{A} : A \rightarrow B \) with complementary channel \( \hat{\mathcal{A}} : A \rightarrow E \) bidegradable if both \( \mathcal{A} \) and \( \hat{\mathcal{A}} \) are degradable, which is equivalent to requiring the existence of channels \( \mathcal{D} : B \rightarrow E \) and \( \mathcal{D}' : E \rightarrow B \) such that \( \mathcal{D} \circ \mathcal{A} = \hat{\mathcal{A}} \) and \( \mathcal{D}' \circ \hat{\mathcal{A}} = \mathcal{A} \). Then, using the Stinespring theorem on such \( \mathcal{A} \) and the data processing inequality for the coherent information [28], we have

\[
Q^{(1)}_\text{ss}(\mathcal{N}) = \sup_{\mathcal{A} \text{ bidegradable}} Q^{(1)}(\mathcal{N} \otimes \mathcal{A}).
\]

Returning to our goal of finding upper bounds for \( Q \), we will make use of Theorem 6, which allows us to calculate the ss-capacity of any degradable channel. If a channel \( \mathcal{N} \) can be written as a convex combination of degradable channels, Theorem 6 together with the convexity of \( Q_\text{ss} \), provides an upper bound for \( Q_\text{ss}(\mathcal{N}) \) and therefore also \( Q(\mathcal{N}) \).

For instance, the depolarizing channel can be written as a convex combination of dephasing-type channels,

\[
\mathcal{B}_p(\rho) = (1 - p)\rho + \frac{p}{3} X\rho X + \frac{p}{3} Y\rho Y + \frac{p}{3} Z\rho Z = \frac{1}{3} X_p(\rho) + \frac{1}{3} Y_p(\rho) + \frac{1}{3} Z_p(\rho),
\]

where \( X_p(\rho) = (1 - p)\rho + pX\rho X \) and similarly for \( Y_p \) and \( Z_p \). From this we conclude that

\[
Q_\text{ss}(\mathcal{B}_p) \leq \frac{1}{3} Q_\text{ss}(X_p) + \frac{1}{3} Q_\text{ss}(Y_p) + \frac{1}{3} Q_\text{ss}(Z_p) = 1 - H_p,
\]

where we have used the fact that \( X_p \) is degradable and has ss-capacity \( 1 - H_p \) (Theorem 6). This reproduces the upper bounds of [7, 19, 20], which have been the best known for small \( N \) and the ss-capacity of any degradable channel. If a channel \( \mathcal{Q} \) channels, Theorem 6, together with the convexity of \( I \), so that \( \mathcal{Q} \) is coherent information, \( \mathcal{Q} \), implies that the maximum value of the left hand side of Eq. (14) is no larger than the maximum

\[
Q(\mathcal{Q}) = \sup_{\mathcal{A}} Q(\mathcal{A} \otimes \mathcal{Q}).
\]

As an aside, we note that the definition of \( Q^{(1)}_\text{ss} \) can be reformulated in terms of degradable channels. In particular, we call a channel \( \mathcal{A} : A \rightarrow B \) with complementary channel \( \hat{\mathcal{A}} : A \rightarrow E \) bidegradable if both \( \mathcal{A} \) and \( \hat{\mathcal{A}} \) are degradable, which is equivalent to requiring the existence of channels \( \mathcal{D} : B \rightarrow E \) and \( \mathcal{D}' : E \rightarrow B \) such that \( \mathcal{D} \circ \mathcal{A} = \hat{\mathcal{A}} \) and \( \mathcal{D}' \circ \hat{\mathcal{A}} = \mathcal{A} \). Then, using the Stinespring theorem on such \( \mathcal{A} \) and the data processing inequality for the coherent information [28], we have

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\]

where \( X_p(\rho) = (1 - p)\rho + pX\rho X \) and similarly for \( Y_p \) and \( Z_p \). From this we conclude that

\[
Q_\text{ss}(\mathcal{B}_p) \leq \frac{1}{3} Q_\text{ss}(X_p) + \frac{1}{3} Q_\text{ss}(Y_p) + \frac{1}{3} Q_\text{ss}(Z_p) = 1 - H_p,
\]

where we have used the fact that \( X_p \) is degradable and has ss-capacity \( 1 - H_p \) (Theorem 6). This reproduces the upper bounds of [7, 19, 20], which have been the best known for small \( p \).

We can also evaluate \( Q_\text{ss}(\mathcal{B}_p) \) for \( p = \frac{1}{4} \) as follows. For this value of \( p \), there is a CP-map which can be composed with the complementary channel, \( \mathcal{B}_p \), to generate \( \mathcal{B}_{\perp} \) [17]. This immediately implies \( Q_\text{ss}(\mathcal{B}_{1/4}) = 0 \), since otherwise both Bob and Eve could both reconstruct the encoded state with high fidelity, giving a violation of the no-cloning theorem. More explicitly, for any state \( \mid \phi \rangle_{A\perp A\perp} \) with the symmetry \( \top \leftrightarrow \perp \) we have, with respect to the state \( (\text{id} \otimes \mathcal{B}_{1/4})\phi \),

\[
I(\mathcal{A})B\top = -I(\mathcal{A})E\top \leq -I(\mathcal{A})B\top,
\]

from which we conclude \( Q_\text{ss}(\mathcal{B}_{1/4}) = 0 \), and where the second step is due to the quantum data processing inequality [28]. This reproduces the bound of [17], and furthermore, because the ss-capacity is convex, we find that

\[
Q(\mathcal{B}_p) \leq Q_\text{ss}(\mathcal{B}_p) \leq \text{conv}(1 - H(p), (1 - 4p)_+),
\]

with the notation

\[
x_+ = \begin{cases} x & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}
\]

It is important to note that the quantum capacity \( Q \) is not known to be convex and, indeed, may well not be—in the two way scenario, both nonadditivity and nonconvexity would imply [26] by the conjecture of [30] that a family of NPT Werner states is bound entangled. Thus, while the two bounds above were already known, it was not clear that the convex hull of these was also an upper bound.

We will now provide a tighter bound for \( Q_\text{ss}(\mathcal{B}_p) \), by expressing the depolarizing channel as a convex combination of amplitude-damping channels, which were shown to be degradable in [7]. The amplitude-damping channel can be expressed as

\[
\Delta_\gamma(\rho) = A_0\rho A_0^\dagger + A_1\rho A_1^\dagger,
\]

where

\[
A_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{1 - \gamma} & 0 \\ 0 & 0 & \sqrt{\gamma} \end{pmatrix} \quad \text{and} \quad A_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}.
\]
From this we find that
\[
\frac{1}{2} \Delta_\gamma (\rho) + \frac{1}{2} Y \Delta_\gamma (Y \rho Y) Y = N_{(q,q,p_z)}(\rho),
\]
(21)

where
\[
N_{(q,q,p_z)}(\rho) = (1 - 2q - p_z) \rho + q X \rho X + q Y \rho Y + p_z Z \rho Z,
\]
(22)

with \( q = \frac{1}{4} \) and \( p_z = \frac{1}{2} (1 - \frac{1}{2} - \sqrt{1 - \gamma}) \). The depolarizing channel can now be expressed as
\[
B_{2q + p_z} = \frac{1}{3} N_{(q,q,p_z)} + \frac{1}{3} N_{(q,p_z,q)} + \frac{1}{3} N_{(p_z,q,q)},
\]
(23)

so that \( B_p \) is a convex combination of amplitude damping channels with \( \gamma_p = 4 \sqrt{1 - p} (1 - \sqrt{1 - p}) \). This gives us an upper bound, shown in Figure 1, of
\[
Q(B_p) \leq Q_{ss}(B_p) \leq \text{conv}(Q(\Delta_{\gamma_p}), (1 - 4p)_+),
\]
(24)

where \( Q(\Delta_{\gamma_p}) \) is, according to [6], given by
\[
Q(\Delta_{\gamma_p}) = \max_{0 \leq t \leq 1} \left[ H_2(t(1 - \gamma_p)) - H_2(t \gamma_p) \right].
\]
(25)

The resulting bound is strictly stronger than the previously known bounds of \( 1 - H(p) \) and \( (1 - 4p)_+ \) for all \( 0.25 > p > 0.04 \).

![Figure 1: Our upper bound evaluated for the depolarizing channel: the straight solid turquoise line comes from no-cloning, the broken turquoise line is the capacity of a dephasing channel, and the broken red line is the capacity of the amplitude damping channel; finally, the solid red line is the convex hull of the first three, our best upper bound on \( Q_{ss}(B_p) \) and \( Q(B_p) \) so far; The blue solid line is the hashing (lower) bound, \( 1 - H(p) - p \log 3 \).](image)

IV. A LOWER BOUND FOR \( Q_{ss} \)

In this section we present a particular state relative to which the quantity optimized in Eq. (7) to give \( Q_{ss} \) is, for the depolarizing channel, strictly larger than hashing lower bound for \( Q_{ss} \) mentioned in the previous section. Letting
\[
|\phi \rangle = \sum_{s,t=0}^1 \sqrt{q_{st} X^s Z^t} \otimes 1|\Phi^+\rangle_A \tilde{A}|st \rangle_F,
\]
(26)
we have
\[
Q_{ss}^{(1)}(\mathcal{N}) = \sup_{\rho_{AAF}} \frac{1}{2} [I(A)BF - I(A)EF] \geq \frac{1}{2} [I(A)BF_{(id_{AF} \otimes \mathcal{E}_p)(\phi)} + I(A)B_{(id_{AF} \otimes \mathcal{E}_p)(\phi)}]
\]
for any choice of \(q_{st}\) with \(\sum_{st} q_{st} = 1\). For the depolarizing channel, the optimal such \(q_{st}\) is of the form
\[
q_{st} = (1 - q, q/3, q/3, q/3),
\]
which leads to entropies
\[
S(BF) = - \left[ \frac{1}{2} - \frac{4pq}{9} - 2\eta_{p,q} \right] \log \left[ \frac{1}{4} - \frac{2pq}{9} - \eta_{p,q} \right] - \left[ \frac{1}{2} - \frac{4pq}{9} + 2\eta_{p,q} \right] \log \left[ \frac{1}{4} - \frac{2pq}{9} + \eta_{p,q} \right] - \frac{8pq}{9} \log \left[ \frac{2pq}{9} \right]
\]
\[
S(AB) = - \left[ 1 - p - q + \frac{4pq}{3} \right] \log \left[ 1 - p - q + \frac{4pq}{3} \right] - \left[ p + q - \frac{4pq}{3} \right] \log \left[ \frac{p + q}{3} - \frac{4pq}{9} \right]
\]
\[
S(B) = 1
\]
\[
S(AFB) = H(p) + p \log 3,
\]
where
\[
\eta_{p,q} = \frac{1}{36} \sqrt{81 - 720pq - 512p^2q^2 + 576qp(p + q)}.
\]
This gives a lower bound of
\[
Q_{ss}(\mathcal{N}) \geq \frac{1}{2} (1 - H(p) - p \log 3)
\]
\[
+ \frac{1}{2} \left( - \left[ \frac{1}{2} - \frac{4pq}{9} - 2\eta_{p,q} \right] \log \left[ \frac{1}{4} - \frac{2pq}{9} - \eta_{p,q} \right] - \left[ \frac{1}{2} - \frac{4pq}{9} + 2\eta_{p,q} \right] \log \left[ \frac{1}{4} - \frac{2pq}{9} + \eta_{p,q} \right] - \frac{8pq}{9} \log \left[ \frac{2pq}{9} \right] \right)
\]
\[
- \frac{1}{2} \left( - \left[ 1 - p - q + \frac{4pq}{3} \right] \log \left[ 1 - p - q + \frac{4pq}{3} \right] - \left[ p + q - \frac{4pq}{3} \right] \log \left[ \frac{p + q}{3} - \frac{4pq}{9} \right] \right),
\]
which, optimized over \(q\), is plotted in Fig. 2. The resulting bound is nonzero up to \(p = 0.213\), which should be compared to the threshold of hashing at \(p = 0.1893\) and of the best known codes for the depolarizing channel at 0.19088 [14].

It is intriguing that the form of Eq. (29) corresponds to a preprocessing of \(\mathcal{N}\)'s input by applying a depolarizing channel whose environment is \(F\), then sending \(F\) through the side channel, with the optimal level of preprocessing noise increasing to \(3/4\) as \(\mathcal{N}\)'s noise level increases.

V. ONE-WAY DISTILLATION WITH SYMMETRIC SIDE CHANNELS

Based on the connection between quantum channel capacities and one-way LOCC assisted entanglement distillation [21, 31], we can define a similar symmetric side channel assisted distillation notion for bipartite states \(\rho_{AB}\):
\[
D_{ss}^{(1)}(\rho) = \sup_{\sigma, S} I(A')BF_{(S \otimes id_{AB}) \rho \otimes \sigma},
\]
where the supremum is over states \(\sigma_{AB\mathcal{E}}\) (such that \(\tilde{B} \simeq \tilde{E}\)) with the property \(\sigma_{\mathcal{A}} = \sigma_{\mathcal{E}}\) and operations on Alice's system \(S: \mathcal{A} \rightarrow \mathcal{A}'\). Observe that these states (or rather their restrictions \(\sigma_{AB}\)) are often called two-shareable in the literature. Note also that w.l.o.g. we may restrict to pure states, at the expense of increasing the dimension of their local supports (which, in any case, is unbounded in the above definition).

For a state \(\rho_{AB}\) with purification \(|\phi\rangle_{ABE}\) and w.r.t. the state \(\omega_{A'BEF} = (T_A \otimes id_{BE})\phi\), we have the analogue of Lemma [4]
\[
D_{ss}^{(1)}(\rho) = \sup_{T: \mathcal{A} \rightarrow \mathcal{A}'} \frac{1}{2} (I(A')BF - I(A')EF).
\]
FIG. 2: Our lower bound for the symmetric side channel capacity of the depolarizing channel: The dotted curve is the hashing lower bound for $Q_{ss}$, which in this case is $1 - H(p) - p \log 3$. The solid curve is Eq. (30), evaluated for the optimal value of $q$. The dashed curve is the optimal value of $q$.

Just as for channels, we find that $D_{ss}^{(1)}$ is additive, convex and indeed a one-way LOCC entanglement monotone, reducing to the entropy of entanglement for pure states, and vanishing for all two-shareable states. Furthermore, $D_{ss}^{(1)}(\rho)$ has an operational meaning—it is the one-way distillable entanglement of $\rho$ when assisted by arbitrary two-shareable states.

The notion of degradability of channels is translated to states as follows: $\rho_{AB}$ is called degradable if, for its purification $\phi_{ABE}$, there exists a quantum channel $D : B \to E$ such that $\phi_{AE} = (\id_A \otimes D)\rho_{AB}$. The analogue of the bidegradable channels are states $\sigma_{ABE}$ such that there are channels degrading both ways, $B \to E$ and $E \to B$.

Analogously to our findings for channels, we can prove that $D^{(1)}_{ss}(\rho) = D^{(1)}(\rho)$ for degradable states, so that the upper bounds in the previous section on the quantum capacity of the depolarizing channels, including Fig. 1, translate into upper bounds on the one-way distillable entanglement of two-qubit Werner states.

VI. QUANTUM VALUE ADDED

In Section III we saw that the ss-capacity of a degradable channel is equal to its unassisted capacity. In fact, we have not been able to show a separation between the ss-capacity and the unassisted capacity for any channel. The question arises: Are there $N$ such that $Q_{ss}(N) > Q(N)$?

Motivated by this question, for any CPTP map $B$, we define the value added of $B$ to be

$$V^{(1)}(B) := \sup_N \left[ Q^{(1)}(N \otimes B) - Q^{(1)}(N) \right]. \quad (33)$$

In words, $V^{(1)}(B)$ is the largest increase in the optimized coherent information that $B$ can provide when used as a side channel for some other $N$. This definition has the appealing property that $V^{(1)}$ is sub-additive, since

$$V^{(1)}(B_1 \otimes B_2) = \sup_N \left[ Q^{(1)}(N \otimes B_1 \otimes B_2) - Q^{(1)}(N) \right]$$

$$\leq \sup_N \left[ Q^{(1)}(N \otimes B_1 \otimes B_2) - Q^{(1)}(N \otimes B_2) \right] + \sup_N \left[ Q^{(1)}(N \otimes B_2) - Q^{(1)}(N) \right]$$

$$\leq V^{(1)}(B_1) + V^{(1)}(B_2).$$

Letting

$$V(B) := \lim_{n \to \infty} \frac{1}{n} V^{(1)}(B \otimes^n),$$

where $B \otimes^n$ denotes the $n$-fold tensor product of $B$.
we have $V(B) \leq V^{(1)}(B)$, and furthermore, for all $\epsilon > 0$ and sufficiently large $n$

$$V^{(1)}(B^{\otimes n}) = \sup_N \left[ Q^{(1)}(\mathcal{N} \otimes B^{\otimes n}) - Q^{(1)}(\mathcal{N}) \right]$$

$$\geq \left[ Q^{(1)}(B^{\otimes n} \otimes B^{\otimes n}) - Q^{(1)}(B^{\otimes n}) \right]$$

$$\geq (2n) (Q(B) - \epsilon) - nQ(B),$$

so that

$$\frac{1}{n} V^{(1)}(B^{\otimes n}) \geq Q(B) - 2\epsilon,$$

which gives us $V^{(1)}(B) \geq V(B) \geq Q(B)$.

In addition to this upper bound for the capacity, $V^{(1)}$ also provides a sufficient condition for $Q_{ss}(\mathcal{N}) = Q(\mathcal{N})$:

$$Q_{ss}(\mathcal{N}) - Q(\mathcal{N}) = \lim_{n \to \infty} \frac{1}{n} \left( \sup_{d} Q^{(1)}(\mathcal{N}^{\otimes n} \otimes \mathcal{A}_{d}) - Q^{(1)}(\mathcal{N}^{\otimes n}) \right)$$

$$\leq \lim_{n \to \infty} \frac{1}{n} \left( \sup_{d} \sup_{\mathcal{M}} \left( Q^{(1)}(\mathcal{M} \otimes \mathcal{A}_{d}) - Q^{(1)}(\mathcal{M}) \right) \right)$$

$$\leq \sup_{d} V^{(1)}(\mathcal{A}_{d}),$$

so that $Q_{ss}(\mathcal{N}) = Q(\mathcal{N})$ for all $\mathcal{N}$ as long as $V^{(1)}(\mathcal{A}_{d}) = 0$ for all $d$. Unfortunately, although Eq. (33) is nominally single-letter, evaluating $V^{(1)}$ seems to be quite difficult, as it contains an optimization over an infinite number of variables.

VII. DISCUSSION

We have studied the capacity of a quantum channel given the assistance of an arbitrary symmetric side channel. The capacity formula we find is in many ways more manageable than the known expression for the (unassisted) quantum capacity, and we are able to establish that the ss-capacity is both convex and additive. By taking advantage of the convexity of $Q_{ss}$ and the fact that $Q_{ss}$ and $Q$ coincide for degradable channels, we presented a general method for finding upper bounds to $Q$ and in particular provided a bound for the capacity of the depolarizing channel that is stronger than any previously known result.

We have left many questions unanswered. The most pressing is whether it is possible to bound the dimension of the symmetric side channel needed to achieve the ss-capacity. Such a bound would allow us to evaluate $Q_{ss}(\mathcal{N})$ efficiently, which we expect would provide very tight bounds on $Q$ in many cases.

So far, we have not been able to find a channel for which the ss-capacity and capacity differ. We expect that such channels exist, and a better understanding of when the two capacities differ may point towards simplifications of the quantum capacity formula in Eq. (1).

It is worth mentioning that we first discovered the unsymmetrized version of the quantity $Q_{ss}^{(1)}$ given in Lemma 1, and that it is an upper bound for $Q$. This was motivated by the quest to find the entanglement analogue of the upper bound on distillable key presented in [32, 33]. It was only later that it became clear that the formula could be made symmetric and interpreted as the quantum capacity of a channel given the family of assistance channels we have considered. The upper bound of [32, 33] can be understood similarly as the one-way distillable key of a ccq-state, assisted by cq-channels mapping symmetrically from Alice’s (classical) data to states of Bob/Eve.

Finally, it should be noted that the approach we have taken here is qualitatively similar to the work of [2, 13, 20] in the two-way scenario. In that work, it was found that enlarging the set of operations allowed for entanglement distillation from LOCC to the easier-to-deal-with set of separable or positive-partial-transpose-(PPT-)preserving operations made it possible to establish tighter bounds on two-way distillable entanglement than was possible by considering LOCC protocols directly. Similarly, we have shown that by augmenting a channel with a zero capacity side channel, a dramatically simplified capacity formula can be found that allows us to establish tighter bounds on the unassisted capacity than were possible by direct considerations. To what extent this approach can be used in general, the reason such an approach works at all, and the tightness of the bounds achieved in this way are all questions that we leave wide open.
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