Supersymmetric Rotating Black Hole in a Compactified Spacetime

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We construct a supersymmetric rotating black hole with asymptotically flat four-dimensional spacetime times a circle, by superposing an infinite number of BMPV black hole solutions at the same distance in one direction. The near horizon structure is the same as that of the five-dimensional BMPV black hole. The rotation of this black hole can exceed the Kerr bound in general relativity ($q \equiv a/G_{4}M = 1$), if the size is small.

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I. INTRODUCTION

One of the most promising approaches to unification of fundamental interactions is a superstring theory, or M-theory \(^{1,2}\). Such unified theories are formulated in dimensions higher than four with gravitational interactions. From this perspective, it is important to study properties of black hole solutions in higher dimensional theories \(^{3,4}\), and it turns out that they show a variety of new and interesting properties. For example, there is no uniqueness theorem for black holes in higher dimensions \(^{4,5}\). In fact, we have a new type of black object, which is the so-called black ring with horizon of a topology of $S^{1} \times S^{2}$.

Those solutions are obtained in the effective supergravity theories in the low-energy limit. Among others, supersymmetric solutions are expected to be important for the following reason. The black hole solutions in the effective supergravity theories will in general receive higher-order corrections in the string coupling constant. However, in the presence of unbroken partial supersymmetry, there are certain properties on which such higher-order effects can be ignored. For instance, the microstate counting of the black holes is performed in the corresponding string solutions at the lowest order in the string coupling, with result in perfect agreement with the Bekenstein-Hawking entropy. The underlying reason for this agreement is the agreement in perfect agreement with the Bekenstein-Hawking entropy.

Since our world is four-dimensional, we should discuss four-dimensional (or effectively four-dimensional) black holes with supersymmetry. In higher dimensions, there is no uniqueness theorem and we can easily find supersymmetric black holes for the case of spherically symmetric and charged black holes. On the other hand, in four-dimensional spacetime, it is known that the Kerr-Newman black hole is the unique solution, but it is not supersymmetric unless rotation vanishes. Furthermore, there is even an argument that the event horizon of a supersymmetric black hole must be non-rotating \(^{6}\). It is thus interesting to find rotating supersymmetric solutions which are (effectively) four-dimensional. In five dimensions, there is no such restriction and it is already known that there is a supersymmetric rotating black hole called the BMPV black hole \(^{7}\), and a supersymmetric black ring \(^{8}\). Recently, Refs. \(^{9-11}\) constructed interesting rotating black hole solutions with asymptotically flat four-dimensional spacetime. Some of them use five-dimensional supersymmetric black ring solutions in Taub-NUT base space \(^{12,13}\). Here we present another example which is obtained by superposing an infinite number of the BMPV black hole solutions at the same distance in one direction, which can be regarded as a rotating supersymmetric black hole in a compactified spacetime. We discuss their properties including thermodynamic quantities. Similar techniques of compactification has been discussed for extreme Reissner-Nordstrom black holes in five dimensions \(^{13}\).

For the case of non-BPS black holes, we can consider deformation of a black hole in a compactified spacetime \(^{14}\) and a transition from black string to black hole \(^{15}\). The localized black hole may be preferred when the compactification radius is large compared with the radius of the sphere of a black string because of the Gregory-Laflamme instability \(^{21}\). It is a subtle question if such a topology change occurs in the gravitational theory. In our case, however, because the solutions are supersymmetric, there is no force between them and it is possible to superpose these without any deformation. No such subtle instability is expected in our case, which may be considered an advantage of our solutions.

This paper is organized as follows. In the next section,
we present our explicit solution for compactified BMPV black hole which is obtained by compactification of one dimension. In Sec. III, we examine its near horizon and asymptotic structures, entropy, rotation and supersymmetry.

II. A COMPACTIFIED BMPV SOLUTION

In the previous paper [22], we showed that five-dimensional rotating spacetimes can be constructed by compactifying 11-dimensional stationary solutions with intersecting branes in M-theory. We started with a generic form of the metric and solved the field equations (the Einstein equations and the equations for form fields). Assuming the intersection rule for branes [23], we derived the equations for each metric. We found that most of the metric components are described by harmonic functions. One metric component $f$, which describes a travelling wave, is determined by the Poisson equation, whose source term is given by the quadratic form of the “gravitoelectromagnetic” field $F_{ij}$. In some specific configuration of branes, e.g., for two intersecting branes (M2⊥M5), the source term vanishes. In these cases, all the harmonic functions are independent. We can recover the BMPV rotating black hole solution in the simplest case.

Since the basic equations are linear (Laplace equations), we can easily construct various solutions by superposing those harmonics. Using this fact, here we construct an effectively four-dimensional rotating black hole with supersymmetry. In order to preserve 1/8 supersymmetry, we have to impose that $F_{ij}$ should be self-dual.

Let us start with the metric in five-dimensions:

$$ds^2_5 = -\Xi^2 \left( dt + A_i dx^i \right)^2 + \Xi^{-1} ds^2_{S^4},$$

where $\Xi = [H_2 H_5(1 + f)]^{-1/3}$. $ds^2_{S^4}$ is a four-dimensional Euclidean space $(x^i) = (x, y, z, w)$. $f$, $H_2 (A = 2, 5)$, and $A_i$ are metric functions, which will be determined later.

Adopting the hyperspherical coordinates:

$$x + iy = \rho \cos \theta e^{i \psi}, \quad z + iw = -\rho \sin \theta e^{i \psi},$$

where $0 \leq \phi, \psi < 2\pi$ and $0 \leq \theta \leq \pi/2$, we have the line element of four-dimensional Euclidean space:

$$ds^2_{S^4} = d\rho^2 + \rho^2 \left( d\theta^2 + \sin^2 \theta d\psi^2 \right).$$

The unknown metric functions $H_2 (A = 2, 5)$, $A_i$ and $f$ satisfy the following equations:

$$\partial^2 H_2 = 0,$$

$$\partial^2 f = 0,$$

$$\partial^2 \partial_i A_{ij} = 0.$$  

The simplest rotating spacetime solution of these equations is given by the so-called BMPV solution

$$H_2 = 1 + Q_H^{(A)} \sum_{n=-\infty}^{\infty} \frac{1}{r^2 + (w + 2\pi n R_5)^2},$$

$$f = Q_0 \sum_{n=-\infty}^{\infty} \frac{1}{r^2 + (w + 2\pi n R_5)^2},$$

$$A_\phi = \frac{J_\phi}{2} \sum_{n=-\infty}^{\infty} \frac{x^2 + y^2}{r^2 + (w + 2\pi n R_5)^2},$$

$$A_\psi = \frac{J_\psi}{2} \sum_{n=-\infty}^{\infty} \frac{z^2 + (w + 2\pi n R_5)^2}{r^2 + (w + 2\pi n R_5)^2},$$

where $Q_H^{(A)}$, $Q_0$, $J_\phi$, and $J_\psi$ are charges and angular momenta. Note that supersymmetry of the BMPV solution imposes the condition $J_\phi = -J_\psi$. The electric fields are given by

$$E_j^{(A)} = \partial_j \left( \frac{1}{H_2} \right).$$

Since the origin of coordinate system can be shifted by any distance, we can move such a black hole to any position. Also because the basic equations (2.4), (2.5), and (2.6) are linear, we can superpose those black hole solutions at different positions. Especially, if we superpose an infinite number of black holes, each of which is separated by the same distance in one direction, we obtain a periodic BMPV black hole solution. It can be regarded as a deformed BMPV black hole in a compactified five-dimensional spacetime. We call it a compactified BMPV black hole (a CBMPV black hole).

Let us now present its explicit solution. Suppose that an infinite number of black holes exist along the $w$-axis with the same coordinate distance $2\pi R_5$, with $R_5$ denoting the compactification radius at infinity. By superposing those black hole solutions, we have

$$H_2 = 1 + Q_H^{(A)} \sum_{n=-\infty}^{\infty} \frac{1}{r^2 + (w + 2\pi n R_5)^2},$$

$$f = Q_0 \sum_{n=-\infty}^{\infty} \frac{1}{r^2 + (w + 2\pi n R_5)^2},$$

$$A_\phi = \frac{J_\phi}{2} \sum_{n=-\infty}^{\infty} \frac{x^2 + y^2}{r^2 + (w + 2\pi n R_5)^2},$$

$$A_\psi = \frac{J_\psi}{2} \sum_{n=-\infty}^{\infty} \frac{z^2 + (w + 2\pi n R_5)^2}{r^2 + (w + 2\pi n R_5)^2},$$

where $r^2 = x^2 + y^2 + z^2$. 
Introducing a new function

\[ F(\xi, \eta) = \sum_{n=\infty}^{1} \frac{1}{\xi^2 + (\eta + 2\pi n)^2} = \frac{\sinh \xi}{2\xi (\cosh \xi - \cos \eta)}, \]  

and setting \( \bar{r} = r/R_5 \) and \( \bar{w} = w/R_5 \), we obtain

\[ H_A = 1 + \frac{Q_H^{(A)}}{2R_5^2} F(\bar{r}, \bar{w}) = 1 + \frac{Q_H^{(A)}}{2R_5^2} \frac{\sinh \bar{r}}{\bar{r} (\cosh \bar{r} - \cos \bar{w})}, \]  

\[ f = \frac{Q_0}{2R_5^2} F(\bar{r}, \bar{w}) = \frac{Q_0}{2R_5^2} \frac{\sinh \bar{r}}{\bar{r} (\cosh \bar{r} - \cos \bar{w})}, \]  

\[ A_\phi = -\frac{J_\phi}{2R_5^2} \frac{\bar{x}^2 + \bar{y}^2}{2\bar{r}} \frac{\partial F(\bar{r}, \bar{w})}{\partial \bar{r}} = \frac{J_\phi}{8R_5^2} \frac{(\bar{x}^2 + \bar{y}^2)}{\bar{r}^2} \left[ \frac{(\cosh \bar{r} \cos \bar{w} - 1) + \sinh \bar{r}}{(\cosh \bar{r} - \cos \bar{w})^2} \right], \]  

\[ A_\psi = \frac{\bar{J}_\psi}{2R_5^2} \frac{(\bar{x}^2 + \bar{y}^2) (\cosh \bar{r} \cos \bar{w} - 1)}{(\cosh \bar{r} - \cos \bar{w})^2} + \frac{(\bar{r}^2 + \bar{z}^2)}{\bar{r}^2} \frac{\sinh \bar{r}}{\bar{r} (\cosh \bar{r} - \cos \bar{w})}. \]  

Since this solution is periodic in the \( w \)-direction with the period \( 2\pi R_5 \), it can be regarded as a deformed BMPV black hole in a compactified five-dimensional spacetime \((-\pi R_5 \leq w \leq \pi R_5\) with supersymmetry \((if \, J_\phi = -J_\psi)\). In the following sections, we analyze the properties of this solution.

### III. SPACETIME STRUCTURE OF A COMPACTIFIED BMPV BLACK HOLE

#### A. horizon

The horizon exists at \((r, w) = (0, 0)\). Setting \( x = \epsilon \cos \theta \cos \phi, \, y = \epsilon \cos \theta \sin \phi, \, z = \epsilon \sin \theta \cos \psi, \, \bar{w} = \epsilon \sin \theta \sin \psi (\epsilon \ll 1)\), we find the behavior of the metrics near horizon to be

\[ H_A = 1 + \frac{Q_H^{(A)}}{R_5^2 \epsilon^2}, \quad f = \frac{Q_0}{R_5^2 \epsilon^2}, \]  

\[ A_\phi = \frac{J_\phi}{2R_5^2 \epsilon^2} \cos^2 \theta, \quad A_\psi = \frac{\bar{J}_\psi}{2R_5^2 \epsilon^2} \sin^2 \theta. \]

This is exactly the same near-horizon structure as that of the BMPV black hole. Hence the horizon structure is not modified by superposition of an infinite number of black holes. This is due to the BPS properties of the solutions which guarantee the no-force condition between the black holes of the solutions. It is just the same as the case of superposition of extreme Reissner-Nordstrom black holes.

The area of the horizon is given by

\[ A_H = 2\pi^2 \sqrt{Q_0 Q_H^{(A)} - J^2/8}, \]

where \( J^2 = (J_\phi^2 + J_\psi^2)/2 \).

#### B. asymptotic structure

The metric describes the five-dimensional spacetime, but it behaves effectively as the four-dimensional spacetime (times a circle) when an observer is far from the black hole. In fact, \( w \)-direction is compactified with the period \( 2\pi R_5 \), while the other spatial directions are not \((0 \leq r < \infty)\). In Fig. 1 we depict equipotential lines for “gravitational potential” \( f \) (or \( H_A \)). The origin \(((r, w) = (0, 0)\)) is horizon of a black hole. The potential depends nontrivially on the compact direction \( w \) near a black hole, but the metric asymptotically approaches flat four-dimensional Minkowski times the \( w \)-circle. Near infinity, the metric becomes independent of the coordinate \( w \).

![Fig. 1: Equipotential lines for "gravitational potential" \( f(r, w) \). The dotted lines (\( w = -\pi R_5 \) and \( w = \pi R_5 \)) are identified to compactify the \( w \)-direction.](image1)
The structure of this black hole is schematically given in Fig. 2 (a). For reference, we also show the structure of another supersymmetric rotating black hole solution with asymptotically flat four-dimensional spacetime, which is obtained by use of a black ring solution in Taub-NUT base space \([12,14,15]\). We call it a RING black hole.

The RING black hole is embedded in a curved Taub-NUT space. The asymptotic region is four-dimensional Minkowski spacetime (Fig. 2 (b)). The CBMPV black hole exists in a compactified 5D spacetime obtained by compactification of the fifth direction (Fig. 2 (a)). Both spacetimes have asymptotically flat 4D Minkowski spacetime with small compactified dimension (its radius is \(R_5\)), but those horizons are five-dimensional.

Let us now examine how the asymptotic four-dimensional spacetime looks like. Introducing the three-dimensional spherical coordinates \((\bar{r}, \Theta, \Phi)\), which are defined by the transformations

\[
\begin{align*}
\bar{x} &= \bar{r} \sin \Theta \cos \Phi, \\
\bar{y} &= \bar{r} \sin \Theta \sin \Phi, \\
\bar{z} &= \bar{r} \cos \Theta,
\end{align*}
\]

our metric is rewritten as

\[
d\bar{s}^2 = -\Xi^2 \left[ d\bar{t} + \bar{A}_\phi d\Phi + \frac{\bar{A}_\psi}{(\bar{r}^2 \cos^2 \Theta + \bar{w}^2)} (-\bar{w} \cos \Theta d\bar{r} + \bar{r} \bar{w} \sin \Theta d\Theta + \bar{r} \cos \Theta d\bar{w}) \right]^2 \\
+ \Xi^{-1} \left( d\bar{r}^2 + \bar{r}^2 d\Omega^2 + d\bar{w}^2 \right),
\]

where

\[
\begin{align*}
\bar{A}_\phi &= \frac{J_\phi}{8R_5^3} \sin^2 \Theta \left[ \frac{\sinh \bar{r}}{\bar{r}(\cosh \bar{r} - \cos \bar{w})} + \frac{\cosh \bar{r} \cos \bar{w} - 1}{(\cosh \bar{r} - \cos \bar{w})^2} \right], \\
\bar{A}_\psi &= \frac{J_\psi}{8R_5^3} \left[ (1 + \cos^2 \Theta) \frac{\sinh \bar{r}}{\bar{r}(\cosh \bar{r} - \cos \bar{w})} - \sin^2 \Theta \frac{\cosh \bar{r} \cos \bar{w} - 1}{(\cosh \bar{r} - \cos \bar{w})^2} \right].
\end{align*}
\]

In the asymptotic region (\(\bar{r} \gg \bar{w}\)), we can make a Kaluza-Klein reduction of the near-asymptotic metric in the \(\bar{w}\) direction. We then find the effective four-dimensional metric, which can be written in the Einstein frame as

\[
d\bar{s}_4^2 = -\Xi^{3/2} \left[ d\bar{t} + \bar{A}_\phi d\Phi \right]^2 \\
+ \Xi^{-3/2} \left[ d\bar{r}^2 + \bar{r}^2 \left( d\Theta^2 + \sin^2 \Theta d\Phi^2 \right) \right],
\]

where

\[
\begin{align*}
H_A &= 1 + \frac{1}{2R_5^3} \frac{Q^{(A)}_H}{\bar{r}}, \\
f &= \frac{1}{2R_5^3} \frac{Q_0}{\bar{r}}, \\
\bar{A}_\phi &= \frac{1}{8R_5^3} \frac{J_\phi}{\bar{r}} \sin^2 \Theta.
\end{align*}
\]

Comparing this with the asymptotic form of the four-dimensional Kerr-Newman metric, we get a gravitational mass of the black hole \(M\) and a rotation parameter \(a\), which is the angular momentum per unit mass:

\[
G_4 M = \frac{(Q_0 + Q^{(2)}_H + Q^{(5)}_H)}{8R_5} = \frac{G_5 M_{\text{ADM}}}{2\pi R_5},
\]

\[
G_4 Ma = \frac{J_\phi}{16R_5},
\]

where \(G_4 = G_5/(2\pi R_5)\) is the four-dimensional gravitational constant and \(M_{\text{ADM}}\) is the ADM mass of the BMPV black hole. The rotation parameter is given by

\[
a = \frac{J_\phi}{2(Q_0 + Q^{(2)}_H + Q^{(5)}_H)}.\]

From the asymptotic form of the electric fields

\[
E^{(A)}_r \sim \frac{1}{2R_5} \frac{Q^{(A)}_H}{\bar{r}},
\]

we also find the charges in four-dimensions are

\[
Q_A = \frac{Q^{(A)}_H}{2R_5}.
\]
The entropy of the present black hole is given by that of the BMPV black hole, i.e.

\[ S_{\text{CBMPV}} = \frac{A_H}{4G_5} = \frac{\pi^2}{2G_5} \sqrt{Q_0 Q_H^{(2)} Q_H^{(5)} - \frac{J^2}{8}} \]  

(3.12)

Setting \( Q_H^{(A)} = \alpha^{(A)} Q_0 \) \((A = 2, 5)\), and assuming supersymmetry, i.e. \( J_\phi = -J_\psi (= J) \), we find

\[ S_{\text{CBMPV}} = \frac{\pi r_g^2}{2G_4} \left[ \frac{1}{2} \left( \frac{1}{\lambda^2} - q^2 \right) \right] \]  

(3.13)

where \( r_g \equiv 2G_4 M \) is the Schwarzschild radius of a black hole mass \( M \), and

\[ \lambda^2 = \frac{(1 + \alpha^{(2)} + \alpha^{(5)})^3}{32\alpha^{(2)}\alpha^{(5)}} \times \frac{r_g}{R_5} \]  

(ratio of the BH size to the compactification scale)

\[ q^2 = \frac{Q_A}{G_4 M^2} \]  

(Kerr rotation parameter)

\[ e_A = \frac{Q_A}{G_4 M} = \frac{4\alpha^{(A)}}{1 + \alpha^{(2)} + \alpha^{(5)}} \]  

\((A = 2, 5)\) (normalized charges)  

(3.14)

For reference, we show the entropy of Kerr-Newman black hole, which is given by

\[ S_{\text{KN}} = \frac{\pi r_g^2}{2G_4} \left[ 1 - \frac{e^2}{2} + \sqrt{1 - q^2 - e^2} \right] \]  

(3.15)

where \( r_g \) and \( q \) are the same as above and \( e \) is a normalized charge of Kerr-Newman black hole.

**D. rotation**

For the Kerr-Newman black hole, we have one constraint for the existence of regular horizon, which is \( q^2 + e^2 \leq 1 \). The equality gives the extreme limit. So we find maximal rotation from this condition, that is \( q^2 = 1 - e^2 \). The Kerr black hole \((e = 0)\) gives maximal rotation \( q = 1 \) in the extreme limit.

For a CBMPV black hole, we have a constraint of \( \lambda^2 \leq 1/q^2 \). Suppose two charges are the same \((Q_2 = Q_5)\). Writing \( \lambda^2 \) in terms of \( e_2 = e_5 = e \), we find a relation between \( q \) and \( e \) in the extreme limit (the equality of the above condition) as

\[ q^2 = e^2(2 - e) \frac{R_5}{r_g} \]  

(3.16)

which is shown in Fig. 3. The maximum value of \( q \) is \( q_{\text{max}} = 32/27 \times (r_g/R_5)^{-1} \) at \( e = 4/3 \). Hence if \( r_g/R_5 < 32/27 \), the value of \( q \) at maximum rotation exceeds unity (the Kerr bound). This fact may be important for the following reason. In four-dimensional general relativity, rotation of a black hole is bounded by a critical value (the Kerr bound \([q = 1]\)), beyond which there appears a naked singularity. However, if we discuss a black hole in superstring/M-theory, we can find an effectively four-dimensional compact object with \( q > 1 \) without a naked singularity. Observation of such a regular compact object with \( q > 1 \) might provide us an indirect evidence of extra dimensions.

**E. supersymmetry**

The asymptotic metric can be considered to describe effectively a four-dimensional rotating object. It contains only one rotation, that is \( A_\phi \). No \( A_\phi \) appears in the effective four-dimensional metric. However, supersymmetry is preserved if and only if \( F_{ij} \) is self-dual,
i.e. \( J_\phi = -J_\psi \). Since the effective metric does not contain the information about \( J_\psi \), one may wonder whether this spacetime is supersymmetric or not. In particular, non-supersymmetric CBMPV black hole \( (J_\phi \neq -J_\psi) \) also gives the same asymptotic structure as that of a supersymmetric one if it has the same value of \( J_\phi \).

To understand this situation, we should not forget the fifth direction. Although the effective spacetime looks four-dimensional, there is a small circle \( S^1 \) with the radius \( R_5 \) in the fifth direction. In the Kaluza-Klein theory, when we compactify the five-dimensional spacetime on four-dimensional one, we find not only four-dimensional gravity but also an electromagnetic field and a scalar field coming from the metric in the compactified dimension. When we consider the far region in the present solution, the four-dimensional metric does not contain \( A_\phi \), but it appears as the “electromagnetic” field. So, when we analyze the asymptotic structure of this spacetime, we have to discuss the Einstein-Maxwell system. We can check that supersymmetry is preserved if and only if the magnitude of rotation \( (A_\phi) \) balances with that of the “electromagnetic” field \( (A_\psi) \).

IV. CONCLUDING REMARKS

In this paper we have presented a supersymmetric rotating solution with asymptotically flat four-dimensional spacetime times a circle. It is constructed by superposing infinite number of BMPV black holes aligned in the fifth direction in five dimensions. This is possible because the solution is supersymmetric and so the field equation governing the solution is linear. We have examined its properties including mass, area and entropy. Considering the five-dimensional solution from the asymptotical four-dimensional point of view, we have also shown that the rotation parameter of maximally rotating black hole exceeds unity (the Kerr bound) if the size of black hole is small \( (r_g < (32/27)R_5) \).

It would be interesting to compare our solution with another supersymmetric rotating black hole solution with asymptotically flat four-dimensional spacetime (the RING black hole). However, because those solutions contain different type of charges, such a comparison may not make sense, even if we fix four-dimensional observables such as the ADM mass \( M \), angular momentum \( J \) (or a rotation parameter \( q = J/G_4 M^2 \) ) of a black hole, and a compactification radius \( R_5 \).

So far, not so many supersymmetric rotating solutions with asymptotically flat four-dimensional spacetime (+ small extra dimension) are known. It would be interesting to find further examples of effective four-dimensional rotating supersymmetric solutions and study their properties.

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