Completeness of non-normalizable modes

Philip D. Mannheim∗ and Ionel Simbotin†

Department of Physics, University of Connecticut, Storrs, CT 06269

(Dated: July 13, 2006)

Abstract

We establish the completeness of some characteristic sets of non-normalizable modes by constructing fully localized square steps out of them, with each such construction expressly displaying the Gibbs phenomenon associated with trying to use a complete basis of modes to fit functions with discontinuous edges. As well as being of interest in and of itself, our study is also of interest to the recently introduced large extra dimension brane-localized gravity program of Randall and Sundrum, since the particular non-normalizable mode bases that we consider (specifically the irregular Bessel functions and the associated Legendre functions of the second kind) are associated with the tensor gravitational fluctuations which occur in those specific brane worlds in which the embedding of a maximally four-symmetric brane in a five-dimensional anti-de Sitter bulk leads to a warp factor which is divergent. Since the brane-world massless four-dimensional graviton has a divergent wave function in these particular cases, its resulting lack of normalizability is thus not seen to be any impediment to its belonging to a complete basis of modes, and consequently its lack of normalizability should not be seen as a criterion for not including it in the spectrum of observable modes. Moreover, because the divergent modes we consider form complete bases, we can even construct propagators out of them in which these modes appear as poles with residues which are expressly finite. Thus even though normalizable modes appear in propagators with residues which are given as their finite normalization constants, non-normalizable modes can just as equally appear in propagators with finite residues too – it is just that such residues will not be associated with bilinear integrals of the modes.

∗Electronic address: philip.mannheim@uconn.edu
†Electronic address: simbotin@phys.uconn.edu


I. INTRODUCTION

In constructing complete bases of mode solutions to wave equations it is very convenient to work with modes which are normalizable since they obey a closure relation. Specifically, if one has some complete orthonormal basis of modes \( f_m(w) \) with eigenvalues labelled by \( m \) and orthonormality relation

\[
\int_{-\infty}^{\infty} dw e^{-2A(w)} f_m(w) f_{m'}(w) = \delta_{m,m'} \quad (1)
\]

where \( e^{-2A(w)} \) is an appropriate normalization measure, the completeness of the basis will then require that any localized function be expandable in terms of the basis modes as

\[
\psi(w) = \sum_m a_m f_m(w) \quad (2)
\]

with coefficients which are given as

\[
a_m = \int_{-\infty}^{\infty} dw e^{-2A(w)} \psi(w) f_m(w) . \quad (3)
\]

With insertion of these coefficients back into Eq. (2) yielding

\[
\psi(w) = \int_{-\infty}^{\infty} dw' \delta(w - w') \psi(w')
= \sum_m \int_{-\infty}^{\infty} dw' e^{-2A(w')} \psi(w') f_m(w') f_m(w) , \quad (4)
\]

the arbitrariness of the choice of \( \psi(w) \) will then require that the basis modes obey a closure relation of the form

\[
\sum_m f_m(w') f_m(w) = e^{2A(w) \delta(w - w')} . \quad (5)
\]

With Eq. (5) being recognized as being a special case of Eq. (2) (viz. the expansion of the extremely localized \( \delta(w) \) in a complete basis of \( f_m(w) \) with coefficients \( a_m = f_m(0) \)), the notions of completeness and closure are often treated interchangeably in the literature, with Eq. (3) not only often being referred to as being a completeness relation, but with it even being regarded as being an essential requirement for a basis to be complete in the first place.

It is the purpose of this paper to show that this need not in fact be the case, and that modes can be complete even when they do not obey Eq. (5) at all. Indeed, the steps which lead from Eq. (1) to Eq. (3) only hold when the basis is in fact one for which the integrals on
the left-hand side of Eq. (1) do in fact exist. With both of Eqs. (1) and (5) involving bilinear functions of the basis modes, but with Eq. (2) only being a linear function of the modes, it is still possible for the summation in Eq. (2) to be well-defined even when the bilinear expressions which appear in Eqs. (1) and (5) are not. Moreover, the wave equations for which the \( f_m(w) \) are the mode solutions are themselves only linear functions of the \( f_m(w) \), and it should thus be immaterial to the completeness of their solutions as to whether or not bilinear integrals of the modes exist. In general then completeness of a basis has to be understood as being the requirement that for localized functions \( \psi(w) \) there exists an expansion of the form of Eq. (2) with finite coefficients \( a_m \) regardless of whether or not the integrals on the left-hand side of Eq. (1) actually exist. Non-normalizable modes whose behavior is so bad as to cause these bilinear integrals to diverge can still be complete in the sense of Eq. (2), with the \( a_m \) coefficients being such as to lead to total destructive interference between the \( f_m(w) \) in the regions where the \( f_m(w) \) diverge. It is thus Eq. (2) which has to be recognized as being the general statement of completeness, and in this paper we shall confirm this by explicitly constructing localized square steps as sums over some characteristic bases of divergent modes. While the existence or not of the normalization integrals of Eq. (1) is immaterial to a differential wave equation, if the solutions to the wave equation are required to belong to a Hilbert space one can restrict to square integrable functions alone, though otherwise there is no reason to discard any non-normalizable solutions. Since wave equations in classical physics do not act in a Hilbert space, in classical physics one is not free to discard non-normalizable modes, and since classical physics wave equations play a prominent role in classical gravity where they are associated with classical gravitational fluctuations around classical gravity backgrounds, it is to classical gravity that we shall look for examples in which to test whether non-normalizable modes can be complete.

II. WAVE EQUATIONS FOR GRAVITATIONAL FLUCTUATIONS

The wave equations we shall explicitly explore are associated with the recently introduced brane-localized gravity program of Randall and Sundrum [2, 3]. As introduced, the brane gravity program provides for the possibility that our four-dimensional universe could be embedded in some infinitely-sized bulk space and yet not conflict with the fact that there is no apparent sign of any such higher-dimensional bulk. Specifically, by taking the higher-
dimensional bulk to possess a very special geometry, viz. the five-dimensional anti-de Sitter geometry $AdS_5$, and by taking our four-dimensional universe to be a brane (viz. membrane) embedded in it, Randall and Sundrum found that under certain circumstances it was then possible for gravitational signals to localize around the brane and not penetrate very far into the bulk, with $AdS_5$ acting as a sort of refractive medium which rapidly attenuates any signals which try to propagate in it. Within the Randall-Sundrum brane world there are six fully soluble set-ups (technically $AdS_5$ bulks with embedded Minkowski, de Sitter or anti-de Sitter branes each with either positive or negative tension $\lambda$ – to be referred to as the $M_4^\pm$, $dS_4^\pm$ and $AdS_4^\pm$ brane worlds in the following), with all six of them having backgrounds which can be described by the generic five-dimensional metric

$$ds^2 = dw^2 + e^{2A(|w|)} q_{\mu\nu}(x^\lambda) dx^\mu dx^\nu$$

where the $w$-independent $q_{\mu\nu}$ is the four-dimensional metric and the so-called warp factor $e^{2A(|w|)}$ is taken to be a function of $|w|$ where $w$ is the fifth coordinate. With the curvature of $AdS_5$ being taken to be given as $-b^2$, in the various cases the explicit background metrics are given as

$$ds^2(M_4^\pm) = dw^2 + e^{-2\epsilon(\lambda)|w|}[dx^2 + dy^2 + dz^2 - dt^2] ,$$

$$ds^2(dS_4^\pm) = dw^2 + \frac{H^2}{b^2}\sinh^2\left[\text{arcsinh}\left(\frac{b}{H}\right) - \epsilon(\lambda)b|w|\right]\left[e^{2Ht}(dx^2 + dy^2 + dz^2) - dt^2\right] ,$$

and

$$ds^2(AdS_4^\pm) = dw^2 + \frac{H^2}{b^2}\cosh^2\left[\text{arccosh}\left(\frac{b}{H}\right) - \epsilon(\lambda)b|w|\right]\left[dx^2 + e^{2Hz}(dy^2 + dz^2) - dt^2\right] ,$$

where $\epsilon(\lambda)$ is the sign of $\lambda$. (The $M_4^\pm$ background metrics are given in [2, 3], the $dS_4^\pm$ background metrics are given in [4, 5], and the $AdS_4^\pm$ background metrics are given in [4].)

For the brane world the gravitational fluctuations around these six backgrounds are most readily treated in the axial gauge where the transverse-traceless tensor fluctuation modes $h_{\mu\nu}^{TT}$ then all obey the generic wave equation (see e.g. [6] where full derivations and relevant citations are given)

$$\left[\frac{\partial^2}{\partial|w|^2} - 4 \left(\frac{dA}{|w|}\right)^2 + e^{-2A} \nabla_\alpha \nabla^\alpha\right] h_{\mu\nu}^{TT} = 0 ,$$

as subject to the constraint (technically the Israel junction condition)

$$\delta(w) \left[\frac{\partial}{\partial|w|} - 2 \frac{dA}{|w|}\right] h_{\mu\nu}^{TT} = 0$$

(11)
at a brane which is located at $w = 0$. In Eq. (10) the tildas in $\tilde{\nabla}_\alpha \tilde{\nabla}^\alpha$ indicate that these particular covariant derivatives are to be evaluated in the geometry associated with the four-dimensional $q_{\mu \nu}$. And with the four-dimensional sector of the theory being separable according to
\[ [\tilde{\nabla}_\alpha \tilde{\nabla}^\alpha - 2kH^2]h_{\mu \nu}^{TT} = m^2h_{\mu \nu}^{TT}, \] (12)
as defined here so that tensor fluctuations with $m^2 = 0$ propagate on the appropriate $dS_4$, $M_4$ or $AdS_4$ lightcones ($k = 1, 0, -1$ respectively), a separation of the modes into the form $h_{\mu \nu}^{TT} = f_m(|w|)e_{\mu \nu}(x^\lambda, m)$ then requires that the $f_m(|w|)$ obey
\[ \left[ \frac{d^2}{d|w|^2} - 4 \left( \frac{dA}{|w|} \right)^2 - 2 \left( \frac{d^2A}{d|w|^2} \right) + e^{-2A}m^2 \right] f_m(|w|) = 0 \] (13)
in each of the six background cases of interest to us the identity $d^2A/d|w|^2 = -kH^2e^{-2A}$ holds), as subject to the constraint
\[ \delta(w) \left[ \frac{d}{d|w|} - 2 \frac{dA}{|w|} \right] f_m(|w|) = 0 . \] (14)

Our task is thus to explore the completeness of solutions to Eqs. (13) and (14), and a reader unfamiliar with the physics of the brane world can start at this point as none of the analysis which ensues will depend on how Eqs. (13) and (14) were first arrived at. What will matter in the following is only that these equations admit of exact solutions, solutions whose large $|w|$ behavior can then explicitly be monitored.

Before actually identifying explicit solutions to Eqs. (13) and (14) for the specific choices of $A$ and $\epsilon(\lambda)$ of interest, we note that via manipulation of Eq. (13) we find that every pair of its solutions have to obey
\[ e^{-2A}(m_1^2 - m_2^2)f_{m_1}f_{m_2} = \frac{d}{d|w|} \left[ f_{m_1} \left( \frac{d}{d|w|} - 2 \frac{dA}{|w|} \right) f_{m_2} - f_{m_2} \left( \frac{d}{d|w|} - 2 \frac{dA}{|w|} \right) f_{m_1} \right] , \] (15)
which with Eq. (14) then requires the modes to obey
\[ (m_1^2 - m_2^2) \int_0^\infty d|w|e^{-2A}f_{m_1}f_{m_2} = \lim_{|w| \to \infty} \left[ f_{m_1} \left( \frac{d}{d|w|} - 2 \frac{dA}{|w|} \right) f_{m_2} - f_{m_2} \left( \frac{d}{d|w|} - 2 \frac{dA}{|w|} \right) f_{m_1} \right] . \] (16)
Orthogonality of modes with different separation constants is thus achieved when the modes are well-enough behaved at $|w| = \infty$ to cause the right-hand side of Eq. (16) to vanish.
with the orthogonality measure then being precisely the one we introduced in Eq. (11),
with modes which diverge badly enough at infinity causing the integral on the left-hand side
to not exist. While one could now proceed to determine the mode solutions and identify for
which particular ones the integral on the left-hand side of Eq. (16) converges or diverges,
before doing so it is instructive to recall that via a sequence of transformations it is possible
to bring Eq. (13) to a more familiar form. Specifically, if we change variables from \( w \)
by setting \( dz = e^{-A(w)}dw \) and define \( f_m = e^{A(z)/2} \hat{f}_m \), the \( \hat{f}_m \) will then obey
\[
-\frac{d^2}{dz^2} + \frac{9}{4} \left( \frac{dA}{dz} \right)^2 + \frac{3}{2} \frac{d^2A}{dz^2} - m^2 \hat{f}_m = 0 ,
\]
while at the same time the normalization integral will change as
\[
\int_0^\infty d|w|e^{-2A(|w|)}f_{m_1}(|w|)f_{m_2}(|w|) \rightarrow \int_{z[0]}^{z[\infty]} dz \hat{f}_{m_1}(z)\hat{f}_{m_2}(z) .
\]
While we thus recognize Eq. (17) as being in the familiar form of a one-dimensional
Schrödinger equation and Eq. (18) as being in the form of its conventional quantum-
mechanical normalization integral, nonetheless, as noted above, since in the cases which
are of interest to us here we are not requiring the \( \hat{f}_m \) modes to belong to a Hilbert space,
we should not discard the non-normalizable solutions to Eq. (17) \[7\]. And having now rec-
ognized the rationale for not discarding non-normalizable solutions, we return to Eqs. (13)
and (14) to actually find and then explore them.

III. COMPLETENESS TESTS FOR THE MINKOWSKI BRANE CASES

A. Positive tension case

For the \( M_4^+ \) case where \( A = -b|w| \) the solutions to Eq. (13) are readily obtained by
setting \( y = me^{b|w|}/b \) as this transformation brings Eq. (13) to the Bessel equation form
\[
\left[ \frac{d^2}{dy^2} + \frac{1}{y} \frac{d}{dy} + 1 - \frac{4}{y^2} \right] f_m(y) = 0 .
\]
Mode solutions with any positive \( m^2 \) are thus given by
\[
f_m(y) = \alpha_m J_2(y) + \beta_m Y_2(y)
\]
where $\alpha_m$ and $\beta_m$ are $y$-independent coefficients, with those solutions with $m^2 = 0$ being given directly from Eq. \[13\] as

$$f_0(y) = \alpha_0 e^{-2b|w|} + \beta_0 e^{2b|w|} .$$

(21)

To satisfy the junction condition of Eq. \[14\] then requires that the various mode coefficients obey

$$\alpha_m J_1(m/b) + \beta_m Y_1(m/b) = 0 , \ \beta_0 = 0 \ ,$$

(22)

with the continuum of $m^2 > 0$ modes thus satisfying the junction condition via an interplay of the two types of Bessel function, and the $m^2 = 0$ mode $f_0(y) = \alpha_0 e^{-2b|w|}$ satisfying it all on its own. In the brane world the $m^2 > 0$ modes are known as the KK (Kaluza-Klein) modes, while the $m^2 = 0$ mode serves as a massless graviton. At large $y$ these solutions behave as

$$f_m \to \left(\frac{2}{\pi y}\right)^{1/2} \left[\alpha_m \cos(y - 5\pi/4) + \beta_m \sin(y - 5\pi/4)\right] , \ f_0 \to \frac{\alpha_0 m^2}{b^2 y^2} .$$

(23)

With all of these modes having wave functions which fall very fast in $|w|$ as we go away from the brane, the gravitational fluctuation modes are thus localized around it, this being the key result of \[3\]. With the measure of the normalization integral being rewriteable as

$$\int_0^\infty |w| e^{2b|w|} = b \int_{1/b}^\infty dx x$$

(24)

on setting $x = e^{b|w|}/b$, we see that the massless graviton wave function is bound state normalizable and that the KK modes possess the same continuum normalization as flat space Bessel functions. Consequently, the totality of massless graviton plus KK continuum modes is complete in exactly the same way as plane waves, with both of Eqs. \[11\] and \[5\] being satisfied (the summation in Eq. \[5\] is understood to contain both discrete and continuous indices). While we thus see that there is no need to perform any explicit completeness test for the modes of $M_4^+$ as everything is standard, a quite different situation will emerge when we consider $M_4^-$.\[\]

**B. Negative tension case**

For the $M_4^-$ case where $A = +b|w|$, the $m^2 > 0$ and the $m^2 = 0$ solutions to Eq. \[13\] are given by

$$f_m(y) = \alpha_m J_2(y) + \beta_m Y_2(y) ,$$

(25)

7
and

\[ f_0(y) = \alpha_0 e^{-2b|w|} + \beta_0 e^{2b|w|}, \]  

(26)

where now \( y = me^{-b|w|}/b \), while to satisfy the junction condition of Eq. (14) this time requires

\[ \alpha_m J_1(m/b) + \beta_m Y_1(m/b) = 0, \quad \alpha_0 = 0. \]  

(27)

Unlike the \( M_4^+ \) case this time \( y \) goes to zero as \( |w| \) goes to infinity, with large \( |w| \) asymptotics now being controlled by the behavior of Bessel functions at small argument rather than large, with the solutions behaving at small \( y \) as

\[ f_m \to \frac{\alpha_m y^2}{8} - \frac{4\beta_m}{\pi y^2}, \quad f_0 \to \frac{\beta_0 m^2}{b^2 y^2}, \]  

(28)

(the \( Y_2(y) \) behave irregularly at small argument). With the measure of the normalization integral now being given as

\[ \int_0^\infty d|w|e^{-2b|w|} = b \int_0^{1/b} dx x \]  

(29)

on setting \( x = e^{-b|w|}/b \), this time we see that it is only the \( J_2(y) \) modes which are normalizable, and that the massless graviton wave function and all the \( Y_2(y) \) modes are not only non-normalizable, they diverge far too violently to even be plane wave normalizable. In order to be able to satisfy the junction condition of Eq. (27) with normalizable modes alone, the convergent \( J_2(y) \) modes would have to satisfy Eq. (27) all by themselves, with the modes then needing to obey \( J_1(m/b) = 0 \). Solutions to this condition exist, and are given as the zeroes, \( j_i \), of the Bessel function \( J_1 \). This set of zeroes is discrete and infinite, with the normalizable modes of the \( M_4^- \) brane world then being given as modes with masses \( m_i = bj_i \). Similarly, the divergent \( Y_2(y) \) modes can satisfy the junction condition all on their own if their masses obey \( m_i = by_i \) where the \( y_i \) are the zeroes of the Bessel function \( Y_1 \), to yield another infinite set of discrete modes. With the divergent massless graviton mode with wave function \( \beta_0 e^{2b|w|} \) also satisfying the junction condition on its own, we thus recognize two classes of basis modes in the \( M_4^- \) brane world, the convergent \( J_2(j_i e^{-b|w|}) \), and the divergent \( e^{2b|w|} \) and \( Y_2(y_i e^{-b|w|}) \). And while our objective is to apply a completeness test to the divergent mode basis, it will be instructive to actually apply a completeness test to the convergent \( M_4^- \) mode basis first.
IV. COMPLETENESS TEST FOR CONVERGENT $M_4^-$ MODES

To test for completeness of a basis we need to determine whether it is possible to expand the typical localized square step $V_J = \hat{V}$, $\alpha \leq e^{-b|w|}/b \leq \beta$, $V_J = 0$ otherwise in terms of the modes of this basis, viz. we seek to find a set of $V_m$ from which we can reconstruct the square step according to

$$V_J(|w|) = \sum_m V_m J_2(me^{-b|w|}/b) . \quad (30)$$

To determine the needed coefficients $V_m$, we apply $\int_0^\infty d|w|e^{-2b|w|}J_2(me^{-b|w|}/b)$ to Eq. (30) and use the orthogonality relations that the asymptotically well-behaved $J_2(me^{-b|w|}/b)$ modes obey. Specifically, with the right-hand side of Eq. (16) vanishing for these modes, the modes will then obey

$$\int_0^\infty d|w|e^{-2b|w|}J_2^2(me^{-b|w|}/b)J_2(m'e^{-b|w|}/b) = 0 \quad (31)$$

when $m$ is not equal to $m'$, with use of some standard properties of Bessel functions obliging them to obey

$$\int_0^\infty d|w|e^{-2b|w|}J_2^2(me^{-b|w|}/b) = b \int_0^{1/b} dx J_2^2(mx) = \frac{b^2}{2} \left[ J_2^2(mx) - J_1(mx)J_3(mx) \right]^{1/b} = \frac{J_2^2(m/b)}{2b} , \quad (32)$$

when $m$ and $m'$ are equal and $m$ is such that $J_1(m/b)$ is zero. Armed with Eqs. (31) and (32) we thus find that $V_J(|w|)$ is to be given by

$$V_J(|w|) = \sum_m \frac{2bB_m}{J_2^2(m/b)} J_2(me^{-b|w|}/b) , \quad (33)$$

where the coefficients $B_m$ are given by

$$B_m = \int_0^\infty d|w|e^{-2b|w|}V_J(|w|)J_2(me^{-b|w|}/b) = -b\hat{V} \int_\alpha^\beta xdxJ_2(mx)$$

$$= -\frac{b\hat{V}}{m^2} \int_\alpha^\beta [2J_1(x) - xJ_0(x)] dx = \frac{b\hat{V}}{m^2} [2J_0(x) + xJ_1(x)] \bigg|_\alpha^\beta$$

$$= \frac{b\hat{V}}{m^2} [2J_0(m\beta) + m\beta J_1(m\beta)] - \frac{b\hat{V}}{m^2} [2J_0(m\alpha) + m\beta J_1(m\alpha)] . \quad (34)$$

With every quantity which appears in Eq. (33) now being known, $V_J(|w|)$ can readily be plotted, and we display it in Fig. (1) as evaluated through the use of the first 1000 modes in the sum (8). As we see, the basis is indeed capable of generating the square step to very high accuracy, with its completeness thus being confirmed.
FIG. 1: The left panel shows a reconstruction of the square step $V_f(|w|) = 1, 1 < |w| < 2, V_f = 0$ otherwise via the $M_4^-$ discrete $J_2(j_k e^{-b|w|})$ mode basis, with the parameter $b$ being set equal to one. The right panel shows a blow-up of the region near the top of the step.

With regard to the plot in Fig. (1), as can be seen from the blow-up of the region near the top of the step, the mode sum expressly displays the Gibbs phenomenon associated with trying to fit a discontinuity with a complete basis, with there being an overshoot (to near $V_f = 1.1$ in the figure) at the top of the discontinuity and an accompanying undershoot at the bottom, an overshoot and undershoot which as required of the Gibbs phenomenon were explicitly found to get narrower (in $|w|$) as the number of modes in the sum was increased, but not to shorten in height, always reaching close to $V_f = 1.1$ in the figure. We regard the recovering of the Gibbs phenomenon as a very good indicator of the reliability of our construction, and together with the quality of the overall fit itself, as providing very good evidence for completeness of the convergent $M_4^-$ mode basis.


V. COMPLETENESS TEST FOR DIVERGENT $M_4$ MODES

To test for completeness of the divergent $Y_2(y_i e^{-b|w|})$ plus $e^{2b|w|}$ mode basis we try to reconstruct the square step via the expansion

$$V_Y(|w|) = \sum_n V_n Y_2(ne^{-b|w|}/b) + V_0 e^{2b|w|} \ .$$

(In Eq. (35) we use $n$ to denote the $y_i$ zeroes of $Y_1(y)$, and shall use $m$ to denote the $j_i$ zeroes of $J_1(y)$.) Now while such a reconstruction might at first be thought unlikely to succeed since every term on the right-hand side of Eq. (35) diverges badly in the large $|w|$ region where we need the summation to vanish, the various terms in Eq. (35) are not diverging arbitrarily but, as can be seen from Eq. (28), are actually all diverging in exactly the same $e^{2b|w|}$ manner. In consequence of this, we are therefore able to adjust the various coefficients in Eq. (35) so as to expressly cancel out the divergent part. However, in order to get $V_Y(|w|)$ to actually vanish rather than merely not diverge outside the step, we will also need to cancel the finite part there as well. Thus, with each $Y_2(y)$ having a leading behavior of the form $-4/\pi y^2 - 1/\pi$ at small argument, i.e. with Eq. (35) behaving as

$$V_Y(|w|) \to e^{2b|w|} \left[ V_0 - \frac{4b^2}{\pi} \sum_n \frac{V_n}{n^2} \right] - \frac{1}{\pi} \sum_n V_n$$

at large $|w|$, we need to impose the two conditions

$$\frac{4b^2}{\pi} \sum_n \frac{V_n}{n^2} = V_0 \ , \ \sum_n V_n = 0 \ .$$

(37)

on the coefficients, with the two leading large $|w|$ terms then being cancelled.

Having thus taken care of the leading behavior at large $|w|$, we now try to proceed as with our analysis of the expansion of $V_f(|w|)$ in convergent modes. However, we cannot simply apply $\int_0^\infty d|w|e^{-2b|w|}Y_2(ne^{-b|w|}/b)$ to Eq. (35) as every overlap integral would diverge. However, we have found it very convenient to apply $\int_0^\infty d|w|e^{-2b|w|}J_2(me^{-b|w|}/b)$ to Eq. (35) instead, where we take the $m/b$ to be the $j_i$ zeroes of $J_1(y)$. With none of the $J_1(m/b)$ zeroes coinciding with any of the zeroes of $Y_1(n/b)$ [10], the needed overlap integrals are given (on setting $x = e^{-b|w|}/b$) by

$$\int_0^\infty d|w|e^{-2b|w|}J_2(me^{-b|w|}/b)Y_2(ne^{-b|w|}/b) = b \int_0^{1/b} dx xJ_2(mx)Y_2(nx)$$

$$= bx \left[ \frac{nY_1(nx)J_2(mx) - mJ_1(mx)Y_2(nx)}{(m^2 - n^2)} \right] \bigg|_0^{1/b} = \frac{2bm^2}{\pi n^2(n^2 - m^2)} \ ,$$

(38)
and
\[ \int_0^\infty d|w|e^{-2b|w|}J_2(me^{-b|w|}/b)e^{2b|w|} = \frac{1}{b} \int_0^{1/b} \frac{dx}{x}J_2(mx) \]
\[ = -\frac{1}{b} \int_0^{1/b} dx \frac{d}{dx} \left( \frac{J_1(mx)}{mx} \right) = \frac{1}{2b} , \] (39)

overlap integrals which despite the badly divergent behavior of the \( Y_2(y) \) and \( e^{2b|w|} \) are nonetheless actually finite due to the compensating convergent behavior of the \( J_2(y) \). On thus applying \( \int_0^\infty d|w|e^{-2b|w|}J_2(me^{-b|w|}/b) \) to Eq. (35), we find that for the square step \( V_Y(|w|) = \hat{V}, \alpha \leq e^{-b|w|}/b \leq \beta, V_Y(|w|) = 0 \) otherwise, the expansion coefficients must thus obey
\[ \frac{V_0}{2b} + \frac{2b}{\pi} \sum_n V_n \frac{m^2}{n^2(n^2 - m^2)} = \frac{V_0}{2b} + \frac{2b}{\pi} \sum_n V_n \left[ \frac{1}{(n^2 - m^2)} - \frac{1}{n^2} \right] \]
\[ = \frac{2b}{\pi} \sum_n \frac{V_n}{(n^2 - m^2)} = B_m \] (40)

for all \( m \), where the \( B_m \) are given by
\[ B_m = -b\hat{V} \int_\alpha^\beta dxJ_2(mx) = \frac{b\hat{V}}{m^2} [2J_0(mx) + mxJ_1(mx)] \bigg|_\alpha^\beta \]
\[ = \frac{b\hat{V}}{m^2} [2J_0(m\beta) + m\beta J_1(m\beta)] - \frac{b\hat{V}}{m^2} [2J_0(m\alpha) + m\alpha J_1(m\alpha)] . \] (41)

With the \( B_m \) being given in closed form, Eq. (40) is thus a set of \( N \) equations for \( N \) unknowns and can be viewed as an eigenvalue equation for the \( V_n \). (While the \( J_2(me^{-b|w|}/b)Y_2(ne^{-b|w|}/b) \) overlap integrals of Eq. (38) are finite, the \( J_2(me^{-b|w|}/b) \) and \( Y_2(ne^{-b|w|}/b) \) modes are not orthogonal, with Eq. (40), unlike Eq. (33), thus not being diagonal in its indices.) The \( V_n \) coefficients can thus be determined, and on being found to be finite and rapidly oscillating in sign, lead, for the case of the first 1000 modes in the basis, to the plot displayed in Fig. (2) (i.e. we restrict to the first 1000 \( y_i \) and the first 1000 \( j_i \) in Eq. (40)). As Fig. (2) thus indicates, and quite spectacularly so, the divergent mode basis is every bit as capable of reconstructing the square step as the convergent one and every bit as capable of recovering the Gibbs phenomenon, and is thus every bit as complete [11]. It is thus invalid to use normalizability as a criterion for discarding modes as non-normalizable modes are fully capable of serving as a complete basis for constructing localized packets [12].

As a final comment, we recall that for the harmonic oscillator wave equation there are two sets of solutions, the sines and the cosines, and both sets are complete. It is hence perfectly
FIG. 2: The left panel shows a reconstruction of the square step $V_Y(|w|) = 1, 1 < |w| < 2, V_Y = 0$ otherwise via the $M_4^{-}$ discrete $Y_2(y_i e^{-b|w|})$ plus $e^{2b|w|}$ mode basis, with the parameter $b$ being set equal to one. The right panel shows a blow-up of the region near the top of the step.

reasonable to expect other second order wave equations to also have two complete sets of bases even if one of them consists entirely of divergent modes.

VI. COMPLETENESS TESTS FOR THE ANTI-DE SITTER BRANE CASES

A. The basis modes

For $AdS_4^+$ brane world with warp factor $e^{A(|w|)} = Hcosh(\sigma - b|w|)/b$ where $cosh\sigma = b/H$, the transformation $y = tanh(b|w| - \sigma)$ brings Eq. \[\text{Eq. (13)}\] to the form

$$\left[(1 - y^2) \frac{d^2}{dy^2} - 2y \frac{d}{dy} + \nu(\nu + 1) - \frac{4}{(1 - y^2)}\right] f_m(y) = 0 . \quad (42)$$

where we have introduced the convenient parameter $\nu$ defined by

$$\nu = \left(\frac{9}{4} + \frac{m^2}{H^2}\right)^{1/2} - \frac{1}{2} , \quad \frac{m^2}{H^2} = (\nu - 1)(\nu + 2) . \quad (43)$$

Equation \[\text{Eq. (42)}\] is recognized as an associated Legendre equation, with its solutions being the associate Legendre functions of the first and second kind, so that for $m \neq 0$ (viz. $\nu \neq 1$) we
can set

\[ f_m(y) = \alpha_m P^2_\nu(y) + \beta_m Q^2_\nu(y) \]  

This solution also applies to one of the \( m = 0 \) solutions as well, viz. \( Q^2_\nu(y) \), a quantity which can be written in terms of the warp factor as \( Q^2_\nu(y) = 2/(1 - y^2) = 2\cosh^2(b|w| - \sigma) = 2b^2e^{2A(|w|)/H^2} \), but misses one other solution since \( P^2_1(y) \) is kinematically zero. This second \( m = 0 \) solution can be found by setting \( \nu = 1 \) in Eq. (42) and solving it directly, to yield

\[ f_0(y) = \alpha_0 \left( \frac{2}{1 + y} - y \right) + \beta_0 Q^2_1(y) \]  

Requiring the modes to also obey the junction condition of Eq. (14) then restricts them according to

\[ \alpha_m P^1_\nu(-\tanh\sigma) + \beta_m Q^1_\nu(-\tanh\sigma) = 0, \quad \alpha_0 = 0 \]  

to thus define the \( AdS^+_4 \) brane world basis modes.

As functions, all of the \( P^1_\nu(y), P^2_\nu(y), Q^1_\nu(y) \) and \( Q^2_\nu(y) \) possess a cut in the complex \( y \) plane which can be located to run from \( y = -\infty \) to \( y = 1 \). For the \( AdS^+_4 \) brane world the parameter \( y = \tanh(b|w| - \sigma) \) lies in the range \( -\tanh\sigma \leq y \leq 1 \), and so in this range the Legendre functions have to be evaluated on the cut (as the real \( P^\nu_\nu(y) = (1/2)[e^{i\pi\mu/2}P^\mu_\nu(y + i\epsilon) + e^{-i\pi\mu/2}P^\mu_\nu(y - i\epsilon)] \), \( Q^\nu_\nu(y) = (e^{-i\pi\mu/2})[e^{-i\pi\mu/2}Q^\mu_\nu(y + i\epsilon) + e^{i\pi\mu/2}Q^\mu_\nu(y - i\epsilon)] \) where they can then be power series expandable via their relation to hypergeometric functions to yield

\[
P^m_\nu(y) = \frac{(-1)^m\Gamma(\nu + m + 1)}{2^m m! \Gamma(\nu - m + 1)} (1 - y^2)^{m/2} F(\nu + m + 1, -\nu + m; m + 1; (1 - y)/2) \\
= \frac{(-1)^m\Gamma(\nu + m + 1)}{2^m m! \Gamma(\nu - m + 1)} (1 - y^2)^{m/2} \left[ 1 + \frac{(\nu + m + 1)(-\nu + m)}{(m + 1)!} \frac{(1 - y)}{2} \right. \\
+ \left. \frac{(\nu + m + 1)(\nu + m + 2)(-\nu + m)(-\nu + m + 1)}{(m + 1)(m + 2)!} \frac{(1 - y)^2}{2^2} + \ldots \right],
\]

\[
Q^m_\nu(y) = \frac{e^{i\pi\nu/2} \Gamma(\nu + 1) \Gamma(\nu + m + 1)}{\Gamma(2\nu + 2)(1 + y)^{\nu + 1 - m/2}(1 - y)^{m/2}} F(\nu - m + 1, \nu + 1; 2\nu + 2; 2/(1 + y)) \\
= \frac{e^{i\pi\nu/2} \Gamma(\nu + 1) \Gamma(\nu + m + 1)}{(1 + y)^{\nu + 1 - m/2}(1 - y)^{m/2}} \left[ \frac{\Gamma(m)}{\Gamma(\nu + 1) \Gamma(\nu + m + 1)} \right. \\
\times \sum_{n=0}^{m-1} \frac{(\nu - m + 1)_n (\nu + 1)_n (y - 1)^n}{(1 - m) n! (y + 1)^n} \\
+ \left. \frac{(-1)^m (y - 1)^m}{\Gamma(\nu - m + 1) \Gamma(\nu + 1) (y + 1)^m} \sum_{n=0}^{\infty} \frac{(\nu + 1)_n (\nu + m + 1)_n (y - 1)^n}{(n + m)! n! (y + 1)^n} \right] \\
\times \left[ \psi(n + 1) + \psi(n + m + 1) - \psi(\nu + 1 + n) - \psi(\nu + m + 1 + n) - \log \left( \frac{1 - y}{1 + y} \right) \right].
\]
when $\mu$ is a general positive integer $m$. (In Eq. [17] $\psi(y)$ denotes $(d\Gamma(y)/dy)/\Gamma(y)$ and $(a)_n$ denotes $\Gamma(a+n)/\Gamma(a)$). From Eq. [17] we see that in the $-\tanh\sigma \leq y \leq 1$ range of interest the $P^2_\nu(y)$ are well-behaved, behaving as $y$ approaches one from below (viz. as $|w| \rightarrow \infty$) as

$$P^2_\nu(y \rightarrow 1) \rightarrow P(\nu) \left[ (1 - y) - \frac{(1 - y)^2(\nu^2 + \nu - 3)}{6} \right]$$

(48)

where

$$P(\nu) = \frac{\nu(\nu^2 - 1)(\nu + 2)}{4}$$

(49)

to thus be fully normalizable and have finite normalization

$$N_\nu = \int_{-\infty}^{\infty} dw e^{-2A[P^2_\nu(|w|)]^2} = 2 \int_{0}^{\infty} d|w|e^{-2A[P^2_\nu(|w|)]^2} = \frac{2b}{H^2} \int_{-\tanh\sigma}^{1} dy [P^2_\nu(y)]^2.$$  

(50)

However, unlike the $P^2_\nu(y)$, the $Q^2_\nu(y)$ all found to diverge at $y = 1$, behaving there as

$$Q^2_\nu(y \rightarrow 1) \rightarrow \frac{1}{(1 - y)} + \frac{(\nu^2 + \nu - 1)}{2} + O((1 - y)\ln(1 - y))$$

(51)

and thus in the $AdS^+_4$ brane world none of the $Q^2_\nu(y)$, and particularly the massless $Q^2_1(y)$ graviton, are normalizable. We shall thus seek to construct complete bases in both the normalizable and non-normalizable sectors.

B. Completeness test for convergent $AdS^+_4$ modes

To construct a complete basis out of normalizable modes alone requires that the normalizable $P^2_\nu(y)$ satisfy Eq. [10] all on their own, with the eigenmodes then needing to satisfy

$$P^1_\nu(-\tanh\sigma) = P^1_\nu(-(1 - H^2/b^2)^{1/2}) = 0.$$  

(52)

For arbitrary $\sigma$ the solutions to Eq. [52] cannot be written down in a closed form, but on noting that for one particular value of $\sigma$, viz. $\sigma = 0$ (i.e. $H = b$), $P^1_\nu(0)$ is known in closed form as

$$P^1_\nu(0) = \frac{2\pi^{1/2}}{\Gamma(\nu/2 + 1/2)\Gamma(-\nu/2)}.$$  

(53)

to thus be zero at $\nu = 2, 4, 6, \ldots$, we see that on solving for an arbitrary given $\sigma$ numerically an infinite discrete set of allowed $\nu$ values will then be found to ensue [13]. The normalizable mode sector of $AdS^+_4$ is thus discrete and infinite, a result first obtained in [14] by directly numerically solving Eq. [13].
To test for completeness of the normalizable $AdS^+_4$ mode basis we need to find a set of coefficients $V_m$ for which the expansion

$$V_P = \sum_m V_m P^2_{\nu}(y)$$

reproduces the square step $V_P = \hat{V}$ when $|w_1| < |w| < |w_2|$, $V_P = 0$ otherwise. With the $P^2_{\nu}(y)$ modes being orthogonal, the coefficients are readily given as

$$V_m = B_m/P^2_{\nu}(y)$$

where $P^2_{\nu}(y)$ is the normalization factor given in Eq. (50), and where some standard properties of the associated Legendre functions allow $B_m$ to be written as

$$B_m = \frac{\hat{V}}{H^2} \int_{y_1}^{y_2} dy \left[ \frac{d}{dy} \left[ (2 - \nu)y P_{\nu} + \nu P_{\nu-1} \right] - 2 P_{\nu} \right]$$

$$= \frac{\hat{V}}{H^2} \int_{y_1}^{y_2} dy \left[ \frac{d}{dy} \left[ \frac{(2 - \nu)}{(2\nu + 1)}[(\nu + 1)P_{\nu + 1} + \nu P_{\nu - 1}] + \nu P_{\nu - 1} - \frac{2}{(2\nu + 1)}(P_{\nu + 1} - P_{\nu - 1}) \right] \right]$$

$$= \frac{\hat{V}}{H^2} \left[ \frac{(\nu + 1)(\nu + 2)P_{\nu - 1} - \nu(\nu - 1)P_{\nu + 1}}{2\nu + 1} \right]_{y_1}^{y_2}.$$  (55)

With every quantity which appears in Eq. (54) now being known, $V_P(|w|)$ can readily be plotted, and we display it in Fig. (3) as evaluated through the use of the first 1000 modes in the sum. As we see, the basis is indeed capable of generating the square step to very high accuracy, and with it expressly displaying the Gibbs phenomenon [15], its completeness is thus confirmed.

C. Completeness test for divergent $AdS^+_4$ modes

With the massless $AdS^+_4$ graviton with divergent warp factor wave function $f_0(y) = \beta_0 Q^2_{1}(y) = 2\beta_0/(1 - y^2)$ obeying the junction condition, it could also belong to a complete basis of divergent $Q^2_{\nu}(y)$ modes (modes which according to Eq. (51) actually diverge in precisely the same $1/(1 - y)$ way near $y = 1$ as the massless graviton itself) if the $Q^2_{\nu}(y)$ modes were to satisfy the junction condition on their own, i.e. if they were to obey

$$Q^1_{\nu}(-\tanh \sigma) = Q^1_{\nu}(-(1 - H^2/b^2)^{1/2}) = 0.$$  (56)

With Eq. (56) being found to possess an infinite set of discrete solutions for the arbitrary $\sigma$, we shall thus seek to expand the localized square step $V_Q = \hat{V}$ when $|w_1| \leq |w| \leq |w_2|$, \ldots
FIG. 3: The left panel shows a reconstruction of the square step $V_P(w) = 1$, $0.1 \leq \tanh(b|w| - \text{arctanh}(0.9)) \leq 0.2$, $V_P(w) = 0$ otherwise, via the $\text{AdS}^4_+\times \text{discrete}$ $P^2_{\nu}(\tanh(b|w| - \sigma))$ mode basis in the typical case where $\tanh\sigma = 0.9$, $H/b = 0.436$, and $b = 1$. The right panel shows a blow-up of the region near the top of the step.

$V_Q = 0$ otherwise, in terms of these solutions as

$$V_Q = \sum_n V_n Q^2_{\nu}(y) + \frac{V_0}{1-y^2}. \quad (57)$$

(For clarity we use $n^2$ here to denote the squared masses $n^2/H^2 = (\nu - 1)(\nu + 2)$ of the $Q^2_{\nu}(y)$ sector modes, and use $m^2$ for the $P^2_{\nu}(y)$ sector.) Given the asymptotic limit exhibited in Eq. (51), in order to first cancel both the leading $1/(1-y)$ term and the next to leading $O(1)$ term from the right-hand side of Eq. (57), we must constrain the $V_n$ coefficients according to

$$\sum_n V_n + \frac{V_0}{2} = 0, \quad \frac{1}{2} \sum_n V_n (\nu^2 + \nu - 1) + \frac{V_0}{4} = 0, \quad (58)$$

to thus enable us to reexpress the square step expansion as

$$V_Q = \sum_n V_n \left[ Q^2_{\nu}(y) - \frac{2}{1-y^2} \right], \quad (59)$$

as subject to the constraint

$$\sum_n V_n \left[ \nu^2 + \nu - 2 \right] = \sum_n V_n \frac{n^2}{H^2} = 0. \quad (60)$$
While we cannot apply \( \int_0^\infty d|w|e^{-2A Q_\nu^2(y)} \) to Eq. (59) as every overlap integral would diverge, finite overlap integrals are obtained if we instead apply \( \int_0^\infty d|w|e^{-2A P_\nu^2(y)} \), where we use \( \nu' \) to label the \( P_\nu^2(y) \) sector so that its squared masses are given by \( m^2/H^2 = (\nu' - 1)(\nu' + 2) \). With none of the \( P_{\nu'}(\pm \tanh \sigma) \) and \( Q_{\nu'}(\pm \tanh \sigma) \) zeroes being found to coincide, via. Eqs. (16), (48) and (51) the needed overlap integrals are found to be of the form

\[
\int_0^\infty d|w|e^{-2A P_{\nu'}^2(y)}Q_{\nu'}^2(y) = 4bP(\nu') \quad \frac{2bP(\nu')}{m^2},
\]

(\( P(\nu') \) is given in Eq. (49)), and are indeed finite, just as required. With the overlap integral which involves the massless graviton mode being given by

\[
\int_0^\infty d|w|e^{-2A P_{\nu'}^2(y)} = 2bP(\nu'),
\]

the application of \( \int_0^\infty d|w|e^{-2A P_{\nu'}^2(y)} \) to Eq. (59) thus yields

\[
4b \sum_n V_n P(\nu') \left[ \frac{1}{m^2 - n^2} \right] = \sum_n V_n b(m^2 + 2H^2)n^2 = B_m,
\]

(63)

where \( B_m \) is the same function that was already given earlier in Eq. (55).

Given Eq. (63), the \( V_n \) coefficients can now be found numerically, and lead, for the case of the first 1000 modes in the basis to the plot displayed in Fig. (4) (i.e. we restrict to the first 1000 \( P_{\nu'}(\pm \tanh \sigma) \) zeroes and the first 1000 \( Q_{\nu'}(\pm \tanh \sigma) \) zeroes). As Fig. (4) thus indicates, the divergent mode basis is every bit as capable of reconstructing the square step as the convergent one and every bit as capable of recovering the Gibbs phenomenon, and is thus every bit as complete [17]. Once again then we see that it is invalid to use normalizability as a criterion for discarding modes, and in this regard we differ from the view of [14] that it is permissible to discard modes such as the massless \( AdS^+_4 \) graviton simply because they are not normalizable [18].

VII. COMPLETENESS TESTS FOR THE DE SITTER BRANE CASES

A. The basis modes

For \( dS_4^\pm \) brane worlds with warp factor \( e^{A(|w|)} = H \sinh(\sigma \mp b|w|)/b \) where \( \sinh \sigma = b/H \), the transformation \( y = \coth(\sigma \mp b|w|) \) brings Eq. (13) to the form

\[
\left[ (1 - y^2) \frac{d^2}{dy^2} - 2y \frac{d}{dy} + \nu(\nu + 1) - \frac{4}{(1 - y^2)} \right] f_m(y) = 0.
\]

(64)
FIG. 4: The left panel shows a reconstruction of the square step $V_Q(w) = 1$, $0.1 \leq \tanh(b|w| - \text{arctanh}(0.9)) \leq 0.2$, $V_Q(w) = 0$ otherwise, via the $AdS_4^+$ discrete $Q_\nu^2(\tanh(b|w| - \sigma))$ plus $\cosh^2(\tanh(b|w| - \sigma))$ mode basis in the typical case where $\tanh\sigma = 0.9$, $H/b = 0.436$, and $b = 1$. The right panel shows a blow-up of the region near the top of the step.

where we have introduced the convenient parameter $\nu$ defined by

$$\nu = \left(\frac{9}{4} - \frac{m^2}{H^2}\right)^{1/2} - \frac{1}{2}, \quad \frac{m^2}{H^2} = (1 - \nu)(\nu + 2).$$

(65)

Recognizing Eq. (64) to be the previously discussed associated Legendre equation, its $m \neq 0$ (viz. $\nu \neq 1$) solutions are given as

$$f_m(y) = \alpha_m P^2_\nu(y) + \beta_m Q^2_\nu(y).$$

(66)

while its $\nu = 1$ solutions are of the form

$$f_0(y) = \alpha_0 \left(\frac{2}{1+y} - y\right) + \beta_0 Q^1_\nu(y).$$

(67)

Requiring the modes to also obey the junction condition of Eq. (14) then restricts them according to

$$\alpha_m P^1_\nu(\coth\sigma) + \beta_m Q^1_\nu(\coth\sigma) = 0, \quad \alpha_0 = 0,$$

(68)

to thus define the $dS_4^+$ brane-world basis modes.
While the $dS_4^+$ and $dS_4^-$ basis modes are quite similar to each other in their generic structure, they differ from each other significantly in one crucial regard. Specifically, unlike the $dS_4^-$ warp factor $e^{A(|w|)} = H\sinh(\sigma + b|w|)/b$ which never vanishes ($\sigma$ having been defined to be positive), the $dS_4^+$ warp factor $e^{A(|w|)} = H\sinh(\sigma - b|w|)/b$ has a zero at $b|w| = \sigma$. With a null signal taking an infinite amount of time to travel from the brane to the location of this zero, this zero serves as a horizon for an observer on the brane [19], with the brane observer only being sensitive to fluctuation modes in the $\sigma \geq b|w| \geq 0$ region. With the $dS_4^+$ parameter $y = \coth(\sigma - b|w|)$ lying in the range $\coth \sigma \leq y \leq \infty$, we see that $y$ is infinite at the $dS_4^+$ horizon. Then, with the associated Legendre functions behaving as $P_{\nu}^2(y) \to O(y^\nu) + O(y^{-\nu-1})$, $Q_{\nu}^2(y) \to O(y^{-\nu-1})$ as $y \to \infty$, the $\nu = 1$ massless $dS_4^+$ graviton and all $dS_4^+$ modes with complex $\nu = -1/2 \pm i(m^2/H^2 - 9/4)^{1/2}$ will be normalizable within the horizon [20]. With the massless graviton and a massive continuum of modes with $m^2/H^2 \geq 9/4$ which satisfy the junction condition of Eq. (68) by an interplay (of the real $P_{\nu}^2(y)$ and the real part of $Q_{\nu}^2(y)$) thus providing a conventional continuum normalized complete basis in the sense of Eqs. (1) to (5), as with the $M_4^+$ brane world, in the $dS_4^+$ brane world there is no need to test explicitly for completeness.

However, for $dS_4^-$ the situation is quite different since there is now no vanishing of the warp factor and no horizon, with the coordinate $|w|$ now extending all the way to infinity, and with the parameter $y = \coth(\sigma + b|w|)$ instead now lying in the $1 \leq y \leq \coth \sigma = (1 + H^2/b^2)^{1/2}$ range. Unlike the previously discussed $AdS_4^+$ brane world case where $y$ approached one from below as $|w|$ went to infinity, in the $dS_4^-$ case $y$ instead approaches one from above in the large $|w|$ limit, with Eqs. (48) and (51) having to be replaced by the limits

$$P_{\nu}^2(y \to 1) \to P(\nu) \left[(y - 1) + \frac{(y - 1)^2(\nu^2 + \nu - 3)}{6}\right]$$

$$Q_{\nu}^2(y \to 1) \to \frac{1}{(y - 1)} - \frac{(\nu^2 + \nu - 1)}{2} + O((y - 1)\ln(y - 1)) ,$$

where $P(\nu) = \nu(\nu^2 - 1)(\nu + 2)/4$ is as given in Eq. (19). Since the $P_{\nu}^2(y)$ are well behaved at $y = 1$, while the $Q_{\nu}^2(y)$ diverge there, as with the $AdS_4^+$ case, the normalizable sector will consist of all $P_{\nu}^2(\coth(\sigma + b|w|))$ modes which satisfy the junction condition on their own according to

$$P_{\nu}^1(\coth \sigma) = P_{\nu}^1((1 + H^2/b^2)^{1/2}) = 0$$

while the non-normalizable sector will consist of the divergent warp factor wave function
$Q^2_{1}(\coth(\sigma+b|w|)) (=2/(y^2-1) \text{ in } y > 1)$ massless graviton and all massive $Q^2_{\nu}(\coth(\sigma+b|w|))$ modes which obey

$$ Q^1_{\nu}(\coth\sigma) = Q^1_{\nu}((1 + H^2/b^2)^{1/2}) = 0 \ . \quad (71) $$

While this pattern is thus quite similar to the situation found in the $AdS^+_4$ case, the $dS^-_4$ brane world differs from it in one key regard, namely that the parameter $y$ is required to be greater or equal to one rather than less than or equal to it, and thus the completeness of its mode bases requires independent testing.

**B. Completeness test for convergent $dS^-_4$ modes**

With the general Eq. (16) taking the form

$$ \left( \frac{m_1^2}{H^2} - \frac{m_2^2}{H^2} \right) \int_{1}^{\coth\sigma} dy f_{m_1}(y)f_{m_2}(y) $$

$$ = \lim_{y \to 1} \left[ (y^2 - 1)f_{m_2}(y)\frac{df_{m_1}(y)}{dy} - (y^2 - 1)f_{m_1}(y)\frac{df_{m_2}(y)}{dy} \right] \quad (72) $$

in the $dS^-_4$ case, and with the $P^2_{\nu}(y)$ modes behaving near $y = 1$ as in Eq. (69), the $P^2_{\nu}(y)$ modes form an orthonormal basis, and we can normalize thus them according to

$$ N_{\nu} = \int_{-\infty}^{\infty} dw e^{-2A[P^2_{\nu}(|w|)]^2} = \frac{2b}{H^2} \int_{1}^{\coth\sigma} dy [P^2_{\nu}(y)]^2 \ . \quad (73) $$

With the $\coth\sigma$ argument of $P^1_{\nu}(\coth\sigma)$ in Eq. (70) being greater than one, the $P^1_{\nu}(\coth\sigma) = 0$ condition has no solutions with real $\nu$. Rather, all of its solutions are of the form $\nu = -1/2 + i\lambda$ where $\lambda$ is real and discrete [21]. According to Eq. (68), for such solutions the associated squared masses obey $m^2/H^2 = 9/4 + \lambda^2$ and are thus nicely positive. Additionally, as noted previously, for the particular choice of $\nu = -1/2 + i\lambda$, the $P^2_{\nu}(y)$ mode wave functions themselves are real.

Having now explicitly identified the $dS^-_4$ normalizable mode basis, to test for completeness we need to find a set of coefficients $V_m$ for which the expansion

$$ \hat{V}_P = \sum_m V_m P^2_{\nu}(y) \quad (74) $$

reproduces the square step $\hat{V}_P = \hat{V}$ when $|w_1| < |w| < |w_2|$, $\hat{V}_P = 0$ otherwise. With the $P^2_{\nu}(y)$ modes being orthogonal, the coefficients are readily given as $V_m = B_m/N_{\nu}$ where $N_{\nu}$
FIG. 5: The left panel shows a reconstruction of the square step $\hat{V}_P(w) = 1$, $1.05 \leq \coth(\text{arccoth}(1.1) + w) \leq 1.06$, $\hat{V}_P(w) = 0$ otherwise, via the $dS_4^-$ discrete $P^2_\nu(\coth(\sigma + b|w|))$ mode basis in the typical case where $\coth(1.1) = 1.1$, $H/b = 0.458$, and $b = 1$. The right panel shows a blow-up of the region near the top of the step.

is the normalization factor given in Eq. (73), where $m$ and $\nu$ are related as in Eq. (65), and where the $B_m$ are given as

$$B_m = \hat{V} \int_{|w_1|}^{(w_2)} d|w|e^{-2A}P^2_\nu(|w|) = -\frac{\hat{V}b}{H^2} \int_{y_1}^{y_2} dy P^2_\nu(y) = -\frac{\hat{V}b}{H^2} \int_{y_1}^{y_2} dy (y^2 - 1) \frac{d^2 P_\nu(y)}{dy^2}$$

$$= -\frac{\hat{V}b}{H^2} \int_{y_1}^{y_2} dy \left[ \frac{d}{dy} \left[ (\nu - 2)yP_\nu - \nu P_{\nu-1} \right] + 2P_\nu \right]$$

$$= -\frac{\hat{V}b}{H^2} \int_{y_1}^{y_2} dy \left[ \frac{\nu(\nu - 1)P_{\nu+1} + (\nu + 1)P_{\nu-1} - \nu P_{\nu-1} + 2}{(2\nu + 1)} \right]$$

$$= -\frac{\hat{V}b}{H^2} \left[ \frac{\nu(\nu - 1)P_{\nu+1} + (\nu + 1)(\nu + 2)P_{\nu-1}}{2\nu + 1} \right] \Big|_{y_1}^{y_2}. \tag{75}$$

Given Eq. (75), $\hat{V}_P(|w|)$ can readily be plotted, and we display it in Fig. (5) as evaluated through the use of the first 500 modes in the sum. As we see, the basis is indeed capable of generating the square step to very high accuracy, and with it also nicely displaying the Gibbs phenomenon, its completeness is thus confirmed.
C. Completeness test for divergent $dS^-_4$ modes

As with the $P^1_\nu(\coth\sigma) = 0$ condition, the solutions to $Q^1_\nu(\coth\sigma) = 0$ are also all of the form $\nu = -1/2 + i\lambda$ where $\lambda$ is again real and discrete, with the solutions to $P^1_\nu(\coth\sigma) = 0$ and $Q^1_\nu(\coth\sigma) = 0$ being found to interlace each other [22]. With it being only the real parts of the $Q^2_\nu(y)$ wave functions with $\nu = -1/2 + i\lambda$ and $y$ real which are independent of the real $P^2_\nu(y)$, the non-normalizable $dS^-_4$ brane world mode basis consists of the massless graviton with its real warp factor wave function plus the real parts of the $Q^2_\nu(y)$ wave functions with the appropriate $\nu = -1/2 + i\lambda$. Then, with the $y \to 1$ limit of Eq. (69) holding for the general $Q^2_\nu(y)$ with arbitrary $\nu$, we see that the real parts of the $Q^2_\nu(y)$ wave functions all have the same $1/(y-1)$ leading behavior at $y = 1$ as the massless graviton itself, with the non-normalizable modes all diverging at $y = 1$ at one and the same rate.

In order to test for completeness in the $\text{Re}[Q^2_\nu(y)]$ plus massless graviton sector, we need to expand the localized square step $\hat{V}_Q = \hat{V}$ when $|w_1| \leq |w| \leq |w_2|$, $\hat{V}_Q = 0$ otherwise, in terms of these solutions as

$$\hat{V}_Q = \sum_n V_n \text{Re}[Q^2_\nu(y)] + \frac{V_0}{y^2 - 1}. \quad (76)$$

(As previously, for clarity we use $n^2$ here to denote the squared masses of the $Q^2_\nu(y)$ sector modes, and use $m^2$ for the $P^2_\nu(y)$ sector.) Given the asymptotic limit exhibited in Eq. (69), in order to cancel both the leading $1/(y-1)$ term and the next to leading $O(1)$ term from the right-hand side of Eq. (76), we must constrain the $V_n$ coefficients according to

$$\sum_n V_n + \frac{V_0}{2} = 0, \quad \frac{1}{2} \sum_n V_n (\nu^2 + \nu - 1) + \frac{V_0}{4} = 0, \quad (77)$$

to thus enable us to reexpress the square step expansion as

$$\hat{V}_Q = \sum_n V_n \left[ \text{Re}[Q^2_\nu(y)] - \frac{2}{y^2 - 1} \right], \quad (78)$$

as subject to the constraint

$$\sum_n V_n \left[ \nu^2 + \nu - 2 \right] = -\sum_n V_n \frac{n^2}{H^2} = 0. \quad (79)$$

On now applying $\int_0^\infty d|w|e^{-2A}P^2_\nu(|w|) = (b/H^2) \int^\coth\sigma dyP^2_\nu(y)$ to Eq. (78) where $\nu^2 + \nu' - 2 = -m^2/H^2$, use of the relations

$$\int^\coth\sigma dyP^2_\nu(y)\text{Re}[Q^2_\nu(y)] = \frac{4H^2P(\nu')}{(m^2 - n^2)}, \quad (80)$$

and
FIG. 6: The left panel shows a reconstruction of the square step $\hat{V}_Q(w) = 1$, $1.05 \leq \coth[\text{arccoth}(1.1) + w] \leq 1.06$, $\hat{V}_Q(w) = 0$ otherwise, via the $dS_4$ discrete $\text{Re}[Q^2_\nu(\coth(\sigma + b|w|))]$ plus $\sinh^2(\coth(\sigma + b|w|))$ mode basis in the typical case where $\coth\sigma = 1.1$, $H/b = 0.458$, and $b = 1$. The right panel shows a blow-up of the region near the top of the step.

$$\int_{\coth} dy \frac{P^2_\nu(y)}{(y^2 - 1)} = 2H^2P(\nu') \frac{m^2}{m^2},$$

which follow from Eqs. (69) and (72) (with $P(\nu') = \nu'/(\nu'^2 - 1)(\nu' + 2)/4$ now being given by $m^2(m^2 - 2H^2)/4H^4$) then yields

$$4b \sum_n V_n P(\nu') \left[ \frac{1}{(m^2 - n^2)} - \frac{1}{m^2} \right] = \sum_n V_n \frac{b(m^2 - 2H^2)n^2}{H^4(m^2 - n^2)} = B_m,$$  

where $B_m$ is the same function that was already given earlier in Eq. (75).

Given Eq. (82), the $V_n$ coefficients can now be found numerically, and lead, for the case of the first 500 modes in the basis to the plot displayed in Fig. (6) (i.e. we restrict to the first 500 $P^1_\nu(\coth\sigma)$ zeroes and the first 500 $\text{Re}[Q^1_\nu(\coth\sigma)]$ zeroes). As Fig. (6) thus indicates, the divergent mode basis is every bit as capable of reconstructing the square step as the convergent one and every bit as capable of recovering the Gibbs phenomenon, and is thus every bit as complete. As with our earlier examples then, we once again confirm that completeness is not at all tied to normalizability.
VIII. FINAL COMMENTS

In this work we have shown that in and of itself the requirement of normalizability of basis modes is not at all needed for completeness, and that one can construct localized steps out of bases whose modes are not normalizable at all. Since the localized steps that we have constructed out of non-normalizable bases involve expansion coefficients $V_n$ which are explicitly found to be finite, this suggests that we should be able to construct propagators involving the modes in which these modes appear as poles which have residues which are themselves finite. Thus in sharp contrast to the situation in which propagators are built out of normalizable modes, for propagators which are built out modes of which are not normalizable, these residues must then not be related to normalization constants or to any bilinear integrals of the modes at all for that matter.

To explicitly construct such divergent mode based propagators we must first introduce explicit source terms. For the case of interest to the brane world the source is typically taken to be a transverse-traceless energy-momentum tensor $S_{\mu\nu}^{TT}$ which is confined to the brane at $w = 0$, with Eqs. (10) and (11) being replaced by (see e.g. [6])

$$\nabla_\alpha \nabla^\alpha h_{\mu\nu}^{TT} = 0,$$

$$\delta(w) \left[ \frac{\partial}{\partial |w|} - 2 \frac{dA}{d|w|} \right] h_{\mu\nu}^{TT} = -\kappa_5^2 \delta(w) S_{\mu\nu}^{TT},$$

where $\kappa_5^2$ is the brane-world gravitational constant.

For the case first of the convergent warp factor $M_4^+$ brane world where Eqs. (83) and (84) reduce to

$$\nabla_\alpha \nabla^\alpha h_{\mu\nu}^{TT} = 0,$$

$$\delta(w) \left[ \frac{\partial}{\partial |w|} + 2b \right] h_{\mu\nu}^{TT} = -\kappa_5^2 \delta(w) S_{\mu\nu}^{TT},$$

on recalling that the Bessel functions obey

$$\left[ \frac{d}{d|w|} + 2b \right] \left[ \alpha_q J_2 \left( \frac{qe^{b|w|}}{b} \right) + \beta_q Y_2 \left( \frac{qe^{b|w|}}{b} \right) \right] = qe^{b|w|} \left[ \alpha_q J_1 \left( \frac{qe^{b|w|}}{b} \right) + \beta_q Y_1 \left( \frac{qe^{b|w|}}{b} \right) \right],$$

an explicit solution to Eqs. (85) and (86) can readily be given, viz. [23]

$$h_{\mu\nu}^{TT}(x, |w|) = -\frac{\kappa_5^2}{(2\pi)^4} \int d^4x' d^4pe^{ip(x-x')} \left[ \frac{\alpha_q J_2(qe^{b|w|}/b) + \beta_q Y_2(qe^{b|w|}/b)}{q[\alpha_q J_1(q/b) + \beta_q Y_1(q/b)]} \right] S_{\mu\nu}^{TT}(x').$$

25
\[ = -2\kappa_5^2 \int d^4x' \hat{G}_{TT}^{TT}(x, x', w, 0; \alpha_q, \beta_q, M_4^+) S_{\mu\nu}^{TT}(x') , \tag{88} \]

where \( q^2 = (p^0)^2 - \vec{p}^2 \) (\( q \) being understood to have the same sign as \( p^0 \) here), and \( \alpha_q \) and \( \beta_q \) are arbitrary constants.

The generalization of this solution to the divergent warp factor \( M_4^- \) brane world where we have

\[ \left[ \frac{\partial^2}{\partial|w|^2} - 4b^2 + e^{-2b|w|}\eta^{\alpha\beta}\partial_\alpha \partial_\beta \right] h_{\mu\nu}^{TT} = 0 , \tag{89} \]

\[ \delta(w) \left[ \frac{\partial}{\partial|w|} - 2b \right] h_{\mu\nu}^{TT} = -\kappa_5^2 \delta(w) S_{\mu\nu}^{TT} , \tag{90} \]

and

\[ \left[ \frac{d}{d|w|} - 2b \right] \left[ \alpha_q J_2 \left( \frac{qe^{-b|w|}}{b} \right) + \beta_q Y_2 \left( \frac{qe^{-b|w|}}{b} \right) \right] = -qe^{-b|w|} \left[ \alpha_q J_1 \left( \frac{qe^{-b|w|}}{b} \right) + \beta_q Y_1 \left( \frac{qe^{-b|w|}}{b} \right) \right] , \tag{91} \]

is of the form

\[ h_{\mu\nu}^{TT}(x, |w|) = \frac{\kappa_5^2}{(2\pi)^4} \int d^4x \int d^4p e^{i p \cdot (x-x')} \frac{[\alpha_q J_2(qe^{-b|w|}/b) + \beta_q Y_2(qe^{-b|w|}/b)]}{q[\alpha_q J_1(q/b) + \beta_q Y_1(q/b)]} S_{\mu\nu}^{TT}(x') = -2\kappa_5^2 \int d^4x' \hat{G}_{TT}^{TT}(x, x', w, 0; \alpha_q, \beta_q, M_4^-) S_{\mu\nu}^{TT}(x') , \tag{92} \]

with \( \alpha_q \) and \( \beta_q \) again being arbitrary constants. That the solution of Eq. (92) satisfies Eq. (89) follows directly, since both \( J_2(qe^{-b|w|}/b) \) and \( Y_2(qe^{-b|w|}/b) \) separately satisfy the Bessel function equation given as Eq. (19) with \( y \) being given by \( y = qe^{-b|w|}/b \); and that the solution satisfies Eq. (90) follows from Eq. (91). For this solution we note that it is the requirement that Eq. (92) obey Eq. (90) (technically the Israel junction condition in the presence of the source) which fixes the overall normalization of the integrand in Eq. (92), with none of the \( \alpha_q \) or \( \beta_q \) coefficients needing to be infinite. In fact the same is true of the \( M_4^+ \) brane world propagator as its overall normalization is fixed by the junction condition of Eq. (86), with the similarity of the \( M_4^+ \) solution of Eq. (88) and the \( M_4^- \) solution of Eq. (92) essentially showing complete insensitivity to the normalizability or lack thereof of basis modes.

In order to be able to make contact with the various bases we used in our construction of localized steps in the divergent warp factor \( M_4^- \) brane world, we need to make specific choices for the \( \alpha_q \) and \( \beta_q \) coefficients which appear in Eq. (92). To make contact with the
convergent \( J_2(qe^{-b|w|}/b) \) modes, we recall that a Taylor series expansion of \( J_1(q/b) \) around any \( j_i \) zero of \( J_1 \) is of the form

\[
J_1(q/b) = \left( \frac{q}{b} - j_i \right) J'_1(j_i) = \left( \frac{q}{b} - j_i \right) \left[ \frac{J_1(j_i)}{j_i} - J_2(j_i) \right] = -\left( \frac{q}{b} - j_i \right) J_2(j_i).
\]

(93)

Thus on setting \( \alpha_q = 1, \beta_q = 0 \) and recalling that each \( j_i \) zero of \( J_1(j_i) \) is also a zero of \( J_1(-j_i) \), we see that the propagator of Eq. (92) contains a set of isolated poles at the zeroes of \( J_1 \) (a pole at \( q = bj_1 \) when \( p^0 \) is positive, and a pole at \( q = -bj_1 \) when \( p^0 \) is negative), with a \( p^0 \) plane contour integration yielding a net pole contribution to the propagator of the form

\[
\hat{G}^{TT}(x, 0, w, 0; \alpha_q = 1, \beta_q = 0, M^-) = -i \sum_i f_i(|w|) f_i(0) \int \frac{d^3p}{(2\pi)^3} \frac{e^{i\vec{p}\cdot\vec{x}}}{2E_i} \left[ e^{-iE_it} - e^{iE_it} \right],
\]

(94)

where

\[
f_i(|w|) = \frac{b^1/2 J_2(j_i e^{-b|w|})}{J_2(j_i)}, \quad E_i = (\vec{p}^2 + b^2 j_i^2)^{1/2},
\]

(95)

and where the summation in Eq. (94) only needs extend over the \( j_i > 0 \) modes. Finally, recalling Eq. (32), viz.

\[
\int_0^\infty d|w| e^{-2b|w|} J_2^2(me^{-b|w|}/b) = \frac{J_2^2(m/b)}{2b},
\]

(96)

we see that the \( f_i(|w|) \) basis modes precisely obey Eqs. (11) and (15), with the pole structure of the \( M^- \) brane-world propagator \( \hat{G}^{TT}(x, 0, w, 0; \alpha_q = 1, \beta_q = 0, M^-) \) nicely recovering the orthonormality and closure structure of the \( J_2^2(me^{-b|w|}/b) \) sector basis modes.

In order to make contact with the non-normalizable \( M^- \) mode sector, we need to take \( \beta_q \) to be non-zero in the \( M^- \) propagator. Recalling that \( J_1(y), J_2(y), Y_1(y) \) and \( Y_2(y) \) respectively behave as \( y/2, y^2/8, -2/\pi y + O(y) \) and \( -4/\pi y^2 - 1/\pi \) near \( y = 0 \), we see that once \( \beta_q \) is non-zero, the integrand \( [\alpha_q J_2(qe^{-b|w|}/b) + \beta_q Y_2(qe^{-b|w|}/b)]/[q(\alpha_q J_1(q/b) + \beta_q Y_1(q/b)] \) will behave as \( 2be^{2b|w|}/q^2 \) near \( q^2 = 0 \) independent of the actual values of \( \alpha_q \) and \( \beta_q \), to thus give rise to a massless graviton pole term contribution of the form

\[
\hat{G}^{TT}(x, 0, w, 0; \alpha_q, \beta_q \neq 0, M^- \text{, graviton}) = ibe^{2b|w|} \int \frac{d^3p}{(2\pi)^3} \frac{e^{i\vec{p}\cdot\vec{x}}}{2|p|} \left[ e^{-i|p|t} - e^{i|p|t} \right].
\]

(97)

Non-normalizable as the \( M^- \) brane-world graviton might be, as we see, it nonetheless appears in the propagator with a finite residue [24].

To make contact with the \( M^- \) brane world divergent \( Y_2(qe^{-b|w|}/b) \) modes we set \( \alpha_q = 0 \) in \( \hat{G}^{TT}(x, 0, w, 0; \alpha_q, \beta_q, M^-) \), and while we immediately then obtain poles at the zeroes of
\( Y_1(q/b) \), since both \( Y_2(qe^{-b|w|}/b) \) and \( Y_1(q/b) \) have branch points at \( q = 0 \), we also obtain a cut discontinuity, with the full singular term evaluating to [9]

\[
\hat{G}^{TT}(x, 0, w, 0; \alpha_q = 0, \beta_q \neq 0, M_4^-) = ibe^{2b|w|} \int \frac{d^3p}{(2\pi)^3 \frac{2}{|p|}} \left[ e^{-ip|x|} - e^{ip|x|} \right]
\]

\[
-i \sum_i \tilde{f}_i(|w|) \tilde{f}_i(0) \int \frac{d^3p}{(2\pi)^3 2E_i} \left[ e^{-iE_i t} - e^{iE_i t} \right]
\]

\[
+ \frac{1}{(2\pi)^3} \int d^3p \left[ e^{-iEp t} - e^{iEp t} \right] \int dm \left[ 1 - 2i \frac{J_2(me^{-b|w|}/b)}{Y_1(m/b)} \right]
\]

\[
\times \left[ \frac{[Y_1(m/b)J_2(me^{-b|w|}/b) - J_1(m/b)Y_2(me^{-b|w|}/b)]}{\pi[4J_1^2(m/b) + Y_1^2(mb)]} \right],
\]

(98)

where

\[
\tilde{f}_i(|w|) = \frac{b^{1/2}Y_2(y_ie^{-b|w|})}{Y_2(y_i)}, \quad E_i = (p^2 + b^2y_i^2)^{1/2}.
\]

(99)

As we again see, despite the lack of normalizability of \( Y_2(me^{-b|w|}/b) \) modes, all the terms which appear in Eq. (98) do so with coefficients which are nonetheless finite.

Further examples of this phenomenon may be found in the other divergent warp factor brane worlds we have been considering. However, unlike the exact propagator solutions of Eqs. (88) and (92), for the \( AdS_4 \) and \( dS_4 \) based brane worlds so far such propagators have only been constructed in low order. Specifically, for the \( AdS_4^+ \) brane world for instance where the background metric of Eq. (3) takes the explicit form

\[
ds^2 = dw^2 + e^{2A(|w|)}[dx^2 + e^{2Hx}(dy^2 + dz^2 - dt^2)]
\]

(100)

with the \( AdS_4^+ \) warp factor \( A(|w|) \) being given in Eq. (3) and \( \lambda \) being positive, to lowest order in \( H \) the appropriate \( AdS_4^+ \) propagator is given as [9]

\[
\hat{G}^{TT}(x, x', w, 0; \hat{\alpha}_\nu, \hat{\beta}_\nu, AdS_4^+) = \frac{1}{2H(2\pi)^3} \int_{-\infty}^{\infty} dp^0 dp^0 dp^3 \int_{0}^{\infty} dp^1 p^1 B_\nu(\tanh(b|w| - \sigma), \hat{\alpha}_\nu, \hat{\beta}_\nu) e^{Hx'/2} e^{Hx'/2}
\]

\[
\times e^{-ip^0(t-t') + ip^2(y-y') + ip^3(z-z')} J_\nu(k e^{-Hx}/H) J_\nu(k e^{-Hx'}/H)
\]

(101)

where \( k \) is given by \( k = [(p^0)^2 - (p^2)^2 - (p^3)^2]^{1/2} \), \( \tau \) and \( \nu \) are given by \( \tau = \nu + 1/2 = [9/4 + k^2/H^2 - (p^1)^2/H^2]^{1/2} \), and the quantity \( B_\nu(\tanh(b|w| - \sigma), \hat{\alpha}_\nu, \hat{\beta}_\nu) \) is given by

\[
B_\nu(\tanh(b|w| - \sigma), \hat{\alpha}_\nu, \hat{\beta}_\nu) = \frac{1}{H(\nu - 1)(\nu + 2)} \left[ \hat{\alpha}_\nu P_\nu^2(\tanh(b|w| - \sigma)) + \hat{\beta}_\nu Q_\nu^2(\tanh(b|w| - \sigma)) \right].
\]

(102)
As constructed the quantity \( B_\nu (\tanh(b|w| - \sigma), \hat{\alpha}_\nu, \hat{\beta}_\nu) \) obeys

\[
\delta(w) \left[ \frac{d}{d|w|} - 2 \frac{dA}{d|w|} \right] B_\nu (\tanh(b|w| - \sigma), \hat{\alpha}_\nu, \hat{\beta}_\nu) = \delta(w) ,
\]

and has a small \( H \) limit into the analogous \( M_4^+ \) integrand, viz.

\[
B_\nu (\tanh(b|w| - \sigma), \hat{\alpha}_\nu, \hat{\beta}_\nu) \rightarrow \frac{[\alpha_\nu J_2(qe^{b|w|}/b) + \beta_\nu Y_2(qe^{b|w|}/b)]}{q[\alpha_\nu J_1(q/b) + \beta_\nu Y_1(q/b)]} ,
\]

where \( \hat{\alpha}_\nu = \alpha_\nu \cos(\nu \pi) + \beta_\nu \sin(\nu \pi) \), \( \hat{\beta}_\nu = (2/\pi) [-\alpha_\nu \sin(\nu \pi) + \beta_\nu \cos(\nu \pi)] \). In the small \( H \) limit \( \hat{G}^{TT} (x, x', w, 0; \hat{\alpha}_\nu, \hat{\beta}_\nu, AdS_4^+) \) obeys

\[
\left[ \frac{\partial^2}{\partial w^2} - 4 \left( \frac{dA}{d|w|} \right)^2 - 4 \frac{dA}{d|w|} \delta(w) + e^{-2A} \hat{\nabla}_\alpha \hat{\nabla}^\alpha \right] \hat{G}^{TT} (x, x', w, 0; \hat{\alpha}_\nu, \hat{\beta}_\nu, AdS_4^+) \]

\[
= e^{Hx} \delta(x - x') \delta(t - t') \delta(y - y') \delta(z - z') \delta(w) ,
\]

with the fluctuation

\[
h^{TT}_{\mu\nu} (x, |w|) = -2 \kappa_5^2 \int d^4 x' e^{-Hx'} \hat{G}^{TT} (x, x', w, 0; \hat{\alpha}_\nu, \hat{\beta}_\nu, AdS_4^+) S^{TT}_{\mu\nu} (x')
\]

thus being an exact \( AdS_4^+ \) brane world small \( H \) solution to the \( AdS_4^+ \) variant of Eqs. (83) and (84) for an arbitrary \( S^{TT}_{\mu\nu} (x') \) source on the brane.

As regards pole terms in the \( AdS_4^+ \) brane-world propagator, since \((\nu - 1)(\nu + 2) = q^2/H^2\), the \((\nu - 1)(\nu + 2)\) term in \( B_\nu (\tanh(b|w| - \sigma), \hat{\alpha}_\nu, \hat{\beta}_\nu) \) generates a massless \( \nu = 1 \) graviton pole in the propagator which is found to be of the form

\[
\hat{G}^{TT} (x, x', w, 0; \hat{\alpha}_\nu, \hat{\beta}_\nu, AdS_4^+ )_{\text{graviton}}
\]

\[
= \frac{be^{2A} \hat{D}_S(x, x', m = 0)}{- (\hat{\alpha}_1/\hat{\beta}_1)(H^2/b^2) + (1 - H^2/b^2)^{1/2} + (H^2/b^2) \arccosh(b/H)} ,
\]

where \( \hat{D}_S(x, x', m) \) is the pure \( AdS_4 \) spacetime propagator which obeys

\[
[\hat{\partial}_x^2 - H \partial_x + e^{-2Hx} (\hat{\partial}_y^2 + \hat{\partial}_z^2 - \hat{\partial}_t^2) - 2H^2 - m^2] \hat{D}_S(x, x', m) = e^{Hx} \delta^4(x - x') .
\]

As we see, despite the lack of normalizability of the graviton wave function, the residue at the \( AdS_4^+ \) massless graviton pole is nonetheless finite \([25]\). Similarly, if we set \( \hat{\alpha}_\nu = 0 \) in Eq. (101) we will immediately generate the divergent \( Q^2_\nu (\tanh(b|w| - \sigma)) \) modes as poles associated with the zeroes of \( Q^1_\nu (-\tanh \sigma) \), with these pole terms also possessing finite residues. Consequently, in the brane world divergent modes are fully capable of appearing with finite
residues in propagators and their lack of normalizability should not be taken as being a criterion for excluding them. In fact, with the $M_4^+$ propagator $\hat{G}^{TT}(x, x', w, 0; \alpha_q, \beta_q, M_4^+)$ of Eq. (88) being causal when we set $\alpha_q = 1$, $\beta_q = i$ (so that it is then based on outgoing Hankel functions), given the small $H$ limit of $B_\nu(\tanh(b|w| - \sigma), \hat{\alpha}_\nu, \hat{\beta}_\nu)$ exhibited in Eq. (103), it will be the $\hat{G}^{TT}(x, x', w, 0; \hat{\alpha}_\nu, \hat{\beta}_\nu, AdS_4^+)$ propagator with $\hat{\alpha}_\nu = e^{i\pi\nu}$, $\hat{\beta}_\nu = (2i/\pi)e^{i\pi\nu}$ which will be the $AdS_4^+$ analog of the outgoing Hankel function based causal $M_4^+$ brane-world propagator, with this particular $AdS_4^+$ brane world propagator explicitly being found to be causal [23]. As such, the causal $AdS_4^+$ brane-world propagator with $\hat{\alpha}_\nu = e^{i\pi\nu}$, $\hat{\beta}_\nu = (2i/\pi)e^{i\pi\nu}$ possesses an explicit massless graviton pole whose residue is finite, with there thus being no justification for excluding it [24].

Acknowledgments

The authors would like to thank Dr. A. H. Guth, Dr. D. I. Kaiser and Dr. A. Nayeri for their active participation in this work, and for their many helpful comments.

[1] Even in quantum mechanics we note that the Schrödinger equation $H|\psi\rangle = E|\psi\rangle$ is an operator equation which acts linearly on the ket vector $|\psi\rangle$, with its existence being independent of what particular dual vector bra $\langle \psi|$ might be used to construct the bilinear norm $\langle \psi|\psi\rangle$. There is thus freedom available in choosing the dual space vectors, with choices for them other than simply as the conjugates of the kets having been found to lead to a sensible probability interpretation in the case of theories with a non-Hermitian potential (C. M. Bender, D. C. Brody and H. F. Jones, Am. J. Phys. 71, 1095 (2003)) or an indefinite metric (the fourth order oscillator theory discussed in P. D. Mannheim and A. Davidson, Phys. Rev. A 71, 042110 (2005)). Even in quantum mechanics then, imposing the finiteness of the $\langle \psi|\psi\rangle$ norm is not the most general requirement that one can consider.


Even in quantum mechanics one does not discard plane wave modes even though they cause the integral on the right-hand side of Eq. (18) to diverge, since divergent as they may be, one can still construct localized wave packets out of them. In this respect then, the point of this paper will be to construct localized configurations out of basis vectors which diverge even more rapidly than plane waves. And while we shall restrict the study of this paper to the classical-mechanical context, we note that within a quantum-mechanical context such localized configurations could still belong to a Hilbert space even if the basis vectors themselves out of which they are built do not.

While Eq. (33) is given in closed form, the actual sum over modes is itself done numerically.

This particular completeness test was carried out in collaboration with Dr. A. H. Guth, Dr. D. I. Kaiser and Dr. A. Nayeri, and grew out of a study of brane-world fluctuations in which they were engaged with one of us (PDM).

The zeroes of \( J_1(y) \) and \( Y_1(y) \) are simple, discrete ones which interlace each other, with first three positive zeroes of \( J_1(y) \) for instance occurring at 3.832, 7.016 and 10.173, and with the \( n \)-th positive zero being well-approximated by \( j_n \approx (n + 1/4)\pi \) when \( n \) is large; while the first three positive zeroes of \( Y_1(y) \) occur at 2.197, 5.430 and 8.596, with the \( n \)-th positive zero being well-approximated by \( y_n \approx (n - 1/4)\pi \) when \( n \) is large. While these particular approximations do not hold at small \( n \), the \( n \) which appears in the \( j_n \approx (n + 1/4)\pi \) and \( y_n \approx (n - 1/4)\pi \) expressions does denote the number of the zero (counting the first positive zero as \( n = 1 \)), so that for \( n \) large or small these expressions give a correct counting of the number of zeroes.

The reconstruction of the square step using the divergent mode basis is so good that the only perceptible difference between Figs. (1) and (2) is that in the region close to \( e^{-w} = 0 \) the \( J_2(me^{-b}w/b) \) contribution is ever so slightly thicker. (The constraints of Eq. (37) force a more rapid convergence on the \( Y_2(ne^{-b}w/b) \) mode sum.)

For the \( M_4^- \) brane world this is just as well, since it could otherwise not contain any massless graviton.

The typical case of \( \tanh \sigma = 0.9 \) (viz. \( H/b = 0.436 \)) yields \( \nu = 1.088, 2.216 \) and 3.362 as the three lowest positive solutions to \( P_1^n(-\tanh \sigma) = 0 \), with the \( n \)-th positive zero being well-approximated by \( \nu_n \approx (n + 1/4)\pi/\arccos(-\tanh \sigma) - 1/2 \) when \( n \) is large.

[15] It is possible that this might perhaps be the first time that the Gibbs phenomenon has explicitly been demonstrated for associated Legendre functions, and especially for the divergent $Q^2_\nu(y)$ modes which we show below.

[16] The typical case of $\tanh\sigma = 0.9$ yields $\nu = 0.536, 1.649$ and $2.788$ as the three lowest positive solutions to $Q^1_\nu(-\tanh\sigma) = 0$, with the $n$-th positive zero being well-approximated by $\nu_n \approx (n - 1/4)\pi/\arccos(-\tanh\sigma) - 1/2$ when $n$ is large, with the zeroes of $P^1_\nu(-\tanh\sigma) = 0$ and $Q^1_\nu(-\tanh\sigma) = 0$ thus interlacing each other. As regards the $Q^1_\nu(-\tanh\sigma) = 0$ solutions, we note additionally that the lowest positive one actually corresponds to an $m^2 < 0$ tachyon since it has $\nu < 1$.

[17] The construction is so good that the only perceptible difference between Figs. (3) and (4) is that in the regions close to the edges of the steps the Gibbs phenomenon overshoot as shown in the Fig. (3) blow-up is ever so slightly closer to 1.1 than the one shown in the blow-up of Fig. (4).

[18] Since the negative tension $AdS^-_4$ brane world with divergent warp factor $e^{A(|w|)} = H\cosh(\sigma + b|w|)/b$ also has convergent $P^2_\nu(y)$ and divergent $Q^2_\nu(y)$ modes (where now $y = \tanh(b|w| + \sigma)$ with range $\tanh\sigma \leq y \leq 1$), its structure is analogous to that of the divergent warp factor $AdS^+_4$ world, and so we do not seek completeness tests for it here.


[20] With the arbitrary hypergeometric function $F(a, b; c, z)$ being equal to one when its argument $z$ is taken to be zero, the large $y$ limits of $P^2_\nu(y)$ and $Q^2_\nu(y)$ are readily obtained from their $|y| > 1$ hypergeometric function representations of the form $P^\mu_\nu(y) = 2^{\nu+1}\Gamma(-2\nu-1)\Gamma^{-1}(-\nu)\Gamma^{-1}(-\nu-\mu)(y+1)^{-\mu/2}\Gamma(y+1)^{-\nu/2}F(\nu+1, \nu-\mu+1; 2\nu+2, 2/(1+y)) + 2^{-\nu}\Gamma(2\nu+1)\Gamma^{-1}(\nu+1)\Gamma^{-1}(\nu-\mu+1)(y+1)^{-\mu/2}\Gamma(y+1)^{-\nu/2}F(-\nu, -\nu-\mu; -2\nu, 2/(1+y))$, $Q^\mu_\nu(y) = e^{i\mu\pi/2}2^{-\nu-1}\pi^{1/2}\Gamma(\nu+\mu+1)\Gamma^{-1}(\nu+3/2)y^{-\nu-\mu-1}(y^2-1)^{\nu/2}F(\nu/2+\mu/2+1, \nu/2+\mu/2+1/2; \nu+3/2, 1/y^2)$. While these representations show that $P^\mu_\nu(y)$ and $Q^\mu_\nu(y)$ will in general be complex in the $|y| > 1$ region, the form for $P^\mu_\nu(y)$ shows that it will actually be real when $y$ and $\mu$ are real and the parameter $\nu$ takes the value $\nu = -1/2 + i\lambda$ where $\lambda$ is real, a value for which the quantity $\nu(\nu+1) = (\nu+1/2)^2 - 1/4$ which appears in the defining equation for the associated Legendre functions of Eq. (64) is then given as the real $\nu(\nu+1) = -\lambda^2 - 1/4$. With Eq. (64) remaining real at $\nu = -1/2 + i\lambda$, for such values of $\nu$ the then real $P^2_\nu(y)$ and the real and imaginary parts of $Q^2_\nu(y)$ will all separately obey it. However, since Eq. (64) can only have
two independent solutions, it must be the case that one of these three classes of solutions is redundant. On noting that no matter what the value of \( \nu \), the divergent part of \( Q^2_\nu(y) \) at \( y = 1 \) is real while \( P^2_\nu(y) \) is well-behaved there, we thus anticipate that when \( y \) is real and greater than one, it must be the (thus well-behaved at \( y = 1 \)) imaginary part of \( Q^2_{\nu+1,i\lambda}(y) \) which must coincide with the real \( P^2_{\nu-1,2+i\lambda}(y) \); and since it is not immediately obvious how one may explicitly check such a connection analytically, we have instead confirmed it numerically. In the following then we can restrict the discussion to the use of \( P^2_{\nu-1,2+i\lambda}(y) \) and \( \text{Re}[Q^2_{\nu-1,2+i\lambda}(y)] \) as basis modes (in both the \( dS^+_4 \) and the \( dS^-_4 \) brane worlds). As well as enabling us to show that \( P^\mu_\nu(y) \) is real for real \( y \), real \( \mu \) and complex \( \nu = -1/2 + i\lambda \), the above representations of the \( P^\mu_\nu(y) \) and \( Q^\mu_\nu(y) \) are also of use for actual computational purposes when \( y \) is greater than one, since for argument \(|z| < 1\) a hypergeometric function can be represented as the absolutely convergent power series \( F(a, b; c, z) = [\Gamma(c)/\Gamma(a)\Gamma(b)] \sum_{n=0}^{\infty} \Gamma(a+n)\Gamma(b+n)z^n/\Gamma(c+n)n! \). Moreover, for large values of the parameter \( \lambda \), the functions \( P^\mu_{\nu-1,2+i\lambda}(y) \) and \( \text{Re}[Q^\mu_{\nu-1,2+i\lambda}(y)] \) can even be approximated by \( P^\mu_{\nu-1,2+i\lambda}(\text{cosh}\theta) = \lambda^{\nu-1/2}(2/\pi\sinh\theta)^{1/2}\cos(\theta + \mu\pi/2 - \pi/4) - \lambda^{\nu-3/2}(2/\pi\sinh\theta)^{1/2}(\mu - 1/2)(\mu + 1/2)\coth\sin(\theta + \mu\pi/2 - \pi/4) \) and \( \text{Re}[Q^\mu_{\nu-1,2+i\lambda}(\text{cosh}\theta)] = \lambda^{\nu-1/2}(2/\pi\sinh\theta)^{1/2}\cos(\theta + \mu\pi/2 + \pi/4) - \lambda^{\nu-3/2}(2/\pi\sinh\theta)^{1/2}(\mu - 1/2)(\mu + 1/2)\coth\sin(\theta + \mu\pi/2 + \pi/4) \). (It is necessary to carry the first non-leading terms here since the oscillatory leading terms can vanish at some specific \( \theta \) values.)

[21] With the \( dS^-_4 \) brane world range for \( y \) being restricted to the finite range \( 1 \leq y \leq \text{coth}\sigma \), in cases in which we restrict to \( \text{coth}\sigma < 3 \), we are actually able to use an extremely compact representation for evaluation of \( P^2_\nu(y) \), \( P^1_\nu(y) \) and \( P_\nu(y) \), viz. the form \( P^m_\nu(y) = (y^2 - 1)^{m/2}\Gamma(\nu + m + 1)F(-\nu + m, \nu + m + 1, m + 1, 1 - y)/[2^m m!\Gamma(\nu - m + 1)] \) which holds for any positive integer \( m \), and the limiting form \( P_\nu(y) = F(-\nu, \nu + 1, 1, 1 - y)/2 \) which holds when \( m = 0 \), as each of these hypergeometric function representations can be written as a power series which is absolutely convergent over the entire \( 1 \leq y \leq 3 \) range. From these representation we find in a typical case with \( \text{coth}\sigma = 1.1 \) that the three lowest positive \( \lambda \) solutions to \( P^1_{\nu-1,2+i\lambda}(\text{coth}\sigma) = 0 \) are given as \( \lambda = 8.624, 15.808 \) and \( 22.930 \), with the \( n \)-th positive solution being well-approximated by \( \lambda_n \approx (n + 1/4)\pi/\arccosh(\text{coth}\sigma) \) when \( n \) is large.

[22] The typical case of \( \text{coth}\sigma = 1.1 \) yields \( \lambda = 4.928, 12.231 \) and \( 19.373 \) as the three lowest positive \( \lambda \) solutions to \( \text{Re}[Q^1_{\nu-1,2+i\lambda}(\text{coth}\sigma)] = 0 \), with the \( n \)-th positive solution being well-approximated by \( \lambda_n \approx (n - 1/4)\pi/\arccosh(\text{coth}\sigma) \) when \( n \) is large.
Despite the fact that the negative tension $M^-_4$ brane world possesses a massless graviton whose residue is finite, we note that its residue appears with an overall minus sign (viz. negative signature) compared to the otherwise identical in structure positive signature massless graviton residue of the positive tension $M^+_4$ brane world (compare the first forms given for $h^{TT}_{\mu\nu}(x,|w|)$ given in Eqs. (88) and (92) which differ by an overall minus sign occasioned by the overall difference in sign between the right-hand sides of Eqs. (87) and (91)). Such negative signature is thought to indicate an instability of the $M^-_4$ brane world. Nonetheless, even though the $M^-_4$ brane world might thus not be of direct physical interest, it can still serve as a useful mathematical laboratory for exploring the completeness properties of bases built out of non-normalizable modes.

Unlike the massless graviton of the negative tension $M^-_4$ brane world, the massless graviton of the positive tension $AdS_4^+$ brane world has a residue with a perfectly acceptable positive signature – as indeed it must since its residue continues into that of the positive signature massless graviton of the positive tension $M^+_4$ brane world in the limit in which $H$ is taken to zero.

From the perspective of the possible physical viability of the $AdS_4^+$ brane world, the need to include non-normalizable modes is actually somewhat unfortunate since they lead to a gravity which is not at all localized to the brane. It was the fact that a restriction to normalized modes did lead to gravitational fluctuation modes which were localized to the brane which prompted the $AdS_4^+$ study of [14].