Thermodynamics of Rotating Solutions in Gauss-Bonnet-Maxwell Gravity and the Counterterm Method

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Abstract

By a suitable transformation, we present the \((n + 1)\)-dimensional charged rotating solutions of Gauss-Bonnet gravity with a complete set of allowed rotation parameters which are real in the whole spacetime. We show that these charged rotating solutions present black hole solutions with two inner and outer event horizons, extreme black holes or naked singularities provided the parameters of the solutions are chosen suitable. Using the surface terms that make the action well-defined for Gauss-Bonnet gravity and the counterterm method for eliminating the divergences in action, we compute finite action of the solutions. We compute the conserved and thermodynamical quantities through the use of free energy and the counterterm method, and find that the two methods give the same results. We also find that these quantities satisfy the first law of thermodynamics. Finally, we perform a stability analysis by computing the heat capacity and the determinant of Hessian matrix of mass with respect to its thermodynamic variables in both the canonical and the grand-canonical ensembles, and show that the system is thermally stable. This is commensurate with the fact that there is no Hawking-Page phase transition for black objects with zero curvature horizon.

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I. INTRODUCTION

The thermodynamics of asymptotically anti-de Sitter (AdS) black holes, which produces an aggregate of ideas from thermodynamics, quantum field theory and classical gravity, continues to attract a great deal of attention. One reason for this is the role of AdS/CFT duality \[1\], and in particular, Witten interpretation of the Hawking-Page phase transition between thermal AdS and asymptotically AdS black hole as the confinement-deconfinement phases of the Yang-Mills theory defined on the asymptotic boundaries of the AdS geometry \[2\]. The fact that thermodynamics of black holes may be modified by using other theories of gravity with higher derivative (HD) curvature terms provides a strong motivation for considering thermodynamics of black holes in HD gravity \[3\]. The appearance of HD gravitational terms can be seen, for example, in the renormalization of quantum field theory in curved spacetime \[4\], or in the construction of low-energy effective action of string theory \[5\]. Among the theories of gravity with higher derivative curvature terms, the Gauss-Bonnet gravity has some special features in some sense. Indeed, in order to have a ghost-free action, the quadratic curvature corrections to the Einstein-Hilbert action should be proportional to the Gauss-Bonnet term \[6, 7\]. This combination also appear naturally in the next-to-leading order term of the heterotic string effective action, and plays a fundamental role in Chern-Simons gravitational theories \[8\]. From a geometric point of view, the combination of the Einstein-Gauss-Bonnet terms constitutes, for five dimensional spacetimes, the most general Lagrangian producing second order field equations, as in the four-dimensional gravity which the Einstein-Hilbert action is the most general Lagrangian producing second order field equations \[9\].

To investigate the thermodynamics of black holes, one should compute the finite conserved and thermodynamic quantities of the system. In general the gravitational action is divergent when evaluated on solutions, as is the Hamiltonian and other associated conserved quantities. A common approach to evaluating thermodynamic quantities has been to carry out all computations relative to some other spacetime that is regarded as the ground state for the class of spacetimes of interest. This could be done by taking the original action for gravity coupled to matter fields and subtracting from it a reference action, which is a functional of the induced metric $\gamma$ on the boundary $\partial M$ \[10\]. Conserved and/or thermodynamic quantities are then computed relative to this boundary, which can then be taken to infinity if
desired. Unfortunately, it suffers from several drawbacks. The choice of reference spacetime is not always unique [11], nor is it always possible to embed a boundary with a given induced metric into the reference background. An extension of this approach which addresses these difficulties was developed based on the conjectured AdS/CFT correspondence for asymptotic AdS spacetimes [12, 13]. Applications of AdS/CFT correspondence also include computations of conserved quantities for black holes with rotation, NUT charge, various topologies, rotating black strings with zero curvature horizons and rotating higher genus black branes [14, 15, 16]. Although the counterterm method applies for the case of a specially infinite boundary, it was also employed for the computation of the conserved and thermodynamic quantities in the case of a finite boundary [17]. It is believed that appending a counterterm, $I_{ct}$, to the action which depends only on the intrinsic geometry of the boundary(ies) can remove the divergences. This requirement, along with general covariance, implies that these terms are functionals of curvature invariants of the induced metric and have no dependence on the extrinsic curvature of the boundary(ies) [13]. An algorithmic procedure exists for constructing $I_{ct}$ for asymptotic AdS in Einstein gravity and so its determination in this theory is unique [18]. But in HD gravity these counterterms have been not introduced till now. Since these counterterms should be functions of intrinsic curvature of boundary, the construction of these terms for spacetimes with flat boundaries is straightforward [19, 20].

The outline of our paper is as follows. We give a brief review of the field equations of Gauss-Bonnet gravity and the counterterm method for calculating conserved quantities in Sec. II. In Sec. III we obtain mass, angular momentum, entropy, temperature, charge, and electric potential of the $(n + 1)$-dimensional black hole solutions with a complete set of rotational parameters and show that these quantities satisfy the first law of thermodynamics. We also perform a local stability analysis of the black holes in the canonical and grand canonical ensembles. We finish our paper with some concluding remarks.
II. THE ACTION AND CONSERVED QUANTITIES

The gravitational action of an \((n+1)\)-dimensional asymptotically anti-de Sitter spacetime \(\mathcal{M}\) with the Gauss-Bonnet term in the presence of an electromagnetic field is:

\[
I_G = -\frac{1}{16\pi} \int_{\mathcal{M}} d^{n+1}x \sqrt{-g} \left\{ R - 2\Lambda + \alpha \left( R_{\mu\nu\sigma\kappa} R^{\mu\nu\sigma\kappa} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \right) - F_{\mu\nu} F^{\mu\nu} \right\} \\
- \frac{1}{8\pi} \int_{\partial\mathcal{M}} d^n x \sqrt{-\gamma} \left\{ \Theta + 2\alpha \left( J - 2G_{ab} \Theta^{ab} \right) \right\} 
\]

(1)

where \(F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}\) is electromagnetic tensor field, \(A_{\mu}\) is the vector potential, \(\Lambda = -\frac{n(n-1)}{2l^2}\) is the cosmological constant, \(\alpha\) is Gauss-Bonnet coefficient with dimensions \((\text{length})^2\) which we assume to be positive as in heterotic string theory, \(R_{\mu\nu\sigma\kappa}\), \(R_{\mu\nu}\) and \(R\) are Riemann tensor, Ricci tensor and Ricci scalar of the manifold \(\mathcal{M}\) respectively, \(\gamma_{ab}\) is induced metric on the boundary \(\partial\mathcal{M}\), \(\Theta\) is trace of extrinsic curvature of the boundary, \(G_{ab}(\gamma)\) is Einstein tensor calculated on the boundary, and \(J\) is trace of:

\[
J_{ab} = \frac{1}{3} (\Theta_{cd} \Theta^{cd} \Theta_{ab} + 2\Theta \Theta^{c} \Theta_{b} - 2\Theta \Theta^{cd} \Theta_{db} - \Theta^2 \Theta_{ab}) 
\]

(2)

The second integral is a surface term which is chosen such that the variational principle is well-defined \([21, 22]\). In order to obtain the field equations by the variation of the volume integral with respect to the fields, one should impose the boundary condition \(\delta A_{\mu} = 0\) on \(\partial\mathcal{M}\). Thus the action \((\ref{eq:action})\) is appropriate to study the grand-canonical ensemble with fixed electric potential \([23]\). To study the canonical ensemble with fixed electric charge one should impose the boundary condition \(\delta(n^a F_{ab}) = 0\), and therefore the gravitational action is \([24]\):

\[
\tilde{I}_G = I_G - \frac{1}{4\pi} \int_{\partial\mathcal{M}_\infty} d^n x \sqrt{-\gamma n_a F^{ab} A_b}, 
\]

(3)

where \(n_a\) is the normal to the boundary \(\partial\mathcal{M}\). Varying the action \((\ref{eq:action})\) or \((\ref{eq:canonical_action})\) with respect to the metric tensor \(g_{\mu\nu}\) and electromagnetic tensor field \(F_{\mu\nu}\), with appropriate boundary condition, the equations of gravitation and electromagnetic fields are obtained as

\[
G_{\mu\nu} + \Lambda g_{\mu\nu} + \alpha G_{\mu\nu}^{(2)} = T_{\mu\nu} 
\]

(4)

\[
\nabla_{\mu} F^{\mu\nu} = 0 
\]

(5)

where \(G_{\mu\nu}\) is the Einstein tensor, \(T_{\mu\nu}^{(\text{em})} = 2F_{\mu}^{\rho} F_{\rho\nu} - \frac{1}{2} F_{\rho\sigma} F^{\rho\sigma} g_{\mu\nu}\) is the energy-momentum tensor of electromagnetic field and \(G_{\mu\nu}^{(2)}\) is second order Lovelock tensor:

\[
G_{\mu\nu}^{(2)} = 2(R_{\mu\sigma\kappa\tau} R^{\sigma\kappa\tau} - 2R_{\mu\nu\rho\sigma} R^{\rho\sigma} - 2R_{\mu\sigma} R^{\sigma}_{\nu} + RR_{\mu\nu}) \\
- \frac{1}{2} \left( R_{\mu\sigma\rho\kappa} R^{\mu\rho\sigma\kappa} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \right) g_{\mu\nu} 
\]

(6)
In general the action $I_G$, is divergent when evaluated on the solutions, as is the Hamiltonian and other associated conserved quantities. Rather than eliminating these divergences by incorporating reference term, a counterterm $I_{ct}$ is added to the action which is functional only of the boundary curvature invariants. For asymptotically AdS spacetimes with flat boundary, all the counterterm containing the curvature invariants of the boundary are zero and therefore the counterterm reduces to a volume term as

$$I_{ct} = -\frac{(n-1)}{8\pi L} \int_{\partial M} d^n x \sqrt{-\gamma} \tag{7}$$

where $L$ is a length factor that depends on the fundamental constants $l$ and $\alpha$, that must reduce to $l$ as $\alpha$ goes to zero. Thus, the total action, $I$, can be written as

$$I = I_G + I_{ct} \tag{8}$$

Having the total finite action, one can use Brown and York definition \[10\] to construct a divergence free stress-energy tensor as

$$T^{ab} = \frac{1}{\kappa} \left\{ \left( \Theta^{ab} - \Theta_i^{ab} \right) + 2\alpha \left( 3J^{ab} - J_{i}^{ab} \right) + \frac{n-1}{L} \gamma^{ab} \right\} \tag{9}$$

To compute the conserved charges of the spacetime, one should choose a spacelike hypersurface $B$ in $\partial M$ with metric $\sigma_{ij}$, and write the boundary metric in ADM form:

$$\gamma_{ab} dx^a dx^a = -N^2 dt^2 + \sigma_{ij} \left( d\phi^i + V^i dt \right) \left( d\phi^j + V^j dt \right),$$

where the coordinates $\phi^i$ are the angular variables parameterizing the hypersurface of constant $r$ around the origin. When there is a Killing vector field $\xi$ on the boundary, then the conserved quantities associated with the stress tensors of Eq. (9) can be written as

$$Q(\xi) = \int_B d^{n-1} \phi \sqrt{\sigma} T_{ab} n^a \xi^b, \tag{10}$$

where $\sigma$ is the determinant of the metric $\sigma_{ij}$ and $n^a$ is the unit normal vector to the boundary $B$. For boundaries with timelike ($\xi = \partial/\partial t$) and rotational Killing vector fields ($\zeta = \partial/\partial \phi$), we obtain

$$M = \int_{B_\infty} d^{n-1} \phi \sqrt{\sigma} T_{ab} n^a \zeta^b, \tag{11}$$

$$J = \int_{B_\infty} d^{n-1} \phi \sqrt{\sigma} T_{ab} n^a \zeta^b, \tag{12}$$
provided the surface $\mathcal{B}$ contains the orbits of $\zeta$. These quantities are, respectively, the conserved mass and angular momentum of the system enclosed by the boundary. Note that they will both be dependent on the location of the boundary $\mathcal{B}$ in the spacetime, although each is independent of the particular choice of foliation $\mathcal{B}$ within the boundary $\partial \mathcal{M}$. In the context of the (A)dS/CFT correspondence, the limit in which the boundary $\mathcal{B}$ becomes infinite ($\mathcal{B}_\infty$) is taken, and the counterterm prescription ensures that the action and conserved charges are finite. No embedding of the surface $\mathcal{B}$ into a reference spacetime is required and the quantities which are computed are intrinsic to the spacetimes.

III. THERMODYNAMICS OF BLACK BRANES

The rotation group in $n+1$ dimensions is $SO(n)$ and therefore the number of independent rotation parameters for a localized object is equal to the number of Casimir operators, which is $[n/2] \equiv k$, where $[n/2]$ is the integer part of $n/2$. Asymptotically AdS solution of the field equations (4) with cylindrical symmetry with $k$ rotation parameter $a_i$, can be written as [25]:

$$ds^2 = -f(\rho) \left( \Xi dt - \sum_{i=1}^{k} a_i d\phi_i \right)^2 + \frac{\rho^2}{l^4} \sum_{i=1}^{k} \left( a_i dt - \Xi l^2 d\phi_i \right)^2$$

$$- \frac{\rho^2}{l^2} \sum_{i=1}^{k} (a_i d\phi_j - a_j d\phi_i)^2 + \frac{d\rho^2}{f(\rho)} + \rho^2 dX^2,$$

$$\Xi^2 = \sum_{i=1}^{k} \left( 1 + \frac{a_i^2}{l^2} \right),$$

$$A_\mu = \sqrt{\frac{(n-1)}{2(n-2)}} \frac{q}{\rho^{n-2}} (\Xi \delta_\mu^0 - a_i \delta_\mu^i), \quad \text{(no sum on } i).$$

where $dX^2$ is the Euclidean metric on the $(n-k-1)$-dimensional submanifold with volume $\omega_{n-k-1}$ and $f(\rho)$ is

$$f(\rho) = \frac{\rho^2}{2(n-2)(n-3)} \left\{ 1 - \sqrt{1 - 4(n-2)(n-3)\alpha \left( \frac{1}{l^2} - \frac{m}{\rho^n} + \frac{q^2}{\rho^{2n-2}} \right)} \right\}, \quad \text{(13)}$$

As we will see in the next section, $m$ and $q$ are related to the total mass and total charge of spacetimes. Although $f(\rho)$ for the uncharged solution ($q = 0$) is real in the whole range $0 \leq \rho < \infty$ provided $\alpha \leq l^2/4(n-2)(n-3)$, for charged solution it is real only in the range
\[ r_0 \leq \rho < \infty \text{ where } r_0 \text{ is the largest real root of the following equation:} \]
\[
\left( \frac{1}{4(n-2)(n-3)} - \frac{\alpha}{l^2} \right) r_0^{(2n-2)} + \alpha m r_0^{(n-2)} - \alpha q^2 = 0 \tag{14}
\]
One may note that the only solution of Eq. (14) for the case of uncharged solution is \( r_0 = 0 \).

In order to restrict the spacetime to the region \( \rho \geq r_0 \), we introduce a new radial coordinate \( r \) as:
\[
r^2 = \rho^2 - r_0^2 \Rightarrow d\rho^2 = \frac{r^2}{r^2 + r_0^2} dr^2.
\]

With this new coordinate, the above metric becomes:
\[
ds^2 = -f(r) \left( \Xi dt - \sum_{i=1}^{k} a_i d\phi_i \right)^2 + \frac{(r^2 + r_0^2)}{l^4} \sum_{i=1}^{k} \left( a_i dt - \Xi l^2 d\phi_i \right)^2
- \frac{r^2 + r_0^2}{l^2} \sum_{i=1}^{k} (a_i d\phi_j - a_j d\phi_i)^2 + \frac{r^2 dr^2}{(r^2 + r_0^2) f(r)} + (r^2 + r_0^2) d\Omega^2, \tag{15}
\]

where now the gauge potential and the metric function are:
\[
A_\mu = \sqrt{\frac{(n-1)}{2(n-2) (r^2 + r_0^2)^{\frac{n-2}{2}}}} ((\Xi \delta^0_\mu - a_i \delta^i_\mu), \quad \text{(no sum on } i) \tag{16}.
\]
\[
f(r) = \frac{r^2 + r_0^2}{2(n-2)(n-3) \alpha} \left\{ 1 - \sqrt{1 - 4(n-2)(n-3) \left( \frac{\alpha}{l^2} - \frac{\alpha m (r^2 + r_0^2)}{2} + \frac{\alpha q^2}{(r^2 + r_0^2)^{n-1}} \right)^2 - 4} \right\} \tag{17}
\]

It is notable to mention that the in the static case \( a_i = 0 \), in contrast with the case of uncharged solution, \( g_{tt} \) is not equal to \( g_{rr}^{-1} \).

### A. Properties of the solutions

One can show that the Kretschmann scalar \( R_{\mu\nu\lambda\kappa} R^{\mu\nu\lambda\kappa} \) diverges at \( r = 0 \), and therefore there is a curvature singularity located at \( r = 0 \). Seeking possible black hole solutions, we turn to looking for the existence of horizons. As in the case of rotating black hole solutions of Einstein gravity, the above metric given by Eqs. (15)-(17) has both Killing and event horizons. The Killing horizon is a null surface whose null generators are tangent to a Killing field. The proof that a stationary black hole event horizon must be a Killing horizon in the four-dimensional Einstein gravity [26] cannot obviously be generalized to higher order gravity. However the result is true for all known static solutions. Although our solution is not static, the Killing vector
\[
\chi = \partial_t + \sum_{i=1}^{k} \Omega_i \partial_{\phi_i}, \tag{18}
\]
is the null generator of the event horizon, where \( k \) denotes the number of rotation parameters. The event horizon defined by the solution of \( g^{rr} = f(r) = 0 \). The metric of Eqs. (15)-(17) has two inner and outer event horizons located at \( r_- \) and \( r_+ \), provided the mass parameter \( m \) is greater than \( m_{\text{ext}} \) given as:

\[
m_{\text{ext}} = 2 \left( \frac{n-1}{n-2} \right) \left( \frac{n}{n-2} \right)^{-\frac{n-2}{2(n-1)}} l^{-\frac{n-2}{n-1}} q^{n-1}.
\]  

(19)

In the case that \( m = m_{\text{ext}} \), we will have an extreme black brane. One can obtain the angular velocity of the event horizon by analytic continuation of the metric. Setting \( a_i \to ia_i \) yields the Euclidean section of (15), whose regularity at \( r = r_+ \) requires that we should identify \( \phi_i \sim \phi_i + \beta_+ \Omega_i \), where \( \Omega_i \)'s are the angular velocities of the outer event horizon. One obtains:

\[
\Omega_i = \frac{a_i}{\Xi l^2}.
\]  

(20)

The temperature may be obtained through the use of definition of surface gravity,

\[
T_+ = \frac{1}{2\pi} \sqrt{-\frac{1}{2} (\nabla_{\mu} \chi_{\nu}) (\nabla^\mu \chi^\nu)},
\]  

(21)

where \( \chi \) is the Killing vector (18). One obtains:

\[
\beta^{-1} = T_+ = \frac{f'(r_+)}{4\pi\Xi} \sqrt{1 + \frac{r_0^2}{r_+^2}} = \frac{n(r_+^2 + r_0^2)^{n-1} - (n-2)l^2q^2}{4\pi\Xi l^2 r_+(r_+^2 + r_0^2)^{n-2}}
\]  

(22)

One may note that the temperature (22) reduces to the temperature of the uncharged solution for the uncharged solutions where \( r_0 = 0 \) [25]. Next, we calculate the electric charge of the solutions. To determine the electric field we should consider the projections of the electromagnetic field tensor on special hypersurfaces. The normal to such hypersurfaces is

\[
u^0 = \frac{1}{N}, \quad \nu^r = 0, \quad \nu^i = \frac{V^i}{N},
\]  

(23)

and the electric field is \( E^\mu = g^{\mu\nu} F_{\rho\nu} u^\nu \), where \( N \) and \( V^i \) are the lapse and shift functions. Then the electric charge per unit volume \( V_{n-1} \) can be found by calculating the flux of the electric field at infinity, yielding

\[
Q = \Xi \frac{1}{4\pi} \sqrt{\frac{(n-1)(n-2)}{2}} q.
\]  

(24)

The electric potential \( \Phi \), measured at infinity with respect to the horizon, is defined by [23, 27]:

\[
\Phi = A_\mu \chi^\mu \big|_{r \to \infty} - A_\mu \chi^\mu \big|_{r = r_+},
\]  

(25)
where $\chi$ is the null generator of the horizon given by Eq. (18). We find
\[
\Phi = \sqrt{\frac{(n-1)}{2(n-2)}} q \Xi(r_+^2 + r_0^2)^{\frac{n-2}{2}}.
\] (26)

B. Action and conserved quantities

The action and conserved quantities associated with the spacetime described by (15) can be obtained via the counterterm method. Using Eqs. (1), (3) and (7), the Euclidean actions will be finite provided
\[
\frac{1}{L} = \frac{1}{3\sqrt{2(n-2)(n-3)}} \frac{l^2 + 4(n-2)(n-3) \alpha - 4\sqrt{l^2 - 4(n-2)(n-3) \alpha}}{l \sqrt{\alpha(l - \sqrt{l^2 - 4(n-2)(n-3) \alpha})}}
\] (27)

where we note that $L$ reduces to $l$ as $\alpha$ goes to zero. Denoting the volume of the hypersurface boundary at constant $t$ and $r$ by $V_{n-1} = (2\pi)^k \omega_{n-k-1}$, the Euclidean actions per unit volume $V_{n-1}$ in canonical and grand-canonical ensembles can then be obtained through the use of Eqs. (1), (3) and (7). We find:
\[
I_n = -\frac{\beta}{16\pi} \frac{(r_+^2 + r_0^2)^{n-1} + q^2 l^2}{l^2 (r_+^2 + r_0^2)^{\frac{n}{2}-1}},
\] (28)
\[
\tilde{I}_n = -\frac{\beta}{16\pi} \frac{(r_+^2 + r_0^2)^{n-1} - (2n-3)q^2 l^2}{l^2 (r_+^2 + r_0^2)^{\frac{n}{2}-1}}.
\] (29)

We first obtain the action in grand canonical ensemble as a function of the intensive quantities $\beta$, $\Omega$ and $\Phi$ by using the expression for the temperature, the angular velocity and the potential given in Eqs. (22), (20) and (26):
\[
I = -\frac{\beta(r_+^2 + r_0^2)^{n-2}}{16\pi l^2 (r_+^2 + r_0^2)^{\frac{n}{2}-1}} \left( (r_+^2 + r_0^2) + \frac{2(n-2)l^2}{(n-1)(1-\Omega^2l^2)} \right),
\] (30)

where $r_+$ in terms of $\beta$, $\Omega$ and $\Phi$ is
\[
r_+ = \pi(n-1) + \sqrt{\pi^2(n-1)^2 - \Lambda \Phi^2 \beta^2 (n-2)^2}.
\] (31)

Since the Euclidean action is related to the free energy in the grand canonical ensemble, the electric charge $Q$, the angular momenta $J_i$, the entropy $S$ and the mass $M$ per unit
volume $V_{n-1}$ can be found using the familiar thermodynamics relations:

$$Q = -\beta^{-1} \frac{\partial I}{\partial \Phi} = \frac{\Xi}{4\pi} \sqrt{(n-1)(n-2)q},$$  \hspace{1cm} (32)

$$S = \left( \frac{\beta}{\partial \beta} - 1 \right) I = \frac{\Xi}{4} (r_+^2 + r_0^2)^{\frac{n-1}{2}},$$  \hspace{1cm} (33)

$$J_i = -\beta^{-1} \frac{\partial I}{\partial \Omega_i} = \frac{1}{16\pi} n\Xi m a_i,$$  \hspace{1cm} (34)

$$M = \left( \frac{\partial}{\partial \beta} - \beta^{-1} \frac{\partial}{\partial \Phi} - \beta^{-1} \sum \Omega_i \frac{\partial}{\partial \Omega_i} \right) I = \frac{1}{16\pi} (n\Xi^2 - 1)m$$  \hspace{1cm} (35)

Note that the charged obtained by use of free energy is the same as that obtained in Eq. (24). Of course, if one computes the mass and angular momentum per unit volume $V_{n-1}$ through the use of counterterm method introduced in Ref. [20] (explained in Sec. II), the same results will be obtained.

Equation (33) shows that the entropy of the black branes satisfies the so-called area law of entropy which states that the black hole entropy equals to one-quarter of horizon area $\Xi$. Although the area law of entropy applies to almost all kind of black holes and black strings of Einstein gravity [29], it is not satisfied in higher derivative gravity [30]. But, for our solutions in Gauss-Bonnet gravity with flat horizon the area law is satisfied.

C. Stability of black branes

We first obtain the mass $M$ as a function of $S$, $Q$ and $J$. Using the expressions (33), (35) and (34) for the entropy, the mass and the angular momentum, and the fact that $f(r_+) = 0$, one obtains by simple algebraic manipulation the Smarr-type formula as:

$$M(S, J, Q) = \frac{(nZ - 1)J}{n\ell \sqrt{Z(Z - 1)}},$$  \hspace{1cm} (36)

where $J^2 = |J|^2 = \sum_i J_i^2$ and $Z = \Xi^2$ is the real positive solution of the following equation:

$$(Z - 1)^{(d-2)} = \frac{Z}{16\pi^2} \left\{ \frac{4\pi(n-1)(n-2)lSJ}{n[(n-1)(n-2)S^2 + 2\pi^2Q^2l^2]} \right\}^{(2n-2)} = 0.$$  \hspace{1cm} (37)

It is worthwhile to mention that the thermodynamic quantities calculated above satisfy the first law of thermodynamics,

$$dM = TdS + \sum_i \Omega_i dJ_i + \Phi dQ.$$  \hspace{1cm} (38)
The stability of a thermodynamic system with respect to the small variations of the thermodynamic coordinates, is usually performed by analyzing the behavior of the entropy $S(M, Q, J)$ around the equilibrium. The local stability in any ensemble requires that $S(M, Q, J)$ be a convex function of their extensive variables or its Legendre transformation must be a concave function of their intensive variables. Thus, the local stability can in principle be carried out by finding the determinant of the Hessian matrix of $S$ with respect to its extensive variables, 

$$H^S_{X_i, X_j} = \begin{vmatrix} \frac{\partial^2 S}{\partial X_i \partial X_j} \end{vmatrix},$$

where $X_i$’s are the thermodynamic variables of the system. Indeed, the system is locally stable if the determinant of Hessian matrix satisfies $H^S_{X_i, X_j} \leq 0$ \[23, 27\]. Also, one can perform the stability analysis through the use of the determinant of Hessian matrix of the energy with respect to its thermodynamic variables, and the stability requirement $H^S_{X_i, X_j} \leq 0$ may be rephrased as $H^M_{Y_i, Y_j} \geq 0$ \[31\].

The number of the thermodynamic variables depends on the ensemble which is used. In the canonical ensemble, the charge and angular momenta are fixed parameters, and therefore the positivity of the heat capacity $C_{Q, J} = T(\partial S/\partial T)_{Q, J}$ is sufficient to assure the local stability. One can show that the latter criteria is:

$$C_{Q, J} = \frac{\Xi}{4} (r_0^2 + r_+^2) \frac{(n-1)}{(n+1)} \left\{ \begin{array}{l} n (r_0^2 + r_+^2)^{(n-1)} - (n - 2) q^2 l^2 \left( (r_0^2 + r_+^2)^{(n-1)} + q^2 l^2 \right) \\ ((n - 2) \Xi^2 + 1) \\ (n - 2) q^4 l^4 \left((3n - 6) \Xi^2 - (n - 3) \right) \\ - 2q^2 l^2 (r_0^2 + r_+^2)^{(n-1)} \left((3n - 6) \Xi^2 - n^2 + 3 \right) + n(r_0^2 + r_+^2)^{(2n-2)} \left((n + 2) \Xi^2 - (n + 1) \right) \end{array} \right\}^{-1}.$$ \[39\]

Figure 1 shows the behavior of the heat capacity as a function of the charge parameter. It shows that $C_{Q, J}$ is positive in various dimensions and goes to zero as $q$ approaches its critical value (extreme black brane). Thus, the $(n + 1)$-dimensional asymptotically AdS charged rotating black brane is locally stable in the canonical ensemble. One may also plot $T$ versus $S$ for fixed $J$ and $Q$, and look for inflection point(s). Indeed, for systems with phase transition the diagram has an inflection point, that demonstrates the region of coexistence of various phases \[23\]. Since there is no an inflection point in Fig. 2 the black brane is stable.

In the grand-canonical ensemble, the stability analysis can be carried out by calculating the determinant of Hessian matrix of the energy with respect to $S, Q$ and $J$. It is a matter
FIG. 1: $C_{J,Q}$ versus $q$ for $l = 12$, $r_+ = 1.4$, $n = 4$ (dashed), $n = 5$ (dash-dotted), and $n = 6$ (dotted).

FIG. 2: $T$ versus $S$ for $l = 0.09$, $q = 0.06$, $a = 0.01$, $n = 4$ (solid line), $n = 5$ (dash-dotted), and $n = 6$ (bold line).

of calculation to show that the black brane is locally stable in the grand-canonical ensemble, since the determinant of Hessian matrix,

$$H_{SQJ}^M = \frac{\pi}{l^2 \Xi^6} \frac{\left(\frac{64}{n} l^2 q^2 + \frac{64}{n-2} (r_0^2 + r_+^2)^{n-1}\right)}{(n-2)\Xi^2 + 1)l^2 q^2 + (r_0^2 + r_+^2)^{n-1}(r_0^2 + r_+^2)^{\frac{3}{2}n-2}}$$

(40)

is positive. It is worthwhile to note that $\alpha$ has not appeared in the thermodynamic quantities computed in this section. Thus, the charged solutions in the Gauss-Bonnet gravity has the same thermodynamic features as the solutions in the Einstein gravity. This fact is also true for the case of static spherically symmetric black holes with zero curvature horizon [32].
IV. CONCLUDING REMARKS

The metric function of charged solution of Gauss-Bonnet gravity introduced in literature is not real for the whole spacetime even for static case \(^33\). In this paper we presented the charged rotating solutions of Gauss-Bonnet gravity which are real in the whole spacetime and investigated the thermodynamics of them. We found that in contrast with the case of uncharged solution, the temperature is not equal to \(f'(r_+)/4\pi \Xi\). The charged solutions may be interpreted as black brane solutions with two inner and outer event horizons for \(m > m_{\text{ext}}\), extreme black holes for \(m = m_{\text{ext}}\) or naked singularity for \(m < m_{\text{ext}}\), where \(m_{\text{ext}}\) is given in Eq. \(^19\). We found that the Killing vectors are the null generators of the event horizon, and therefore the event horizon is a Killing horizon for the stationary solution of the Gauss-Bonnet gravity explored in this paper. We computed the finite action of the charged rotating solutions through the use of counterterm method. By using the relation between the action and free energy, we compute the conserved and thermodynamic quantities of the solutions and showed that they satisfy the first law of thermodynamics. Finally, we obtained a Smarr-type formula for the mass of the black brane solution as a function of the entropy, the charge and the angular momenta of the black brane and studied the phase behavior of the \((n+1)\)-dimensional charged rotating black branes. We showed that there is no Hawking-Page phase transition in spite of the charge and angular momenta of the branes. Indeed, we calculated the heat capacity and the determinant of the Hessian matrix of the mass with respect to \(S\), \(J\) and \(Q\) of the black branes and found that they are positive for all the phase space, which means that the brane is stable for all the allowed values of the metric parameters discussed in Sec. \(\text{III}\). It is worth to mention that the charged rotating solutions of first order Lovelock gravity \(^{15}\), of second order Lovelock gravity introduced in this paper, of third order Lovelock gravity presented in Ref. \(^{20}\) and the static solutions of continued Lovelock gravity \(^{19}\) are stable. Thus, we conjecture that the Lovelock terms do not have any effects on the stability of black branes with flat horizon. This is commensurate with the fact that there is no Hawking-Page transition for a black object whose horizon is diffeomorphic to \(\mathbb{R}^p\) and therefore the system is always in the high temperature phase \(^2\). Of course this kind of solutions in the presence of a dilaton field is not stable even in Einstein gravity \(^34\).

Although the thermodynamics of charged rotating solutions of Gauss-Bonnet gravity with
flat horizon have been investigated in this paper, no such a kind of solutions with spherical horizon have been presented till now. Thus, it is worth constructing these kinds of solutions and investigating their thermodynamics. The charged rotating black holes with spherical horizon in Einstein gravity are not stable in the whole phase space \[23\], and therefore it is interesting to investigate the effects of Gauss-Bonnet term on the thermodynamics of such kinds of solutions.

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